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**Assignment 17** (Unit spheres)
Use Matlab to plot in the plane the set of points

$$S_p := \{ x \in \mathbb{R}^2 : \|x\|_p = 1 \}$$

for $p \in \{1, 2, \infty\}$.

**Solution**
see zip-file.

**Assignment 18** (Monte-Carlo quadrature)
Consider the following integral

$$\int_{-\infty}^{\infty} f(x) e^{-\frac{x^2}{2}} \, dx$$

for a function $f$.

(a) Write a Matlab function `mc` that uses $N$ points and the Monte-Carlo quadrature to approximate this integral.

```
function result = mc(f,N)
```

(b) Compare theoretically the following two experiments:

(1) `mc(f,2*N)`

(2) `mean( [mc(f,N), mc(f,N)] )`

(c) Write a Matlab function `mc_composite` for implementing (2) with $k$ (instead of 2) calls of `mc`. How much memory is used?

```
function result = mc_composite(f,N,k)
```

(d) Do a convergence plot for $f(x) = \cos(x)$. Use your method from (c) with reasonable values of $k$ and $N$.

**Solution**
see zip-file.
(b) Let's compare the two expermiments with the (random) numbers $a_1, a_2, \ldots, a_{2n}$. Then we have

$$\frac{1}{2}\left(\frac{a_1 + a_2 + \cdots + a_n}{n} + \frac{a_{n+1} + a_{n+2} + \cdots + a_{2n}}{n}\right) = \frac{a_1 + a_2 + \cdots + a_{2n}}{2n}.$$

Hence: There is no difference between (1) and (2).

**Assignment 19** (Norms)
Given $x \in \mathbb{R}^m$ and $A \in \mathbb{R}^{n\times m}$. Verify the following inequalities:

(a) $\|x\|_\infty \le \|x\|_2$

(b) $\|x\|_2 \le \sqrt{m}\|x\|_\infty$

(c) $\|A\|_\infty \le \sqrt{m}\|A\|_2$

(d) $\|A\|_2 \le \sqrt{n}\|A\|_\infty$

**Solution**

(a)

$$\|x\|_\infty = \max_i |x_i| = \max_i \sqrt{x_i^2} \leq \sqrt{x_1^2 + \cdots + x_m^2} = \|x\|_2.$$

(b)

$$\frac{\|x\|_2^2}{\|x\|_\infty^2} = \sum_i \frac{x_i^2}{\|x\|_\infty^2} \leq \sum_i 1 = m.$$

By taking the square root on both sides we obtain the claimed inequality.

(c) Using (a) and (b)

$$\|A\|_\infty = \max_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} \leq \max_{x\neq 0} \frac{\|Ax\|_2}{\frac{1}{\sqrt{m}}\|x\|_2} = \sqrt{m} \max_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \sqrt{m}\|A\|_2.$$

(d) Again using (a) and (b) for the vector $Ax \in \mathbb{R}^n$

$$\|A\|_2 = \max_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} \leq \max_{x\neq 0} \frac{\sqrt{n}\|Ax\|_\infty}{\|x\|_\infty} = \sqrt{n}\|A\|_\infty.$$