**Exercise 1: 1.12.3.10 Intersection of sigma algebras**

$\mathcal{F}_1$ and $\mathcal{F}_2$ are two sigma algebras of subsets of $\Omega$. Show that

$$\mathcal{F}_1 \cap \mathcal{F}_2 \tag{1}$$

is a sigma algebra of subsets of $\Omega$.

*Solution.*

**Exercise 2: 2.6.5.6 Use Chen's Lemma**

$X \in \mathrm{Po}(\lambda)$. Show that

$$\mathrm{E}\left[X^n\right] = \lambda \sum_{k=0}^{n-1} \binom{n-1}{k} \mathrm{E}\left[X^k\right]. \tag{2}$$

Aid: Use Chen's Lemma with suitable $H(x)$.

*Solution.*

*Lemma* 0.1. **Chen's Lemma** $X \in \mathrm{Po}(\lambda)$ and $H(x)$ is a bounded Borel function, then

$$\mathrm{E}\left[XH(X)\right] = \lambda \mathrm{E}\left[H(X+1)\right]. \tag{3}$$

**Exercise 3: 3.8.3.1 Joint Distributions & Conditional Expectations**

Let $(X, Y)$ is a bivariate random variable, where $X$ is discrete and $Y$ is continuous. $(X, Y)$ has a joint probability mass - and density function given by

$$f_{X,Y}(k, y) = \begin{cases} \frac{\partial P(X=k, Y \leq y)}{\partial y} = \lambda \frac{(\lambda y)^k}{k!} e^{-2\lambda y} & , k \in \mathbb{Z}_{\geq 0}, \ y \in [0, \infty) \\ 0 & . \end{cases} \tag{4}$$

(a) Check that

$$\sum_{k=0}^{\infty} \int_0^{\infty} f_{X,Y}(k, y) dy = \int_0^{\infty} \sum_{k=0}^{\infty} f_{X,Y}(k, y) dy = 1 \tag{5}$$

(b) Compute the mixed moment $\mathrm{E}[XY]$ defined as

$$\mathrm{E}[XY] = \sum_{k=0}^{\infty} \int_0^{\infty} f_{X,Y}(k, y) dy. \tag{6}$$

*Answer:* $\frac{2}{\lambda}$

(c) Find the marginal p.m.f. of $X$. *Answer:* $X \in \mathrm{Ge}(1/2)$

(d) Compute the marginal density of $Y$ here defined as

$$f_Y(y) = \begin{cases} \sum_{k=0}^{\infty} f_{X,Y}(k, y) & , y \in [0, \infty) \\ 0 & . \end{cases} \tag{7}$$

*Answer:* $Y \in \mathrm{Exp}(\frac{1}{\lambda})$

(e) Find

$$p_{X|Y}(k|y) = p(X = k|Y = y), \ k \in \mathbb{Z}_{\geq 0} \tag{8}$$

*Answer:* $X|Y = y \in \mathrm{Po}(\lambda y)$.

(f) Compute $\mathrm{E}[X|Y = y]$ and then $\mathrm{E}[XY]$ using double expectation. Compare your results with (b).

*Solution.*

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**Exercise 4: 3.8.3.14 Computing moments**

Let $(X, Y)$ be a bivariate r.v. such that

$$Y|X = x \in \mathrm{Fs}(x), \quad f_X(x) = 3x^2, \quad x \in [0, 1]. \tag{9}$$

Compute $\mathrm{E}[Y]$, $\mathrm{Var}[Y]$, $\mathrm{Cov}(X, Y)$ and the p.m.f. of $Y$. *Answer*: $\mathrm{E}[Y] = \frac{3}{2}$, $\mathrm{Var}[Y] = \frac{9}{4}$, $\mathrm{Cov}(X, Y) = -\frac{1}{8}$, and $p_Y(k) = \frac{18}{(k+3)(k+2)(k+1)k}$, $k \geq 1$.

*Solution.*

**Exercise 5: 4.7.2.4** Equidistribution

Let $\{X_k\}_{k=1}^n$ be independent and identically distributed. Furthermore $\{a_k\}_{k=1}^n$, $a_k \in \mathbb{R}$. Set

$$Y_1 = \sum_k a_k X_k \tag{10}$$

and

$$Y_2 = \sum_k a_{n-k+1} X_k. \tag{11}$$

Show that

$$Y_1 \stackrel{d}{=} Y_2. \tag{12}$$

*Solution.*

**Exercise 6: 5.8.3.11** Title of the problem

*Solution.*

**Exercise 7: 7.6.1.1** Title of the problem

*Solution.*