Why Teleports Solve the Problem?

$$r^{(t+1)} = Mr^{(t)}$$

Markov chains

- Set of states ***X***
- Transition matrix ***P*** where $P_{ij} = P(X_t = i \mid X_{t-1} = j)$
- **π** specifying the stationary probability of being at each state $x \in X$
- Goal is to find **π** such that **π = P π**

Why is This Analogy Useful?

- **Theory of Markov chains**

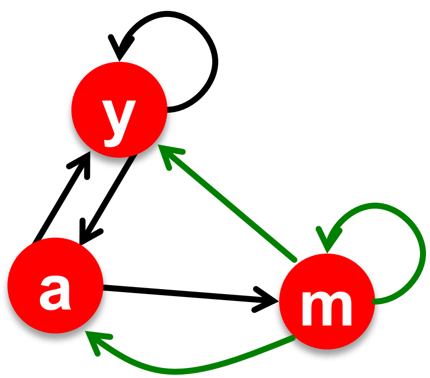
- **Fact:** For **any start vector**, the power method applied to a Markov transition matrix ***P*** will **converge** to a **unique** positive stationary vector as long as ***P*** is **stochastic**, **irreducible** and **aperiodic**.

Make M Stochastic

- **Stochastic:** Every column sums to **1**
- **Solution:** Add **green** links

$$A = M + a^T\left(\frac{1}{n}e\right)$$

- a_i =1 if node *i* has out deg 0, =0 else
- **e** vector of all 1s



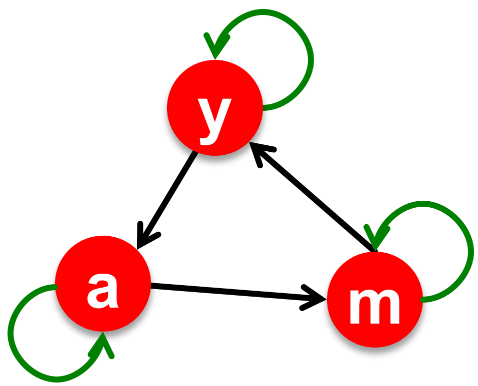
	y	a	m
y	½	½	1/3
a	½	0	1/3
m	0	½	1/3

$$r_y = r_y/2 + r_a/2 + r_m/3$$
$$r_a = r_y/2 + r_m/3$$
$$r_m = r_a/2 + r_m/3$$

Make M Aperiodic

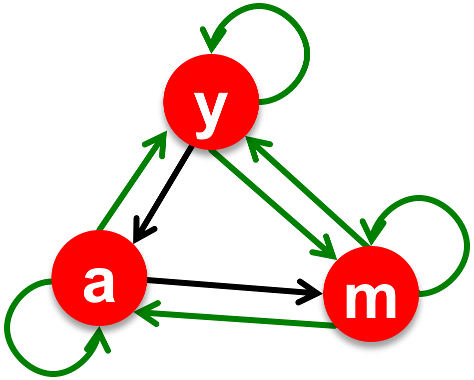
- A chain is **periodic** if there exists $k > 1$ such that the interval between two visits to some state s is always a multiple of k

- **Solution:** Add **green** links



Make *M* Irreducible

- From any state, there is a non-zero probability of going from any one state to any another
- **Solution:** Add **green** links



Solution: Random Jumps

- **Google's solution that does it all:**
  - Makes ***M*** **stochastic, aperiodic, irreducible**
- **At each step, random surfer has two options:**
  - With probability β, follow a link at random
  - With probability $1-\beta$, jump to some random page
- **PageRank equation** [Brin-Page, 98]

$$r_j = \sum_{i \to j} \beta \frac{r_i}{d_i} + (1-\beta)\frac{1}{n}$$

d_i ... out-degree of node i

The above formulation assumes that M has no dead ends. We can either preprocess matrix M **(bad!)** or explicitly follow random teleport links with probability 1.0 from dead-ends.

The Google Matrix

- **PageRank equation** [Brin-Page, 98]

$$r_j = \sum_{i \to j} \beta \frac{r_i}{d_i} + (1-\beta)\frac{1}{n}$$

- **The Google Matrix *A*:**

$$A = \beta\, M + (1-\beta)\frac{1}{n}\boldsymbol{e}\cdot\boldsymbol{e}^T$$

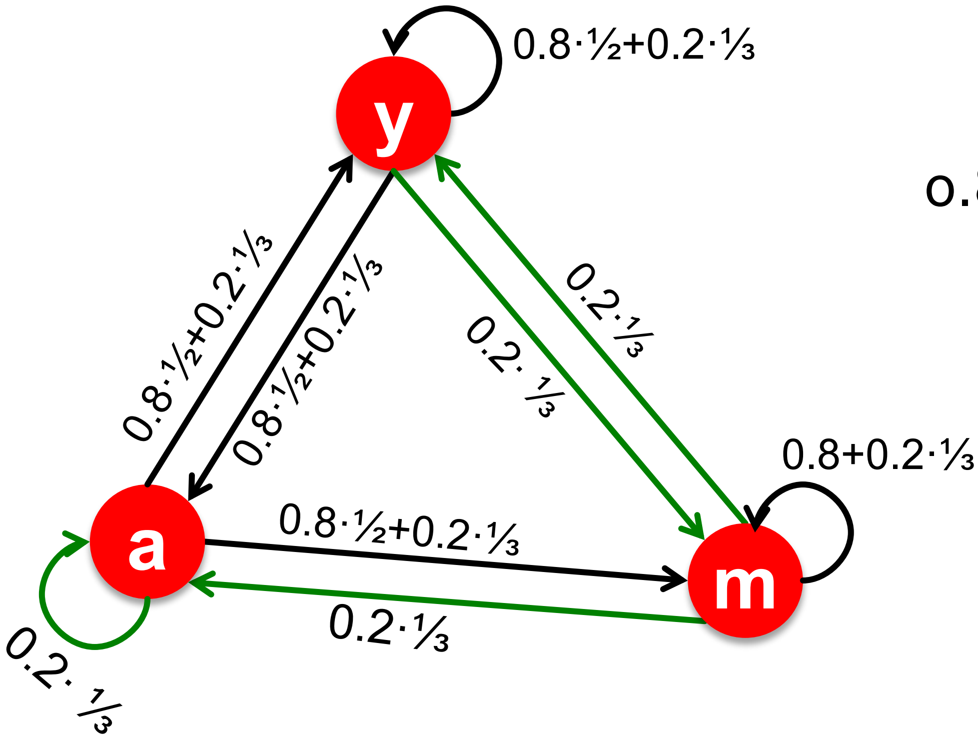
e vector of all 1s

- ***A* is stochastic, aperiodic and irreducible, so**

$$\boldsymbol{r}^{(t+1)} = \boldsymbol{A}\cdot\boldsymbol{r}^{(t)}$$

- **What is β ?**
  - In practice $\beta = 0.8, 0.9$ (make 5 steps and jump)

Random Teleports ($\beta = 0.8$)



$$\underbrace{0.8\begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 1 \end{bmatrix}}_{\mathbf{M}} + 0.2\underbrace{\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}}_{1/n\cdot\mathbf{1}\cdot\mathbf{1}^T}$$

$$\mathbf{A} = \begin{array}{c} y \\ a \\ m \end{array}\begin{bmatrix} 7/15 & 7/15 & 1/15 \\ 7/15 & 1/15 & 1/15 \\ 1/15 & 7/15 & 13/15 \end{bmatrix}$$

$$\begin{array}{c} y \\ a \\ m \end{array} = \begin{array}{c} 1/3 \\ 1/3 \\ 1/3 \end{array} \quad \begin{array}{c} 0.33 \\ 0.20 \\ 0.46 \end{array} \quad \begin{array}{c} 0.24 \\ 0.20 \\ 0.52 \end{array} \quad \begin{array}{c} 0.26 \\ 0.18 \\ 0.56 \end{array} \quad \cdots \quad \begin{array}{c} 7/33 \\ 5/33 \\ 21/33 \end{array}$$