OLS Properties

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1. Positive Definite, Trace, Idempotent, Inverse, Matrix Derivative

1.1 Positive Definite

- X'X = Square Matrix = Postive Definite = Convex = Can be minimized
- In OLS assumptions, we requires (X'X)$^{-1}$ $\rightarrow$ 0 to derive consistency of Cov(β)

1.2 Trace

- Trace(scalar $\times$X) = scalar $\times$ Trace(X)
- Trace(X+Z) = Trace(X) + Trace(Z)
- Trace(X') = Trace(C)
- Trace(AB) = Trace(BA)
- We need it when Cov(Y) = $\sigma^2 I$ = Cov(ε) = E($\varepsilon\varepsilon'$) , where E[Trace(e'Me)] = Trace[M]E(e'e)]= (n-k)E(e'e), where n = sample number, k = variable numbers = column dim(X)

1.3 Idempotent Matrix

- We need it when Cov(Y) = $\sigma^2 I$ = Cov(ε) = E($\varepsilon\varepsilon'$) , where E[Trace(e'M'Me)] = E[Trace(e'Me)]
- P $\subset$Idempotent Matrix, then P'P = P
- P $\subset$Idempotent Matrix, then I – P = M, M $\subset$Idempotent Matrix

1.4 Inverse Matrix

- A = (A^{-1})$^{-1}$
- (A')$^{-1}$= (A^{-1})'
- (AB)$^{-1}$=B^{-1}A^{-1}
- |A^{-1}| = $\frac{1}{|A|}$

1.5 Matrix Derivative

- From Prof. Ron's Book p.440, Useful for OLS Loss minimization
- **z** : (k x 1) vector, **A** : (k x j) matrix, **w** : (j x 1) vector
- $\frac{\vartheta z' Aw}{\vartheta z} = Aw$
- $\frac{\vartheta z' Aw}{\vartheta w} = A'w$
- $\frac{\vartheta z' Az}{\vartheta z} = 2Az$, where k=j and A is a symmetric (A'=A)
- $\frac{\vartheta z' w}{\vartheta z} = w$ (k=j)
- $\frac{\vartheta z' w}{\vartheta w} = z$, (k=j)

2. Summarize the classical assumptions of GLM

- E(Y) = xβ and E(ε)=0
- Cov(Y) = $\sigma^2 I$ = Cov(ε) = E($\varepsilon\varepsilon'$)
- x is a fixed matrix of values with rank(x) = k, where k = unknown parameters

3. OLS objective function

- S = e'e, OLS minize S
- S = e'e = (y–xβ)'(y–xβ)
- FOC β: –2X'y + 2X'X$\hat{\beta}$=0 $\mapsto$ X'y = X'X$\hat{\beta}$ $\mapsto$ (X'X)$^{-1}$X'y =$\hat{\beta}$
- SOC β: 2X'X is positive definite

4. Gauss-Markov

- $\hat{\beta}$ in OLS is unbiased and efficient within the class of linear, unbiased estimator
- $\hat{\beta}$ = (X'X)$^{-1}$X'y =(X'X)$^{-1}$X'(Xβ+e) = (X'X)$^{-1}$X'Xβ+(X'X)$^{-1}$X'e
  = β+(X'X)$^{-1}$X'e
- E($\hat{\beta}$) = E(β+(X'X)$^{-1}$X'e) = β + 0 = β , unbias
- Cov($\hat{\beta}$) = E[($\hat{\beta}$–E($\hat{\beta}$))($\hat{\beta}$–E($\hat{\beta}$))'] = E[($\hat{\beta}$–β)($\hat{\beta}$–β)']= E[(X'X)$^{-1}$X'ee'X(X'X)$^{-1}$]
  = (X'X)$^{-1}$X'E(ee')X(X'X)$^{-1}$, where E(ee') = $\sigma^2 I$
  = σ^2(X'X)$^{-1}$X'X(X'X)$^{-1}$= σ^2(X'X)$^{-1}$
- Cov(Y) = E(e'e) = E(e'M'Me)=E(e'Me)=E(tr(e'Me))=E(tr(M)e'e)=tr(M)E(e'e)
  =(n–k)σ^2
- $\frac{\hat{e}'\hat{e}}{n-k} = s^2$, sample variance to make it unbiased estimator of $\sigma^2 \mapsto E(\frac{\hat{e}'\hat{e}}{n-k})$ = E(s^2) = σ^2

5. Consistent estimator of θ

- If $\hat{\theta}$ converges in probability to θ, then we say it is a consistent estimator
- plim $\hat{\beta} \to \beta$, consistent estimator as follows:
  plim$\hat{\beta}$ $= \beta+$ plim(X'X)$^{-1}$X'e $= \beta+$ plim(X'X$\frac{n}{n}$)$^{-1}$X'e $=\beta+$ plim(X'X$\frac{1}{n}$)$^{-1}\frac{X'e}{n}$
  $=\beta$ + plim(X'X$\frac{1}{n}$)$\cdot$0 $=\beta$
- plim $s^2 = \frac{\hat{e}'\hat{e}}{n-k} = \to \sigma^2$, consistent estimator as follows:
  $s^2 = \frac{\hat{e}'\hat{e}}{n-k} = \frac{e'Me}{n-k} = \frac{e'(I-P)e}{n-k} = \frac{e'e-e'X(X'X)^{-1}X'e}{n-k} = \frac{e'e}{n-k} - \frac{e'X(X'X)^{-1}X'e}{n-k}$
  plim $s^2 \to$ plim$(\frac{e'e}{n-k})$ − plim $(\frac{e'X(X'X)^{-1}X'e}{n-k}) \to \sigma^2-$ plim $(\frac{e'X(X'X)^{-1}X'e}{n-k})$
  $\to \sigma^2$, where plim $(\frac{e'X(X'X)^{-1}X'e}{n-k}) = 0$ as $\frac{(X'X)^{-1}}{n} \to 0$

6. R^2

- Simple regression case, 1 exogeneous variable“
- R^2 measures the precentage of variations in $\hat{Y}$ can be explained by the regression model
- SST = total sum of squares = (Y−$\bar{Y}$)(Y−$\bar{Y}$)'
- SSE = error sum of squares = (Y−$\hat{Y}$)(Y−$\hat{Y}$)'
- SSR = sum of square residuals = SST – SSE = ($\hat{Y}$−$\bar{Y}$)($\hat{Y}$−$\bar{Y}$)'
- $R^2 = \frac{SSR}{SST} = \frac{SST-SSE}{SST} = 1 - \frac{SSE}{SST}$

Reference

- Mittelhammer, R.C. and Mittelhammer, R.C., 1996. Mathematical statistics for economics and business (Vol. 78). New York: Springer.
- Fomby, T.B., Hill, R.C. and Johnson, S.R., 2012. Advanced econometric methods. Springer Science & Business Media.