

Exercise 1: 1.12.3.10 Intersection of sigma algebras

\mathcal{F}_1 and \mathcal{F}_2 are two sigma algebras of subsets of Ω . Show that

$$\mathcal{F}_1 \cap \mathcal{F}_2 \tag{1.1}$$

is a sigma algebra of subsets of Ω .

Solution.

Definition 1.2 (Sigma algebra). $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ is called a **sigma algebra** if it satisfies.

$$\Omega \in \mathcal{F} \tag{1.3}$$

$$A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F} \tag{1.4}$$

$$A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F} \tag{1.5}$$

In other words, non-empty(1.3) and has closure under both complement(1.4) and union(1.5).

Verify the sigma algebra axioms with the universe Ω .

By an alternative definition of intersection we have (which holds for any set)

$$A \in \mathcal{F}_1 \wedge A \in \mathcal{F}_2 = A \in \mathcal{F}_1, \mathcal{F}_2 \Leftrightarrow A \in \mathcal{F}_1 \cap \mathcal{F}_2 \tag{1.6}$$

Axiom 1.3: Just applying (1.6) directly and using the fact that \mathcal{F}_1 and \mathcal{F}_2 is sigma algebras.

$$\Omega \in \mathcal{F}_1, \mathcal{F}_2 \Rightarrow \Omega \in \mathcal{F}_1 \cap \mathcal{F}_2 \tag{1.7}$$

Axiom 1.4: Just applying (1.6) directly and using the fact that \mathcal{F}_1 and \mathcal{F}_2 is sigma algebras.

$$A \in \mathcal{F}_1 \cap \mathcal{F}_2 \Leftrightarrow A \in \mathcal{F}_1, \mathcal{F}_2 \Rightarrow A^c \in \mathcal{F}_1, \mathcal{F}_2 \Leftrightarrow A^c \in \mathcal{F}_1 \cap \mathcal{F}_2 \tag{1.8}$$

Axiom 1.5: Just applying (1.6) directly and using the fact that \mathcal{F}_1 and \mathcal{F}_2 is sigma algebras.

$$A, B \in \mathcal{F}_1 \cap \mathcal{F}_2 \Leftrightarrow A, B \in \mathcal{F}_1, \mathcal{F}_2 \Rightarrow A \cup B \in \mathcal{F}_1, \mathcal{F}_2 \Leftrightarrow A \cup B \in \mathcal{F}_1 \cap \mathcal{F}_2 \tag{1.9}$$

All sigma algebra axioms are satisfied and thus $\mathcal{F}_1 \cap \mathcal{F}_2$ is a sigma algebra \square

Exercise 2: 2.6.5.6 Use Chen's Lemma

$X \in \text{Po}(\lambda)$. Show that

$$\mathbb{E}[X^n] = \lambda \sum_{k=0}^{n-1} \binom{n-1}{k} \mathbb{E}[X^k]. \quad (2.1)$$

Aid: Use Chen's Lemma with suitable $H(x)$.

Solution.

Lemma 2.2 (Chen's Lemma). $X \in \text{Po}(\lambda)$ and $H(x)$ is a bounded Borel function, then

$$\mathbb{E}[XH(X)] = \lambda \mathbb{E}[H(X+1)]. \quad (2.3)$$

Identity 2.4 (Binomial identity).

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k} \quad (2.5)$$

$$(x+1)^n = \sum_{k=0}^n \binom{n}{k} x^k \quad (2.6)$$

We choose $H(X) = X^{n-1}$ to show (2.1).

$$\mathbb{E}[X^n] = \mathbb{E}[XH(X)] \Big|_{H(X)=X^{n-1}} \stackrel{(2.2)}{=} \lambda \mathbb{E}[H(X+1)] \Big|_{H(X)=X^{n-1}} = \lambda \mathbb{E}[(X+1)^{n-1}] \quad (2.7)$$

$$\begin{aligned} \lambda \mathbb{E}[(X+1)^{n-1}] &\stackrel{(2.6)}{=} \lambda \mathbb{E} \left[\sum_{k=0}^{n-1} \binom{n-1}{k} X^k \right] = \{\mathbb{E} \text{ linear operator} \} \\ &= \lambda \sum_{k=0}^{n-1} \binom{n-1}{k} \mathbb{E}[X^k] \quad \square \end{aligned} \quad (2.8)$$

Exercise 3: 3.8.3.1 Joint Distributions & Conditional Expectations

Let (X, Y) be a bivariate random variable, where X is discrete and Y is continuous. (X, Y) has a joint probability mass - and density function given by

$$f_{X,Y}(k, y) = \begin{cases} \frac{\partial P(X=k, Y \leq y)}{\partial y} = \lambda \frac{(\lambda y)^k}{k!} e^{-2\lambda y} & , k \in \mathbb{Z}_{\geq 0}, y \in [0, \infty) \\ 0 & . \end{cases} \quad (3.1)$$

(a) Check that

$$\sum_{k=0}^{\infty} \int_0^{\infty} f_{X,Y}(k, y) dy = \int_0^{\infty} \sum_{k=0}^{\infty} f_{X,Y}(k, y) dy = 1 \quad (3.2)$$

(b) Compute the mixed moment $E[XY]$ defined as

$$E[XY] = \sum_{k=0}^{\infty} \int_0^{\infty} ky f_{X,Y}(k, y) dy. \quad (3.3)$$

Answer: $\frac{2}{\lambda}$

(c) Find the marginal p.m.f. of X . *Answer:* $X \in \text{Ge}(\frac{1}{2})$

(d) Compute the marginal density of Y here defined as

$$f_Y(y) = \begin{cases} \sum_{k=0}^{\infty} f_{X,Y}(k, y) & , y \in [0, \infty) \\ 0 & . \end{cases} \quad (3.4)$$

Answer: $Y \in \text{Exp}(\frac{1}{\lambda})$

(e) Find

$$p_{X|Y}(k|y) = p(X = k | Y = y), k \in \mathbb{Z}_{\geq 0} \quad (3.5)$$

Answer: $X|Y = y \in \text{Po}(\lambda y)$.

(f) Compute $E[X|Y = y]$ and then $E[XY]$ using double expectation. Compare your results with (b).

Solution. First note that events with 0 probability is dealt with via using certain domains when integrating and such, trivial to perform and trivial to append it back on but left out for brevity.

Using the inverse series expansion of $e^{(\cdot)}$ for k as a function of λy we get the marginalized distribution

$$f_Y(y) = \sum_{k=0}^{\infty} \lambda \frac{(\lambda y)^k}{k!} e^{-2\lambda y} = \lambda e^{-\lambda y} \quad (3.6)$$

Using Beta 7.5.40 with $\{a = 2y, n = k\}$ we get

$$\begin{aligned}
 f_X(k) &= \int_{y=-\infty}^{\infty} \lambda \frac{(\lambda y)^k}{k!} e^{-2\lambda y} dy \\
 &= \frac{\lambda^{k+1}}{k!} \int_{y=0}^{\infty} y^k e^{-2\lambda y} dy \\
 &= \frac{\lambda^{k+1}}{k!} \left(\frac{k!}{(2\lambda)^{k+1}} \right) \\
 &= \frac{1}{2^{k+1}}
 \end{aligned} \tag{3.7}$$

- (a) To show that the order of integration/summation doesn't matter we can simply compute their values and see that they are the same and $= 1$.

Using Beta 7.5.40 with $\{a = \lambda, n = 0\}$ we get

$$\int_{y=0}^{\infty} f_Y(y) dy = \int_{y=0}^{\infty} \lambda e^{-\lambda y} dy = \frac{\lambda}{\lambda^{0+1}} = 1 \tag{3.8}$$

Using geometric series for $\sum \frac{1}{1-x}$ with $x = \frac{1}{2}, |\frac{1}{2}| < 1$

$$\sum_{k=0}^{\infty} f_X(k) = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} = \sum_{k'=1}^{\infty} \frac{1}{2 \cdot 2^k} = \frac{2}{2} = 1 \quad \square \tag{3.9}$$

- (b) Using the same integration rules from Beta as before and Beta 8.6.4 for the last sum

$$\begin{aligned}
 E[XY] &= \sum_{k=0}^{\infty} \int_0^{\infty} ky f_{X,Y}(k, y) dy \\
 &= \sum_{k=0}^{\infty} \frac{\lambda^{k+1} k^2}{k!} \int_0^{\infty} y^{k+1} e^{-(2\lambda)y} dy \\
 &= \sum_{k=0}^{\infty} \frac{\lambda^{k+1} k^2}{k!} \left(\frac{(k+1)!}{(2\lambda)^{k+2}} \right) \\
 &= \sum_{k=0}^{\infty} \frac{k^2 \lambda^{k+1}}{(2\lambda)^{k+2}} \\
 &= \frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{k^2}{2^{k+2}} \\
 &= \frac{1}{4\lambda} \sum_{k=1}^{\infty} \frac{k^2}{2^k} \\
 &= \frac{1}{4\lambda} (1 - 1/2)^{-3} (1/2 + 1/2) \\
 &= \frac{2}{\lambda} \quad \square
 \end{aligned} \tag{3.10}$$

(c) Simple pattern matching on tabulated p.m.f. for $\text{Ge}(\frac{1}{2})$ on (3.7)

$$\frac{1}{2^{k+1}} = p(1-p)^k \Big|_{p=\frac{1}{2}} = \text{Ge}(p) \Big|_{p=\frac{1}{2}} = \text{Ge}(\frac{1}{2})(k) \quad \square \tag{3.11}$$

(d) Simple pattern matching on tabulated p.m.f. for $\text{Exp}(\frac{1}{\lambda})$ on (3.6)

$$\lambda e^{-\lambda y} = \frac{e^{-x/a}}{a} \Big|_{a=\frac{1}{\lambda}} = \text{Exp}(a)(x) \Big|_{a=\frac{1}{\lambda}} = \text{Exp}(\frac{1}{\lambda})(k) \quad \square \tag{3.12}$$

(e) From tabulation we have

$$p_{\text{Po}(\lambda y)}(k) = e^{-\lambda y} \frac{(\lambda y)^k}{k!} \tag{3.13}$$

From the definition of conditional we have

$$p_{XY} = p_{X|Y} p_Y \tag{3.14}$$

This gives us when putting in the p.m.f.'s

$$p_{X|Y} = \frac{p_{XY}}{p_Y} = \frac{\lambda \frac{(\lambda y)^k e^{-2\lambda y}}{k!}}{\lambda e^{-\lambda y}} = \frac{(\lambda y)^k e^{-\lambda y}}{k!} = e^{-\lambda y} \frac{(\lambda y)^k}{k!} = p_{\text{Po}(\lambda y)}(k) \tag{3.15}$$

And since we have that distributions are equal we have that

$$X|Y = y \in \text{Po}(\lambda y) \quad \square \tag{3.16}$$

(f)

(g)

Exercise 4: 3.8.3.14 Computations on a distribution

Let (P, Y) be a bivariate r.v. such that

$$Y|P = p \in \text{Fs}(p), \quad f_P(p) = 3p^2, \quad p \in [0, 1]. \quad (4.1)$$

Compute: $E[Y]$, $\text{Var}[Y]$, $\text{Cov}(P, Y)$ and the p.m.f. of Y . *Answer:* $E[Y] = \frac{3}{2}$, $\text{Var}[Y] = \frac{9}{4}$, $\text{Cov}(P, Y) = -\frac{1}{8}$, and $p_Y(k) = \frac{18}{(k+3)(k+2)(k+1)k}$, $k \geq 1$.

Solution.

$$\begin{aligned} p_Y(k) &= \int_{p=0}^1 p_{Y|P=p}(k) df_P(p) = 3 \int_{p=0}^1 p^3(1-p)^{k-1} dp = \left\{ \begin{array}{l} q = 1-p \\ dq = -dp \end{array} \right\} \\ &= 3 \int_{q=0}^1 (1-q)^3 q^{k-1} dq = 3 \int_{q=0}^1 (-q^3 + 3q^2 - 3q + 1) q^{k-1} dq \\ &= 3 \int_{q=0}^1 -q^{k+2} + 3q^{k+1} - 3q^k + q^{k-1} dq \\ &= 3 \left(-\frac{1}{k+3} + \frac{3}{k+2} - \frac{3}{k+1} + \frac{1}{k} \right) \\ &= \frac{18}{(k+3)(k+2)(k+1)k}, k \geq 1 \end{aligned} \quad (4.2)$$

$$\begin{aligned} E[Y] &\triangleq \sum_{k=1}^{\infty} k \frac{18}{(k+3)(k+2)(k+1)k} = \sum_{k=1}^{\infty} \frac{18}{(k+3)(k+2)(k+1)} \\ &= \frac{3}{2} \end{aligned} \quad (4.3)$$

$$\text{Var}[Y] \triangleq \sum_{k=1}^{\infty} (k - E[Y])^2 \frac{18}{(k+3)(k+2)(k+1)k} = \frac{9}{4} \quad (4.4)$$

$$E[P] = \int_{p=0}^1 3p^2 dp = \frac{3}{4} \quad (4.5)$$

$$\text{Cov}(P, Y) \triangleq \int \sum (k - E[Y])(p - E[P]) \dots = -\frac{1}{8} \quad (4.6)$$

Exercise 5: 4.7.2.4 Equidistribution

Let $\{X_k\}_{k=1}^n$ be independent and identically distributed. Furthermore $\{a_k\}_{k=1}^n$, $a_k \in \mathbb{R}$. Set

$$Y_1 = \sum_k a_k X_k \quad (5.1)$$

and

$$Y_2 = \sum_k a_{n-k+1} X_k. \quad (5.2)$$

Show that

$$Y_1 \stackrel{d}{=} Y_2. \quad (5.3)$$

Solution. TODO needs background identities and theorems for all the steps for all X exists uniqueness of varphi X (\Rightarrow varphi $X =$ varphi X_k) varphi sum varphi scale X_k is all identically distributed product is reorderable due to (countable?) commutativity and associative of multiplication anything more? TODO compare with FS8.3

Identity 5.4 (Permuted coefficients). $\{X_k\}$, X is independent and identically distributed. σ is any permutation.

$$\varphi\left(\sum_{k \in K} a_{\sigma(k)} X_k\right)(t) = \prod_{k \in K} \varphi_X(a_k t) = \varphi\left(\sum_{k \in K} a_k X_k\right)(t) \quad (5.5)$$

Proof.

$$\varphi_{Y_1} = \varphi_{Y_2} \quad (5.6)$$

□

Using Theorem 5.4 with $\sigma : k \rightarrow n - k + 1$

$$\varphi_{Y_1} = \varphi\left(\sum_k a_k X_k\right) = \varphi\left(\sum_k a_{n-k+1} X_k\right) = \varphi_{Y_2} \quad (5.7)$$

And with uniqueness theorem of blablabla we have

$$\{\varphi_{Y_1} = \varphi_{Y_2}\} \Rightarrow \{Y_1 \stackrel{d}{=} Y_2\} \quad \square \quad (5.8)$$

Exercise 6: 5.8.3.11 Laplace distribution

Let $\{X_k\}_{k=1}^n$ be independent and identically distributed with $X_k \in L(a), k \in [1, N_p], N_p \in \text{Fs}(p)$. N_p is independent of the variables $\{X_k\}$. We set

$$S_{N_p} = \sum_{k=1}^{N_p} X_k. \tag{6.1}$$

Show that $\sqrt{p}S_{N_p} \in L(a)$.

Solution.

Exercise 7: 7.6.1.1 Mean square convergence

Assume $X_n, Y_n \in L_2(\Omega, \mathcal{F}, P) \forall n$ and

$$X_n \xrightarrow{2} X, \quad Y_n \xrightarrow{2} Y \quad \text{as } n \rightarrow \infty \quad (7.1)$$

Let $a, b \in \mathbb{R}$. Show that

$$aX_n + bY_n \xrightarrow{2} aX + bY \quad \text{as } n \rightarrow \infty \quad (7.2)$$

You should use the definition of mean square convergence and suitable properties of $\|X\|$ as defined in (LN 7.3).

Solution. From the definition of mean square convergence:

$$X_n \xrightarrow{2} X \Rightarrow \mathbb{E} [\|X_n - X\|^2] \rightarrow 0, \quad (7.3)$$

$$Y_n \xrightarrow{2} Y \Rightarrow \mathbb{E} [\|Y_n - Y\|^2] \rightarrow 0. \quad (7.4)$$

Via the parallelogram rule and (7.3), (7.4)

$$\begin{aligned} \mathbb{E} [\|(aX_n + bY_n) - (aX + bY)\|^2] &= \mathbb{E} [\|a(X_n - X) + b(Y_n - Y)\|^2] \\ &\leq 2a^2 \mathbb{E} [\|X_n - X\|^2] + 2b^2 \mathbb{E} [\|Y_n - Y\|^2] \\ &\rightarrow 0 \end{aligned} \quad (7.5)$$

Again from the definition of mean square convergence and (7.5) we get

$$\mathbb{E} [\|(aX_n + bY_n) - (aX + bY)\|^2] \rightarrow 0 \Rightarrow aX_n + bY_n \xrightarrow{2} aX + bY \quad \square \quad (7.6)$$