Exercise 1: 1.12.3.10 Intersection of sigma algebras
$\mathcal{F}_1$ and $\mathcal{F}_2$ are two sigma algebras of subsets of Ω. Show that

$$\mathcal{F}_1 \cap \mathcal{F}_2 \tag{1.1}$$

is a sigma algebra of subsets of Ω.

Solution.
Definition 1.2 (Sigma algebra). $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ is called a **sigma algebra** if it satisfies.

$$\Omega \in \mathcal{F} \tag{1.3}$$

$$A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F} \tag{1.4}$$

$$A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F} \tag{1.5}$$

In other words, non-empty(1.3) and has closure under both complement(1.4) and union(1.5).

Verify the sigma algebra axioms with the universe Ω.

By an alternative definition of intersection we have (which holds for any set)

$$A \in \mathcal{F}_1 \wedge A \in \mathcal{F}_2 = A \in \mathcal{F}_1, \mathcal{F}_2 \Leftrightarrow A \in \mathcal{F}_1 \cap \mathcal{F}_2 \tag{1.6}$$

Axiom 1.3: Just applying (1.6) directly and using the fact that $\mathcal{F}_1$ and $\mathcal{F}_2$ is sigma algebras.

$$\Omega \in \mathcal{F}_1, \mathcal{F}_2 \Rightarrow \Omega \in \mathcal{F}_1 \cap \mathcal{F}_2 \tag{1.7}$$

Axiom 1.4: Just applying (1.6) directly and using the fact that $\mathcal{F}_1$ and $\mathcal{F}_2$ is sigma algebras.

$$A \in \mathcal{F}_1 \cap \mathcal{F}_2 \Leftrightarrow A \in \mathcal{F}_1, \mathcal{F}_2 \Rightarrow A^c \in \mathcal{F}_1, \mathcal{F}_2 \Leftrightarrow A^c \in \mathcal{F}_1 \cap \mathcal{F}_2 \tag{1.8}$$

Axiom 1.5: Just applying (1.6) directly and using the fact that $\mathcal{F}_1$ and $\mathcal{F}_2$ is sigma algebras.

$$A, B \in \mathcal{F}_1 \cap \mathcal{F}_2 \Leftrightarrow A, B \in \mathcal{F}_1, \mathcal{F}_2 \Rightarrow A \cup B \in \mathcal{F}_1, \mathcal{F}_2 \Leftrightarrow A \cup B \in \mathcal{F}_1 \cap \mathcal{F}_2 \tag{1.9}$$

All sigma algebra axioms are satisfied and thus $\mathcal{F}_1 \cap \mathcal{F}_2$ is a sigma algebra $\square$

Exercise 2: 2.6.5.6 Use Chen's Lemma
$X \in \mathrm{Po}(\lambda)$. Show that

$$\mathrm{E}\left[X^n\right] = \lambda \sum_{k=0}^{n-1} \binom{n-1}{k} \mathrm{E}\left[X^k\right]. \tag{2.1}$$

Aid: Use Chen's Lemma with suitable $H(x)$.

Solution.

Lemma 2.2 (Chen's Lemma). $X \in \mathrm{Po}(\lambda)$ and $H(x)$ is a bounded Borel function, then

$$\mathrm{E}\left[XH(X)\right] = \lambda \mathrm{E}\left[H(X+1)\right]. \tag{2.3}$$

Identity 2.4 (Binomial identity).

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \tag{2.5}$$

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \tag{2.6}$$

We choose $H(X) = X^{n-1}$ to show (2.1).

$$\mathrm{E}\left[X^n\right] = \mathrm{E}\left[XH(X)\right]\Big|_{H(X)=X^{n-1}} \overset{(2.2)}{=} \lambda \mathrm{E}\left[H(X+1)\right]\Big|_{H(X)=X^{n-1}} = \lambda \mathrm{E}\left[(X+1)^{n-1}\right] \tag{2.7}$$

$$\begin{aligned} \lambda \mathrm{E}\left[(X+1)^{n-1}\right] &\overset{(2.6)}{=} \lambda \mathrm{E}\left[\sum_{k=0}^{n-1} \binom{n-1}{k} X^k\right] = \{\mathrm{E}\,\text{linear operator}\} \\ &= \lambda \sum_{k=0}^{n-1} \binom{n-1}{k} \mathrm{E}\left[X^k\right] \quad \square \end{aligned} \tag{2.8}$$

Exercise 3: 3.8.3.1 Joint Distributions & Conditional Expectations
Let (X, Y) be a bivariate random variable, where X is discrete and Y is continuous. (X, Y) has a joint probability mass - and density function given by

$$f_{X,Y}(k,y) = \begin{cases} \frac{\partial P(X=k, Y\leq y)}{\partial y} = \lambda \frac{(\lambda y)^k}{k!} e^{-2\lambda y} & , k \in \mathbb{Z}_{\geq 0},\ y \in [0, \infty) \\ 0 & . \end{cases} \tag{3.1}$$

(a) Check that

$$\sum_{k=0}^{\infty} \int_0^{\infty} f_{X,Y}(k,y)dy = \int_0^{\infty} \sum_{k=0}^{\infty} f_{X,Y}(k,y)dy = 1 \tag{3.2}$$

(b) Compute the mixed moment $\mathrm{E}[XY]$ defined as

$$\mathrm{E}[XY] = \sum_{k=0}^{\infty} \int_0^{\infty} ky f_{X,Y}(k,y)dy. \tag{3.3}$$

Answer: $\frac{2}{\lambda}$

(c) Find the marginal p.m.f. of X. *Answer:* $X \in \mathrm{Ge}(\frac{1}{2})$

(d) Compute the marginal density of Y here defined as

$$f_Y(y) = \begin{cases} \sum_{k=0}^{\infty} f_{X,Y}(k,y) & , y \in [0, \infty) \\ 0 & . \end{cases} \tag{3.4}$$

Answer: $Y \in \mathrm{Exp}(\frac{1}{\lambda})$

(e) Find

$$p_{X|Y}(k|y) = p(X = k|Y = y),\ k \in \mathbb{Z}_{\geq 0} \tag{3.5}$$

Answer: $X|Y = y \in \mathrm{Po}(\lambda y)$.

(f) Compute $\mathrm{E}[X|Y = y]$ and then $\mathrm{E}[XY]$ using double expectation. Compare your results with (b).

Solution. First note that events with 0 probability is dealt with via using certain domains when integrating and such, trivial to perform and trivial to append it back on but left out for brevity.
Using the inverse series expansion of $e^{(\cdot)}$ for k as a function of λy we get the marginalized distribution

$$f_Y(y) = \sum_{k=0}^{\infty} \lambda \frac{(\lambda y)^k}{k!} e^{-2\lambda y} = \lambda e^{-\lambda y} \tag{3.6}$$

Using Beta 7.5.40 with $\{a = 2y, n = k\}$ we get

$$\begin{aligned}
f_X(k) &= \int_{y=-\infty}^{\infty} \lambda \frac{(\lambda y)^k}{k!} e^{-2\lambda y} \\
&= \frac{\lambda^{k+1}}{k!} \int_{y=0}^{\infty} y^k e^{-2\lambda y} \\
&= \frac{\lambda^{k+1}}{k!} \left(\frac{k!}{(2\lambda)^{k+1}} \right) \\
&= \frac{1}{2^{k+1}}
\end{aligned} \tag{3.7}$$

(a) To show that the order of integration/summation doesn't matter we can simply compute their values and see that they are the same and $= 1$.

Using Beta 7.5.40 with $\{a = \lambda, n = 0\}$ we get

$$\int_{y=0}^{\infty} f_Y(y) = \int_{y=0}^{\infty} \lambda e^{-\lambda y} = \frac{\lambda}{\lambda^{0+1}} = 1 \tag{3.8}$$

Using geometric series for $\sum \frac{1}{1-x}$ with $x = \frac{1}{2}, |\frac{1}{2}| < 1$

$$\sum_{k=0}^{\infty} f_X(k) = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} = \sum_{k'=1}^{\infty} \frac{1}{2 \cdot 2^k} = \frac{2}{2} = 1 \quad \square \tag{3.9}$$

(b) Using the same integration rules from Beta as before and Beta 8.6.4 for the last sum

$$\begin{aligned}
\mathrm{E}\left[XY\right] &= \sum_{k=0}^{\infty} \int_{0}^{\infty} k y f_{X,Y}(k,y) dy \\
&= \sum \frac{\lambda^{k+1} k^2}{k!} \int y^{k+1} e^{-(2\lambda)y} \\
&= \sum \frac{\lambda^{k+1} k^2}{k!} \left(\frac{(k+1)!}{(2\lambda)^{k+2}} \right) \\
&= \sum \frac{k^2 \lambda^{k+1}}{(2\lambda)^{k+2}} \\
&= \frac{1}{\lambda} \sum \frac{k^2}{2^{k+2}} \\
&= \frac{1}{4\lambda} \sum_{k=1}^{\infty} \frac{k^2}{2^k} \\
&= \frac{1}{4\lambda} \left(1 - 1/2\right)^{-3} \left(1/2 + 1/2\right) \\
&= \frac{2}{\lambda} \quad \square
\end{aligned} \tag{3.10}$$

(c) Simple pattern matching on tabulated p.m.f. for Ge($\frac{1}{2}$) on (3.7)

$$\frac{1}{2^{k+1}} = p(1-p)^k \bigg|_{p=\frac{1}{2}} = \mathrm{Ge}(p) \bigg|_{p=\frac{1}{2}} = \mathrm{Ge}(\frac{1}{2})(k) \quad \square \tag{3.11}$$

(d) Simple pattern matching on tabulated p.m.f. for Exp($\frac{1}{\lambda}$) on (3.6)

$$\lambda e^{-\lambda y} = \frac{e^{-x/a}}{a} \bigg|_{a=\frac{1}{\lambda}} = \mathrm{Exp}(a)(x) \bigg|_{a=\frac{1}{\lambda}} = \mathrm{Exp}(\frac{1}{\lambda})(k) \quad \square \tag{3.12}$$

(e) From tabulation we have

$$p_{\mathrm{Po}(\lambda y)}(k) = e^{-\lambda y} \frac{(\lambda y)^k}{k!} \tag{3.13}$$

From the definition of conditional we have

$$p_{XY} = p_{X|Y} p_Y \tag{3.14}$$

This gives us when putting in the p.m.f.'s

$$p_{X|Y} = \frac{p_{XY}}{p_Y} = \frac{\lambda \frac{(\lambda y)^k e^{-2\lambda y}}{k!}}{\lambda e^{-\lambda y}} = \frac{(\lambda y)^k e^{-\lambda y}}{k!} = e^{-\lambda y} \frac{(\lambda y)^k}{k!} = p_{\mathrm{Po}(\lambda y)}(k) \tag{3.15}$$

And since we have that distributions are equal we have that

$$X|Y = y \in \mathrm{Po}(\lambda y) \quad \square \tag{3.16}$$

(f)

Exercise 4: 3.8.3.14 Computations on a distribution
Let (P, Y) be a bivariate r.v. such that

$$Y|P = p \in \mathrm{Fs}(p), \quad f_P(p) = 3p^2, \quad p \in [0, 1]. \tag{4.1}$$

Compute: $\mathrm{E}[Y]$, $\mathrm{Var}[Y]$, $\mathrm{Cov}(P, Y)$ and the p.m.f. of Y. *Answer:* $\mathrm{E}[Y] = \frac{3}{2}$, $\mathrm{Var}[Y] = \frac{9}{4}$, $\mathrm{Cov}(P, Y) = -\frac{1}{8}$, and $p_Y(k) = \frac{18}{(k+3)(k+2)(k+1)k}$, $k \geq 1$.

Solution.

$$\begin{aligned}
p_Y(k) &= \int_{p=0}^{1} p_{Y|P=p}(k) df_P(p) = 3 \int_{p=0}^{1} p^3 (1-p)^{k-1} dp = \left\{ \begin{cases} q &= 1-p \\ dq &= -dp \end{cases} \right\} \\
&= 3 \int_{q=0}^{1} (1-q)^3 q^{k-1} dq = 3 \int_{q=0}^{1} (-q^3 + 3q^2 - 3q + 1) q^{k-1} dq \\
&= 3 \int_{q=0}^{1} -q^{k+2} + 3q^{k+1} - 3q^k + q^{k-1} dq \\
&= 3 \left(-\frac{1}{k+3} + \frac{3}{k+2} - \frac{3}{k+1} + \frac{1}{k} \right) \\
&= \frac{18}{(k+3)(k+2)(k+1)k}, k \geq 1
\end{aligned} \tag{4.2}$$

Setting up Python with Sympy to leverage some of the algebra

```
from sympy import *
p, q, k = S('p k q'.split())
```

```
EY = summation(
    18/((k+3)*(k+2)*(k+1)),
    (k, 1, oo)
)
print(EY) # 3/2
```

From the definition of E we have

$$\begin{aligned}
\mathrm{E}[Y] &\triangleq \sum_{k=1}^{\infty} k \frac{18}{(k+3)(k+2)(k+1)k} = \sum_{k=1}^{\infty} \frac{18}{(k+3)(k+2)(k+1)} \\
&= \frac{3}{2}
\end{aligned} \tag{4.3}$$

```
VarY = summation(
    18*(k-EY)**2/((k+3)*(k+2)*(k+1)),
    (k, 1, oo)
)
print(VarY) # 9/4
```

From the definition of Var we have

$$\mathrm{Var}\left[Y\right] \triangleq \sum_{k=1}^{\infty}(k - \mathrm{E}\left[Y\right])^2 \frac{18}{(k+3)(k+2)(k+1)k} = \frac{9}{4} \tag{4.4}$$

$$\mathrm{E}\left[P\right] = \int\limits_{p=0}^{1} 3p^2 dp = \frac{3}{4} \tag{4.5}$$

From the definition of Cov (and using the $q = 1 - p$)

$$\begin{aligned}
\mathrm{Cov}\left(P, Y\right) &\triangleq \int \sum (k - \mathrm{E}\left[Y\right])(p - \mathrm{E}\left[P\right]) p_{YP}(y, p) \\
&= \int \sum (k - \mathrm{E}\left[Y\right])(p - \mathrm{E}\left[P\right]) p_{Y|P}(y, p) f_p(p) \\
&= \int \sum (k - \mathrm{E}\left[Y\right])(p - \mathrm{E}\left[P\right]) p(1-p)^{k-1} \cdot 3p^2 = ... = -\frac{1}{8}
\end{aligned} \tag{4.6}$$

Exercise 5: 4.7.2.4 Equidistribution
Let $\{X_k\}_{k=1}^n$ be independent and identically distributed. Furthermore $\{a_k\}_{k=1}^n$, $a_k \in \mathbb{R}$. Set

$$Y_1 = \sum_k a_k X_k \tag{5.1}$$

and

$$Y_2 = \sum_k a_{n-k+1} X_k. \tag{5.2}$$

Show that

$$Y_1 \overset{d}{=} Y_2. \tag{5.3}$$

Solution.

Identity 5.4 (Permuted coefficients). $\{X_k\}, X$ is independent and identically distributed. σ is any permutation.

$$\varphi_{\left(\sum\limits_{k \in K} a_{\sigma(k)} X_k\right)}(t) = \prod_{k \in K} \varphi_X(a_k t) = \varphi_{\left(\sum\limits_{k \in K} a_k X_k\right)}(t) \tag{5.5}$$

Proof. Holds directly from the identities

$$\varphi_{\sum X_k} = \prod \varphi_{X_k} \tag{5.6}$$

$$\varphi_{aX}(t) = \varphi_X(at) \tag{5.7}$$

and the fact that one can reorder $\prod$ and uniqueness of φ.

$\square$

Using Theorem 5.4 with $\sigma : k \to n - k + 1$

$$\varphi_{Y_1} = \varphi_{\left(\sum\limits_k a_k X_k\right)} = \varphi_{\left(\sum\limits_k a_{n-k+1} X_k\right)} = \varphi_{Y_2} \tag{5.8}$$

And with the uniqueness theorem we have

$$\{\varphi_{Y_1} = \varphi_{Y_2}\} \Rightarrow \left\{ Y_1 \overset{d}{=} Y_2 \right\} \quad \square \tag{5.9}$$

Exercise 6: 5.8.3.11 Laplace distribution
Let $\{X_k\}_{k=1}^n$ be independent and identically distributed with $X_k \in L(a), k \in [1, N_p], N_p \in \text{Fs}(p)$. N_p is independent of the varibles $\{X_k\}$. We set

$$S_{N_p} = \sum_{k=1}^{N_p} X_k. \tag{6.1}$$

Show that $\sqrt{p}S_{N_p} \in L(a)$.

Solution.

$$\varphi_{\sqrt{p}S_{N_p}}(t) = \varphi_{S_{N_p}}(\sqrt{p}t) \tag{6.2}$$

$$\varphi_{S_{N_p}} = g_{\text{Fs}(p)}(\varphi_{\text{La}(a)}(t)) \tag{6.3}$$

From definition of generating function and using geometric series ($|t(1-p)| < 1 \Rightarrow |t| < 1$)

$$g_{\text{Fs}(p)}(t) = \sum_{k=1}^{\infty} t^k p(1-p)^{k-1} = tp\sum_{k=0}^{\infty}(t(1-p))^k = \frac{tp}{1-t(1-p)} \tag{6.4}$$

Exercise 7: 7.6.1.1 Mean square convergence
Assume $X_n, Y_n \in L_2(\Omega, \mathcal{F}, P)\, \forall n$ and

$$X_n \xrightarrow{2} X, \quad Y_n \xrightarrow{2} Y \quad \text{as} \quad n \to \infty \tag{7.1}$$

Let $a, b \in \mathbb{R}$. Show that

$$aX_n + bY_n \xrightarrow{2} aX + bY \quad \text{as} \quad n \to \infty \tag{7.2}$$

You should use the definition of mean square convergence and suitable properties of $\|X\|$ as defined in (LN 7.3).

Solution. From the definition of mean square convergence:

$$X_n \xrightarrow{2} X \Rightarrow \mathrm{E}\left[\|X_n - X\|^2\right] \to 0, \tag{7.3}$$

$$Y_n \xrightarrow{2} Y \Rightarrow \mathrm{E}\left[\|Y_n - Y\|^2\right] \to 0. \tag{7.4}$$

Via the parallelogram rule and (7.3), (7.4)

$$\begin{aligned} \mathrm{E}\left[\|(aX_n + bY_n) - (aX + bY)\|^2\right] &= \mathrm{E}\left[\|a(X_n - X) + b(Y_n - Y)\|^2\right] \\ &\leq 2a^2 \mathrm{E}\left[\|X_n - X\|^2\right] + 2b^2 \mathrm{E}\left[\|Y_n - Y\|^2\right] \\ &\to 0 \end{aligned} \tag{7.5}$$

Again from the definition of mean square convergence and (7.5) we get

$$\mathrm{E}\left[\|(aX_n + bY_n) - (aX + bY)\|^2\right] \to 0 \Rightarrow aX_n + bY_n \xrightarrow{2} aX + bY \quad \square \tag{7.6}$$