FINM 348: MODERN APPLIED OPTIMIZATION
FALL 2024
PROBLEM SET 1

1. (a) Let $m, n \in \mathbb{N}$, i.e., positive integers. Find all local and global optimizers, if any, of the following functions on $\mathbb{R}$,

$$f(x) = \left(1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}\right) e^{-x}, \quad g(x) = x^m (1-x)^n, \quad h(x) = \sin^{2m} x \cdot \cos^{2n} x.$$

(b) Find the local and global optimizers, if any, of f on $\mathbb{R}$ and of g on $[-1, 1]$,

$$f(x) = x^{1/3}(1-x)^{2/3}, \quad g(x) = x \sin^{-1} x + \sqrt{1-x^2}.$$

(c) Find the global minimum of the following functions on $\mathbb{R}$,

$$f(x) = -\frac{1}{1+|x|} - \frac{1}{1+|x-1|}, \quad g(x) = \begin{cases} e^{-\frac{1}{|x|}}\left(\sqrt{2} + \frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

2. Every problem in the world is an optimization problem. The key here is to figure out how to formulate it as a *univariate* optimization problem. In the following, every problem should be solved via a *univariate* optimization problem.

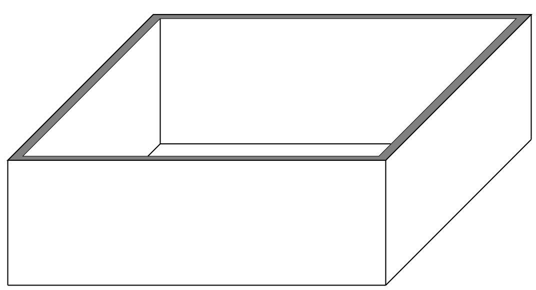
(a) Which number is larger, π^e or e^π?

(b) Show that for any $p \geq 1$ and $a, b \in [0, 1]$,

$$|a^p - b^p| \leq p|a - b|.$$

(c) Write down an expression for the area of a rectangle inscribed in the unit circle as a differentiable univariate function $f(x)$. Hence show that the largest inscribed rectangle in a circle must be a square.

(d) A volume of 76 cubic meters of iron is to be moulded into the shape of a box with a square base, rectangular sides, and no lid. The base and sides must be exactly one meter thick. Find its maximum capacity.



(e) Light travels at different speeds in different media (e.g., air and water). Consider the following scenario depicted in Figure 1. Let v_1 be the speed of light in air and v_2 be the speed of light in water. Write down an expression for time required for light to travel from a point in air to a point in water. Show that the minimum is attained when

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}.$$

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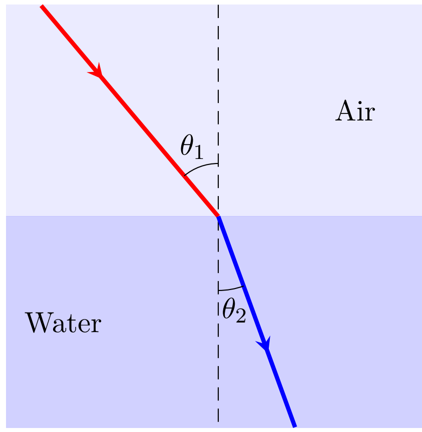


FIGURE 1. Light passing through different media.

3. In all of the following you will need to justify your answers.

(a) Let $\alpha_{n+1} = \alpha_n + \alpha_{n-1}$ with $\alpha_0 = 0$ and $\alpha_1 = 1$. Let $\phi_n := \alpha_{n+1}/\alpha_n$. Is $(\phi_n)_{n=1}^{\infty}$ convergent? If so, is the convergence sublinear, linear, superlinear, or quadratic?

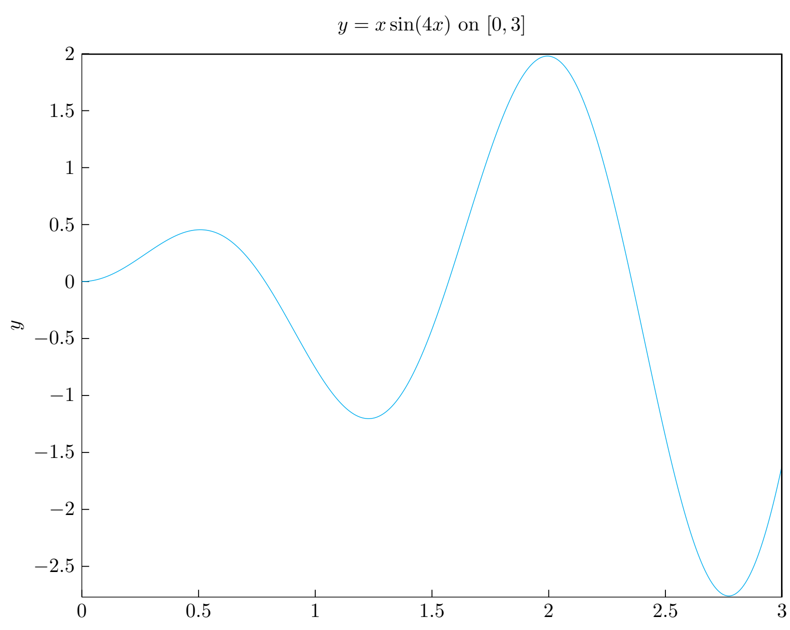
(b) Design an algorithm for computing reciprocal of a positive real number $a > 0$ that requires only $+, -, \times$ (obviously you can't use $\div$). For what values of x_0 do the algorithm converges? Apply your algorithm to find the decimal expansion of $1/12$ to 10 decimal digits of accuracy starting from $x_0 = 0.1$ and $x_0 = 1$. Discuss your results.

(c) Find the number of roots of $2x = 3\sin x$ in $\mathbb{R}$ and determine the largest root to three decimal accuracy with only evaluations of sin, cos, and $+, -, \times, \div$.

4. We will look at methods for local minimizing a univariate function over an interval that are close in spirit to the bisection method for root finding. Suppose we want a local minimizer of

$$\min_{x\in[0,3]} x\sin(4x) \tag{4.1}$$

up to an accuracy of $\varepsilon > 0$. The graph of $f(x) = x\sin(4x)$ is given below:



To find a local minimizer we divide the interval into four equal subintervals

$$[0,3] = [0,0.75] \cup [0.75,1.5] \cup [1.5,2.25] \cup [2.25,3]$$

and evaluate the function at the three interior points $x_1 = 0.75$, $x_2 = 1.5$, $x_3 = 2.25$. See the first row of the table below. Select the interval centered at the interior point with the smallest value (if there is a tie, choose arbitrarily). in this case $x_2 = 1.5$ has the smallest value $f(x_2) \approx -0.4191$ and so we select the interval $[0.75, 2.25]$. Repeat the procedure with the new interval until we obtain an interval with length $\leq \varepsilon$. For example if we set $\varepsilon = 0.02$, then this algorithm gives us a local minimizer $x_* \approx 1.231$ with local minimum is $f(x_*) \approx -1.20354$.

Current Interval	Interior Points			$f(x) = x\sin(4x)$		
	x_1	x_2	x_3	$f(x_1)$	$f(x_2)$	$f(x_3)$
$[0,3]$	0.75	1.5	2.25	0.1058	-0.4191	0.9273
$[0.75,2.25]$	1.125	1.5	1.875	-1.100	-0.4191	1.759
$[0.75,1.5]$	0.9375	1.125	1.313	-0.5358	-1.100	-1.126
$[1.125,1.5]$	1.219	1.313	1.406	-1.203	-1.126	-0.8611
$[1.125,1.313]$	1.172	1.219	1.266	-1.172	-1.203	-1.189
$[1.172,1.266]$	1.196	1.219	1.243	-1.193	-1.203	-1.201
$[1.196,1.243]$	1.208	1.219	1.231	-1.199	-1.20272	-1.20354
$[1.219,1.243]$	1.225	1.231	1.237	-1.20350	-1.20354	-1.2028
$[1.225,1.237]$						

All implementations below should work for any function f on any interval $[a,b]$, with any accuracy $\varepsilon > 0$, and for both minimization and maximization.

(a) Implement the procedure described above as Algorithm A. Show rigorously, i.e., not just by looking at the graph, that the global minimizer of (4.1) lies in $[\frac{7}{8}\pi, 3]$. Now apply Algorithm A to find the global minimizer to within $\varepsilon = 0.01$.

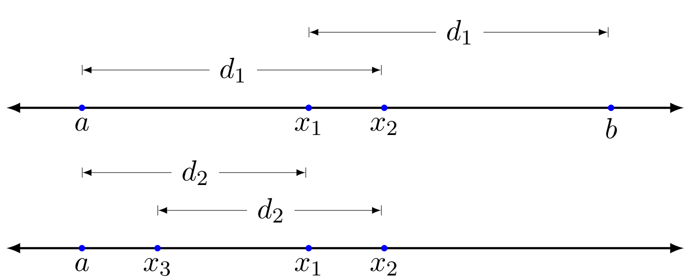
(b) We will now reduce to just two interior points with $a < x_1 < x_2 < b$ but will consider overlapping subintervals $[a, x_2]$ and $[x_1, b]$. We want the lengths of these subintervals to be equal to the r times the length of the original interval for some $r \in (\frac{1}{2}, 1)$, i.e.,

$$x_2 - a = b - x_1 = r(b-a).$$

We will call this common value d_1. Similar to Algorithm A, we will evaluate $f(x_1)$ and $f(x_2)$ and pick our new interval to be the one where the point with the smaller value is an interior point. For example, if $f(x_1) < f(x_2)$, then the new interval is $[a, x_2]$. The next point x_3 will be chosen so that

$$x_2 - x_3 = x_1 - a = r^2(b-a).$$

We will call this common value d_2. The relations are as in the figure below.



Note that if we repeat this procedure, the length of the intervals at kth step will satisfy

$$d_k = r d_{k-1}.$$

Show that we must have $r = 1/\phi$ where $\phi = (1+\sqrt{5})/2$ is the Golden ratio. Implement this procedure as Algorithm B.

(c) We next allow the value r in Algorithm B to change from step to step. Let α_k and ϕ_k be as in Problem 3a. Find the smallest $n \in \mathbb{N}$ so that

$$\alpha_n \varepsilon > b - a$$

and run n steps of Algorithm B but with $r_k = 1/\phi_k$, i.e., instead of reducing the length of the interval at every step by a constant r, we now reduce it by r_k so that

$$d_k = r_k d_{k-1}.$$

Implement this procedure as Algorithm C. What is the relation between Algorithms B and C when $n \to \infty$?

(d) Apply Algorithms A, B, and C to solve the following problems

$$\max_{x\in[0,20]} x(5\pi - x), \qquad \min_{x\in(0,2]} x + \frac{4}{x}, \qquad \max_{x\in[0,\pi]} x^2 \sin x, \qquad \min_{x\in[\frac{3}{2},\frac{9}{2}]} \frac{-1}{(x-1)^2}\left[\log x - \frac{2(x-1)}{x+1}\right]$$

to within accuracy of $\varepsilon = 1$ for the first problem and $\varepsilon = 0.1$ for the other three. Compare the cost of each algorithm in terms of the number of function evaluations required to reach the required level of accuracy for each problem.