Groups and Rings - SF2729

Homework 1

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Exercise 1. Give a table for binary operation on the set $\{e, a, b\}$ of three elements satisfying axioms $\mathcal{G}_2$ (existence of a unit) and $\mathcal{G}_3$ (existence of inverses) but not the axiom $\mathcal{G}_1$ (associativity)

Solution. e (identity element) in each row/colon of the table $\Rightarrow \mathcal{G}_2 \wedge \mathcal{G}_3$ Then we just make the same element appear twice $\Rightarrow$ table unable to represent a group, that together with $\mathcal{G}_{1:3} =$ Group $\Rightarrow \neg\mathcal{G}_1$ $\square$

*	e	a	b
e	e	a	b
a	a	e	b
b	b	a	e

Table 1: Example of a $\mathcal{G}_2 \wedge \mathcal{G}_3 \wedge \neg\mathcal{G}_1$

Exercise 2. Let G be a group with a finit number of elements. Show that $\forall a \in G \exists n \in \mathbb{Z}_+ : a^n = e$ where the symbol a^n denotes the multiplication of a n-times.

Solution. Since $|G| < \infty$ not all a^n can be different for all positive n. Thus $\exists n_1 > n_2 : a^{n_1} = a^{n_2}$. Using the group axioms one gets, $a^{n_1 - n_2} = e$ which shows that it exists a $n = n_1 - n_2 > 0$ which satisfies this $\forall a$ in the finite group. $\square$

Exercise 3. $a, b \in G$ **(where G is a group) Show** $(a * b)^2 = a^2 * b^2 \Rightarrow a * b = b * a$

Solution.

$$\begin{aligned}
(a*b)^2 &= a^2 * b^2 \\
(a*b)*(a*b) &= (a*a)*(b*b) && \text{(Associativity of groups)} \\
a*((b*a)*b) &= a*((a*b)*b) && \text{(Cancel out a then b)} \\
b*a &= a*b
\end{aligned}$$

$\square$

Exercise 4. Let G be a group and let g be one fixed element of G. Show that the map $i_g : G \to G$ such that $i_g(x) = gxg^{-1}$ for $x \in G$ is an isomorphism.

Solution. Let $a, b \in G$
$i_g(a) = i_g(b) \Rightarrow g * a * g^{-1} = g * b * g^{-1}$ cancelate (since $g, g^{-1} \in G$) $\Rightarrow a = b$ and thus i_g is one-to-one.
$i_g(g^{-1} * a * g) = g * g^{-1} * a * g * g^{-1} = a$ i_g maps G onto G.
$i_g(a*b) = g*a*b*g^{-1} = g*a*(g^{-1}*g)*b*g^{-1} = (g*a*g^{-1})*(g*b*g^{-1}) = i_g(a)*i_g(b)$ and thereby satisfies the homomorphism.
$\therefore$ i_g is an homomorphism, it's one-to-one and onto and are therefore a isomorphism. $\square$