

Exercise 1: 1.12.3.10 Intersection of sigma algebras

\mathcal{F}_1 and \mathcal{F}_2 are two sigma algebras of subsets of Ω . Show that

$$\mathcal{F}_1 \cap \mathcal{F}_2 \tag{1.1}$$

is a sigma algebra of subsets of Ω .

Solution.

Definition 1.2 (Sigma algebra). A collection \mathcal{A} of subsets of Ω is called a **sigma algebra** if it satisfies.

$$\Omega \in \mathcal{A} \tag{1.3}$$

$$A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A} \tag{1.4}$$

$$A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A} \tag{1.5}$$

In other words, non-empty(1.3) and has closure under both complement(1.4) and union(1.4).

Identity 1.6 (De Morgan's rule).

$$A \cap B = (A^c \cup B^c)^c \tag{1.7}$$

TODO introduce \mathcal{F} is this on the right level? corner cases? compare to the general case

$$A, B \in \mathcal{A} \xrightarrow{(1.4)} A^c, B^c \in \mathcal{A} \xrightarrow{(1.5)} (A^c \cup B^c) \in \mathcal{A} \xrightarrow{(1.7)} (A \cap B)^c \in \mathcal{A} \xrightarrow{(1.4)} A \cap B \in \mathcal{A} \quad \square \tag{1.8}$$

Exercise 2: 2.6.5.6 Use Chen's Lemma

$X \in \text{Po}(\lambda)$. Show that

$$\mathbb{E}[X^n] = \lambda \sum_{k=0}^{n-1} \binom{n-1}{k} \mathbb{E}[X^k]. \quad (2.1)$$

Aid: Use Chen's Lemma with suitable $H(x)$.

Solution.

Lemma 2.2 (Chen's Lemma). $X \in \text{Po}(\lambda)$ and $H(x)$ is a bounded Borel function, then

$$\mathbb{E}[XH(X)] = \lambda \mathbb{E}[H(X+1)]. \quad (2.3)$$

Identity 2.4 (Binomial identity).

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k} \quad (2.5)$$

$$(x+1)^n = \sum_{k=0}^n \binom{n}{k} x^k \quad (2.6)$$

We choose $H(X) = X^{n-1}$ to show (2.1).

$$\mathbb{E}[X^n] = \mathbb{E}[XH(X)] \Big|_{H(X)=X^{n-1}} \stackrel{(2.2)}{=} \lambda \mathbb{E}[H(X+1)] \Big|_{H(X)=X^{n-1}} = \lambda \mathbb{E}[(X+1)^{n-1}] \quad (2.7)$$

$$\lambda \mathbb{E}[(X+1)^{n-1}] \stackrel{(2.6)}{=} \lambda \mathbb{E} \left[\sum_{k=0}^{n-1} \binom{n-1}{k} X^k \right] = \{\text{E linear operator}\} = \lambda \sum_{k=0}^{n-1} \binom{n-1}{k} \mathbb{E}[X^k] \quad \square \quad (2.8)$$

Exercise 3: 3.8.3.1 Joint Distributions & Conditional Expectations

Let (X, Y) be a bivariate random variable, where X is discrete and Y is continuous. (X, Y) has a joint probability mass - and density function given by

$$f_{X,Y}(k, y) = \begin{cases} \frac{\partial P(X=k, Y \leq y)}{\partial y} = \lambda \frac{(\lambda y)^k}{k!} e^{-2\lambda y} & , k \in \mathbb{Z}_{\geq 0}, y \in [0, \infty) \\ 0 & . \end{cases} \quad (3.1)$$

(a) Check that

$$\sum_{k=0}^{\infty} \int_0^{\infty} f_{X,Y}(k, y) dy = \int_0^{\infty} \sum_{k=0}^{\infty} f_{X,Y}(k, y) dy = 1 \quad (3.2)$$

(b) Compute the mixed moment $E[XY]$ defined as

$$E[XY] = \sum_{k=0}^{\infty} \int_0^{\infty} ky f_{X,Y}(k, y) dy. \quad (3.3)$$

Answer: $\frac{2}{\lambda}$

(c) Find the marginal p.m.f. of X . *Answer:* $X \in \text{Ge}(\frac{1}{2})$

(d) Compute the marginal density of Y here defined as

$$f_Y(y) = \begin{cases} \sum_{k=0}^{\infty} f_{X,Y}(k, y) & , y \in [0, \infty) \\ 0 & . \end{cases} \quad (3.4)$$

Answer: $Y \in \text{Exp}(\frac{1}{\lambda})$

(e) Find

$$p_{X|Y}(k|y) = p(X = k|Y = y), k \in \mathbb{Z}_{\geq 0} \quad (3.5)$$

Answer: $X|Y = y \in \text{Po}(\lambda y)$.

(f) Compute $E[X|Y = y]$ and then $E[XY]$ using double expectation. Compare your results with (b).

Solution. (a)

(b)

(c)

(d)

(e)

(f)

(g)

Exercise 4: 3.8.3.14 Computations on a distribution

Let (P, Y) be a bivariate r.v. such that

$$Y|P = p \in \text{Fs}(p), \quad f_P(p) = 3p^2, \quad p \in [0, 1]. \quad (4.1)$$

Compute: $E[Y]$, $\text{Var}[Y]$, $\text{Cov}(P, Y)$ and the p.m.f. of Y . *Answer:* $E[Y] = \frac{3}{2}$, $\text{Var}[Y] = \frac{9}{4}$, $\text{Cov}(P, Y) = -\frac{1}{8}$, and $p_Y(k) = \frac{18}{(k+3)(k+2)(k+1)k}$, $k \geq 1$.

Solution.

$$\begin{aligned} p_Y(k) &= \int_{p=0}^1 p_{Y|P=p}(k) df_P(p) = 3 \int_{p=0}^1 p^3(1-p)^{k-1} dp = \left\{ \begin{array}{l} q = 1-p \\ dq = -dp \end{array} \right\} \\ &= 3 \int_{q=0}^1 (1-q)^3 q^{k-1} dq = 3 \int_{q=0}^1 (-q^3 + 3q^2 - 3q + 1) q^{k-1} dq \\ &= 3 \int_{q=0}^1 -q^{k+2} + 3q^{k+1} - 3q^k + q^{k-1} dq \\ &= 3 \left(-\frac{1}{k+3} + \frac{3}{k+2} - \frac{3}{k+1} + \frac{1}{k} \right) \\ &= \frac{18}{(k+3)(k+2)(k+1)k}, k \geq 1 \end{aligned} \quad (4.2)$$

$$\begin{aligned} E[Y] &\triangleq \sum_{k=1}^{\infty} k \frac{18}{(k+3)(k+2)(k+1)k} = \sum_{k=1}^{\infty} \frac{18}{(k+3)(k+2)(k+1)} \\ &= \frac{3}{2} \end{aligned} \quad (4.3)$$

$$\text{Var}[Y] \triangleq \sum_{k=1}^{\infty} (k - E[Y])^2 \frac{18}{(k+3)(k+2)(k+1)k} = \frac{9}{4} \quad (4.4)$$

$$E[P] = \int_{p=0}^1 3p^2 dp = \frac{3}{4} \quad (4.5)$$

$$\text{Cov}(P, Y) \triangleq \int \sum (k - E[Y])(p - E[P]) \dots = -\frac{1}{8} \quad (4.6)$$

Exercise 5: 4.7.2.4 Equidistribution

Let $\{X_k\}_{k=1}^n$ be independent and identically distributed. Furthermore $\{a_k\}_{k=1}^n$, $a_k \in \mathbb{R}$. Set

$$Y_1 = \sum_k a_k X_k \quad (5.1)$$

and

$$Y_2 = \sum_k a_{n-k+1} X_k. \quad (5.2)$$

Show that

$$Y_1 \stackrel{d}{=} Y_2. \quad (5.3)$$

Solution. TODO needs background identities and theorems for all the steps for all X exists uniqueness of varphiX (\Rightarrow varphiX = varphiXk) varphi sum varphi scale Xk is all identically distributed product is reorderable due to (countable?) commutativity and associative of multiplication anything more? TODO compare with FS8.3

Identity 5.4 (Permuted coefficients). $\{X_k\}$, X is independent and identically distributed. σ is any permutation.

$$\varphi\left(\sum_{k \in K} a_{\sigma(k)} X_k\right)(t) = \prod_{k \in K} \varphi_X(a_k t) = \varphi\left(\sum_{k \in K} a_k X_k\right)(t) \quad (5.5)$$

Proof.

$$\varphi_{TODO} \quad (5.6)$$

□

Using Theorem 5.4 with $\sigma : k \rightarrow n - k + 1$

$$\varphi_{Y_1} = \varphi\left(\sum_k a_k X_k\right) = \varphi\left(\sum_k a_{n-k+1} X_k\right) = \varphi_{Y_2} \quad (5.7)$$

And with uniqueness theorem of blablabla we have

$$\{\varphi_{Y_1} = \varphi_{Y_2}\} \Rightarrow \{Y_1 \stackrel{d}{=} Y_2\} \quad \square \quad (5.8)$$

Exercise 6: 5.8.3.11 Laplace distribution

Let $\{X_k\}_{k=1}^n$ be independent and identically distributed with $X_k \in L(a), k \in [1, N_p], N_p \in \text{Fs}(p)$. N_p is independent of the variables $\{X_k\}$. We set

$$S_{N_p} = \sum_{k=1}^{N_p} X_k. \quad (6.1)$$

Show that $\sqrt{p}S_{N_p} \in L(a)$.

Solution.

Exercise 7: 7.6.1.1 Mean square convergence

Assume $X_n, Y_n \in L_2(\Omega, \mathcal{F}, P) \forall n$ and

$$X_n \xrightarrow{2} X, \quad Y_n \xrightarrow{2} Y \quad \text{as } n \rightarrow \infty \quad (7.1)$$

Let $a, b \in \mathbb{R}$. Show that

$$aX_n + bY_n \xrightarrow{2} aX + bY \quad \text{as } n \rightarrow \infty \quad (7.2)$$

You should use the definition of mean square convergence and suitable properties of $\|X\|$ as defined in (LN 7.3).

Solution. FS12.2 first "if"-statement but for 2 instead of without... (close)