Numerical Programming 1 CSE (MA3305)
Exercise Sheet 1

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Assignment 1

```
>> x = [1000.12345; 1000.23456; 1000.34567];
>> sum(x.^2)-sum(x)^2/3
ans =
     2.469086367636919e-02
```

How many digits are (at best) correct? Give a short explanation.

Solution

Because of $\mathtt{sum(x)} \approx 3\,000$ we have $\mathtt{sum(x)}^2\mathtt{/3} \approx (3\,000^2)/3 = 3\,000\,000$ and *seven* digits in front of the decimal point. These 7 digits and *one other* digit after the decimal point are *cancelled*. Hence $16 - (7 + 1) = 8$ digits are (at best) correct.

Assignment 2

Replace the following expression by an equivalent one, sucht that the evaluation using floating-point arithmetic is free of cancellation:

$$\frac{1 - \cos x}{x} \qquad x \neq 0, \quad |x| \ll 1.$$

Compare the expression above and your expression in Matlab.

Solution

With $1 - \cos x = 2\sin^2(x/2)$ we write the Matlab function

```
function erg=sol(x)
erg = (2*sin(x/2).^2)./x;
erg(x==0)=0;
end
```

and get

x	(1-cos(x))/x	(2*sin(x/2)^2)/x
1.000000000000000e-04	4.999999969612645e-05	4.999999995833334e-05
9.999999999999999e-06	5.000000413701856e-06	4.999999999958332e-06
1.000000000000000e-06	5.000444502911705e-07	4.999999999999583e-07
1.000000000000000e-07	4.996003610813204e-08	4.999999999999996e-08
1.000000000000000e-08	0.000000000000000e+00	5.000000000000000e-09
1.000000000000000e-09	0.000000000000000e+00	5.000000000000000e-10

Assignment 3

Consider

$$f(x) = \sqrt{x+1} - \sqrt{x} \qquad \text{for } x \in [0, \infty[$$

$$g(x) = \arcsin(2(1 - \cos(x))/x^2) \qquad \text{for } x \in \mathbb{R}.$$

(a) What values of x result in cancellation?

(b) Find equivalent expressions for f and g without cancellation.

(c) Compare the expressions above and your expressions in Matlab.

Solution
(a) f: for $x \gg 1$.
g: for $|x| \ll 1$.
(b) We use

$$\sqrt{x+1}-\sqrt{x} = \frac{(\sqrt{x+1}-\sqrt{x})(\sqrt{x+1}+\sqrt{x})}{\sqrt{x+1}+\sqrt{x}} = \frac{1}{\sqrt{x+1}+\sqrt{x}}.$$

With $1-\cos(x) = 2\sin^2(x/2)$ we write

$$\arcsin(2(1-\cos(x))/x^2) = \arcsin\left(\left(\frac{2\sin(x/2)}{x}\right)^2\right)$$

(c)

```
    x                     | sqrt(x+1)-sqrt(x)      |1/(sqrt(x+1)+sqrt(x))   |
=============================================================================
 1.000000000000000e+48   | 0.000000000000000e+00 | 5.000000000000001e-25 |
 1.000000000000000e+49   | 0.000000000000000e+00 | 1.581138830084190e-25 |
 1.000000000000000e+50   | 0.000000000000000e+00 | 5.000000000000000e-26 |
 1.000000000000000e+51   | 0.000000000000000e+00 | 1.581138830084190e-26 |
 1.000000000000000e+52   | 0.000000000000000e+00 | 4.999999999999999e-27 |
 1.000000000000000e+53   | 0.000000000000000e+00 | 1.581138830084190e-27 |

     x                     |  given expression    |asin((2*sin(x/2)./x)^2)|
=============================================================================
 1.000000000000000e-04   | 1.570686077341591e+00 | 1.570755501964158e+00 |
 9.999999999999999e-06   | 1.570796326794897e+00 | 1.570792244311823e+00 |
 1.000000000000000e-06   | 1.570796326794897e+00 | 1.570795918437825e+00 |
 1.000000000000000e-07   | 1.530811721401428e+00 | 1.570796282091413e+00 |
 1.000000000000000e-08   | 0.000000000000000e+00 | 1.570796326794897e+00 |
 1.000000000000000e-09   | 0.000000000000000e+00 | 1.570796326794897e+00 |
 1.000000000000000e-10   | 0.000000000000000e+00 | 1.570796326794897e+00 |
```

Assignment 4

Consider for$|x| \ll 1$ the two expressions:

$$\log(x+1) \qquad \text{and} \qquad \frac{\log(1+x)}{(1+x)-1}\cdot x$$

and the following divisions in submaps:

$$x \xrightarrow{h} 1+x =: a \quad \xrightarrow{g} \log(a)$$

$$x \xrightarrow{h} \begin{pmatrix} 1+x =: a \\ x =: b \end{pmatrix} \xrightarrow{g} \frac{\log(a)}{a-1}b$$

(a) Explain why the first algorithm is unstable?

(b) Draw (in Matlab) a plot of the graph of both expressions.

(c) Give a heuristic argument, why the second algorithm is stable. [Hint: Use the Taylor expansion of log.]

Solution
(a) For $f(x) = \log(1+x)$ we have

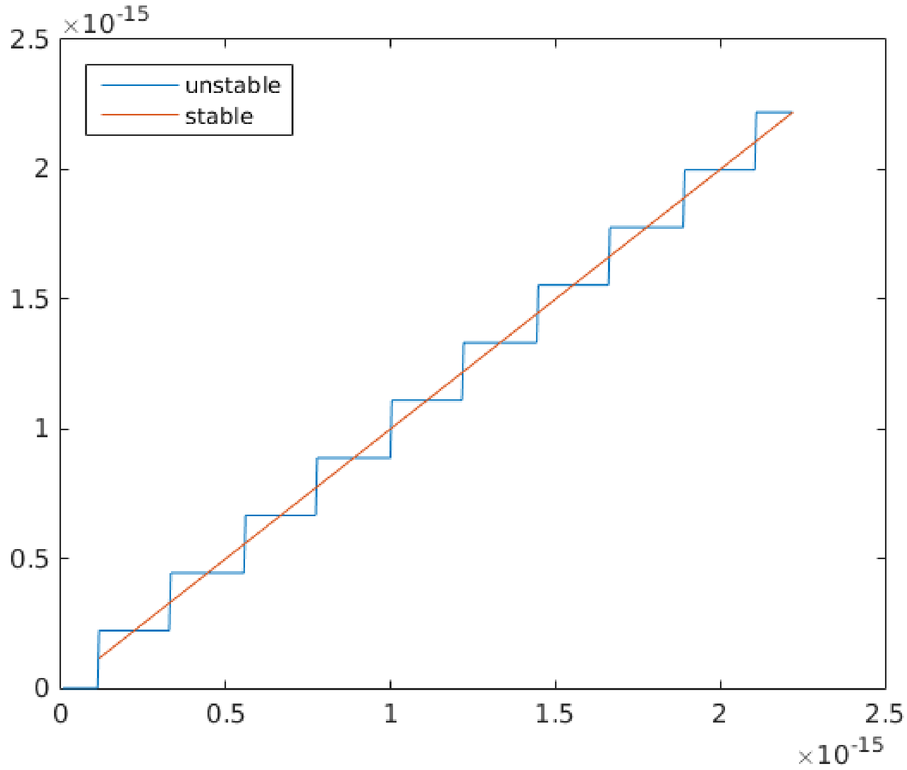
$$\kappa_f(x) = \frac{f'(x)x}{f(x)} = \frac{x}{(x+1)\log(1+x)} \to 1$$

for $x \to 0$, but

$$\kappa_g(a) = \frac{g'(a)a}{g(a)} = \frac{1}{\log a} \to \infty$$

for $a \to 1$. Hence $\kappa_g \gg \kappa_f$ for $|x| \ll 1$.

(b)



(c) $\log(1 \hat{+} x) = 1 \hat{+} x - 1 + O(x^2)$. Hence for $|x| \ll 1$ the algorithm returns a good approximation to x.