Groups and Rings - SF2729

Homework 5 (Rings)

Jim Holmström - 890503-7571

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Exercise 1. Let E be an extension field of F, and let $\alpha, \beta \in E$. Suppose α is transcendental over F but algebraic over $F(\beta)$. Show that β is algebraic $F(\alpha)$.

Solution. α algebraic over $F(\beta) \Rightarrow \exists \alpha : poly(\alpha) = 0, poly \neq 0$, generally

$$c_0 + c_1\alpha + c_2\alpha^2 + \cdots + c_n\alpha^n = 0, c_i \in F(\beta) \tag{1}$$

$$c_i \in F(\beta) \Leftrightarrow c_i = c_{0i} + c_{1i}\beta + c_{2i}\beta^2 + \cdots + c_{mi}\beta^m, c_{ij} \in F \tag{2}$$

By observing the expressions in index notation

$$\sum_j \left(\sum_i c_{ij}\beta^i \right) \alpha^j = \sum_{i,j} c_{ij}\beta^i\alpha^j = \sum_{i,j} c_{ij}\alpha^j\beta^i = \sum_i \left(\sum_j c_{ij}\alpha^j \right) \beta^j \tag{3}$$

and written out more explicity it becomes

$$\begin{aligned}
&(c_{00} + c_{10}\beta + c_{20}\beta^2 + \cdots + c_{m0}\beta^m)+ \\
&(c_{01} + c_{11}\beta + c_{21}\beta^2 + \cdots + c_{m1}\beta^m)\alpha+ \\
&(c_{02} + c_{12}\beta + c_{22}\beta^2 + \cdots + c_{m2}\beta^m)\alpha^2+ \\
&\cdots \\
&(c_{0n} + c_{1n}\beta + c_{2n}\beta^2 + \cdots + c_{mn}\beta^m)\alpha^n
\end{aligned} \tag{4}$$

rearranged to the expression

$$\begin{aligned}
&(c_{00} + c_{01}\alpha + c_{02}\alpha^2 + \cdots + c_{0n}\alpha^n)+ \\
&(c_{10} + c_{11}\alpha + c_{12}\alpha^2 + \cdots + c_{1n}\alpha^n)\beta+ \\
&(c_{20} + c_{21}\alpha + c_{22}\alpha^2 + \cdots + c_{2n}\alpha^n)\beta^2+ \\
&\cdots \\
&(c_{m0} + c_{m1}\alpha + c_{m2}\alpha^2 + \cdots + c_{mn}\alpha^n)\beta^m
\end{aligned} \tag{5}$$

going back to the index notation we have

$$\left(\sum_j c_{ij}\alpha^j\right) \in F(\alpha) \tag{6}$$

and this shows that β is algebraic over $F(\alpha)$ $\square$

Exercise 2. Let E be a finite extension field of F. Let D be an integral domain : $F \subseteq D \subseteq E$. Show that D is a field.

Solution. To show this we only need to show that $\alpha \in D\backslash\{0\} \Rightarrow \alpha^{-1} \in D$ since we know that D is commutative and has unity from it's integral domain properties. E is finite extension over $F \Rightarrow \alpha$ algebraic over F. With $deg(\alpha, F) = n$ Theorem 30.23 gives us

$$F(\alpha) = \{a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_{n-1}\alpha^{n-1} | a_i \in F\} \tag{7}$$

$\alpha^{-1} \in F(\alpha)$, that is α^{-1} can be written as a polynomial of α with coeffs in F, and is in D $\square$