Exercise 1: 1.12.3.10 Intersection of sigma algebras
$\mathcal{F}_1$ and $\mathcal{F}_2$ are two sigma algebras of subsets of Ω. Show that

$$\mathcal{F}_1 \cap \mathcal{F}_2 \tag{1.1}$$

is a sigma algebra of subsets of Ω.

Solution.

Definition 1.2 (Sigma algebra). A collection $\mathcal{A}$ of subsets of Ω is called a **sigma algebra** if it satisfies.

$$\Omega \in \mathcal{A} \tag{1.3}$$

$$A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A} \tag{1.4}$$

$$A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A} \tag{1.5}$$

In other words, non-empty(1.3) and has closure under both complement(1.4) and union(1.4).

Identity 1.6 (De Morgan's rule).

$$A \cap B = (A^c \cup B^c)^c \tag{1.7}$$

TODO introduce $\mathcal{F}$ is this on the right level? corner casees? compare to the general case

$$A, B \in \mathcal{A} \overset{(1.4)}{\Longrightarrow} A^c, B^c \in \mathcal{A} \overset{(1.5)}{\Longrightarrow} (A^c \cup B^c) \in \mathcal{A} \overset{(1.7)}{\Longrightarrow} (A \cap B)^c \in \mathcal{A} \overset{(1.4)}{\Longrightarrow} A \cap B \in \mathcal{A} \quad \square \tag{1.8}$$

Exercise 2: 2.6.5.6 Use Chen's Lemma
$X \in \mathrm{Po}(\lambda)$. Show that

$$\mathrm{E}[X^n] = \lambda \sum_{k=0}^{n-1} \binom{n-1}{k} \mathrm{E}[X^k]. \tag{2.1}$$

Aid: Use Chen's Lemma with suitable $H(x)$.

Solution.

Lemma 2.2 (Chen's Lemma). $X \in \mathrm{Po}(\lambda)$ and $H(x)$ is a bounded Borel function, then

$$\mathrm{E}[XH(X)] = \lambda \mathrm{E}[H(X+1)]. \tag{2.3}$$

Identity 2.4 (Binomial identity).

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \tag{2.5}$$

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \tag{2.6}$$

We choose $H(X) = X^{n-1}$ to show (2.1).

$$\mathrm{E}[X^n] = \mathrm{E}[XH(X)]\Big|_{H(X)=X^{n-1}} \overset{(2.2)}{=} \lambda \mathrm{E}[H(X+1)]\Big|_{H(X)=X^{n-1}} = \lambda \mathrm{E}[(X+1)^{n-1}] \tag{2.7}$$

$$\lambda \mathrm{E}[(X+1)^{n-1}] \overset{(2.6)}{=} \lambda \mathrm{E}\left[\sum_{k=0}^{n-1} \binom{n-1}{k} X^k\right] = \{\mathrm{E}\,\text{linear operator}\} = \lambda \sum_{k=0}^{n-1} \binom{n-1}{k} \mathrm{E}[X^k] \quad \square \tag{2.8}$$

Exercise 3: 3.8.3.1 Joint Distributions & Conditional Expectations
Let (X, Y) be a bivariate random variable, where X is discrete and Y is continuous. (X, Y) has a joint probability mass - and density function given by

$$f_{X,Y}(k,y) = \begin{cases} \frac{\partial P(X=k, Y\leq y)}{\partial y} = \lambda \frac{(\lambda y)^k}{k!} e^{-2\lambda y} & , k \in \mathbb{Z}_{\geq 0},\ y \in [0,\infty) \\ 0 & . \end{cases} \tag{3.1}$$

(a) Check that

$$\sum_{k=0}^{\infty} \int_0^{\infty} f_{X,Y}(k,y)dy = \int_0^{\infty} \sum_{k=0}^{\infty} f_{X,Y}(k,y)dy = 1 \tag{3.2}$$

(b) Compute the mixed moment $\mathrm{E}[XY]$ defined as

$$\mathrm{E}[XY] = \sum_{k=0}^{\infty} \int_0^{\infty} f_{X,Y}(k,y)dy. \tag{3.3}$$

Answer: $\frac{2}{\lambda}$

(c) Find the marginal p.m.f. of X. *Answer:* $X \in \mathrm{Ge}(\frac{1}{2})$

(d) Compute the marginal density of Y here defined as

$$f_Y(y) = \begin{cases} \sum_{k=0}^{\infty} f_{X,Y}(k,y) & , y \in [0,\infty) \\ 0 & . \end{cases} \tag{3.4}$$

Answer: $Y \in \mathrm{Exp}(\frac{1}{\lambda})$

(e) Find

$$p_{X|Y}(k|y) = p(X = k|Y = y),\ k \in \mathbb{Z}_{\geq 0} \tag{3.5}$$

Answer: $X|Y = y \in \mathrm{Po}(\lambda y)$.

(f) Compute $\mathrm{E}[X|Y = y]$ and then $\mathrm{E}[XY]$ using double expectation. Compare your results with (b).

Solution. (a)

(b)

(c)

(d)

(e)

(f)

(g)

Exercise 4: 3.8.3.14 Computations on a distribution
Let (X, Y) be a bivariate r.v. such that

$$Y|X = x \in \text{Fs}(x), \quad f_X(x) = 3x^2, \quad x \in [0, 1]. \tag{4.1}$$

Compute $\mathrm{E}[Y]$, $\mathrm{Var}[Y]$, $\mathrm{Cov}(X, Y)$ and the p.m.f. of Y. *Answer:* $\mathrm{E}[Y] = \frac{3}{2}$, $\mathrm{Var}[Y] = \frac{9}{4}$, $\mathrm{Cov}(X, Y) = -\frac{1}{8}$, and $p_Y(k) = \frac{18}{(k+3)(k+2)(k+1)k}$, $k \geq 1$.

Solution.

Exercise 5: 4.7.2.4 Equidistribution
Let $\{X_k\}_{k=1}^n$ be independent and identically distributed. Furthermore $\{a_k\}_{k=1}^n$, $a_k \in \mathbb{R}$. Set

$$Y_1 = \sum_k a_k X_k \tag{5.1}$$

and

$$Y_2 = \sum_k a_{n-k+1} X_k. \tag{5.2}$$

Show that

$$Y_1 \stackrel{d}{=} Y_2. \tag{5.3}$$

Solution. TODO needs background identies and theorems for all the steps forall X exists unq varphiX uniqness of varphiX (=> varphiX = varphiXk) varphi sum varphi scale Xk is all identically distributed product is reorderable due to (countable?) commutativity and associative of multiplication anything more?

Identity 5.4 (Permuted coefficients). $\{X_k\}, X$ is independent and identically distributed. σ is any permutation.

$$\varphi_{\left(\sum\limits_{k \in K} a_{\sigma(k)} X_k\right)}(t) = \prod_{k \in K} \varphi_X(a_k t) = \varphi_{\left(\sum\limits_{k \in K} a_k X_k\right)}(t) \tag{5.5}$$

Proof.

$$\varphi TODO \tag{5.6}$$

□

$$\varphi_{Y_1} = \varphi_{\left(\sum\limits_k a_k X_k\right)} \tag{5.7}$$

Exercise 6: 5.8.3.11 Laplace distribution
Let $\{X_k\}_{k=1}^{n}$ be independent and identically distributed with $X_k \in L(a), k \in [1, N_p], N_p \in \mathrm{Fs}(p)$. N_p is independent of the varibles $\{X_k\}$. We set

$$S_{N_p} = \sum_{k=1}^{N_p} X_k. \tag{6.1}$$

Show that $\sqrt{p}S_{N_p} \in L(a)$.

Solution.

Exercise 7: 7.6.1.1 Mean square convergence
Assume $X_n, Y_n \in L_2(\Omega, \mathcal{F}, P)\, \forall n$ and

$$X_n \xrightarrow{2} X, \quad Y_n \xrightarrow{2} Y \quad \text{as} \quad n \to \infty \tag{7.1}$$

Let $a, b \in \mathbb{R}$. Show that

$$aX_n + bY_n \xrightarrow{2} aX + bY \quad \text{as} \quad n \to \infty \tag{7.2}$$

You should use the definition of mean square convergence and suitable properties of $\|X\|$ as defined in (LN 7.3).

Solution.