Assignment 26 (Rayleigh quotient iteration)
The Rayleigh quotient iteration does the same as the inverse iteration with shift, except that the shift is given by the Rayleigh quotient and changes which each iteration step:
Starting with an initial vector $u_0 \in \mathbb{C}^n$ with $\|u_0\| = 1$ iterate for $k = 0, 1, 2, \ldots$:

- Compute the Rayleigh quotient $\mu_k = \langle u_k, Au_k \rangle$.
- Solve $(A - \mu_k Id)v_{k+1} = u_k$.
- Normalize $u_{k+1} = \frac{v_{k+1}}{\|v_{k+1}\|}$.

Implement the Rayleigh quotient iteration and create convergence plots for some examples.

Solution
see zip-file.

Assignment 27 (Sylvester equation)
Consider for given $A, B, C \in \mathbb{C}^{n\times n}$ the Sylvester equation

$$AX - XB = C \tag{1}$$

for the unknown matrix $X \in \mathbb{C}^{n\times n}$.

(a) Show: With the two Schur decompositions $A = URU'$ and $B = VSV'$ equation (1) can be rewritten in the form

$$RY - YS = E \tag{2}$$

with $E = U'CV$ and $Y = U'XV$.

(b) Find an algorithm for computing Y. Hint: Write (2) as n equations for the *columns* of Y.

(c) Show: (1) has a unique solution if and only if A and B do not share any eigenvalues (i. e. there is no $\lambda \in \mathbb{C}$ such that λ is eigenvalue of A and B).

(d) Write a Matlab function to compute X. Use the result of (b) to first compute Y. (Use Matlab's `schur` command.) Give the complexity of your algorithm.

Solution
(a) Plugging in the two Schur decompositions in (1)

$$URU'X - XVSV' = C \qquad \Leftrightarrow \qquad R\underbrace{U'XV}_{=Y} - \underbrace{U'XV}_{=Y}S = \underbrace{U'CV}_{=E}.$$

Let's write $Y = [y_1|\cdots|y_m]$, $E = [e_1|\cdots|e_m]$ and $S = (s_{ij})$ with $s_{ij} = 0$ for $i > j$. Then the j-th column of (2) can be written as

$$Ry_j - \sum_{i\leq j} s_{ij} y_i = e_j$$

or

$$(R - s_{jj}I)y_j = e_j + \sum_{i<j} s_{ij}y_i, \qquad j = 1, \ldots, m. \tag{3}$$

Hence for calculating y_j only $y_1, \ldots, y_{j-1}$ is needed. One solves (3) for $j = 1$ first, then for $j = 2$, etc.

(b) The eigenvalues of A and B can be found in the diagonals of R and S, respectively. If A and B do not have an common eigenvalue, then 3 is always uniquely solveable. In this case also 1 has an unique solution.
If there exists an eigenvalue λ of A with $s_{jj} = \lambda$ then $(R - s_{jj}I)$ is singular. Then (2) and (1) have no unique solution.

(c)

```
function X=sylvester(A,B,C)
n=length(A);
[U,R]=schur(A,'complex');
[V,S]=schur(B,'complex');
E=U'*C*V;

Y=zeros(n); I=eye(n);
for j=1:n
  Y(:,j)=(R-S(j,j)*I)\(E(:,j) + Y(:,1:j-1)*S(1:j-1,j));
end
X=U*Y*V';
```

Complexity: $O(n^3)$ because of the two Schur decompositions and the matrix-matrix multiplications.