

# Lab Course Scientific Computing

## Worksheet 1

distributed: 21.10.2015

due: 02.11.2015, 3:00 pm, submission on Moodle

oral examination: 03.11.2015 (exact slots will be announced)

We examine the following ordinary differential equation describing the dynamics of the population of a certain species:

$$\dot{p} = \left(1 - \frac{p}{10}\right) \cdot p \quad (1)$$

with initial condition

$$p(0) = 1. \quad (2)$$

The analytical solution is given by

$$p(t) = \frac{10}{1 + 9e^{-t}}.$$

We use this rather simple equation with a known exact solution to examine the properties of different numerical methods.

- a) Use `matlab` to plot the function  $p(t)$  in a graph.
- b) Consider a general initial value problem

$$\dot{y} = f(y), \quad y(0) = y_0.$$

Implement the following explicit numerical methods with variable stepsize  $\delta t$  and end time  $t_{end}$

- 1) explicit Euler method,

- 2) method of Heun,
- 3) Runge-Kutta method (fourth order)

as a `matlab` function depending on the right hand side  $f(y)$ , the initial value  $y_0$ , the stepsize  $\delta t$  and the end time  $t_{end}$ . The output of the function shall be a vector containing all computed approximate values for  $y$ .

- c) For each of the three methods implemented, compute approximate solutions for equation (1) with initial conditions (2), end time  $t_{end} = 5$ , and with time steps  $\delta t = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ . For each case, compute the approximation error

$$E = \sqrt{\frac{\delta t}{5} \sum_k (p_k - p_{k,exact})^2},$$

where  $p_k$  denotes the approximation,  $p_{exact,k}$  the exact solution at  $t = \delta t \cdot k$ .

Plot your solutions in one graph per method (together with the given solution from a)) and write down the errors in the tabulars below.

- d) For each of the three methods, determine the factor by which the error is reduced if the step size  $\delta t$  is halved. Write down the results in the tabular below.
- e) In general, we do not know the exact solution of an equation we have to solve numerically (otherwise, we would not have to use a numerical method, in fact ;)). To anyhow guess the accuracy of a method, we can use the difference between our best approximation (the one with the smallest time step  $\delta t$ ) and the other approximations:

$$\tilde{E} = \sqrt{\frac{\delta t}{5} \sum_k (p_k - p_{k,best})^2},$$

where  $p_k$  denotes the approximation with time step  $\delta t$ ,  $p_{best,k}$  the best approximation at  $t = \delta t \cdot k$ .

Compute  $\tilde{E}$  for all time steps and methods used, write down the results in the tabulars below and compare them to the exact error.

explicit Euler method ( $q = 1$ )				
$\delta t$				
error				
error red.				
error app.				

method of Heun ( $q = 2$ )				
$\delta t$				
error				
error red.				
error app.				

Runge-Kutta method ( $q = 4$ )				
$\delta t$				
error				
error red.				
error app.				

**Questions:**

1) By which factor is the error reduced for each halving of  $\delta t$  if you apply a

- first order ( $O(\delta t)$ ),
- second order ( $O(\delta t^2)$ ),
- third order ( $O(\delta t^3)$ ),
- fourth order ( $O(\delta t^4)$ )

method.

2) For which integer  $q$  can you conclude that the error of the

- a) explicit Euler method,
- b) method of Heun,
- c) Runge-Kutta method (fourth order)

behaves like  $O(\delta t^q)$ ?

3) Is a higher order method always more accurate than a lower order method (for the same stepsize  $\delta t$ )?

4) Assume you have to compute the solution up to a certain prescribed accuracy limit and that you see that you can do with less time steps if you use the Runge-Kutta-method than if you use Euler or the method of Heun. Can you conclude in this case that the Runge-Kutta method is the most efficient one of the three alternatives?