

Exercise 1: 1.12.3.10 Intersection of sigma algebras

\mathcal{F}_1 and \mathcal{F}_2 are two sigma algebras of subsets of Ω . Show that

$$\mathcal{F}_1 \cap \mathcal{F}_2 \tag{1.1}$$

is a sigma algebra of subsets of Ω .

Solution.

Definition 1.2 (Sigma algebra). A collection \mathcal{A} of subsets of Ω is called a **sigma algebra** if it satisfies.

$$\Omega \in \mathcal{A} \tag{1.3}$$

$$A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A} \tag{1.4}$$

$$A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A} \tag{1.5}$$

In other words, non-empty(1.3) and has closure under both complement(1.4) and union(1.4).

Identity 1.6 (De Morgan's rule).

$$A \cap B = (A^c \cup B^c)^c \tag{1.7}$$

TODO introduce \mathcal{F} is this on the right level? corner cases? compare to the general case

$$A, B \in \mathcal{A} \xrightarrow{(1.4)} A^c, B^c \in \mathcal{A} \xrightarrow{(1.5)} (A^c \cup B^c) \in \mathcal{A} \xrightarrow{(1.7)} (A \cap B)^c \in \mathcal{A} \xrightarrow{(1.4)} A \cap B \in \mathcal{A} \quad \square \tag{1.8}$$

Exercise 2: 2.6.5.6 Use Chen's Lemma

$X \in \text{Po}(\lambda)$. Show that

$$\mathbb{E}[X^n] = \lambda \sum_{k=0}^{n-1} \binom{n-1}{k} \mathbb{E}[X^k]. \quad (2.1)$$

Aid: Use Chen's Lemma with suitable $H(x)$.

Solution.

Lemma 2.2 (Chen's Lemma). $X \in \text{Po}(\lambda)$ and $H(x)$ is a bounded Borel function, then

$$\mathbb{E}[XH(X)] = \lambda \mathbb{E}[H(X+1)]. \quad (2.3)$$

Identity 2.4 (Binomial identity).

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k} \quad (2.5)$$

$$(x+1)^n = \sum_{k=0}^n \binom{n}{k} x^k \quad (2.6)$$

We choose $H(X) = X^{n-1}$ to show (2.1).

$$\mathbb{E}[X^n] = \mathbb{E}[XH(X)] \Big|_{H(X)=X^{n-1}} \stackrel{(2.2)}{=} \lambda \mathbb{E}[H(X+1)] \Big|_{H(X)=X^{n-1}} = \lambda \mathbb{E}[(X+1)^{n-1}] \quad (2.7)$$

$$\lambda \mathbb{E}[(X+1)^{n-1}] \stackrel{(2.6)}{=} \lambda \mathbb{E} \left[\sum_{k=0}^{n-1} \binom{n-1}{k} X^k \right] = \{\text{E linear operator}\} = \lambda \sum_{k=0}^{n-1} \binom{n-1}{k} \mathbb{E}[X^k] \quad \square \quad (2.8)$$

Exercise 3: 3.8.3.1 Joint Distributions & Conditional Expectations

Let (X, Y) be a bivariate random variable, where X is discrete and Y is continuous. (X, Y) has a joint probability mass - and density function given by

$$f_{X,Y}(k, y) = \begin{cases} \frac{\partial P(X=k, Y \leq y)}{\partial y} = \lambda \frac{(\lambda y)^k}{k!} e^{-2\lambda y} & , k \in \mathbb{Z}_{\geq 0}, y \in [0, \infty) \\ 0 & . \end{cases} \quad (3.1)$$

(a) Check that

$$\sum_{k=0}^{\infty} \int_0^{\infty} f_{X,Y}(k, y) dy = \int_0^{\infty} \sum_{k=0}^{\infty} f_{X,Y}(k, y) dy = 1 \quad (3.2)$$

(b) Compute the mixed moment $E[XY]$ defined as

$$E[XY] = \sum_{k=0}^{\infty} \int_0^{\infty} f_{X,Y}(k, y) dy. \quad (3.3)$$

Answer: $\frac{2}{\lambda}$

(c) Find the marginal p.m.f. of X . *Answer:* $X \in \text{Ge}(\frac{1}{2})$

(d) Compute the marginal density of Y here defined as

$$f_Y(y) = \begin{cases} \sum_{k=0}^{\infty} f_{X,Y}(k, y) & , y \in [0, \infty) \\ 0 & . \end{cases} \quad (3.4)$$

Answer: $Y \in \text{Exp}(\frac{1}{\lambda})$

(e) Find

$$p_{X|Y}(k|y) = p(X = k|Y = y), k \in \mathbb{Z}_{\geq 0} \quad (3.5)$$

Answer: $X|Y = y \in \text{Po}(\lambda y)$.

(f) Compute $E[X|Y = y]$ and then $E[XY]$ using double expectation. Compare your results with (b).

Solution. (a)

(b)

(c)

(d)

(e)

(f)

(g)

Exercise 4: 3.8.3.14 Computations on a distribution

Let (X, Y) be a bivariate r.v. such that

$$Y|X = x \in \text{Fs}(x), \quad f_X(x) = 3x^2, \quad x \in [0, 1]. \quad (4.1)$$

Compute $E[Y]$, $\text{Var}[Y]$, $\text{Cov}(X, Y)$ and the p.m.f. of Y . *Answer:* $E[Y] = \frac{3}{2}$, $\text{Var}[Y] = \frac{9}{4}$, $\text{Cov}(X, Y) = -\frac{1}{8}$, and $p_Y(k) = \frac{18}{(k+3)(k+2)(k+1)k}$, $k \geq 1$.

Solution.

Exercise 5: 4.7.2.4 Equidistribution

Let $\{X_k\}_{k=1}^n$ be independent and identically distributed. Furthermore $\{a_k\}_{k=1}^n$, $a_k \in \mathbb{R}$. Set

$$Y_1 = \sum_k a_k X_k \quad (5.1)$$

and

$$Y_2 = \sum_k a_{n-k+1} X_k. \quad (5.2)$$

Show that

$$Y_1 \stackrel{d}{=} Y_2. \quad (5.3)$$

Solution.

Exercise 6: 5.8.3.11 Laplace distribution

Let $\{X_k\}_{k=1}^n$ be independent and identically distributed with $X_k \in L(a), k \in [1, N_p], N_p \in \text{Fs}(p)$. N_p is independent of the variables $\{X_k\}$. We set

$$S_{N_p} = \sum_{k=1}^{N_p} X_k. \quad (6.1)$$

Show that $\sqrt{p}S_{N_p} \in L(a)$.

Solution.

Exercise 7: 7.6.1.1 Mean square convergence

Assume $X_n, Y_n \in L_2(\Omega, \mathcal{F}, P) \forall n$ and

$$X_n \xrightarrow{2} X, \quad Y_n \xrightarrow{2} Y \quad \text{as } n \rightarrow \infty \quad (7.1)$$

Let $a, b \in \mathbb{R}$. Show that

$$aX_n + bY_n \xrightarrow{2} aX + bY \quad \text{as } n \rightarrow \infty \quad (7.2)$$

You should use the definition of mean square convergence and suitable properties of $\|X\|$ as defined in (LN 7.3).

Solution.