IE 529 - Homework 1.2: Random variables

Due Wednesday, Sept. 7

I. Let $X_1, X_2, \ldots, X_n$ be a set of **i.i.d.** random variables, with $X_i \in \mathcal{N}(\mu, \sigma)$, (i.e., all r.v.'s are normally distributed, with mean μ and variance σ^2) and μ, σ both finite. Suppose s^2 is the sample variance of the $\{X_i\}$.

a. Show that the random var. defined as

$$W := \frac{(n-1)s^2}{\sigma^2} \quad \text{is } \chi^2_{(n-1)} - \text{distributed.}$$

b. Show that the random var. defined as

$$U := \frac{\hat{x} - \mu}{s/\sqrt{n}} \quad \text{is } T_{(n-1)} - \text{distributed.}$$

(Note: $\hat{x}$ denotes the sample mean).

II. Let $X_1, X_2, \ldots, X_{n_1}$ be a set of **i.i.d.** random variables, with $X_i \in \mathcal{N}(\mu_1, 1)$, and let $Y_1, Y_2, \ldots, Y_{n_2}$ be a set of **i.i.d.** random variables, with $Y_i \in \mathcal{N}(\mu_2, 1)$, and further suppose X_i and Y_j are independent for all i, j. Define a new random var. as

$$W := \sum_{i=1}^{n_1} (X_i - \hat{x})^2 \; + \; \sum_{i=1}^{n_2} (Y_i - \hat{y})^2.$$

a. What is the distribution of W?

b. What is $E(W)$? and $\mathrm{Var}(W)$?

III. Suppose X is exponentially distributed with mean λ (i.e., $f_X(x) = \frac{1}{\lambda} e^{-\frac{1}{\lambda}x}$ for $x \geq 0$; 0 elsewhere). Define a new random var. as

$$Y := \frac{2X}{\lambda}.$$

Show that Y has a $\chi^2_{(2)}$ - distribution.

IV. Let $\{X_i\},\ i = 1, 2, \ldots, n, \ldots$ be independent Poisson random variables with respective rates $\{\lambda_i\},\ i = 1, 2, \ldots, n, \ldots$. Show that if

$$\sum_{i=1}^{\infty} \lambda_i \text{ converges,}$$

then

$$\sum_{i=1}^{\infty} X_i \text{ converges a.s.}$$

V. Suppose $X_n \xrightarrow{d.} X$ and $Y_n \xrightarrow{p.} c$ where c is a constant.

Show:

$$X_n Y_n \xrightarrow{d.} cX.$$

VI. Suppose $X_1, X_2, \ldots, X_n, \ldots$ is a series of random variables, where $|X_i| \leq Y,\ \forall i$, and where $E(Y) < \infty$.

Show: If

$$X_n \xrightarrow{p.} X,$$

then $E(|X_n - X|) \to 0$ as $n \to \infty$.

VII. True or False: Provide a brief explanation.

a. The standard deviation of the sample mean, $\hat{X}$, increases as the sample size increases.

b. The CLT allows us to claim that the sample mean, $\hat{X}$, is normally distributed under certain assumptions.

c. The standard deviation of the sample mean, $\hat{X}$, is approximately equal to that for the population, σ.

d. Suppose $X \in N(8, \sigma)$, then $P(\hat{X} > 4) < P(X > 4)$.