

**Exercise 1: 1.12.3.10 Intersection of sigma algebras**

$\mathcal{F}_1$  and  $\mathcal{F}_2$  are two sigma algebras of subsets of  $\Omega$ . Show that

$$\mathcal{F}_1 \cap \mathcal{F}_2 \tag{1.1}$$

is a sigma algebra of subsets of  $\Omega$ .

*Solution.*

*Definition 1.2* (Sigma algebra).  $\mathcal{F} \subseteq \mathcal{P}(\Omega)$  is called a **sigma algebra** if it satisfies.

$$\Omega \in \mathcal{F} \tag{1.3}$$

$$A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F} \tag{1.4}$$

$$A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F} \tag{1.5}$$

In other words, non-empty(1.3) and has closure under both complement(1.4) and union(1.5).

Verify the sigma algebra axioms with the universe  $\Omega$ .

By an alternative definition of intersection we have (which holds for any set)

$$A \in \mathcal{F}_1 \wedge A \in \mathcal{F}_2 = A \in \mathcal{F}_1, \mathcal{F}_2 \Leftrightarrow A \in \mathcal{F}_1 \cap \mathcal{F}_2 \tag{1.6}$$

Axiom 1.3: Just applying (1.6) directly and using the fact that  $\mathcal{F}_1$  and  $\mathcal{F}_2$  is sigma algebras.

$$\Omega \in \mathcal{F}_1, \mathcal{F}_2 \Rightarrow \Omega \in \mathcal{F}_1 \cap \mathcal{F}_2 \tag{1.7}$$

Axiom 1.4: Just applying (1.6) directly and using the fact that  $\mathcal{F}_1$  and  $\mathcal{F}_2$  is sigma algebras.

$$A \in \mathcal{F}_1 \cap \mathcal{F}_2 \Leftrightarrow A \in \mathcal{F}_1, \mathcal{F}_2 \Rightarrow A^c \in \mathcal{F}_1, \mathcal{F}_2 \Leftrightarrow A^c \in \mathcal{F}_1 \cap \mathcal{F}_2 \tag{1.8}$$

Axiom 1.5: Just applying (1.6) directly and using the fact that  $\mathcal{F}_1$  and  $\mathcal{F}_2$  is sigma algebras.

$$A, B \in \mathcal{F}_1 \cap \mathcal{F}_2 \Leftrightarrow A, B \in \mathcal{F}_1, \mathcal{F}_2 \Rightarrow A \cup B \in \mathcal{F}_1, \mathcal{F}_2 \Leftrightarrow A \cup B \in \mathcal{F}_1 \cap \mathcal{F}_2 \tag{1.9}$$

All sigma algebra axioms are satisfied and thus  $\mathcal{F}_1 \cap \mathcal{F}_2$  is a sigma algebra  $\square$

**Exercise 2: 2.6.5.6 Use Chen's Lemma**

$X \in \text{Po}(\lambda)$ . Show that

$$\mathbb{E}[X^n] = \lambda \sum_{k=0}^{n-1} \binom{n-1}{k} \mathbb{E}[X^k]. \quad (2.1)$$

*Aid:* Use Chen's Lemma with suitable  $H(x)$ .

*Solution.*

*Lemma 2.2* (Chen's Lemma).  $X \in \text{Po}(\lambda)$  and  $H(x)$  is a bounded Borel function, then

$$\mathbb{E}[XH(X)] = \lambda \mathbb{E}[H(X+1)]. \quad (2.3)$$

*Identity 2.4* (Binomial identity).

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k} \quad (2.5)$$

$$(x+1)^n = \sum_{k=0}^n \binom{n}{k} x^k \quad (2.6)$$

We choose  $H(X) = X^{n-1}$  to show (2.1).

$$\mathbb{E}[X^n] = \mathbb{E}[XH(X)] \Big|_{H(X)=X^{n-1}} \stackrel{(2.2)}{=} \lambda \mathbb{E}[H(X+1)] \Big|_{H(X)=X^{n-1}} = \lambda \mathbb{E}[(X+1)^{n-1}] \quad (2.7)$$

$$\lambda \mathbb{E}[(X+1)^{n-1}] \stackrel{(2.6)}{=} \lambda \mathbb{E} \left[ \sum_{k=0}^{n-1} \binom{n-1}{k} X^k \right] = \{\text{E linear operator}\} = \lambda \sum_{k=0}^{n-1} \binom{n-1}{k} \mathbb{E}[X^k] \quad \square \quad (2.8)$$

**Exercise 3: 3.8.3.1 Joint Distributions & Conditional Expectations**

Let  $(X, Y)$  be a bivariate random variable, where  $X$  is discrete and  $Y$  is continuous.  $(X, Y)$  has a joint probability mass - and density function given by

$$f_{X,Y}(k, y) = \begin{cases} \frac{\partial P(X=k, Y \leq y)}{\partial y} = \lambda \frac{(\lambda y)^k}{k!} e^{-2\lambda y} & , k \in \mathbb{Z}_{\geq 0}, y \in [0, \infty) \\ 0 & . \end{cases} \quad (3.1)$$

(a) Check that

$$\sum_{k=0}^{\infty} \int_0^{\infty} f_{X,Y}(k, y) dy = \int_0^{\infty} \sum_{k=0}^{\infty} f_{X,Y}(k, y) dy = 1 \quad (3.2)$$

(b) Compute the mixed moment  $E[XY]$  defined as

$$E[XY] = \sum_{k=0}^{\infty} \int_0^{\infty} ky f_{X,Y}(k, y) dy. \quad (3.3)$$

*Answer:*  $\frac{2}{\lambda}$

(c) Find the marginal p.m.f. of  $X$ . *Answer:*  $X \in \text{Ge}(\frac{1}{2})$

(d) Compute the marginal density of  $Y$  here defined as

$$f_Y(y) = \begin{cases} \sum_{k=0}^{\infty} f_{X,Y}(k, y) & , y \in [0, \infty) \\ 0 & . \end{cases} \quad (3.4)$$

*Answer:*  $Y \in \text{Exp}(\frac{1}{\lambda})$

(e) Find

$$p_{X|Y}(k|y) = p(X = k|Y = y), k \in \mathbb{Z}_{\geq 0} \quad (3.5)$$

*Answer:*  $X|Y = y \in \text{Po}(\lambda y)$ .

(f) Compute  $E[X|Y = y]$  and then  $E[XY]$  using double expectation. Compare your results with (b).

*Solution.* (a)

(b)

(c)

(d)

(e)

(f)

(g)

**Exercise 4: 3.8.3.14 Computations on a distribution**

Let  $(P, Y)$  be a bivariate r.v. such that

$$Y|P = p \in \text{Fs}(p), \quad f_P(p) = 3p^2, \quad p \in [0, 1]. \quad (4.1)$$

Compute:  $E[Y]$ ,  $\text{Var}[Y]$ ,  $\text{Cov}(P, Y)$  and the p.m.f. of  $Y$ . *Answer:*  $E[Y] = \frac{3}{2}$ ,  $\text{Var}[Y] = \frac{9}{4}$ ,  $\text{Cov}(P, Y) = -\frac{1}{8}$ , and  $p_Y(k) = \frac{18}{(k+3)(k+2)(k+1)k}$ ,  $k \geq 1$ .

*Solution.*

$$\begin{aligned} p_Y(k) &= \int_{p=0}^1 p_{Y|P=p}(k) df_P(p) = 3 \int_{p=0}^1 p^3(1-p)^{k-1} dp = \left\{ \begin{array}{l} q = 1-p \\ dq = -dp \end{array} \right\} \\ &= 3 \int_{q=0}^1 (1-q)^3 q^{k-1} dq = 3 \int_{q=0}^1 (-q^3 + 3q^2 - 3q + 1) q^{k-1} dq \\ &= 3 \int_{q=0}^1 -q^{k+2} + 3q^{k+1} - 3q^k + q^{k-1} dq \\ &= 3 \left( -\frac{1}{k+3} + \frac{3}{k+2} - \frac{3}{k+1} + \frac{1}{k} \right) \\ &= \frac{18}{(k+3)(k+2)(k+1)k}, k \geq 1 \end{aligned} \quad (4.2)$$

$$\begin{aligned} E[Y] &\triangleq \sum_{k=1}^{\infty} k \frac{18}{(k+3)(k+2)(k+1)k} = \sum_{k=1}^{\infty} \frac{18}{(k+3)(k+2)(k+1)} \\ &= \frac{3}{2} \end{aligned} \quad (4.3)$$

$$\text{Var}[Y] \triangleq \sum_{k=1}^{\infty} (k - E[Y])^2 \frac{18}{(k+3)(k+2)(k+1)k} = \frac{9}{4} \quad (4.4)$$

$$E[P] = \int_{p=0}^1 3p^2 dp = \frac{3}{4} \quad (4.5)$$

$$\text{Cov}(P, Y) \triangleq \int \sum (k - E[Y])(p - E[P]) \dots = -\frac{1}{8} \quad (4.6)$$

**Exercise 5: 4.7.2.4 Equidistribution**

Let  $\{X_k\}_{k=1}^n$  be independent and identically distributed. Furthermore  $\{a_k\}_{k=1}^n$ ,  $a_k \in \mathbb{R}$ . Set

$$Y_1 = \sum_k a_k X_k \quad (5.1)$$

and

$$Y_2 = \sum_k a_{n-k+1} X_k. \quad (5.2)$$

Show that

$$Y_1 \stackrel{d}{=} Y_2. \quad (5.3)$$

*Solution.* TODO needs background identities and theorems for all the steps for all  $X$  exists uniqueness of  $\varphi_X$  ( $\Rightarrow \varphi_X = \varphi_{Xk}$ )  $\varphi_{\sum} \varphi_{\text{scale}}$   $Xk$  is all identically distributed product is reorderable due to (countable?) commutativity and associative of multiplication anything more? TODO compare with FS8.3

*Identity 5.4* (Permuted coefficients).  $\{X_k\}$ ,  $X$  is independent and identically distributed.  $\sigma$  is any permutation.

$$\varphi\left(\sum_{k \in K} a_{\sigma(k)} X_k\right)(t) = \prod_{k \in K} \varphi_X(a_k t) = \varphi\left(\sum_{k \in K} a_k X_k\right)(t) \quad (5.5)$$

*Proof.*

$$\varphi_{\text{TODO}} \quad (5.6)$$

□

Using Theorem 5.4 with  $\sigma : k \rightarrow n - k + 1$

$$\varphi_{Y_1} = \varphi\left(\sum_k a_k X_k\right) = \varphi\left(\sum_k a_{n-k+1} X_k\right) = \varphi_{Y_2} \quad (5.7)$$

And with uniqueness theorem of blablabla we have

$$\{\varphi_{Y_1} = \varphi_{Y_2}\} \Rightarrow \{Y_1 \stackrel{d}{=} Y_2\} \quad \square \quad (5.8)$$

**Exercise 6: 5.8.3.11 Laplace distribution**

Let  $\{X_k\}_{k=1}^n$  be independent and identically distributed with  $X_k \in L(a), k \in [1, N_p], N_p \in \text{Fs}(p)$ .  $N_p$  is independent of the variables  $\{X_k\}$ . We set

$$S_{N_p} = \sum_{k=1}^{N_p} X_k. \tag{6.1}$$

Show that  $\sqrt{p}S_{N_p} \in L(a)$ .

*Solution.*

**Exercise 7: 7.6.1.1 Mean square convergence**

Assume  $X_n, Y_n \in L_2(\Omega, \mathcal{F}, P) \forall n$  and

$$X_n \xrightarrow{2} X, \quad Y_n \xrightarrow{2} Y \quad \text{as } n \rightarrow \infty \quad (7.1)$$

Let  $a, b \in \mathbb{R}$ . Show that

$$aX_n + bY_n \xrightarrow{2} aX + bY \quad \text{as } n \rightarrow \infty \quad (7.2)$$

You should use the definition of mean square convergence and suitable properties of  $\|X\|$  as defined in (LN 7.3).

*Solution.* From the definition of mean square convergence:

$$X_n \xrightarrow{2} X \Rightarrow \mathbb{E} [\|X_n - X\|^2] \rightarrow 0, \quad (7.3)$$

$$Y_n \xrightarrow{2} Y \Rightarrow \mathbb{E} [\|Y_n - Y\|^2] \rightarrow 0. \quad (7.4)$$

Via the parallelogram rule and (7.3), (7.4)

$$\begin{aligned} \mathbb{E} [\|(aX_n + bY_n) - (aX + bY)\|^2] &= \mathbb{E} [\|a(X_n - X) + b(Y_n - Y)\|^2] \\ &\leq 2a^2 \mathbb{E} [\|X_n - X\|^2] + 2b^2 \mathbb{E} [\|Y_n - Y\|^2] \\ &\rightarrow 0 \end{aligned} \quad (7.5)$$

Again from the definition of mean square convergence and (7.5) we get

$$\mathbb{E} [\|(aX_n + bY_n) - (aX + bY)\|^2] \rightarrow 0 \Rightarrow aX_n + bY_n \xrightarrow{2} aX + bY \quad \square \quad (7.6)$$