Homework 1
Statistical Methods in Applied Computer Science
DD2447

Jim Holmström 890503-7571
jimho@kth.se

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Exercise 2.16 Mean, mode, variance for the beta distribution
Suppose $\theta \sim Beta(a, b)$. Derive the mean, mode and variance.

Solution. Denote $Beta(a, b) = Beta_{a,b}$ to simplify the expressions later on. The mean of a random variable sampled from $Beta(a, b)$ is defined as:

$$\mathrm{E}\,\theta = \int_{\Omega} \theta' Beta_{a,b}(\theta') d\theta' \tag{1}$$

where

$$Beta_{a,b}(\theta) = \frac{\theta^{a-1}(1-\theta)^{b-1}}{B(a,b)}, B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{2}$$

and Ω is the support $[0, 1]$

Lemma 0.1.

$$B(a+1,b) = B(b,a) \bigwedge B(a+1,b) = \frac{a}{a+b} B(a,b) \tag{3}$$

Proof. The first relation is given by the symmetry in the definition of B together with a commutative multiplication. For the second relation we first need to show the well known identity

$$\Gamma(z+1) = z\Gamma(z) \tag{4}$$

By the definition of $\Gamma(z) \triangleq \int\limits_0^\infty u^{z-1} e^{-u} du$ and by using partial integration we get

$$\Gamma(z+1) = \int u^z e^{-u} du = \big[-u^z e^{-u}\big]_{u=0}^{\infty} + \int\limits_0^\infty z u^{z-1} e^{-u} du = z \int\limits_0^\infty u^{z-1} e^{-u} du = z\Gamma(z) \tag{5}$$

Having this relation we can show that

$$B(a+1,b)=\frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+1+b)}=\frac{a\Gamma(a)\Gamma(b)}{(a+b)\Gamma(a+b)}=\frac{a}{a+b}B(a,b) \tag{6}$$

□

Starting out by deriving the mean:

$$\begin{aligned}\mathrm{E}\,\theta=\int\theta' Beta_{a,b}(\theta')d\theta'=\int\frac{B(a+1,b)Beta_{a+1,b}(\theta')}{B(a,b)}d\theta'=\\ =\frac{B(a+1,b)}{B(a,b)}\int Beta_{a+1,b}(\theta')d\theta'=\frac{B(a+1,b)}{B(a,b)}=\{\text{Using Lemma 0.1}\}=\\ =\frac{aB(a,b)}{(a+b)B(a,b)}=\frac{a}{a+b}\end{aligned} \tag{7}$$

Next we derive the mode which is the most occurring θ

$$Mode(\theta)=\underset{\theta'}{argmax}(Beta_{a,b}(\theta')) \tag{8}$$

To find it we use

$$\frac{\partial Beta_{a,b}(\theta)}{\partial\theta}=0 \tag{9}$$

$$\begin{aligned}0=B(a,b)\frac{\partial Beta_{a,b}(\theta)}{\partial\theta}=\{\text{prod. rule}\}=\\ =(a-1)\theta^{a-2}(1-\theta)^{b-1}-(b-1)\theta^{a-1}(1-\theta)^{b-2}\end{aligned} \tag{10}$$

which gives us the equation

$$(a-1)\theta^{a-2}(1-\theta)^{b-1}=(b-1)\theta^{a-1}(1-\theta)^{b-2} \tag{11}$$

$$(a-1)(1-\theta)=(b-1)\theta \tag{12}$$

$$\theta=\frac{1}{1+\frac{b-1}{a-1}}=\frac{a-1}{a-1+b-1}=\frac{a-1}{a+b-2} \tag{13}$$

Finally we deal with the variance:

$$\begin{aligned}\mathrm{Var}\,(\theta)=\mathrm{E}\,\theta^2-(\mathrm{E}\,\theta)^2=\int\theta'^2 Beta_{a,b}(\theta')d\theta'-(\mathrm{E}\,\theta)^2=\\ =\frac{B(a+2,b)}{B(a,b)}-(\mathrm{E}\,\theta)^2=\frac{(a+1)}{(a+b+1)}\frac{a}{(a+b)}-\left(\frac{a}{a+b}\right)^2=\\ =\frac{a}{a+b}\left(\frac{(a+1)(a+b)-a(a+b+1)}{(a+b)(a+b+1)}\right)=\frac{ab}{(a+b)^2(a+b+1)}\end{aligned} \tag{14}$$

Exercise 3.6 MLE for the Poisson distribution
The Poisson pmf is defined as $Poi(x|\lambda) = e^{-\lambda}\frac{\lambda^x}{x!}$ for $x \in \{0, 1, 2, ..\}$ where $\lambda > 0$ is the rate parameter. Derive the MLE.

Solution.

Exercise 3.7 Bayesian analysis of the Poisson distribution
In exercise 3.6, we defined the Poisson distribution with rate λ and derived its MLE. Here we perform a conjugate Bayesian analysis.

a. Derive the posterior $p(\lambda|D)$ assuming a conjugate prior $p(\lambda) = Ga(\lambda|a, b) \propto \lambda^{a-1}e^{-\lambda b}$. Hint: the posterior is also a Gamma distribution.

b. What does the posterior mean tend to as $a \to 0$ and $b \to 0$? (Recall that the mean of a $Ga(a, b)$ distribution is a/b.)

Solution.