Bairstow Method

Bairstow Method is an iterative method used to find both the real and complex roots of a polynomial. It is based on the idea of synthetic division of the given polynomial by a quadratic function and can be used to find all the roots of a polynomial. Given a polynomial say,

$$f_n(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n \tag{B.1}$$

Bairstow's method divides the polynomial by a quadratic function.

$$x^2 - rx - s. \tag{B.2}$$

Now the quotient will be a polynomial $f_{n-2}(x)$, *i.e.*

$$f_{n-2}(x) = b_2 + b_3 x + b_4 x^2 + \dots + b_{n-1} x^{n-3} + b_n x^{n-2} \tag{B.3}$$

and the remainder is a linear function $R(x)$, i.e.

$$R(x) = b_1(x - r) + b_0 \tag{B.4}$$

Since the quotient $f_{n-2}(x)$ and the remainder $R(x)$ are obtained by standard synthetic division the co-efficients $b_i \quad (i = 0..n)$ can be obtained by the following recurrence relation.

$$b_n = a_n \tag{B.5a}$$

$$b_{n-1} = a_{n-1} + r b_n \tag{B.5b}$$

$$b_i = a_i + r b_{i+1} + s b_{i+2} \quad \text{for} \quad i = n-2 \quad to \quad 0 \tag{B.5c}$$

If $x^2 - rx - s$ is an exact factor of $f_n(x)$ then the remainder $R(x)$ is zero and the real/complex roots of $x^2 - rx - s$ are the roots of $f_n(x)$. It may be noted that $x^2 - rx - s$ is considered based on some guess values for r, s. So Bairstow's method reduces to determining the values of r and s such that $R(x)$ is zero. For finding such values Bairstow's method uses a strategy similar to Newton Raphson's method.

Since both b_0 and b_1 are functions of r and s we can have Taylor series expansion of b_0, b_1 as:

$$b_1(r + \Delta r, \quad s + \Delta s) = b_1 + \frac{\partial b_1}{\partial r}\Delta r + \frac{\partial b_1}{\partial s}\Delta s + O(\Delta r^2, \Delta s^2) \tag{B.6a}$$

$$b_0(r + \Delta r, \quad s + \Delta s) = b_0 + \frac{\partial b_0}{\partial r}\Delta r + \frac{\partial b_0}{\partial s}\Delta s + O(\Delta r^2, \Delta s^2) \tag{B.6b}$$

For $\Delta s, \Delta r << 1$, $O(\Delta r^2, \Delta s^2)$ terms ≈ 0 i.e. second and higher order terms may be neglected, so that $(\Delta r, \Delta s)$ the improvement over guess value (r, s) may be obtained by equating (B.6a),(B.6b) to zero i.e.

$$\frac{\partial b_1}{\partial r}\Delta r + \frac{\partial b_1}{\partial s}\Delta s = -b_1 \tag{B.7a}$$

$$\frac{\partial b_0}{\partial r}\Delta r + \frac{\partial b_0}{\partial s}\Delta s = -b_0 \tag{B.7b}$$

To solve the system of equations $(B.7a) - (B.7b)$, we need the partial derivatives of b_0, b_1 w.r.t. r and s. Bairstow has shown that these partial derivatives can be obtained by synthetic division of $f_{n-2}(x)$, which amounts to using the recurrence relation $(B.5a) - (B.5c)$ replacing $a_i's$ with $b_i's$ and $b_i's$ with $c_i's$ i.e.

$$c_n = b_n \tag{B.8a}$$

$$c_{n-1} = b_{n-1} + rc_n \tag{B.8b}$$

$$c_i = b_i + rc_{i+1} + sc_{i+2} \tag{B.8c}$$

for $i = 1,2,\ldots, n-2$

where

$$\frac{\partial b_0}{\partial r} = c_1, \quad \frac{\partial b_o}{\partial s} = \frac{\partial b_1}{\partial r} = c_2 \quad and \quad \frac{\partial b_1}{\partial s} = c_3 \tag{B.9}$$

∴. The system of equations (B.7a)-(B.7b) may be written as.

$$c_2\Delta r + c_3\Delta s = -b_1 \tag{B.10a}$$

$$c_1\Delta r + c_2\Delta s = -b_o \tag{B.10b}$$

These equations can be solved for $(\Delta r, \Delta s)$ and turn be used to improve guess value (r, s) to $(r + \Delta r, s + \Delta s)$.

Now we can calculate the percentage of approximate errors in (r,s) by

$$|\varepsilon_{a,r}| = |\frac{\Delta r}{r}| \times 100; \quad \varepsilon_{a,s} = |\frac{\Delta s}{s}| \times 100 \tag{B.11}$$

If $|\varepsilon_{a,r}| > \varepsilon_s$ or $|\varepsilon_{a,s}| > \varepsilon_s$, where ε_s is the iteration stopping error, then we repeat the process with the new guess i.e. $(r + \Delta r, s + \Delta s)$. Otherwise the roots of $f_n(x)$ can be determined by

$$x = \frac{r \pm \sqrt{r^2 + 4s}}{2} \tag{B.12}$$

If we want to find all the roots of $f_n(x)$ then at this point we have the following three possibilities:

1. If the quotient polynomial $f_{n-2}(x)$ is a third (or higher) order polynomial then we can

again apply the Bairstow's method to the quotient polynomial. The previous values of (r, s) can serve as the starting guesses for this application.

2. If the quotient polynomial $f_{n-2}(x)$ is a quadratic function then use (B.12) to obtain the remaining two roots of $f_n(x)$.
3. If the quotient polynomial $f_{n-2}(x)$ is a linear function say $ax+b=0$ then the remaining single root is given by $x = -\frac{b}{a}$

Example:
Find all the roots of the polynomial

$$f_4(x) = x^4 - 5x^3 + 10x^2 - 10x + 4$$

by Bairstow method . With the initial values $r = 0.5, \quad s = -0.5 \quad and \quad \varepsilon_s = 0.01.$

Solution:
Set *iteration=1*

$$a_0 = 4, \quad a_1 = -10, \quad a_2 = 10, \quad a_3 = -5 \quad a_4 = 1$$

Using the recurrence relations (B.5a)-(B.5c) and (B.8a)-(B.8c) we get

$$b_4 = 1, \quad b_3 = -4.5, \quad b_2 = 7.25, \quad b_1 = -4.125, \quad b_0 = -1.6875$$

$$c_4 = 1, \quad c_3 = -4 \quad c_2 = 4.75, \quad c_1 = 0.25$$

$\therefore$ the simultaneous equations for Δr and Δs are:

$$4.75\Delta r \ - \ 4\Delta s = 4.125$$

$$0.25\Delta r + 4.75\,\Delta s = 1.6875$$

on solving we get $\Delta r = 1.1180371, \quad \Delta s = 0.296419084$

$$\therefore r = 0.5 + \Delta r = 1.6180371$$

$$s = -0.5 + \Delta s = -0.203580916$$

and

$$|\varepsilon_{a,r}| = \left|\frac{1.1180371}{1.6180371}\right| \times 100 = 69.0983582$$

$$|\varepsilon_a, s| = \left|\frac{0.296419084}{-0.203580916}\right| \times 100 = 145.602585$$

Set *iteration=2*

$$b_4 = 1.0, \quad b_3 = -3.38196278, \quad b_2 = 4.32427788, \quad b_1 = -2.31465483, \quad b_0 = -0.625537872$$

$$c_4 = 1.0, \quad c_3 = -1.76392567, \quad c_2 = 1.26659977, \quad c_1 = 0.0938522071$$

$\therefore$ now we have to solve

$$1.26659977\Delta r - 1.76392567\Delta s = 2.31465483$$
$$0.0938522071\Delta r + 1.26659977\Delta s = 0.625537872$$

On solving we get $\Delta r = 2.27996969, \quad \Delta s = 0.324931115$

$$\therefore r = 1.6180371 + \Delta r = 3.89800692$$
$$s = -0.203580916 + \Delta s = 0.121350199$$

$$|\varepsilon_{a,r}| = \left|\frac{2.27996969}{3.89800692}\right| \times 100 = 58.490654$$

$$|\varepsilon_{a,s}| = \left|\frac{0.32493115}{0.121350199}\right| \times 100 = 267.763153$$

Now proceeding in the above manner in about ten iteration we get $r = 3, \quad s = -2$ with

$$|\varepsilon_{a,r}| \sim 7.95 \times 10^{-6} < \varepsilon_s = 0.01$$

$$|\varepsilon_{a,s}| \sim 5.96 \times 10^{-6} < \varepsilon_s = 0.01$$

Now on using $x = \dfrac{r \pm \sqrt{r^2 + 4s}}{2} \quad (i.e. \quad eqn. \quad B.12)$ we get $x = \dfrac{3 \pm \sqrt{9-8}}{2} = 2, 1$

So at this point Quotient is a quadratic equation

$$f_2(x) = x^2 + 2x + 2$$

Roots of $f_2(x)$ are: $x = 1 - i, \quad 1 + i$

$\therefore$ Roots $f_4(x)$ are $= 1 - i, \quad 1 + i, \quad 1, \quad 2$

i.e $f_4(x) = (x-(1-i))(x-(1+i))(x-1)(x-2).$

Exercises:

(1) Use initial approximation $r_0 = 0.5$, $s_0 = 0.5$ to find a quadratic factor of the form $x^2 + rx + s$ of the polynomial equation

$$x^4 + x^3 + 2x^2 + x + 1 = 0$$

using Bairstow method and hence find all its roots.

(2) Use initial approximaton $r_0 = 2$, $s_0 = 2$ to find a quadratic factor of the form $x^2 + rx + s$ of the polynomial equation

$$x^4 - 3x^3 + 20x^2 + 44x + 54 = 0$$

using Bairstow method and hence find all the roots.

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