Bairstow method

From Encyclopedia of Mathematics
Jump to: navigation, search

A well-known and widely-used process for determining the roots of a given polynomial with real coefficients. The method determines a second-degree divisor of the given polynomial iteratively, and hence by using the formula for the roots of second-degree polynomials one can calculate an approximation of two roots of the given polynomial. An advantage of the method is that it uses real arithmetic only.

Let

$$P(x)=a_0x^n+a_1x^{n-1}+\dots+a_n,\ a_0\neq 0,$$

be a given polynomial with real coefficients and $n\geq 3$. After division by x^2+ux+v one finds

$$P(x)= \\ =P_1(x,u,v)(x^2+ux+v)+F(u,v)x+G(u,v), \tag{a1}$$

where $F(u,v)$ and $G(u,v)$ are the coefficients of the remainder. It is now necessary to find two real numbers, u^* and v^*, so that $x^2+u^*x+v^*$ is a divisor of P. This is equivalent to the solution of a system of non-linear equations

$$F(u,v)=0,\ G(u,v)=0\,. \tag{a2}$$

Bairstow's method is nothing else than a Newton method applied to (a2). More precisely,

$$\begin{pmatrix} u_{k+1} \\ v_{k+1} \end{pmatrix}=\begin{pmatrix} u_k \\ v_k \end{pmatrix}-D_{F,G}(u_k,v_k)^{-1}\cdot\begin{pmatrix} F(u_k,v_k) \\ G(u_k,v_k) \end{pmatrix}, \tag{a3}$$

where

$$D_{F,G}(u,v)=\begin{pmatrix} D_1F(u,v) & D_2F(u,v) \\ D_1G(u,v) & D_2G(u,v) \end{pmatrix} \tag{a4}$$

is the Jacobian matrix (cf. Jacobi matrix) and D_1F, D_2F, D_1G, and D_2G denote the partial derivatives of F and G with respect to the first and second argument, respectively. If $D_{F,G}(u_k,v_k)$ is not invertible, then the method is undefined.

It is important that $D_{F,G}(u,v)$ is determined by the remainders of P and P_1 after division by x^2+ux+v. Express the polynomial P as

$$P(x)=P_2(x,u,v)(x^2+ux+v)^2+ \\ +(F_1(u,v)x+G_1(u,v))(x^2+ux+v)+$$

$$+F(u,v)x+G(u,v)\ .$$

Then the Jacobian matrix (a4) has the following elements:

$$D_{F,G}(u,v)=\begin{pmatrix} uF_1(u,v)-G_1(u,v) & -F_1(u,v) \\ uF_1(u,v) & -G_1(u,v) \end{pmatrix}.$$

For the proof see [a1].

The values F, G and F_1, G_1 can be found by means of a Horner-type scheme (cf. Horner scheme). Using

$$P_1(x)=b_0x^{n-2}+b_1x^{n-3}+\dots+b_{n-2}$$

and

$$P_2(x)=c_0x^{n-4}+c_2x^{n-5}+\dots+c_{n-4}$$

one finds the same recursion for F, G, b_i and F_1, G_1, c_i:

$$b_0=a_0,\qquad b_1=a_1-b_0u,$$
$$b_i=a_i-b_{i-1}u-b_{i-2}v(i=2,\dots,n-2),$$
$$F(u,v)=a_{n-1}-b_{n-2}u-b_{n-3}v,$$
$$G(u,v)=a_n-b_{n-2}u,$$
$$c_0=b_0,\qquad c_1=b_1-c_0u,$$
$$c_i=b_i-c_{i-1}u-c_{i-2}v(i=2,\dots,n-4),$$
$$F_1(u,v)=b_{n-3}-c_{n-4}u-c_{n-5}v,$$
$$G_1(u,v)=b_{n-2}-c_{n-4}u\ .$$

The main assumption in local convergence theorems for the Newton method is the non-singularity of the Jacobian matrix at the root. The following theorem gives a necessary and sufficient condition of algebraic character for this to hold.

Let u and v be arbitrary real numbers. The Jacobian matrix $D_{F,G}(u,v)$ introduced in (a4) is regular if and only if x^2+ux+v and the polynomial $P_1(x,u,v)$ defined by (a1) have no common root. The rank of $D_{F,G}(u,v)$ is one if and only if the number of the common roots is one. The Jacobian matrix is zero if and only if x^2+ux+v is a divisor of $P_1(x,u,v)$. For the proof see [a2].

Using the local convergence theorem for the Newton method one obtains, as a corollary, the local convergence theorem for the Bairstow method:

Let

$$P(x)=P_1(x,u^*,v^*)(x^2+u^*x+v^*)$$

and suppose that the two factors on the right-hand side have no common root. Then there exists a positive number d such that the sequence (u_k,v_k) produced by Bairstow's method will converge (quadratically) to

(u^*, v^*), provided

$$|u_0 - u^*| < d, \qquad |v_0 - v^*| < d.$$

References

[a1] J. Stoer, R. Bulirsch, "Introduction to numerical analysis" , Springer (1980) pp. 285–287

[a2] T. Fiala, A. Krebsz, "On the convergence and divergence of Bairstow's method" *Numerische Math.* , **50** (1987) pp. 477–482

How to Cite This Entry:

Bairstow method. A. Krebsz (originator), *Encyclopedia of Mathematics.* URL: http://encyclopediaofmath.org/index.php?title=Bairstow_method&oldid=14717

This text originally appeared in Encyclopedia of Mathematics - ISBN 1402006098