Groups and Rings - SF2729

Homework 5

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Exercise 1. Let $n \geq 1$ and S_n be the permutation group. Describe all group homomorphisms $f : S_n \to Z_3$.

Solution. For $n > 2$:

Homomorphism: $\forall a, b \in G, \Phi(ab) = \Phi(a)\Phi(b) \in \mathbb{Z}_3$
also always holds for $a = b \Rightarrow \Phi(a^2) = \Phi(a)^2 = \{$By matching inverse and square table of $\mathbb{Z}_3\} = \Phi(a)^{-1}$ and this gives us(by left multiplying with $\Phi(a)$ and using the homomorphism property)

$$\Phi(a^3) = 0 \tag{1}$$

Φ is a homomorphism then $\Phi[G] \leq G$ by the first isomorphism theorem.
The only subgroups of $\mathbb{Z}_3$ is the trivial and the non-proper one. The first isomorphism theorem gives that $im(\Phi) = \mathbb{Z}_3 \Rightarrow ker(\Phi) = \{e\}$ Assume that $im(\Phi) = \mathbb{Z}_3$ and show a contradiction.
$ker(\Phi) = \{e\} \Rightarrow \left(\Phi(a) = 0 \Rightarrow a = e\right)$
Choose a non-trivial $a : a^3 \neq e$ (that is any element with order $\neq 3$ nor 1, and since $|S_n| = n!$ it will exist) But we have the property from (1) which also holds for the above choosed a which gives us:
$\exists b \in S_n \backslash \{e\} : \Phi(b) = 0$ which is a contradication and the initial assumption that $im(\Phi) = \mathbb{Z}_3$ must thus be false leaving us with $im(\Phi) = 0$ or in other terms $\Phi[S_n] = \{0\}$ that is all elements are mapped to 0 which gives us with the trivial homomorphism $\Phi(a) = 0$ $\square$

And for $n \leq 2$ simply show it "By hand" for S_1 and S_2

$\Phi(e)\Phi(a)\Phi(e*a) = \Phi(a) \Rightarrow \Phi(e) = e'$ Holds forall group homomorphisms. Basicly saying that identity is mapped to identity.
$S_1 = \{e\}$,$\Phi(e) = e'$ which for Z_3 is 0.

$S_2 = \{e, (12)\}$,$\Phi(e) = 0$ and (12) has order 2 and thus $\Phi((12)(12) = e) = \Phi^2((12)) = 0$
And with $\{0^2 = 0, 1^2 = 2, 2^2 = 4 = 1\}$ we have that $\Phi^2 = 0 \Rightarrow \Phi = 0$

Which show that the homomorphism must be $\Phi(a) = 0$ for S_1 and S_2 also. $\square$

Exercise 2. Let G be a finite group. Consider its center $Z(G) = \{g \in G : ga = ag \forall a \in G\}$.
Show that if $G/Z(G)$ is cyclic, then G is abelian.

Solution. $G/Z(G)$ is cyclic $\Rightarrow \exists g \in G/Z(G) : G/Z(G) = \langle g \rangle$
Thus $\exists h \in G : G/Z(G) = \langle hZ(G) \rangle$
that is all cosets of $Z(G)$ is on the form $(hZ(G))^i = h^i Z(G)$.
$x, y \in G$ suppose $x \in h^m Z(G), y \in h^n Z(G)$ that is x and y belongs to cosets. $\exists z_1, z_2 \in Z(G) : x = h^m z_1 and y = h^n z_2$

$$\begin{aligned}
xy &= h^m z_1 h^n z_2 = (z_1 \in Z(G) \Rightarrow z_1 \text{commutes} \forall h \in G) \\
xy &= h^m h^n z_1 z_2 \\
xy &= h^{m+n} z_1 z_2
\end{aligned}$$

We have the same thing in the same way with yx and thus $xy = yx$ $\forall x, y \in G$ making G abelian $\square$.