Groups and Rings - SF2729

Homework 4 (rings)

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Exercise 1.

Solution.

$$\{ACC \Leftrightarrow MC \Leftrightarrow FBC\} \Leftrightarrow \{ACC \Rightarrow MC, MC \Rightarrow FBC, FBC \Rightarrow ACC\} \tag{1}$$

$ACC \Rightarrow MC:$ Instead show it by a rewrite

$$\{ACC \Rightarrow MC\} \Leftrightarrow \{\neg MC \Rightarrow \neg ACC\} \tag{2}$$

If $\neg MC$ then $\exists S$ of ideals with no ideal not properly contained in any other ideal of S so the sequence of ideals extends to infinity $\Rightarrow \neg ACC$

$MC \Rightarrow FBC:$ As above show it by a rewrite

$$\{MC \Rightarrow FBC\} \Leftrightarrow \{\neg FBC \Rightarrow \neg MC\} \tag{3}$$

If $\neg FBC \exists N \lhd R$ having no finite generating set. Let

$$b_1 \in N, \langle b_1 \rangle = N_1, r \in R \tag{4}$$

By the properties of ideal we have

$$rb_1 \in N \forall r \Leftarrow \langle b_1 \rangle = N_1 \subset N \tag{5}$$

$N_1 \subset N$ since otherwise it would contradict FBC by b_1 generating N. Let

$$b_2 \in N \ N_2, \langle b_1, b_2 \rangle = N_2 \tag{6}$$

then $N_2 \subset N$ for the same reason as for "$N_1 \subset N$".

$$b_2 \notin N_1 \Rightarrow N_1 \subset N_2 \subset N \tag{7}$$

N is not finitely generated we can continue with the above to infinity $\Rightarrow N_1 \subset N_2 \subset \ldots \subset N$ That is $N_i \subset N_{i+1}$ which $\Rightarrow \neg MC$

$FBC \Rightarrow ACC:$ Let the chain of ideals in R be

$$N_1 \subseteq N_2 \subseteq \ldots, N_i \lhd R \tag{8}$$

$$\bigcup N_i = N \lhd R \tag{9}$$

Let $B_N = \{b_i | i \in [1, n]\}$ be a finite basis for N then suppose $b_i \in N_i$ and let r be the maximum subscript i then $B_n \subseteq N_r$ and we have (9) (all ideals containing the elements from B_N) $\Rightarrow N_r = R$ and $N_r = N_{r+i} \Rightarrow ACC$

Using (1) to show the wanted expression $\square$

Exercise 2.

Solution. a) Let $\gamma + \langle\alpha\rangle$ be a coset of $\mathbb{Z}[i] / \langle\alpha\rangle$. The division algorithm gives us

$$\gamma = \alpha\sigma + \rho, \rho = 0 \vee N(\rho) < N(\alpha) \tag{10}$$

then

$$\gamma + \langle\alpha\rangle = \rho + \sigma\alpha + \langle\alpha\rangle \tag{11}$$

we have $\sigma\alpha \in \langle\alpha\rangle$

$$\gamma + \langle\alpha\rangle = \rho + \langle\alpha\rangle \tag{12}$$

thus all cosets contains a representative with norm $< N(\alpha)$.

$$\#\{a \in \mathbb{Z}[i] \, | norm < N(\alpha)\} < \inf \Rightarrow |\mathbb{Z}[i] / \langle\alpha\rangle| < \inf \tag{13}$$

$\square$

b) Let

$$\pi \text{ irreducible in } \mathbb{Z}[i], \langle\mu\rangle \text{ ideal in } \mathbb{Z}[i] : \langle\pi\rangle \subseteq \langle\mu\rangle \tag{14}$$

$$\mathbb{Z}[i] \text{ is PID} \Rightarrow \text{ideal is principal } \forall \text{ideal} \tag{15}$$

then $\pi \in \langle\mu\rangle$ or $\pi = \mu\beta$, π irreducible $\Rightarrow \{\mu \text{ is a unit}\} \vee \{\beta \text{ is a unit}\}$.

(μ is a unit) $\langle\mu\rangle = \mathbb{Z}[i]$

(β is a unit) $\mu = \pi\beta^{-1}$ so $\mu \in \langle\pi\rangle, \langle\mu\rangle = \langle\pi\rangle$

this shows that $\langle\pi\rangle$ is a maximal ideal of $\mathbb{Z}[i] \Rightarrow \mathbb{Z}[i] / \langle\pi\rangle$ is a field. $\square$

c) i.

$$\mathbb{Z}[i] / \langle 3 \rangle \tag{16}$$

$3, 3i \in \langle 3 \rangle \Rightarrow \forall$ cosets is on the form $a + bi \quad a, b \in \{0, 1, 2\}$ and thus there $|\{0, 1, 2\} \times \{0, 1, 2\}| = 3^2 = 9$ unique elements. And the characteristic is

$$\text{argmin}_{c>0}(\sum^{c} 1 = 0) \tag{17}$$

and we have $\sum^3 1 = 0$ and thus the characteristic is 3 $\square$

ii.

$$\mathbb{Z}[i]/\langle 1+i\rangle \tag{18}$$

We know from previous exercise that each coset contains representative with norm less than $N(1+i)=2$.
The only nonzero elements with norm less than 2 are $\{\pm 1, \pm i\}$. Observing

$$\pm i = \pm(-1+(1+i)) \tag{19}$$

Which gives us the possible nontrivial cosets $\pm 1 + \langle 1+i\rangle$ but we have that

$$1+\langle 1+i\rangle - (-1+\langle 1+i\rangle) = 2+\langle 1+i\rangle = (1+i)(1-i)+\langle 1+i\rangle \tag{20}$$

Which gives us the cosets $\langle 1+i\rangle, 1+\langle 1+i\rangle$ and thus the ring has order 2 and characteristic 2 $\square$

iii.

$$\mathbb{Z}[i]/\langle 1+2i\rangle \tag{21}$$

We know from previous exercise that each coset contains representative with norm less than $N(1+2i)=5$.
Elements are on the form $a+bi \quad (a,b)\in\{0,\pm 1\}^2$ or $\{\pm 2\}\times\{0\}$ or $\{0\}\times\{\pm 2\}$
Breaking out $(1+2i)$ out of all these possible we get that every coset has the representatives $\{0,\pm 1,\pm 2\}$ that is the ring has 5 elements and characteristic of 5
$\square$