Groups and Rings - SF2729

Homework 5

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Exercise 1. Let $n \geq 1$ and S_n be the permutation group. Describe all group homomorphisms $f : S_n \to Z_3$.

Solution. For $n > 2$

Show it "By hand" for S_1 and S_2

$\Phi(e)\Phi(a)\Phi(e*a) = \Phi(a) \Rightarrow \Phi(e) = e'$ Holds forall group homomorphisms. Basicly saying that identity is mapped to identity.
$S_1 = \{e\}$,$\Phi(e) = e'$ which for Z_3 is 0.
$S_2 = \{e, (12)\}$,$\Phi(e) = 0$ and (12) has order 2 and thus $\Phi((12)(12) = e) = \Phi^2((12)) = 0$
And with $\{0^2 = 0, 1^2 = 2, 2^2 = 4 = 1\}$ we have that $\Phi^2 = 0 \Rightarrow \Phi = 0$

Which show that the homomorphism must be $\Phi(a) = 0$ for S_1 and S_2

Exercise 2. Let G be a finite group. Consider its center $Z(G) = \{g \in G : ga = ag \forall a \in G\}$.
Show that if $G/Z(G)$ is cyclic, then G is abelian.

Solution. $G/Z(G)$ is cyclic $\Rightarrow \exists g \in G/Z(G) : G/Z(G) = \langle g \rangle$
Thus $\exists h \in G : G/Z(G) = \langle hZ(G) \rangle$
that is all cosets of $Z(G)$ is on the form $(hZ(G))^i = h^i Z(G)$.
$x, y \in G$ suppose $x \in h^m Z(G), y \in h^n Z(G)$ that is x and y belongs to cosets. $\exists z_1, z_2 \in Z(G) : x = h^m z_1 and y = h^n z_2$

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\begin{aligned}
xy &= h^m z_1 h^n z_2 = (z_1 \in Z(G) \Rightarrow z_1 \text{commutes} \forall h \in G)\\
xy &= h^m h^n z_1 z_2\\
xy &= h^{m+n} z_1 z_2
\end{aligned}
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We have the same thing in the same way with yx and thus $xy = yx$ $\forall x, y \in G$ making G abelian $\quad \square$.