Groups and Rings - SF2729

Homework3 (Rings)

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Exercise 1. Let R be a commutative ring with unity of prime characteristic p. Show that the map $\phi_p : R \to R$ given by $\phi_p(a) = a^p$ is a homomorphism.

Solution. In a commutative ring

$$(a+b)^n = \sum \binom{n}{k} a^i b^{n-i} \tag{1}$$

holds. Where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{2}$$

and we can see that we have if n is prime.

$$p \Big| \binom{p}{i} \quad \forall i \in [1, p-1] \tag{3}$$

and hence the terms

$$\binom{p}{i} a^i b^{p-i} = 0 \tag{4}$$

in a commutative ring with characteristic p. This gives us the "freshman's dream"

$$(a+b)^p = a^p + b^p \tag{5}$$

With these facts its easy to show that ϕ is a homomorphism.

$$\phi_p(a+b) = (a+b)^p = a^p + b^p = \phi_p(a) + \phi_p(b) \tag{6}$$

and trivially since R is commutative

$$\phi_p(ab) = (ab)^p = a^p b^p = \phi_p(a)\phi_p(b) \tag{7}$$

and thus ϕ_p is a homomorphism. $\square$

Exercise 2. Prove that if F is a field, every proper nontrivial prime ideal of $F[x]$ is maximal.

Solution. By theorem 26.24 from the book every ideal of $F[x]$ is principal. Suppose $\langle f(x) \rangle \neq 0$ is a proper prime ideal of $F[x] \Rightarrow$ all polynomial in $\langle f(x) \rangle$ has degree equal or greater than the degree of $f(x)$. Thus if we have $f = gh$ in $F[x]$ with both g, h degrees less than f, neither g nor h can be in $\langle f(x) \rangle$. Which is a contradiction on the fact that we said it was an prime ideal $\Rightarrow$ no such factorization in $F[x]$ could exist $\Rightarrow$ f irreducible in $F[x]$ and by theorem 26.25 from the book $\langle f(x) \rangle$ is therefore a maximal ideal of $F[x]$. □