Question 3

The logistic model for q classes can be defined as:

$$P(Y_i = k \mid X_i) = \frac{e^{x_i B_k}}{\sum_{k'=1}^{q} e^{x_i B_k}} = P_{ik}$$

where x_i = i^{th} row of the feature matrix, X

B_k = k^{th} column of the parameter matrix, B

As calculated in class, we want to maximize $L(B)$ using log-likelihood

$$L(B) = \prod_{i=1}^{n} \prod_{k=1}^{q} P(y[i]=k \mid X[i,:])^{\mathbb{1}[i,k]}$$

lets take log of $L(B)$

$$\log L(B) = \log \left(\prod_{i=1}^{n} \prod_{k=1}^{q} P(y[i]=k) \mid X[i,:]) \right)^{\mathbb{1}[i,k]}$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{q} \mathbb{1}[i,k] \cdot \log P(y[i]=k / X[i,:])$$

Substituting the P_{ik} function

$$= \sum_{i=1}^{n} \sum_{k=1}^{q} \mathbb{1}[i,k] \cdot \log \left(\frac{e^{X[i,:]\, B[:,k]}}{\sum_{k'=1}^{q} e^{X[i,:]\, B[:,k']}} \right)$$

simplifying wrt log

$$= \sum_{i=1}^{n} \sum_{k=1}^{q} \mathbb{1}[i,k] \cdot \left(X[i,:]\, B[:,k] - \log \left(\sum_{k'=1}^{q} e^{X[i,:]\, B[:,k']} \right) \right)$$

Let's calculate the gradient for this $\log L(B)$ function.

We will do it partially for each term

$$\frac{d}{dB} \left(X[i,:]\; B[:,k] \right) = X[i,j] \quad \text{— } 1$$

$$\frac{d}{dB} \left(\log \left(\sum_{k'=1}^{q} e^{X[i,:]\, B[:,k']} \right) \right) = \frac{X[i,j] \cdot e^{X[i,:]\, B[:,k']}}{\sum_{k'=1}^{q} e^{X[i,:]\, B[:,k']}}$$

$$= X[i,j] \cdot P_{ik} \quad \text{— } 2$$

Lets substitute these for the entire derivative

Furthermore $\mathbb{1}[i,k]$ is treated as a constant.

$$\frac{d}{dB} \log L(B) = \sum_{i=1}^{n} \left(\mathbb{1}[i,k] \cdot X[i,j] - X_{i,j}\, P_{ik} \right)$$

taking $X[i,j]$ as common

$$= \sum_{i=1}^{n} X[i,j]\left(\mathbb{1}[i,k] - P_{ik}\right)$$

Converting to matrix form, we get

$$\boxed{\frac{d \log L(B)}{dB} = X^T(I - P)}$$

Question 1A

Given

$$X = [x_1, x_2, \ldots, x_n]^T$$

$$y = [y_1, y_2, \ldots, y_n]^T$$

the model can be defined as

$y = XB$ where B is the parameter matrix to be estimated.

Then Mean Square Error $L(B)$ is given by

$$L(B) = \frac{1}{2n} \sum_{i=1}^{n} (y_i - Bx_i)^2$$

Since, the goal is to minimize MSE wrt B

We can iteratively update B using

$$B^{(k+1)} = B^{(k)} - \alpha \frac{dL(B)}{dB}$$

Assume that the iterative update rule for the parameter, B behaves like a fixed point iteration

$\boxed{B_{t+1} = B_t - \alpha\, l(B)}$ where $l(B)$ is the derived loss function for MSE.

and since $\frac{d\, l(B)}{dB} = -2X^Te$

$$= -2 \sum_{i=1}^{n} x_i (y_i - x_i B)$$

$$MSE(B + 2\alpha(X^Te)) < MSE(B)$$

the MSE after updating B is given by

$$MSE(B + 2\alpha(X^Te)) = \frac{1}{n} \sum_{i=1}^{n} (y_i - x_i(B + 2\alpha X^T e))^2$$

$$= \frac{1}{n} \sum_{i=1}^{n} y_i^2 + x_i^2 (B + 2\alpha X^Te)^2 - 2y_i x_i (B + 2\alpha X^Te)$$

$$= \frac{1}{n} \sum_{i=1}^{n} y_i^2 + x_i^2 \left(B^2 + (2\alpha X^T e)^2 + 4\alpha X^T e\right)\right) - 2 y_i x_i (B + 2\alpha X^T e)$$

$$= \frac{1}{n} \sum y_i^2 + x_i^2 B^2 + x_i^2 (2\alpha X^T e)^2 + 4\alpha X^T e - 2 y_i x_i (B + 2\alpha X^T e)$$

// // //

MSE for B

$$MSE(B) = \frac{1}{n} \sum_{i=1}^{n} (y_i - x_i B)^2$$

$$= \frac{1}{n} \sum y_i^2 + (x_i B)^2 - 2 x_i B y_i$$

// // //

$$\sum x_i^2 (2\alpha X^T e)^2 + 4\alpha X^T e - 2 y_i x_i (B + 2\alpha X^T e) < \sum -2 x_i B y_i$$

$$\sum x_i^2 (2\alpha X^T e)^2 + 4\alpha X^T e - 2 y_i x_i B - 4\alpha X^T e < \sum -2 x_i B y_i$$

// //

Which we can solve for α

$$\alpha < \sum_{i=1}^{n} \frac{x_i y_i}{x_i^2}$$

Question 1 B

$$MSE = \frac{1}{n} \sum_{i=1}^{n} (y - xB)^2$$

This is a quadratic function, therefore the plot will be parabolic.

Since the LSE of B minimizes the MSE, we set the derivative of MSE wrt B to 0

$$\frac{d}{dB} l(B) = \frac{2}{n} \left(B \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_i y_i \right) = 0$$

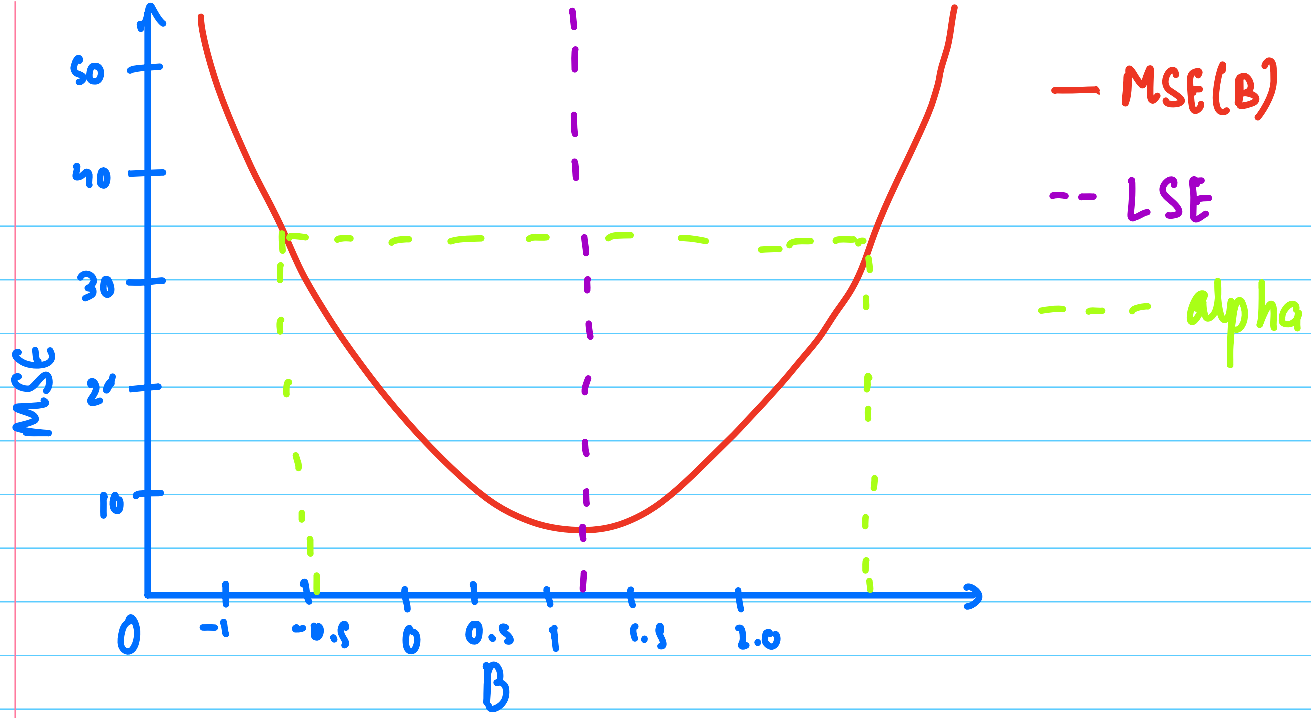
$$B \sum x_i^2 - \sum x_i y_i = 0$$

$$B \sum x_i^2 = \sum x_i y_i$$

$$B_{LSE} = \sum \frac{x_i y_i}{x_i^2}$$

Case 1: $\alpha < B_{LSE}$ at this the parameter B will slowly approach B_{LSE}

Case 2: $\alpha > B_{LSE}$



MSE(B)
LSE
alpha
MSE
B
50
40
30
20
10
0
-1
-0.5
0
0.5
1
1.5
2.0