Exercise 1: 1.12.3.10 Intersection of sigma algebras
$\mathcal{F}_1$ and $\mathcal{F}_2$ are two sigma algebras of subsets of Ω. Show that

$$\mathcal{F}_1 \cap \mathcal{F}_2$$

is a sigma algebra of subsets of Ω.

Solution.

Exercise 2: 2.6.5.6 Use Chen's Lemma
$X \in \mathrm{Po}(\lambda)$. Show that

$$\mathrm{E}[X^n] = \lambda \sum_{k=0}^{n-1} \binom{n-1}{k} \mathrm{E}[X^k]. \tag{1}$$

Aid: Use Chen's Lemma with suitable $H(x)$.

Solution.

Lemma 0.1. **Chen's Lemma** $X \in \mathrm{Po}(\lambda)$ and $H(x)$ is a bounded Borel function, then

$$\mathrm{E}[XH(X)] = \lambda \mathrm{E}[H(X+1)].$$

Exercise 3: 3.8.3.1 Joint Distributions & Conditional Expectations
Let (X, Y) is a bivariate random variable, where X is discrete and Y is continuous. (X, Y) has a joint probability mass - and density function given by

$$f_{X,Y}(k, y) = \begin{cases} \frac{\partial P(X=k, Y \leq y)}{\partial y} = \lambda \frac{(\lambda y)^k}{k!} e^{-2\lambda y} & , k \in \mathbb{Z}_{\geq 0},\ y \in [0, \infty) \\ 0 & . \end{cases}$$

(a) Check that

$$\sum_{k=0}^{\infty} \int_{0}^{\infty} f_{X,Y}(k, y) dy = \int_{0}^{\infty} \sum_{k=0}^{\infty} f_{X,Y}(k, y) dy = 1$$

(b) Compute the mixed moment $\mathrm{E}[XY]$ defined as

$$\mathrm{E}[XY] = \sum_{k=0}^{\infty} \int_{0}^{\infty} f_{X,Y}(k, y) dy.$$

Answer: $\frac{2}{\lambda}$

(c) Find the marginal p.m.f. of X. *Answer*: $X \in \mathrm{Ge}(1/2)$

(d) Compute the marginal density of Y here defined as

$$f_Y(y) = \begin{cases} \sum_{k=0}^{\infty} f_{X,Y}(k, y) & , y \in [0, \infty) \\ 0 & . \end{cases}$$

Answer: $Y \in \mathrm{Exp}(\frac{1}{\lambda})$

(e) Find

$$p_{X|Y}(k|y) = p(X = k|Y = y),\ k \in \mathbb{Z}_{\geq 0}$$

Answer: $X|Y = y \in \mathrm{Po}(\lambda y)$.

(f) Compute $\mathrm{E}[X|Y = y]$ and then $\mathrm{E}[XY]$ using double expectation. Compare your results with (b).

Solution.

Exercise 4: 3.8.3.14 Title of the problem

Solution.

Exercise 5: 4.7.2.4 Title of the problem

Solution.

Exercise 6: 5.8.3.11 Title of the problem

Solution.

Exercise 7: 7.6.1.1 Title of the problem

Solution.