Theory of LSH

Distance Measures
LS Families of Hash Functions
S-Curves
LSH for Different Distance Measures

Mining of Massive Datasets

Leskovec, Rajaraman, and Ullman

Stanford University

Distance Measures

- Generalized LSH is based on some kind of "distance" between points.
  - Similar points are "close."
- Jaccard similarity is not a distance; 1 minus Jaccard similarity is.
- Two major classes of distance measure:

1. *Euclidean*
2. *Non-Euclidean*

Euclidean Vs. Non-Euclidean

- A *Euclidean space* has some number of real-valued dimensions and "dense" points.
  - There is a notion of "average" of two points.
  - A *Euclidean distance* is based on the locations of points in such a space.
- Any other space is *Non-Euclidean*.
  - Distance measures for non-Euclidean spaces are based on properties of points, but not their "location" in a space.

Axioms of a Distance Measure

- *d* is a *distance measure* if it is a function from pairs of points to real numbers such that:
  1. $d(x,y) \geq 0$.
  2. $d(x,y) = 0$ iff $x = y$.
  3. $d(x,y) = d(y,x)$.
  4. $d(x,y) \leq d(x,z) + d(z,y)$ (*triangle inequality*).

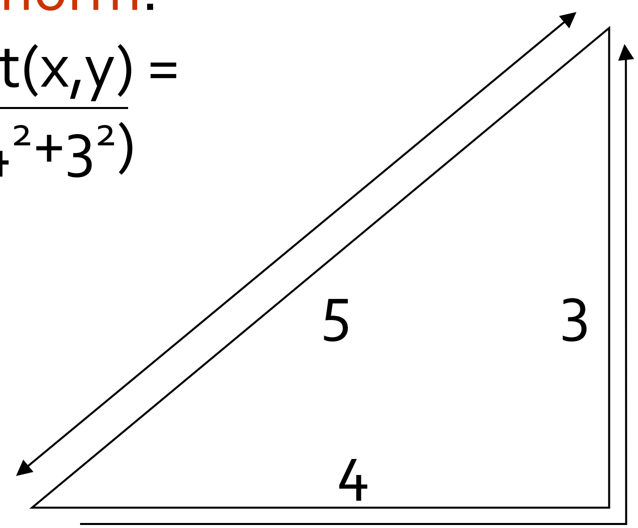
Some Euclidean Distances

- *L_2 norm*: d(x,y) = square root of the sum of the squares of the differences between *x* and *y* in each dimension.
  - The most common notion of "distance."
- *L_1 norm*: sum of the differences in each dimension.
  - *Manhattan distance* = distance if you had to travel along coordinates only.

Examples of Euclidean Distances

b = (9,8)

L_2-norm:
dist(x,y) =
$\sqrt{(4^2+3^2)}$
= 5



L_1-norm:
dist(x,y) =
4+3 = 7

a = (5,5)

More Euclidean Distances

- *L_∞ norm*: d(x,y) = the maximum of the differences between *x* and *y* in any dimension.
- Note: the maximum is the limit as *r* goes to ∞ of the L_r norm: what you get by taking the r^{th} power of the differences, summing and taking the r^{th} root.

Non-Euclidean Distances

- *Jaccard distance* for sets = 1 minus Jaccard similarity.
- *Cosine distance* for vectors = angle between the vectors.
- *Edit distance* for strings = number of inserts and deletes to change one string into another.
- *Hamming Distance* for bit vectors = the number of positions in which they differ.

Example: Jaccard Distance

- Consider $x = \{1,2,3,4\}$ and $y = \{1,3,5\}$
- Size of intersection = 2; size of union = 5, Jaccard similarity (not distance) = 2/5.
- $d(x,y) = 1 - (\text{Jaccard similarity}) = 3/5$.

Why J.D. Is a Distance Measure

- $d(x,y) \geq 0$ because $|x \cap y| \leq |x \cup y|$.
- $d(x,x) = 0$ because $x \cap x = x \cup x$.
  - And if $x \neq y$, then the size of $x \cap y$ is strictly less than the size of $x \cup y$.
- $d(x,y) = d(y,x)$ because union and intersection are symmetric.
- $d(x,y) \leq d(x,z) + d(z,y)$ trickier – next slide.

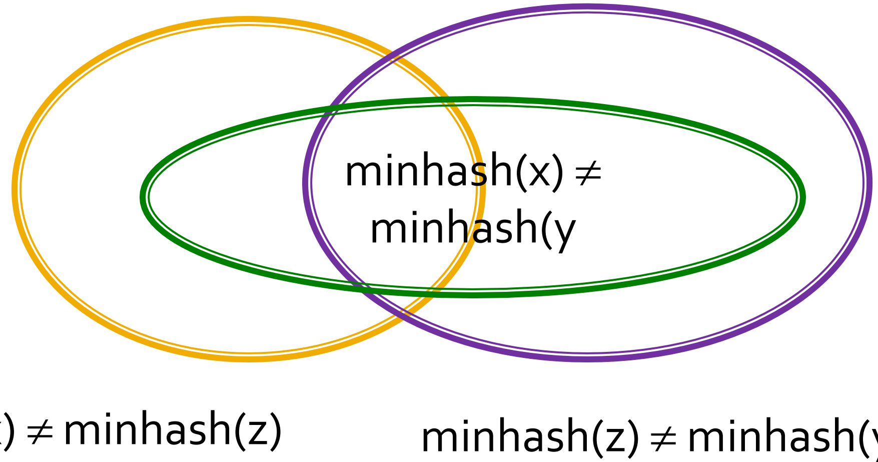
Triangle Inequality for J.D.

$$1 - \frac{|x \cap z|}{|x \cup z|} + 1 - \frac{|y \cap z|}{|y \cup z|} \geq 1 - \frac{|x \cap y|}{|x \cup y|}$$

- Remember: $|a \cap b|/|a \cup b|$ = probability that minhash(a) = minhash(b).
- Thus, $1 - |a \cap b|/|a \cup b|$ = probability that minhash(a) $\neq$ minhash(b).

Triangle Inequality – (2)

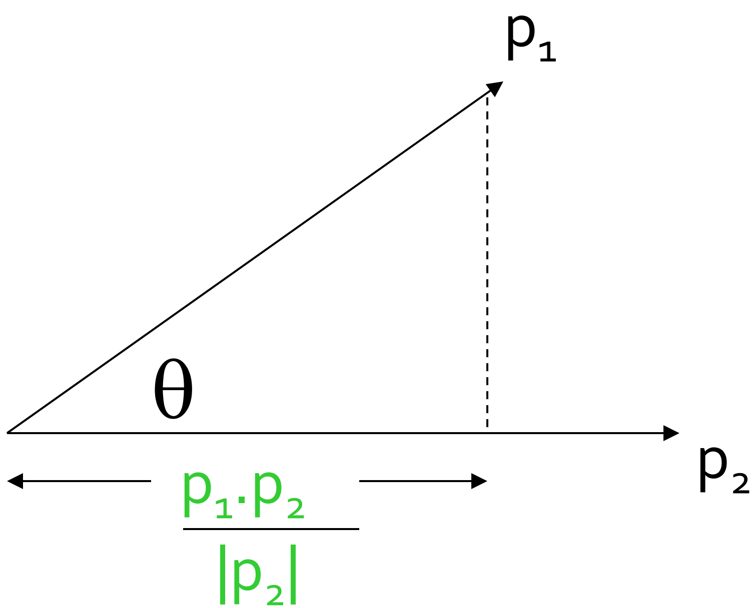
- Claim: prob[minhash(x) $\neq$ minhash(y)] $\leq$ prob[minhash(x) $\neq$ minhash(z)] + prob[minhash(z) $\neq$ minhash(y)]
- Proof: whenever minhash(x) $\neq$ minhash(y), at least one of minhash(x) $\neq$ minhash(z) and minhash(z) $\neq$ minhash(y) must be true.



Cosine Distance

- Think of a point as a vector from the origin (0,0,…,0) to its location.
- Two points' vectors make an angle, whose cosine is the normalized dot-product of the vectors: $p_1.p_2/|p_2||p_1|$.
  - Example: $p_1 = 00111$; $p_2 = 10011$.
  - $p_1.p_2 = 2$; $|p_1| = |p_2| = \sqrt{3}$.
  - $\cos(\theta) = 2/3$; θ is about 48 degrees.

Cosine-Measure Diagram



$d\ (p_1, p_2) = \theta = \arccos(p_1 . p_2 / |p_2||p_1|)$

Why C.D. Is a Distance Measure

- d(x,x) = 0 because arccos(1) = 0.
- d(x,y) $\geq$ 0 because any two intersecting vectors make an angle in the range 0 to 180 degrees.
- d(x,y) = d(y,x) by symmetry.
- Triangle inequality: physical reasoning. If I rotate an angle from *x* to *z* and then from *z* to *y*, I can't rotate less than from *x* to *y*.

Edit Distance

- The *edit distance* of two strings is the number of inserts and deletes of characters needed to turn one into the other.
- An equivalent definition: $d(x,y) = |x| + |y| - 2|LCS(x,y)|$.
  - LCS = *longest common subsequence* = any longest string obtained both by deleting from *x* and deleting from *y*.

Example

- x = *abcde* ; y = *bcduve*.
- Turn x into y by deleting *a*, then inserting *u* and *v* after *d*.
  - Edit distance = 3.
- Or, LCS(x,y) = *bcde*.
- Note: |x| + |y| - 2|LCS(x,y)| = 5 + 6 –2*4 = 3 = edit distance.

Why Edit Distance Is a Distance Measure

- d(x,x) = 0 because 0 edits suffice.
- d(x,y) $\geq$ 0: no notion of negative edits.
- d(x,y) = d(y,x) because insert/delete are inverses of each other.
- Triangle inequality: changing *x* to *z* and then to *y* is one way to change *x* to *y*.

Hamming Distance

- *Hamming distance* is the number of positions in which bit-vectors differ.
- Example: $p_1 = 10101$; $p_2 = 10011$.
- $d(p_1, p_2) = 2$ because the bit-vectors differ in the 3^{rd} and 4^{th} positions.

Why Hamming Distance Is a Distance Measure

- d(x,x) = 0 since no positions differ.
- d(x,y) $\geq$ 0 since strings cannot differ in a negative number of positions.
- d(x,y) = d(y,x) by symmetry of "different from."
- Triangle inequality: changing *x* to *z* and then to *y* is one way to change *x* to *y*.