Groups and Rings - SF2729

Homework 5 (Rings)

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Exercise 1. Let E be an extension field of F, and let $\alpha, \beta \in E$. Suppose α is transcendental over F but algebraic over $F(\beta)$. Show that β is algebraic $F(\alpha)$.

Solution. α algebraic over $F(\beta) \Rightarrow \exists \alpha : poly(\alpha) = 0, poly \neq 0$, generally

$$c_0 + c_1\alpha + c_2\alpha^2 + \cdots + c_n\alpha^n = 0, c_i \in F(\beta) \tag{1}$$

$c_i \in F(\beta) \Leftrightarrow c_i = c_{0j} + c_{1j}\beta + c_{2j}\beta^2 + \cdots + c_{mj}\beta^m$ with $c_i \in F$.

$$(c_{0bajs}+c_{1bajs}\beta+c_{2bajs}\beta^2+\cdots+c_{mbajs}\beta^m_{0hmm}+(c_{0bajs}+c_{1bajs}\beta+c_{2bajs}\beta^2+\cdots+c_{mbajs}\beta^m_{1hmm}\alpha+(c_{0bajs}+c_{1bajs}\beta \tag{2}$$

Exercise 2. Let E be a finite extension field of F. Let D be an integral domain : $F \subseteq D \subseteq E$. Show that D is a field.

Solution. To show this we only need to show that $\alpha \in D\backslash\{0\} \Rightarrow \alpha^{-1} \in D$ since we know that D is commutative and has unity from it's integral domain properties. E is finite extension over $F \Rightarrow \alpha$ algebraic over F. With $deg(\alpha, F) = n$ Theorem 30.23 gives us

$$F(\alpha) = \{a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_{n-1}\alpha^{n-1} | a_i \in F\} \tag{3}$$

$\alpha^{-1} \in F(\alpha)$, that is α^{-1} can be written as a polynomial of α with coeffs in F, and is in $D \quad \square$