Exercise 1: 1.12.3.10 Intersection of sigma algebras
$\mathcal{F}_1$ and $\mathcal{F}_2$ are two sigma algebras of subsets of Ω. Show that

$$\mathcal{F}_1 \cap \mathcal{F}_2 \tag{1.1}$$

is a sigma algebra of subsets of Ω.

Solution.

Definition 1.2 (Sigma algebra). A collection $\mathcal{A}$ of subsets of Ω is called a **sigma algebra** if it satisfies.

$$\Omega \in \mathcal{A} \tag{1.3}$$

$$A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A} \tag{1.4}$$

$$A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A} \tag{1.5}$$

In other words, non-empty(1.3) and has closure under both complement(1.4) and union(1.4).

Identity 1.6 (De Morgan's rule).

$$A \cap B = (A^c \cup B^c)^c \tag{1.7}$$

TODO introduce $\mathcal{F}$ is this on the right level? corner casees? compare to the general case

$$A, B \in \mathcal{A} \overset{(1.4)}{\Longrightarrow} A^c, B^c \in \mathcal{A} \overset{(1.5)}{\Longrightarrow} (A^c \cup B^c) \in \mathcal{A} \overset{(1.7)}{\Longrightarrow} (A \cap B)^c \in \mathcal{A} \overset{(1.4)}{\Longrightarrow} A \cap B \in \mathcal{A} \quad \square \tag{1.8}$$

Exercise 2: 2.6.5.6 Use Chen's Lemma
$X \in \mathrm{Po}(\lambda)$. Show that

$$\mathrm{E}\left[X^n\right] = \lambda \sum_{k=0}^{n-1} \binom{n-1}{k} \mathrm{E}\left[X^k\right]. \tag{2.1}$$

Aid: Use Chen's Lemma with suitable $H(x)$.

Solution.

Lemma 2.2 (Chen's Lemma). $X \in \mathrm{Po}(\lambda)$ and $H(x)$ is a bounded Borel function, then

$$\mathrm{E}\left[XH(X)\right] = \lambda \mathrm{E}\left[H(X+1)\right]. \tag{2.3}$$

Identity 2.4 (Binomial identity).

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \tag{2.5}$$

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \tag{2.6}$$

We choose $H(X) = X^{n-1}$ to show (2.1).

$$\mathrm{E}\left[X^n\right] = \mathrm{E}\left[XH(X)\right]\Big|_{H(X)=X^{n-1}} \stackrel{(2.2)}{=} \lambda \mathrm{E}\left[H(X+1)\right]\Big|_{H(X)=X^{n-1}} = \lambda \mathrm{E}\left[(X+1)^{n-1}\right] \tag{2.7}$$

$$\lambda \mathrm{E}\left[(X+1)^{n-1}\right] \stackrel{(2.6)}{=} \lambda \mathrm{E}\left[\sum_{k=0}^{n-1} \binom{n-1}{k} X^k\right] = \{\mathrm{E}\text{ linear operator}\} = \lambda \sum_{k=0}^{n-1} \binom{n-1}{k} \mathrm{E}\left[X^k\right] \quad \square \tag{2.8}$$

Exercise 3: 3.8.3.1 Joint Distributions & Conditional Expectations
Let (X, Y) be a bivariate random variable, where X is discrete and Y is continuous. (X, Y) has a joint probability mass - and density function given by

$$f_{X,Y}(k,y) = \begin{cases} \frac{\partial P(X=k, Y \leq y)}{\partial y} = \lambda \frac{(\lambda y)^k}{k!} e^{-2\lambda y} & , k \in \mathbb{Z}_{\geq 0}, \, y \in [0, \infty) \\ 0 & . \end{cases} \tag{3.1}$$

(a) Check that

$$\sum_{k=0}^{\infty} \int_0^{\infty} f_{X,Y}(k,y) dy = \int_0^{\infty} \sum_{k=0}^{\infty} f_{X,Y}(k,y) dy = 1 \tag{3.2}$$

(b) Compute the mixed moment $\mathrm{E}[XY]$ defined as

$$\mathrm{E}[XY] = \sum_{k=0}^{\infty} \int_0^{\infty} k y f_{X,Y}(k,y) dy. \tag{3.3}$$

Answer: $\frac{2}{\lambda}$

(c) Find the marginal p.m.f. of X. *Answer:* $X \in \mathrm{Ge}(\frac{1}{2})$

(d) Compute the marginal density of Y here defined as

$$f_Y(y) = \begin{cases} \sum_{k=0}^{\infty} f_{X,Y}(k,y) & , y \in [0, \infty) \\ 0 & . \end{cases} \tag{3.4}$$

Answer: $Y \in \mathrm{Exp}(\frac{1}{\lambda})$

(e) Find

$$p_{X|Y}(k|y) = p(X = k | Y = y), \, k \in \mathbb{Z}_{\geq 0} \tag{3.5}$$

Answer: $X|Y = y \in \mathrm{Po}(\lambda y)$.

(f) Compute $\mathrm{E}[X|Y = y]$ and then $\mathrm{E}[XY]$ using double expectation. Compare your results with (b).

Solution. (a)

(b)

(c)

(d)

(e)

(f)

(g)

Exercise 4: 3.8.3.14 Computations on a distribution
Let (P, Y) be a bivariate r.v. such that

$$Y|P = p \in \mathrm{Fs}(p), \quad f_P(p) = 3p^2, \quad p \in [0,1]. \tag{4.1}$$

Compute: $\mathrm{E}[Y]$, $\mathrm{Var}[Y]$, $\mathrm{Cov}(P,Y)$ and the p.m.f. of Y. *Answer:* $\mathrm{E}[Y] = \frac{3}{2}$, $\mathrm{Var}[Y] = \frac{9}{4}$, $\mathrm{Cov}(P,Y) = -\frac{1}{8}$, and $p_Y(k) = \frac{18}{(k+3)(k+2)(k+1)k}$, $k \geq 1$.

Solution.

$$\begin{aligned}
p_Y(k) &= \int_{p=0}^{1} p_{Y|P=p}(k) df_P(p) = 3\int_{p=0}^{1} p^3(1-p)^{k-1}dp = \left\{ \begin{cases} q &= 1-p \\ dq &= -dp \end{cases} \right\} \\
&= 3\int_{q=0}^{1} (1-q)^3 q^{k-1} dq = 3\int_{q=0}^{1} (-q^3 + 3q^2 - 3q + 1) q^{k-1} dq \\
&= 3\int_{q=0}^{1} -q^{k+2} + 3q^{k+1} - 3q^k + q^{k-1} dq \\
&= 3\left(-\frac{1}{k+3} + \frac{3}{k+2} - \frac{3}{k+1} + \frac{1}{k}\right) \\
&= \frac{18}{(k+3)(k+2)(k+1)k}, k \geq 1
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
\mathrm{E}[Y] &\triangleq \sum_{k=1}^{\infty} k \frac{18}{(k+3)(k+2)(k+1)k} = \sum_{k=1}^{\infty} \frac{18}{(k+3)(k+2)(k+1)} \\
&= \frac{3}{2}
\end{aligned} \tag{4.3}$$

$$\mathrm{Var}[Y] \triangleq \sum_{k=1}^{\infty} (k - \mathrm{E}[Y])^2 \frac{18}{(k+3)(k+2)(k+1)k} = \frac{9}{4} \tag{4.4}$$

$$\mathrm{E}[P] = \int_{p=0}^{1} 3p^2 dp = \frac{3}{4} \tag{4.5}$$

$$\mathrm{Cov}(P,Y) \triangleq \int \sum (k - \mathrm{E}[Y])(p - \mathrm{E}[P]) ... = -\frac{1}{8} \tag{4.6}$$

Exercise 5: 4.7.2.4 Equidistribution
Let $\{X_k\}_{k=1}^n$ be independent and identically distributed. Furthermore $\{a_k\}_{k=1}^n$, $a_k \in \mathbb{R}$. Set

$$Y_1 = \sum_k a_k X_k \tag{5.1}$$

and

$$Y_2 = \sum_k a_{n-k+1} X_k. \tag{5.2}$$

Show that

$$Y_1 \stackrel{d}{=} Y_2. \tag{5.3}$$

Solution. TODO needs background identies and theorems for all the steps forall X exists unq varphiX uniqness of varphiX (=> varphiX = varphiXk) varphi sum varphi scale Xk is all identically distributed product is reorderable due to (countable?) commutativity and associative of multiplication anything more? TODO compare with FS8.3

Identity 5.4 (Permuted coefficients). $\{X_k\}, X$ is independent and identically distributed. σ is any permutation.

$$\varphi_{\left(\sum_{k\in K} a_{\sigma(k)} X_k\right)}(t) = \prod_{k\in K} \varphi_X(a_k t) = \varphi_{\left(\sum_{k\in K} a_k X_k\right)}(t) \tag{5.5}$$

Proof.

$$\varphi TODO \tag{5.6}$$

□

Using Theorem 5.4 with $\sigma : k \to n-k+1$

$$\varphi_{Y_1} = \varphi_{\left(\sum_k a_k X_k\right)} = \varphi_{\left(\sum_k a_{n-k+1} X_k\right)} = \varphi_{Y_2} \tag{5.7}$$

And with uniqueness theorem of blablabla we have

$$\{\varphi_{Y_1} = \varphi_{Y_2}\} \Rightarrow \left\{Y_1 \stackrel{d}{=} Y_2\right\} \quad \square \tag{5.8}$$

Exercise 6: 5.8.3.11 Laplace distribution
Let $\{X_k\}_{k=1}^{n}$ be independent and identically distributed with $X_k \in L(a), k \in [1, N_p], N_p \in \mathrm{Fs}(p)$. N_p is independent of the varibles $\{X_k\}$. We set

$$S_{N_p} = \sum_{k=1}^{N_p} X_k. \tag{6.1}$$

Show that $\sqrt{p} S_{N_p} \in L(a)$.

Solution.

Exercise 7: 7.6.1.1 Mean square convergence
Assume $X_n, Y_n \in L_2(\Omega, \mathcal{F}, P)\, \forall n$ and

$$X_n \xrightarrow{2} X, \quad Y_n \xrightarrow{2} Y \quad \text{as} \quad n \to \infty \tag{7.1}$$

Let $a, b \in \mathbb{R}$. Show that

$$aX_n + bY_n \xrightarrow{2} aX + bY \quad \text{as} \quad n \to \infty \tag{7.2}$$

You should use the definition of mean square convergence and suitable properties of $\|X\|$ as defined in (LN 7.3).

Solution. From the definition of mean square convergence:

$$X_n \xrightarrow{2} X \Rightarrow \mathrm{E}\left[\|X_n - X\|^2\right] \to 0, \tag{7.3}$$

$$Y_n \xrightarrow{2} Y \Rightarrow \mathrm{E}\left[\|Y_n - Y\|^2\right] \to 0. \tag{7.4}$$

Via the parallelogram rule and (7.3), (7.4)

$$\begin{aligned} \mathrm{E}\left[\|(aX_n + bY_n) - (aX + bY)\|^2\right] &= \mathrm{E}\left[\|a(X_n - X) + b(Y_n - Y)\|^2\right] \\ &\leq 2a^2\mathrm{E}\left[\|X_n - X\|^2\right] + 2b^2\mathrm{E}\left[\|Y_n - Y\|^2\right] \\ &\to 0 \end{aligned} \tag{7.5}$$

Again from the definition of mean square convergence and (7.5) we get

$$\mathrm{E}\left[\|(aX_n + bY_n) - (aX + bY)\|^2\right] \to 0 \Rightarrow aX_n + bY_n \xrightarrow{2} aX + bY \quad \square \tag{7.6}$$