

# Groups and Rings - SF2729

## Homework 5

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**Exercise 1.** Let  $n \geq 1$  and  $S_n$  be the permutation group. Describe all group homomorphisms  $f : S_n \rightarrow \mathbb{Z}_3$ .

*Solution.* For  $n > 2$ :

Homomorphism:  $\forall a, b \in G, \Phi(ab) = \Phi(a)\Phi(b) \in \mathbb{Z}_3$

also always holds for  $a = b \Rightarrow \Phi(a^2) = \Phi(a)^2 = \{\text{By matching inverse and square table of } \mathbb{Z}_3\} = \Phi(a)^{-1}$  and this gives us (by left multiplying with  $\Phi(a)$  and using the homomorphism property)

$$\Phi(a^3) = 0 \tag{1}$$

$\Phi$  is a homomorphism then  $\Phi[G] \leq G$  by the first isomorphism theorem.

The only subgroups of  $\mathbb{Z}_3$  is the trivial and the non-proper one. The first isomorphism theorem gives that  $\text{im}(\Phi) = \mathbb{Z}_3 \Rightarrow \ker(\Phi) = \{e\}$  Assume that  $\text{im}(\Phi) = \mathbb{Z}_3$  and show a contradiction.

$$\ker(\Phi) = \{e\} \Rightarrow \left( \Phi(a) = 0 \Rightarrow a = e \right)$$

Choose a non-trivial  $a : a^3 \neq e$  (that is any element with order  $\neq 3$  nor 1, and since  $|S_n| = n!$  it will exist) But we have the property from (1) which also holds for the above choosed  $a$  which gives us:

$\exists b \in S_n \setminus \{e\} : \Phi(b) = 0$  which is a contradiction and the initial assumption that  $\text{im}(\Phi) = \mathbb{Z}_3$  must thus be false leaving us with  $\text{im}(\Phi) = 0$  or in other terms  $\Phi[S_n] = \{0\}$  that is all elements are mapped to 0 which gives us with the trivial homomorphism  $\Phi(a) = 0 \quad \square$

And for  $n \leq 2$  simply show it "By hand" for  $S_1$  and  $S_2$

$\Phi(e)\Phi(a)\Phi(e*a) = \Phi(a) \Rightarrow \Phi(e) = e'$  Holds for all group homomorphisms. Basically saying that identity is mapped to identity.

$S_1 = \{e\}, \Phi(e) = e'$  which for  $\mathbb{Z}_3$  is 0.

$S_2 = \{e, (12)\}$ ,  $\Phi(e) = 0$  and  $(12)$  has order 2 and thus  $\Phi((12)(12) = e) = \Phi^2((12)) = 0$   
 And with  $\{0^2 = 0, 1^2 = 2, 2^2 = 4 = 1\}$  we have that  $\Phi^2 = 0 \Rightarrow \Phi = 0$

Which show that the homomorphism must be  $\Phi(a) = 0$  for  $S_1$  and  $S_2$  also.  $\square$

**Exercise 2.** Let  $G$  be a finite group. Consider its center  $Z(G) = \{g \in G : ga = ag \forall a \in G\}$ .

**Show that if  $G/Z(G)$  is cyclic, then  $G$  is abelian.**

*Solution.*  $G/Z(G)$  is cyclic  $\Rightarrow \exists g \in G/Z(G) : G/Z(G) = \langle g \rangle$

Thus  $\exists h \in G : G/Z(G) = \langle hZ(G) \rangle$

that is all cosets of  $Z(G)$  is on the form  $(hZ(G))^i = h^i Z(G)$ .

$x, y \in G$  suppose  $x \in h^m Z(G), y \in h^n Z(G)$  that is  $x$  and  $y$  belongs to cosets.  $\exists z_1, z_2 \in Z(G) : x = h^m z_1 \text{ and } y = h^n z_2$

$$xy = h^m z_1 h^n z_2 = (z_1 \in Z(G) \Rightarrow z_1 \text{ commutes } \forall h \in G)$$

$$xy = h^m h^n z_1 z_2$$

$$xy = h^{m+n} z_1 z_2$$

We have the same thing in the same way with  $yx$  and thus  $xy = yx \forall x, y \in G$  making  $G$  abelian  $\square$ .