신입생 세미나 HW1

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3.4.2 Triangluar Arrays

Our next step is to generalize the central limit theorem to:

Theorem 3.4.10. The Lindeberg-Feller theorem. *For each n, let $X_{n,m}$, $1 \leq m \leq n$, be independent random variables with $EX_{n,m} = 0$.*
Suppose

(i) $\sum_{m=1}^{n} EX_{n,m}^2 \to \sigma^2 > 0$

(ii) *For all $\epsilon > 0$,* $\lim_{n\to\infty} \sum_{m=1}^{n} E(|X_{n,m}|^2; |X_{n,m}| > \epsilon) = 0$.

Then $S_n = X_{n,1} + ... + X_{n,n} \Rightarrow \sigma\chi$ as $n \to \infty$.

Remarks. In words, the theorem says that a sum of a large number of small independent effects has approximately a normal distribution. To see that Theorem 3.4.10 contains our first central limit theorem, let $Y_1, Y_2, ...$ be i.i.d. with $EY_i = 0$ and $EY_i^2 = \sigma^2 \in (0, \infty)$, and let $X_{n,m} = Y_m/n^{1/2}$. Then $\sum_{m=1}^{n} EX_{n,m}^2 = \sigma^2$ and if $\epsilon > 0$

$$\begin{aligned}\sum_{m=1}^{n} E(|X_{n,m}|^2; |X_{n,m}| > \epsilon) &= nE(|Y_1/n^{1/2}|^2; |Y_1/n^{1/2}| > \epsilon) \\ &= E(|Y_1|^2; |Y_1| > \epsilon n^{1/2}) \to 0\end{aligned}$$

by the dominated convergence theorem since $EY_1^2 < \infty$.

Proof. Let $\varphi_{n,m}(t) = E\exp(itX_{n,m})$, $\sigma_{n,m}^2 = EX_{n,m}^2$. By Theorem 3.3.17, it suffices to show that

$$\prod_{m=1}^{n} \varphi_{n,m}(t) \to exp(-t^2\sigma^2/2)$$

Let $z_{n,m} = \varphi_{n,m}(t)$ and $\omega_{n,m} = (1 - t^2\sigma_{n,m}^2/2)$. By (3.3.3)

$$\begin{aligned}|z_{n,m} - \omega_{n,m}| &\leq E(|tX_n, m|^3 \wedge 2|tX_{n,m}|^2) \\ &\leq E(|tX_{n,m}|^3; |X_{n,m}| \leq \epsilon) + E(2|tX_{n,m}|^2; |X_{n,m}| > \epsilon) \\ &\leq \epsilon t^3 E(|X_{n,m}|^2; |X_{n,m} \leq \epsilon|) + 2t^2 E(|X_{n,m}|^2; |X_{n,m}| > \epsilon)\end{aligned}$$

Summing $m = 1$ to n, letting $n \to \infty$, and using (i) and (ii) gives

$$\limsup_{n\to\infty} \sum_{m=1}^{n} |z_{n,m} - \omega_{n,m}| \leq \epsilon t^3 \sigma^2$$

Since $\epsilon > 0$ is arbitrary, it follows that the sequence converges to 0. Our next step is to use Lemma 3.4.3 with $\theta = 1$ to get

$$\left| \prod_{m=1}^{n} \varphi_{n,m}(t) - \prod_{m=1}^{n} (1 - t^2\sigma_{n,m}^2/2) \right| \to 0$$

To check the hypotheses of Lemma 3.4.3, note that since $\varphi_{n,m}$ is a ch.f. $|\varphi_{n,m}(t)| \leq 1$ for all n, m. For the terms in the second product we note that

$$\sigma_{n,m}^2 \leq \epsilon^2 + E(|X_{n,m}|^2; |X_{n,m}| > \epsilon)$$

and ϵ is arbitrary so (ii) implies $\sup_m \sigma_{n,m}^2 \to 0$ and thus if n is large $1 \geq 1 - t^2\sigma_{n,m}^2/2 > -1$ for all m.

To complete the proof now, we apply Exercise 3.1.1 with $c_{m,n} = -t^2\sigma_{n,m}^2/2$. We have just shown $\sup_m \sigma_{n,m}^2 \to 0$. (i) implies

$$\sum_{m=1}^{n} c_{m,n} \to -\sigma^2 t^2/2$$

so $\prod_{m=1}^{n}(1 - t^2\sigma_{n,m}^2/2) \to \exp(-t^2\sigma^2/2)$ and the proof is complete.

□