Groups and Rings - SF2729

Homework 3

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Exercise 1. Prove that $Z(G) \leq G$ and that it's a commutative group.

Solution. $Z(G)$ is called the center of G in algebra.

Associativity is trivially inherited from G.
e satisfies $ex = xe \forall x \in G \Rightarrow e \in Z(G)$.
With $x, y \in G$ we have $(xy)g = x(yg) = x(gy) = (xg)y = (gx)y = g(xy) \forall g \in G \Rightarrow xy \in Z(G)$ i.e., $Z(G)$ is closed under the group-operation.
$x \in Z(G)$ then $gx = xg \forall g \in G$ multiplying both from left and right in the equation with, from the original group's, inverse of x i.e, x' wich gives $x'g = gx'$ which gives x' the property needed to satisfy $Z(G)$ and thus $x' \in Z(G)$
This shows that $Z(G)$ is a group. $\square$

From the definition of the center we have: $Z(G) = \{z | \forall g \in G, zg = gz\}$ and thus we have that $xy = yx \forall x, y \in Z(G)$ since we have $xy = yx \forall x \in Z(G) \forall y \in G \supset Z(G)$ from the definition of the center we have that it's abelian. $\square$

Exercise 2. Show $Z(S_3) = \{e\}$.

Solution. $S_3 = \{Id, (12), (13), (23), (123), (132)\}$.
$|S_3| = 6 = 3!$

Knowing $\forall$ groups G, $Z(G)$ is commutative (and a group) and thus are $Z(S_3)$ also. Trivial properties used thru out the calculations: $(ab) = (ba)$,$(ab)^2 = Id$ and $\prod_{i=z}^{b}(ai) = (ab..z)$

$(ab)(ac) = (acb)$ but $(ac)(ab) = (abc)$ and thus all the elements on this form has a corresponding element which makes it fail to be in $Z(G)$ which is $(12), (23)$ and (23).

Showing backwards that:

$$
\begin{aligned}
(ab)(abc) &\neq (abc)(ab) \\
(ab)(ac)(ab) &\neq (ac)(ab)(ab) = (ac) \\
(ab)(ac) &\neq (ac)(ab)
\end{aligned}
$$

Which we have shown in previous calculations. So all elements that is on the form above is non-commutative. (123) against (12) and (132) against (13) is non-commutative and thus $(123), (132) \notin Z(G)$ since it $\exists g \in G$ such that the commutative property doesn't hold.

This leaves us with e which trivially holds the property for the center and also we know that $Z(G)$ is a group and thus must have a unit. $Z(S_3) = \{e\} \quad \square$