Groups and Rings - SF2729

Homework 6

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March 7, 2012

Exercise 1. Let $H = \langle (12) \rangle \leq S_3$ and $K = \langle (123) \rangle \leq S_3$
Consider the S_3-set given by $S_3/H \times S_3/K$ Write this S_3-set as a disjoint union of transitive S_3-sets.

Solution. $h = (12)$ and $k = (123)$

General rules used:
$b\langle b \rangle = \langle b \rangle$
$b^{-1}\langle b \rangle = \langle b \rangle$ ($\langle b \rangle$ is cyclic)

$$S_3/H = \{H, (13)H, (23)H\}$$

$$S_3/K = \{K, (12)K\}$$

$$\begin{aligned} S_3/H \times S_3/K = & \\ & \{(H, K) && ,(H, (12)K), \\ & ((13)H, K) && ,((13)H, (12)K), \\ & ((23)H, K) && ,((23)H, (12)K)\} \end{aligned}$$

Transitive if $\forall x_1, x_2 \in S_3{-}\text{set} \exists g \in S_3 : gx_1 = x_2$

$(12)(H, K) = (hH = h\langle h \rangle, hK) = (H, (12)K)$
$(13)((13)H, (12)K) = (H, (13)(12)K = (123)K = K) = (H, K)$
$(23)((23)H, (12)K) = (H, (23)(12)K = (132)K = k^{-1}K = K) = (H, K)$
$(12)((23)H, (12)K) = ((12)(23)H = (123)H = (123)(12)H = (13)H, K) = ((13)H, K)$

$(23)((23)H, K) = (H, (23)k^{-1}K = (23)(132)K = (12)K) = (H, (12)K)$

Since G is closed under the operation one can take any combination of these g_i (or its inverse if you want to go in the other direction) to get from $\forall x_1$ to $\forall x_2$ with some $g = g_2 g_1 \in G$ and all elements are therefore transitive. (One can easily show that these are connected by drawing an arrow from $\rightarrow$ to (and a backarrow with the inverse permutation) the elements in the five equations above in the listing of the set above.)

So the resulting disjoint union of transitive S_3-sets being simply (without union):

$$
\begin{aligned}
S_3/H \times S_3/K = & \\
& \{(H, K) && ,(H, (12)K), \\
& ((13)H, K) && ,((13)H, (12)K), \\
& ((23)H, K) && ,((23)H, (12)K)\}
\end{aligned}
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