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Assignment 22
Improve the Gauss-Seidel and SOR code from the lecture: Reuse as many matrix-vector multiplications as possible.

Solution
If we have the splitting $A = M - N$ and the iteration

$$Mx^{k+1} = \lambda(Nx^k + b) + (1 - \lambda)Mx^k$$

then for the residuum r^{k+1} we obtain

$$\begin{aligned}
r^{k+1} &= b - Ax^{k+1} = b - Mx^{k+1} + Nx^{k+1} \\
&= Nx^{k+1} + b - \lambda(Nx^k + b) - (1 - \lambda)Mx^k \\
&= Nx^{k+1} - \lambda Nx^k + (1 - \lambda)(b - Mx^k) \\
&= Nx^{k+1} - \lambda Nx^k + (1 - \lambda)(b - Ax^k - Nx^k) = Nx^{k+1} - Nx^k + (1 - \lambda)r^k.
\end{aligned}$$

The matrix-vector multiplication Nx^k is also needed for calculating x^{k+1} and it can be reused. If we save Nx^k, Nx^{k+1} and r^k then r^{k+1} can be computed without an additional matrix-vector multiplication.
Code: see zip-file.

Assignment 23
Compare the following iterative methods

1. relaxed Jacobi (with different relaxation parameters)
2. Gauss-Seidel,
3. SOR (with different relaxation parameters) and
4. CG:

Generate test equations $Ax = b$ with sparse $A \in \mathbb{R}^{n \times n}$ for different n and for different condition numbers κ_A.
For each test problem: Draw in one plot all the backward errors of the methods and draw in another plot all the forward errors of the methods.

Solution
see zip-file.

Assignment 24 (Richardson-Iteration)
Consider the fixpoint iteration

$$x^{k+1} = x^k + b - Ax^k$$

for solving the linear system $Ax = b$ with $A \in \mathbb{R}^{n \times n}$ and $x, b \in \mathbb{R}^n$.

(a) What is the splitting $A = M - N$ from which this fixpoint iteration results?

(b) Write down the relaxed fixpoint iteration.

Solution
(a) We have

$$x^{k+1} = x^k + b - Ax^k \qquad \Leftrightarrow \qquad Ix^{k+1} = (I - A)x^k + b.$$

So $M = I$ and $N = I - A$ and the splitting is $A = I - (I - A) = M - N$.

(b) Using the relaxation parameter λ we get the relaxed version

$$x^{k+1} = \lambda(x^k + b - Ax^k) + (1 - \lambda)x^k = x^k + \lambda(b - Ax^k) = x^k + \lambda r^k.$$