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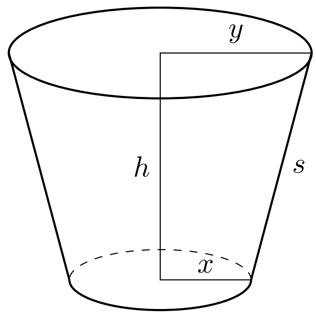
Assignment 25

The surface M and the volume V of a bucket is given by

$$M = \pi(sx + sy + x^2), \quad V = \frac{\pi}{3}(x^2 + xy + y^2)h.$$



Here x is the radius of the bucket's buttom, y is the radius of the top opening, h the height and s the side length of the bucket.
We want to calculate x_* and y_* such that the volume V_* is maximal with the restriction $M = \pi$.

(a) (objective function) Show (please with the help of the computer) that the optimal solution vector (x_*, y_*) is the zero of the function

$$F(x,y) = \begin{pmatrix} (1-4x^2)(x+2y-2y^3) + 4xy^2(x^2-1) + 3xy^4 \\ 2x(1-x^2)^2 + y(1-2x^2+4x^4) + 6xy^2(x^2-y^2) - 3y^5 \end{pmatrix}.$$

(b) (start values) Use the additional restriction $x = y$ (cylindric bucket) to calculate the corresponding maximal volume V_0 and the optimal radius $x_0 = y_0$ for this case.

(c) (automatic differentiation) Download autodiff autodiff[1] from MATLABCentral and install it.

(d) (Newton-Iteration) Use the program `NewtonRelax` from the lecture to calculate the wanted optimum (x_*, y_*) up to a tolerance of 10^{-15}. Use the starting values (x_0, y_0), see (b), for the Newton-Iteration and the `autodiff` tool for the automatic differentiation.

Solution

(a) With the requirement $M = \pi$ we can express s with x and y:

$$M = \pi \iff sx + sy + x^2 = 1 \iff s = (1-x^2)/(x+y).$$

With Pythagoras' formula: $s^2 = (y-x)^2 + h^2$, i.e. $h = \sqrt{s^2 - (y-x)^2}$. With this expression for h one can eliminate h in V:

$$V = V(x,y) = \frac{\pi}{3}(x^2 + xy + y^2)\sqrt{\left(\frac{1-x^2}{x+y}\right)^2 - (y-x)^2}.$$

For finding a maximum we need the zeros of the gradient $\nabla V(x,y)$: The following MATLAB commands

```
syms x y
V = pi/3*(x^2+x*y+y^2)*sqrt(((1-x^2)/(x+y))^2-(y-x)^2);
dx=simplify(diff(V,x));
dy=simplify(diff(V,y));
```

[1] `http://tinyurl.com/MATLABautodiff`

return

$$\partial_x V(x,y) = \frac{\left(-4\,x^3+4\,y^2x^3-8\,x^2y+8\,x^2y^3+x+3\,xy^4-4\,xy^2+2\,y-2\,y^3\right)x\pi}{3\cdot\sqrt{\frac{(-1+y^2)(-y^2+2\,x^2-1)}{(x+y)^2}}\,(x+y)^3}$$

and

$$\partial_y V(x,y) = \frac{\left(2\,x^5+4\,x^4y-4\,x^3+6\,y^2x^3-2\,x^2y+2\,x-6\,xy^4-3\,y^5+y\right)y\pi}{3\cdot\sqrt{\frac{(-1+y^2)(-y^2+2\,x^2-1)}{(x+y)^2}}\,(x+y)^3}.$$

Neither $x=0$ nor $y=0$ are candidates for the maximum. Hence we look for (x_*,y_*) that gives zero in the parenthized expressions in the numerators: $\tilde{F}(x_*,y_*)=0$ with

$$\tilde{F}_1(x,y) := -4\,x^3+4\,y^2x^3-8\,x^2y+8\,x^2y^3+x+3\,xy^4-4\,xy^2+2\,y-2\,y^3$$

and

$$\tilde{F}_2(x,y) := 2\,x^5+4\,x^4y-4\,x^3+6\,y^2x^3-2\,x^2y+2\,x-6\,xy^4-3\,y^5+y.$$

To see $F\equiv\tilde{F}$:

```
Ftilde = numden([dx/x;dy/y])/pi;
F      = F_func([x;y]);
simplify(Ftilde-F)
ans =
 0
 0
```

(b) Because $M=\pi$ and $s\geq 0$ require $x<1$ we get for $x=y$ the volumen $V=V(x,x)=\frac{\pi}{2}(x-x^3)$. This cubic function $[0,\infty[\to\mathbb{R}$, $x\mapsto\frac{\pi}{2}(x-x^3)$ attains its maximum at $x_0=1/\sqrt{3}$ with $V_0=V(x_0)\approx 0.6046$.