IE 529 - Homework 2: Estimators. Matrix analysis.

Due Monday, October 3rd

I. Estimators:

a. Determine the maximum likelihood estimator of θ when $X_1, X_2, \ldots, X_n$ is a random sample (i.i.d.) with density function

$$f(x) = \frac{1}{2}e^{-|x-\theta|}, \quad -\infty < x < \infty$$

b. Sketch of proof for Bayes' estimation:

Suppose $\overline{x}$ is the sample mean for a random sample of size n taken from a normal distribution with <u>unknown</u> mean (denoted μ) and <u>known</u> variance σ^2 (i.e., $x_1, x_2, \ldots, x_n \in \mathcal{N}(\mu, \sigma^2)$); further make the *prior* assumption that the distribution for the mean is also normal, i.e., $\mu \in \mathcal{N}(\nu, \rho^2)$.

Show that the *posterior* distribution for the population mean μ is also normal, with mean $\mu*$ and standard deviation σ^* given by

$$\mu^* = \left(\frac{\rho^2}{\rho^2 + \frac{\sigma^2}{n}}\right)\overline{x} + \left(\frac{\frac{\sigma^2}{n}}{\rho^2 + \frac{\sigma^2}{n}}\right)\nu; \text{ and } \sigma^* = \sqrt{\frac{\rho^2\sigma^2}{n\rho^2 + \sigma^2}}.$$

Hints: recall we know the following density functions apply:

$$f(x_1, x_2, \ldots, x_n|\mu) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \left(\frac{1}{\sigma}\right)^n \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2\right\};$$

$$f(\mu) = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \left(\frac{1}{\rho}\right) \exp\left\{-\frac{1}{2\rho^2}(\mu - \nu)^2\right\}.$$

The posterior distribution is $f(\mu|x_1, x_2, \ldots, x_n)$; show that this has a normal form, i.e., is $\mathcal{N}(\mu^*, \sigma^{*2})$.

II. Linear Algebra:

a. Show that the following statements are true:
For $x,\ y \in \mathbf{R}^n$,

$$\text{(i) } \|y\|_\infty = \max_{\|x\|_1 \neq 0} \left(\frac{|y^* x|}{\|x\|_1} \right), \qquad \text{(ii) } \|y\|_1 = \max_{\|x\|_\infty \neq 0} \left(\frac{|y^* x|}{\|x\|_\infty} \right)$$

b. Suppose $a_1, a_2, \ldots, a_n$ are fixed positive real numbers. Determine which of the following are proper vector norms on $\mathbf{R}^n$ (i.e., which of the following satisfy the four conditions required of functions to be vector norms).

1. $\|x\| := \max_i\{a_i|x_i|\}$
2. $\|x\| := \sum_{i=1}^n a_i|x_i|$

c. For this problem we will prove that the induced matrix 2-norm for a matrix $A \in \mathbf{R}^{m\times n}$ is given by the maximum singular value, $\sigma_1(A)$.

In particular, show that

$$\begin{aligned} \max f(x) &= \|Ax\|_2^2 = x^T A^T A x \\ \text{subj. to } x^T x &= 1 \end{aligned},$$

is given by σ_1^2. Hint: consider using the SVD of the matrix A.

III. Please see the posted Matlab file, PCAdata.mat, or the csv version for Python. Compute the SVD of the original matrix, and using the SVD discuss what you think a proper PCA should reveal.

Write code to compute a PCA. Namely,

1. compute and subtract off the mean from the data,
2. compute the covariance matrix (i.e., A*A'), including scaling by 1/(n-1), and then
3. compute an eigenvalue decomposition, and sort both the eigenvalues and associated eigenvectors in descending order.

Plot and discuss (i) the principal components; and (ii) how this process may be done more directly using a SVD of the (de-biased, scaled) data.

IV. Seminar summaries:

a. Provide a summary of each talk attended (or viewed) from the *S*ymposium on Frontiers of Big Data. You should watch a minimum of 45 minutes of talks, and submit 3 paragraphs (min) summarizing the presentations.

b. Provide a summary of the session/talks attended from the *A*llerton Conference. You should either attend one of the tutorial sessions on Tuesday, or at least one full session (approx. 6 talks) at the conference (held at Allerton House, in Monticello). Please submit a 1/2 - 1 page summary of the presentation(s) you attended.

Please provide the titles and speaker names for all talks summarized.