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Cramer's rule

Cramer's rule is a way of solving a system of linear equations using determinants. Consider the following system of equations:

$$\begin{aligned} 4x_1 + 5x_2 - 2x_3 &= 11 \\ 7x_1 + 12x_2 - 6x_3 &= 49 \\ 2x_1 + 6x_2 + 0x_3 &= 8 \end{aligned}$$

The above system of equations can be written in matrix form as Ax = b, where A is the coefficient matrix (the matrix made up by the coefficients of the variables on the left-hand side of the equation), x represents the variables in the system of equations, and b represents the values on the right-hand side of the equation:

$$A = \begin{bmatrix} 4 & 5 & -2 \\ 7 & 12 & -6 \\ 2 & 6 & 0 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, b = \begin{bmatrix} 11 \\ 49 \\ 8 \end{bmatrix}$$

Given the above, Cramer's rule states that the solution to the system of equations can be found as:

$$x_i = \frac{det(A_i)}{det(A)}$$

where A_i is a new matrix formed by replacing the ith column of A with the b vector. Referencing matrix A above,

$$A_1 = \begin{bmatrix} 11 & 5 & -2 \\ 49 & 12 & -6 \\ 8 & 6 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 4 & 11 & -2 \\ 7 & 49 & -6 \\ 2 & 8 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 4 & 5 & 11 \\ 7 & 12 & 49 \\ 2 & 6 & 8 \end{bmatrix}$$

Computing the determinant of each of these matrices makes it possible to find the solution for the desired variable. For example, to find x_1, compute the determinant of A_1 then divide it by the determinant of A. There are a number of different ways to compute the determinants of square matrices; refer to the determinant page if necessary. For this example, we will use cofactor expansion to find the determinants:

$$det(A) = \begin{vmatrix} 4 & 5 & -2 \\ 7 & 12 & -6 \\ 2 & 6 & 0 \end{vmatrix}$$

$$= 4\begin{vmatrix} 12 & -6 \\ 6 & 0 \end{vmatrix} - 5\begin{vmatrix} 7 & -6 \\ 2 & 0 \end{vmatrix} + (-2)\begin{vmatrix} 7 & 12 \\ 2 & 6 \end{vmatrix}$$

$$= 4[(12)(0) - (-6)(6)] - 5[(7)(0) - (-6)(2)] - 2[(7)(6) - (12)(2)]$$

$$= 144 - 60 - 36$$

Matrix

$$= 48$$

$$det(A_1) = \begin{vmatrix} 11 & 5 & -2 \\ 49 & 12 & -6 \\ 8 & 6 & 0 \end{vmatrix}$$

$$= 11\begin{vmatrix} 12 & -6 \\ 6 & 0 \end{vmatrix} - 5\begin{vmatrix} 49 & -6 \\ 8 & 0 \end{vmatrix} + (-2)\begin{vmatrix} 49 & 12 \\ 8 & 6 \end{vmatrix}$$

$$= 11[(12)(0) - (-6)(6)] - 5[(49)(0) - (-6)(8)] - 2[(49)(6) - (12)(8)]$$

$$= 396 + 240 - 396$$

$$= 240$$

Thus:

$$x_1 = \frac{det(A_1)}{det(A)} = -\frac{240}{48} = -5$$

The determinants of A_2 and A_3 can be calculated in the same manner:

$$det(A_2) = \begin{vmatrix} 4 & 11 & -2 \\ 7 & 49 & -6 \\ 2 & 8 & 0 \end{vmatrix}$$

$$= 4\begin{vmatrix} 49 & -6 \\ 8 & 0 \end{vmatrix} - 11\begin{vmatrix} 7 & -6 \\ 2 & 0 \end{vmatrix} + (-2)\begin{vmatrix} 7 & 49 \\ 2 & 8 \end{vmatrix}$$

$$= 4(48) - 11(12) - 2(-42)$$

$$= 144$$

$$det(A_3) = \begin{vmatrix} 4 & 5 & 11 \\ 7 & 12 & 49 \\ 2 & 6 & 8 \end{vmatrix}$$

$$= 4\begin{vmatrix} 12 & 49 \\ 6 & 8 \end{vmatrix} - 5\begin{vmatrix} 7 & 49 \\ 2 & 8 \end{vmatrix} + 11\begin{vmatrix} 7 & 12 \\ 2 & 6 \end{vmatrix}$$

$$= 4(-198) - 5(-42) + 11(18)$$

$$= -384$$

The solutions to x_1 and x_2 can then be calculated as:

$$x_2 = \frac{det(A_2)}{det(A)} = \frac{144}{48} = 3$$

$$x_3 = \frac{det(A_3)}{det(A)} = -\frac{384}{48} = -8$$

We can then test the solutions with each of the equations in the system:

$$4(-5) + 5(3) - 2(-8) = 11$$

$$7(-5) + 12(3) - 6(-8) = 49$$

$$2(-5)+6(3)+0(-8)=8$$

The general form of Cramer's Rule, using matrix notation can be written as follows. A system of linear equations,

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2 \\ \vdots \qquad \vdots \qquad \vdots \qquad\qquad \vdots \quad \vdots \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n = b_3, \end{array}$$

can be written in the form Ax = b in matrix form as,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots a_{1n} & \\ a_{21} & a_{22} & a_{23} & \ldots a_{2n} & \\ a_{31} & a_{32} & a_{33} & \ldots a_{3n} & \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \ldots a_{nn} & \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix},$$

and Cramer's rule is as stated above,

$$x_i = \frac{det(A_i)}{det(A)},$$

where A_i is the new matrix formed by replacing the i*th* column of A with the b column vector, and v_i represents the i*th* column vector in a:

$$A_i \begin{bmatrix} v_1 & \ldots & v_{i-1} & b & v_{i+1} & \ldots & v_n \end{bmatrix}$$

Proof of Cramer's rule

We reinterpret the matrix-vector equation Ax = b as

$$x_1[v_1]+\ldots+x_n[v_n]=[b]$$

In other words, $b = x_1v_1 + ... + x_nv_n$ where each v_i is the *i*th column of matrix A (see Matrix Multiplication). If we plug this expression for b into A_i, the matrix made by replacing the *i*th column of A with b, we get:

$$A_i = \begin{bmatrix} v_1 & \ldots & v_{i-1} & \begin{matrix} x_1v_1 \\ + \ \ldots \\ + \ x_iv_i \\ + \ \ldots \\ + \, x_nv_n \end{matrix} & v_{i+1} & \ldots & v_n \end{bmatrix}$$

From the, properties of determinants, we can perform column operations of the type (i) + k(j) → (i), where k is a scalar, without changing the determinant. Therefore we can use the columns containing $v_1, ..., v_{i-1}, v_{i+1}, ... v_n$ to subtract out every term in $x_1v_1 + ... + x_nv_n$ except for x_iv_i. In other words,

$$det(A_i) = det(\begin{bmatrix} v_1 & \ldots & v_{i-1} & x_iv_i & v_{i+1} & \ldots & v_n \end{bmatrix})$$

From another property of determinants, a column of type k(i) → (i) has the same effect of multiplying the determinant by k. Therefore we can pull the scalar factor x_i from the i*th* column which contains x_iv_i. In other words,

$$det(\begin{bmatrix} v_1 & \ldots & v_{i-1} & x_iv_i & v_{i+1} & \ldots & v_n \end{bmatrix})$$

=

$$x_i det(\begin{bmatrix} v_1 & \dots & v_{i-1} & v_i & v_{i+1} & \dots & v_n \end{bmatrix})$$

Since

$$A = \begin{bmatrix} v_1 & \dots & v_{i-1} & v_i & v_{i+1} & \dots & v_n \end{bmatrix},$$

we can combine the above to give us $det(A_i) = x_i det(A)$. Dividing by det(A) gives us $x_i = \frac{det(A_i)}{det(A)}$, which is the original statement of Cramer's rule.

Limitations of Cramer's rule

- Because we are dividing by det(A) to get $x_i = \frac{det(A_i)}{det(A)}$, Cramer's rule only works if det(A) ≠ 0. If det(A) = 0, Cramer's rule cannot be used because a unique solution doesn't exist since there would be infinitely many solutions, or no solution at all.
- Cramer's rule is slow because we have to evaluate a determinant for each x_i. When we evaluate each $det(A_i)$ we have to perform Gaussian elimination on each A_i for a total of n times. In comparison, if we were to use the augmented matrix, [A|b], we would only need to perform Gaussian elimination once to solve Ax = b.