IE 529 - Homework 1.2: Random variables

Due Friday, Sept. 9

I. Let $X_1, X_2, \dots, X_n$ be a set of **i.i.d.** random variables, with $X_i \in \mathcal{N}(\mu, \sigma)$, (i.e., all r.v.'s are normally distributed, with mean μ and variance σ^2) and μ, σ both finite. Suppose s^2 is the sample variance of the $\{X_i\}$.

a. Show that the random var. defined as

$$W := \frac{(n-1)s^2}{\sigma^2} \quad \text{is } \chi^2_{(n-1)} - \text{distributed.}$$

Proof: Define $Y_i := \frac{x_i - \mu}{\sigma}$, $\hat{x} := \frac{1}{n}\sum_{i=1}^n X_i$, and

$$\bar{Y} := \frac{1}{n}\sum_{i=1}^n Y_i = \frac{\hat{x} - \mu}{\sigma}.$$

By construction, $Y_i \in N(0,1)$. So

$$\sum_{i=1}^n (Y_i - \bar{Y})^2 = \sum_{i=1}^n \left(\frac{X_i - \hat{x}}{\sigma}\right)^2 = \frac{(n-1)s^2}{\sigma^2}.$$

The joint density of $Y_1, \dots, Y_n$ is $f(y) = \frac{1}{\sqrt{(2\pi)^n}} exp\left\{-\frac{1}{2}\sum_{i=1}^n y_i^2\right\}$. The function f has spherical symmetry, that is, if $A = [a_{ij}]$ is an orthogonal rotation of $\mathbb{R}^n$ and

$$Y_i = \sum_{i=1}^n Z_j a_{ji} \quad \text{and} \quad \sum_{i=1}^n Y_i^2 = \sum_{i=1}^n Z_i^2, \tag{1}$$

then $Z_1, \dots, Z_n$ are independent $N(0,1)$ variables also. Now choose

$$Z_1 = \frac{1}{\sqrt{n}}\sum_{i=1}^n Y_i = \sqrt{n}\bar{Y} \tag{2}$$

With a little effort it can be seen that $Z_1 \in N(0,1)$. Then let $Z_2, \dots, Z_n$ be any collection of variables such that (1) holds, where A is orthogonal. From (1)-(2),

$$\begin{aligned}
\sum_{i=2}^n Z_i^2 &= \sum_{i=1}^n Y_i^2 - \frac{1}{n}\left(\sum_{i=1}^n Y_i^2\right)^2 \\
&= \sum_{i=1}^n Y_i^2 - \frac{2}{n}\sum_{i=1}^n\sum_{j=1}^n Y_i Y_j + \frac{1}{n^2}\sum_{i=1}^n\left(\sum_{j=1}^n Y_j^2\right)^2 \\
&= \sum_{i=1}^n \left(Y_i - \frac{1}{n}\sum_{j=1}^n Y_j\right)^2 = \frac{(n-1)s^2}{\sigma^2} = W.
\end{aligned}$$

Therefore W is the sum of squares $n-1$ independent $N(0,1)$ variables and thus is $\chi^2_{(n-1)}-$ distributed. ■

b. Show that the random var. defined as

$$U := \frac{\hat{x}-\mu}{s/\sqrt{n}} \quad \text{is } T_{(n-1)} - \text{distributed.}$$

(Note: $\hat{x}$ denotes the sample mean).

Proof: Note

$$U = \frac{\hat{x}-\mu}{s/\sqrt{n}} = \frac{\hat{x}-\mu}{\sigma/\sqrt{n}}\frac{\sigma}{s}.$$

From Problem I., namely (2), we know that $\frac{\hat{x}-\mu}{\sigma/\sqrt{n}} = Z_1 \in N(0,1)$. So

$$U = \frac{Z_1}{\sqrt{s^2/\sigma^2}} = \frac{Z_1}{\sqrt{\frac{1}{n-1}\frac{(n-1)s^2}{\sigma^2}}}.$$

Since from Problem I. $\frac{(n-1)s^2}{\sigma^2} \in \chi^2_{(n-1)}$−distributed, we have that

$$U = \frac{Z_1}{\sqrt{\frac{\chi^2_{(n-1)}}{n-1}}}.$$

Therefore U is $T_{(n-1)}$−distributed. ∎

II. Let $X_1, X_2, \ldots, X_{n_1}$ be a set of **i.i.d.** random variables, with $X_i \in \mathcal{N}(\mu_1, 1)$, and let $Y_1, Y_2, \ldots, Y_{n_2}$ be a set of **i.i.d.** random variables, with $Y_i \in \mathcal{N}(\mu_2, 1)$, and further suppose X_i and Y_j are independent for all i, j. Define a new random var. as

$$W := \sum_{i=1}^{n_1}(X_i - \hat{x})^2 \; + \; \sum_{i=1}^{n_2}(Y_i - \hat{y})^2.$$

a. What is the distribution of W?

Note this is just an application of Problem I. a.

$$W \in \chi^2_{(n_1+n_2-2)} - \text{distributed.}$$

b. What is $E(W)$? and $\mathrm{Var}(W)$?

$$E(W) = n_1 + n_2 - 2 \text{ and } \mathrm{Var}(W) = 2(n_1 + n_2 - 2).$$

III. Suppose X is exponentially distributed with mean λ (i.e., $f_X(x) = \frac{1}{\lambda}e^{-\frac{1}{\lambda}x}$ for $x \geq 0$; 0 elsewhere). Define a new random var. as

$$Y := \frac{2X}{\lambda}.$$

Show that Y has a $\chi^2_{(2)}$ - distribution.

Proof: First find the cdf of Y:

$$P(Y \leq y) = P(X \leq \frac{\lambda}{2}y) = \int_0^{\frac{\lambda}{2}y} \frac{1}{y}e^{-\frac{1}{\lambda}x}dx = 1 - e^{-\frac{1}{\lambda}\frac{\lambda}{2}y} = 1 - e^{-\frac{1}{2}y}.$$

So the pdf is:

$$f_y(y) = \frac{1}{2}e^{-\frac{1}{2}y}.$$

This is the pdf of $\chi^2_{(2)}$ distribution. ■

IV. Let $\{X_i\},\ i = 1, 2, \ldots, n, \ldots$ be independent Poisson random variables with respective rates $\{\lambda_i\},\ i = 1, 2, \ldots, n, \ldots$. Show that if

$$\sum_{i=1}^{\infty} \lambda_i \text{ converges,}$$

then

$$\sum_{i=1}^{\infty} X_i \text{ converges a.s.}$$

Proof: Let $S_n = \sum_{i=1}^{n} X_i$. The sum of independent Poisson r.v.'s is again a Poisson r.v. Therefore S_n is a Poisson r.v. with mean $\sum_{i=1}^{n} \lambda_i$, i.e. $E(S_n) = \sum_{i=1}^{n} \lambda_i$. So

$$\lim_{n\to\infty} E(S_n) = \lim_{n\to\infty} \sum_{i=1}^{n} \lambda_i$$

By the problem statement, the RHS converges and we can call it λ.

So

$$\lim_{n\to\infty} E(\sum_{i=1}^{n} X_i) = \lambda.$$

Assuming we can exchange the infinite sum and expectation,

$$E(\lim_{n\to\infty}\sum_{i=1}^{n} X_i) = \lambda.$$

By definition,

$$\lim_{n\to\infty}\sum_{i=1}^{n} X_i = \sum_{i=1}^{\infty} X_i.$$

Therefore

$$E(\sum_{i=1}^{n} X_i) = \lambda,$$

which implies

$$P(\sum_{i=1}^{n} X_i < \infty) = 1.$$

$\blacksquare$

V. Suppose $X_n \xrightarrow{d.} X$ and $Y_n \xrightarrow{p.} c$ where c is a constant.
Show:

$$X_n Y_n \xrightarrow{d.} cX.$$

Proof: Note

$$\begin{aligned} P(X_nY_n \le z) &= P(cX_n + X_n(Y_n - c) \le z) \\ &= P(cX_n + X_n(Y_n - c) \le z, |Y_n - c| \le \epsilon) + P(cX_n + X_n(Y_n - c) \le z, |Y_n - c| > \epsilon). \end{aligned}$$

Since $Y_n \xrightarrow{p.} c$, $\lim\limits_{n\to\infty} P(|Y_n - c| > \epsilon) = 0$. This implies $P(|Y_n - c| \le \epsilon) = 1$. Therefore

$$\lim_{n\to\infty} P(X_nY_n \le z) = P(cX_n \le z).$$

Note that since $X_n \xrightarrow{p.} X$, $\lim\limits_{n\to\infty} P(X_n \le x) = P(X \le x)$,

$$\begin{aligned} \lim_{n\to\infty} P(cX_n \le z) &= \lim_{n\to\infty} P\left(X_n \le \frac{z}{c}\right) \\ &= P\left(X \le \frac{z}{c}\right) \\ &= P\left(cX \le z\right). \end{aligned}$$

Therefore $X_nY_n \xrightarrow{d.} cX$. $\blacksquare$

VI. Suppose $X_1, X_2, \ldots, X_n, \ldots$ is a series of random variables, where $|X_i| \le Y$, $\forall i$, and where $E(Y) < \infty$.

Show: If

$$X_n \xrightarrow{p.} X,$$

then $E(|X_n - X|) \to 0$ as $n \to \infty$.

Proof: To show this we just apply Fatou's lemma to both $(X_n - X)$ and to $(X - X_n)$; each is bounded below by $-2Y$. Consider the positive and negative parts $(X_n - X)_+ = max\{0, (X_n - X)\}$ and $(X_n - X)_- = max\{0, (X - X_n)\}$ separately; each is dominated by $2Y$ and converges to zero. Then use

$$E|X_n - X| \le E(X_n - X)_+ + E(X_n - X)_- \xrightarrow{n\to\infty} 0.$$

■

VII. True or False: Provide a brief explanation.

a. The standard deviation of the sample mean, $\hat{X}$, increases as the sample size increases.

False: by the Central Limit Theorem (CLT) the standard deviation decreases as n increases.

b. The CLT allows us to claim that the sample mean, $\hat{X}$, is normally distributed under certain assumptions.

True: this is practically the statement of the CLT.

c. The standard deviation of the sample mean, $\hat{X}$, is approximately equal to that for the population, σ.

False: the standard deviation of the sample mean is $\frac{\sigma}{\sqrt{n}}$, which for $n > 1$ is not equal to σ.

d. Suppose $X \in N(8, \sigma)$, then $P(\hat{X} > 4) < P(X > 4)$.

False: If $X \in N(8, \sigma)$, then $\hat{X} \in N(8, \frac{\sigma}{\sqrt{n}})$. Therefore

$$P(\hat{X} > 4) = P\left(\frac{\hat{X} - \mu}{\sigma\sqrt{n}} > \frac{4 - 8}{\sigma\sqrt{n}}\right) = P\left(Z > \frac{-4\sqrt{n}}{\sigma}\right),$$

which is greater than

$$P(X > 4) = P\left(Z > \frac{-4}{\sigma}\right).$$