Exercise 1: 1.12.3.10 Intersection of sigma algebras
$\mathcal{F}_1$ and $\mathcal{F}_2$ are two sigma algebras of subsets of Ω. Show that

$$\mathcal{F}_1 \cap \mathcal{F}_2 \tag{1}$$

is a sigma algebra of subsets of Ω.

Solution.

Exercise 2: 2.6.5.6 Use Chen's Lemma
$X \in \text{Po}(\lambda)$. Show that

$$\mathrm{E}\,[X^n] = \lambda \sum_{k=0}^{n-1} \binom{n-1}{k} \mathrm{E}\,[X^k]. \tag{2}$$

Aid: Use Chen's Lemma with suitable $H(x)$.

Solution.

Lemma 0.1. **Chen's Lemma** $X \in \text{Po}(\lambda)$ and $H(x)$ is a bounded Borel function, then

$$\mathrm{E}\,[XH(X)] = \lambda \mathrm{E}\,[H(X+1)]. \tag{3}$$

Exercise 3: 3.8.3.1 Joint Distributions & Conditional Expectations
Let (X, Y) be a bivariate random variable, where X is discrete and Y is continuous. (X, Y) has a joint probability mass - and density function given by

$$f_{X,Y}(k, y) = \begin{cases} \frac{\partial P(X=k, Y \leq y)}{\partial y} = \lambda \frac{(\lambda y)^k}{k!} e^{-2\lambda y} & , k \in \mathbb{Z}_{\geq 0},\, y \in [0, \infty) \\ 0 & . \end{cases} \tag{4}$$

(a) Check that

$$\sum_{k=0}^{\infty} \int_{0}^{\infty} f_{X,Y}(k, y) dy = \int_{0}^{\infty} \sum_{k=0}^{\infty} f_{X,Y}(k, y) dy = 1 \tag{5}$$

(b) Compute the mixed moment $\mathrm{E}[XY]$ defined as

$$\mathrm{E}[XY] = \sum_{k=0}^{\infty} \int_{0}^{\infty} f_{X,Y}(k, y) dy. \tag{6}$$

Answer: $\frac{2}{\lambda}$

(c) Find the marginal p.m.f. of X. *Answer:* $X \in \mathrm{Ge}(\frac{1}{2})$

(d) Compute the marginal density of Y here defined as

$$f_Y(y) = \begin{cases} \sum_{k=0}^{\infty} f_{X,Y}(k, y) & , y \in [0, \infty) \\ 0 & . \end{cases} \tag{7}$$

Answer: $Y \in \mathrm{Exp}(\frac{1}{\lambda})$

(e) Find

$$p_{X|Y}(k|y) = p(X = k|Y = y),\, k \in \mathbb{Z}_{\geq 0} \tag{8}$$

Answer: $X|Y = y \in \mathrm{Po}(\lambda y)$.

(f) Compute $\mathrm{E}[X|Y = y]$ and then $\mathrm{E}[XY]$ using double expectation. Compare your results with (b).

Solution. (a) (b) (c) (d) (e) (f)

Exercise 4: 3.8.3.14 Computations on a distribution
Let (X, Y) be a bivariate r.v. such that

$$Y|X = x \in \mathrm{Fs}(x), \quad f_X(x) = 3x^2, \quad x \in [0, 1]. \tag{9}$$

Compute $\mathrm{E}[Y]$, $\mathrm{Var}[Y]$, $\mathrm{Cov}(X, Y)$ and the p.m.f. of Y. *Answer:* $\mathrm{E}[Y] = \frac{3}{2}$, $\mathrm{Var}[Y] = \frac{9}{4}$, $\mathrm{Cov}(X, Y) = -\frac{1}{8}$, and $p_Y(k) = \frac{18}{(k+3)(k+2)(k+1)k}$, $k \geq 1$.

Solution.

Exercise 5: 4.7.2.4 Equidistribution
Let $\{X_k\}_{k=1}^n$ be independent and identically distributed. Furthermore $\{a_k\}_{k=1}^n$, $a_k \in \mathbb{R}$. Set

$$Y_1 = \sum_k a_k X_k \tag{10}$$

and

$$Y_2 = \sum_k a_{n-k+1} X_k. \tag{11}$$

Show that

$$Y_1 \stackrel{d}{=} Y_2. \tag{12}$$

Solution.

Exercise 6: 5.8.3.11 Laplace distribution
Let $\{X_k\}_{k=1}^{n}$ be independent and identically distributed with $X_k \in L(a), k \in [1, N_p], N_p \in \mathrm{Fs}(p)$. N_p is independent of the varibles $\{X_k\}$. We set

$$S_{N_p} = \sum_{k=1}^{N_p} X_k. \tag{13}$$

Show that $\sqrt{p} S_{N_p} \in L(a)$.

Solution.

Exercise 7: 7.6.1.1 Mean square convergence
Assume $X_n, Y_n \in L_2(\Omega, \mathcal{F}, P)\, \forall n$ and

$$X_n \xrightarrow{2} X, \quad Y_n \xrightarrow{2} Y \quad \text{as} \quad n \to \infty \tag{14}$$

Let $a, b \in \mathbb{R}$. Show that

$$aX_n + bY_n \xrightarrow{2} aX + bY \quad \text{as} \quad n \to \infty \tag{15}$$

You should use the definition of mean square convergence and suitable properties of $\|X\|$ as defined in (LN 7.3).

Solution.