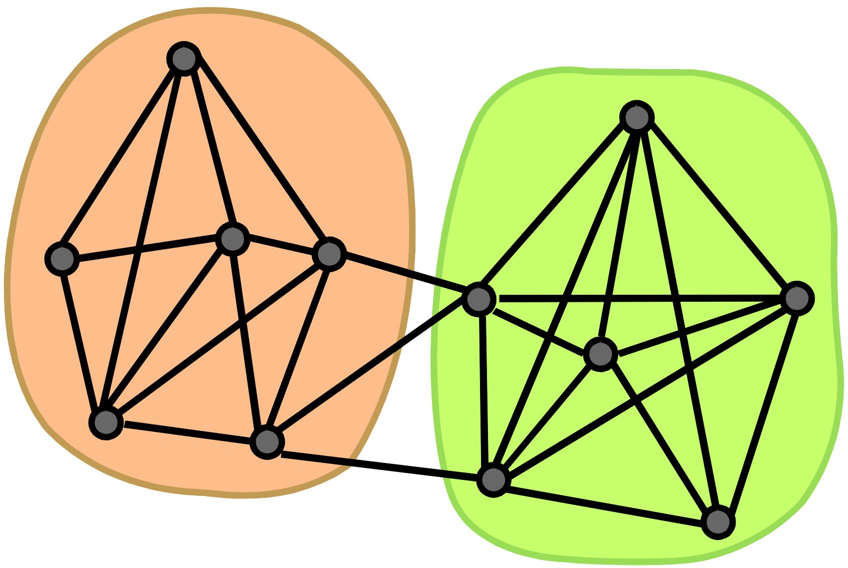
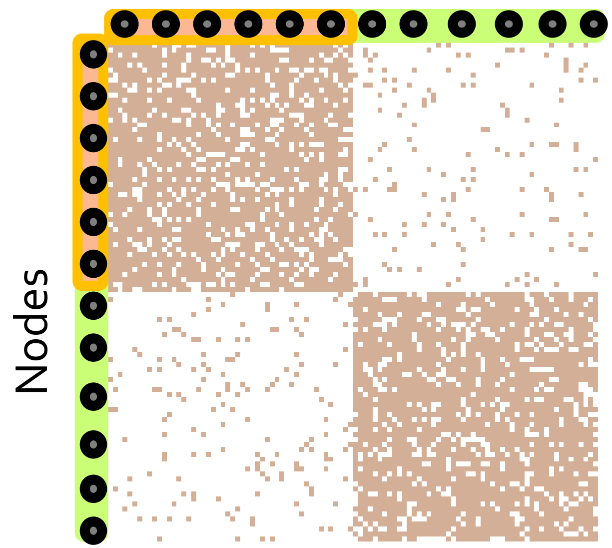
Spectral Graph Partitioning: Graph Laplacian Matrix

Mining of Massive Datasets
Leskovec, Rajaraman, and Ullman
Stanford University

Finding Clusters



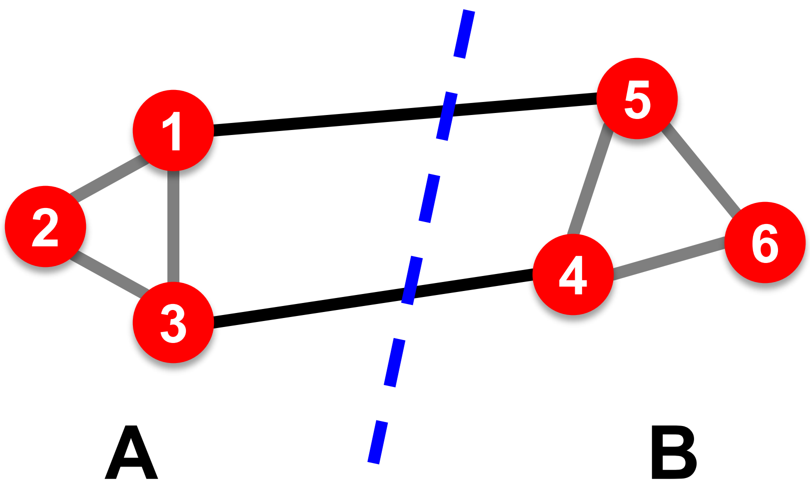
Network



Adjacency matrix

Graph Partitioning

- **Task:** Partition the graph into two pieces such the resulting pieces have low conductance



- **How do we efficiently find a good partition?**
  - **Problem:** Computing optimal cut is NP-hard

Spectral Graph Partitioning

- A: adjacency matrix of undirected **G**
  - A_{ij} **=1** if $(\boldsymbol{i}, \boldsymbol{j})$ is an edge, else **0**
- $\boldsymbol{x}$ is a vector in $\Re^n$ with components $(\boldsymbol{x_1}, \dots, \boldsymbol{x_n})$
  - Think of it as a label/value of each node of $\boldsymbol{G}$
- **What is the meaning of $A \cdot x$?**

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$y_i = \sum_{j=1}^{n} A_{ij} xj = \sum_{(i,j) \in E} x_j$$

- **Entry y_i is a sum of labels x_j of neighbors of i**

What is the meaning of $A \cdot x$?

- j^{th} **coordinate of $A \cdot x$:**
  - Sum of the x-values of neighbors of j
  - Make this a new value at node j

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\boldsymbol{A \cdot x = \lambda \cdot x}$$

- **Spectral Graph Theory:**
  - Analyze the “spectrum” of matrix representing $\boldsymbol{G}$
  - **Spectrum:** Eigenvectors $\boldsymbol{x_i}$ of a graph, ordered by the magnitude (strength) of their corresponding eigenvalues $\boldsymbol{\lambda_i}$:

$$\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$$

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$$