IE 529 - Homework 2: Estimators. Matrix analysis.

Due Friday, September 29th

I. Estimators:

a. Determine the maximum likelihood estimator of θ when $X_1, X_2, \ldots, X_n$ is a random sample (i.i.d.) with density function

$$f(x) = \frac{1}{2}e^{-|x-\theta|}, \quad -\infty < x < \infty$$

b. Sketch of proof for Bayes' estimation:

Suppose $\overline{x}$ is the sample mean for a random sample of size n taken from a normal distribution with <u>unknown</u> mean (denoted μ) and <u>known</u> variance σ^2 (i.e., $x_1, x_2, \ldots, x_n \in \mathcal{N}(\mu, \sigma^2)$); further make the *prior* assumption that the distribution for the mean is also normal, i.e., $\mu \in \mathcal{N}(\nu, \rho^2)$.

Show that the *posterior* distribution for the population mean μ is also normal, with mean $\mu*$ and standard deviation σ^* given by

$$\mu^* = \left(\frac{\rho^2}{\rho^2 + \frac{\sigma^2}{n}}\right)\overline{x} + \left(\frac{\frac{\sigma^2}{n}}{\rho^2 + \frac{\sigma^2}{n}}\right)\nu; \text{ and } \sigma^* = \sqrt{\frac{\rho^2\sigma^2}{n\rho^2 + \sigma^2}}.$$

Hints: recall we know the following density functions apply:

$$f(x_1, x_2, \ldots, x_n|\mu) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \left(\frac{1}{\sigma}\right)^n \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2\right\};$$

$$f(\mu) = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \left(\frac{1}{\rho}\right) \exp\left\{-\frac{1}{2\rho^2}(\mu - \nu)^2\right\}.$$

The posterior distribution is $f(\mu|x_1, x_2, \ldots, x_n)$; show that this has a normal form, i.e., is $\mathcal{N}(\mu^*, \sigma^{*2})$.

II. Linear Algebra:

a. Show that the following statements are true:
For $x,\ y \in \mathbf{R}^n$,

$$\text{(i) } \|y\|_\infty = \max_{\|x\|_1 \neq 0} \left(\frac{|y^* x|}{\|x\|_1} \right), \qquad \text{(ii) } \|y\|_1 = \max_{\|x\|_\infty \neq 0} \left(\frac{|y^* x|}{\|x\|_\infty} \right)$$

b. Suppose $a_1, a_2, \ldots, a_n$ are fixed positive real numbers. Determine which of the following are proper vector norms on $\mathbf{R}^n$ (i.e., which of the following satisfy the four conditions required of functions to be vector norms).

1. $\|x\| := \max_i\{a_i|x_i|\}$
2. $\|x\| := \sum_{i=1}^n a_i|x_i|$

c. For this problem we will prove that the induced matrix 2-norm for a matrix $A \in \mathbf{R}^{m\times n}$ is given by the maximum singular value, $\sigma_1(A)$.

In particular, show that

$$\begin{aligned} \max f(x) &= \|Ax\|_2^2 = x^T A^T A x \\ \text{subj. to } x^T x &= 1 \end{aligned},$$

is given by σ_1^2. Hint: consider using the SVD of the matrix A.

III. SVD/PCA coding exercise (Matlab or Python): Please watch the course Compass site for details, which will be posted shortly.

IV. Seminar summaries:

a. Provide a summary of each talk attended (or viewed) from the *S*ymposium on Frontiers of Big Data. You should watch a minimum of 45 minutes of talks, and submit 3 paragraphs (min) summarizing the presentations.

b. Provide a summary of the session/talks attended from the *A*llerton Conference. You should either attend one of the tutorial sessions on Tuesday, or at least one full session (approx. 6 talks) at the conference (held at Allerton House, in Monticello). Please submit a 1/2 - 1 page summary of the presentation(s) you attended.

Please provide the titles and speaker names for all talks summarized.