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Assignment 14 (Improvement of `quad24`)
Improve the function `quad24` from the lecture (see Screen Write-up):

(a) Reuse as many f evaluations as possible.

(b) Add an additional input argument `Iscal` and use it to support the relative error conecpt.

(c) Improve the error estimation: Instead of using $\delta := |I_{\text{Simpson}} - I_{\text{Trapez}}|$ as error estimation use an error estimator of the form

$$||[\varepsilon]|| := ||[\varepsilon_{\text{Simpson}}]|| := \frac{\delta^k}{|I_{\text{Trapez}}|^\ell}$$

with appropriate k and ℓ. [Hint: $||[\varepsilon]||$ has to have the right h-order and has to scale linearly in α if f ist scaled with α: $f \to \alpha f$.]

(d) see Assignment 15.

Solution
see zip-file.
(c) We know: Simpson quadrature has order 4. Thus the error is an $O(h^5)$. With $\delta = O(h^3)$ (because of the trapezoidal rule) and $|I_{\text{Trapez}}| = O(h)$ we infer the condition

$$O(h^5) \stackrel{!}{=} ||[\varepsilon]|| = O(h^{3k-\ell}) \qquad \Leftrightarrow \qquad 5 = 3k - \ell.$$

We know for $f \to \alpha f$ we have $I_{\text{Trapez}} \to \alpha I_{\text{Trapez}}$ and $\delta \to \alpha\delta$. To get the right scaling we need

$$\alpha^1 \stackrel{!}{=} \alpha^{k-\ell} \qquad \Leftrightarrow \qquad 1 = k - \ell.$$

The unique solution to this 2 linear equations (in 2 unknowns) is: $k = 2$ and $\ell = 1$.

Assignment 15 (Fooling `quad24`)
Construct a non-negative function f on the interval $[0, 1]$ with $\int_0^1 f(x)\,dx \gg 0$ for which `quad24` returns the result 0 for *every* tolerance `tol`.
How can this problem be cured?

Solution
We check on what grid points an f gets evaluated by `quad24` for $a = 0$, $b = 1$. The initial step size is $h = 1/100$ and in the first step f is evaluated at $t_1 = 0$, $t_2 = h/2 = 1/200$ and $t_3 = h = 1/100$. If f is zero there, then `quad24` cannot distinquish f from the zero function in the first interval. The new step size will be $h = 99/100$ and we have $t_4 = 1/100 + h/2 = 101/200$ and $t_5 = 1$.
So with

$$f(t) := \prod_{k=1}^{5} (t - t_k)^2 = t^2(t-1)^2\left(t - \frac{1}{200}\right)^2\left(t - \frac{1}{100}\right)^2\left(t - \frac{101}{200}\right)^2,$$

we have $\int_0^1 f(t)\,dt > 0$ and `quad24` returns 0.
To cure this problem one should use a initial step size depending on the tolerance or choose the initial step size by random.

Assignment 16 Use the original `quad24`, your improved `quad24` and the adaptive Gauss-Quadarature `AdaptiveGausssQuadrature` from the lecture with $m = 0, 1, 2$ and compare them for the integral problem

$$\int_0^1 \sqrt{x} \log x \, dx$$

for tolerances $10^{-2}, 5 \cdot 10^{-3}, 10^{-3}, \ldots, 10^{-15}$: Plot (log-log-scale) the relative error versus the number of f-evalutations for all methods in one plot. Do a second plot: Plot the error versus the estimated error for all methods.

Solution
see zip-file.