FREE BOUNDARY PROBLEMS VIA SAKAI'S THEOREM

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Abstract. A Schwarz function on an open domain Ω is a holomorphic function satisfying $S(\zeta) = \overline{\zeta}$ on Γ, which is part of the boundary of Ω. Sakai in 1991 gave a complete characterization of the boundary of a domain admitting a Schwarz function. In fact, if Ω is simply connected and $\Gamma = \partial\Omega \cap D(\zeta, r)$, then Γ has to be regular real analytic. Here we try to describe Γ when the boundary condition is slightly relaxed. In particular, we are interested in three different scenarios over a simply connected domain Ω: When $f_1(\zeta) = \overline{\zeta} f_2(\zeta)$ on Γ, with f_1, f_2 holomorphic and continuous up to the boundary, when $\mathcal{U}/\mathcal{V}$ equals certain real analytic function on Γ with $\mathcal{U}, \mathcal{V}$ positive and harmonic on Ω and vanishing on Γ, and when $S(\zeta) = \Phi(\zeta, \overline{\zeta})$ on Γ, with Φ a holomorphic function of two variables. It turns out the boundary piece Γ can be, respectively, anything from real analytic to just C^1, regular except finitely many points, or regular except for a measure zero set.

Contents

1. Introduction

Let Ω be an open subset of the disk $D(\zeta_0, r) = \{z \in \mathbb{C} : |z - \zeta_0| < r\}$ $(r > 0)$ where $\zeta_0 \in \Gamma = \partial\Omega \cap D(\zeta_0, r)$ is a non-isolated boundary point. A Schwarz function of Ω on $D(\zeta_0, r)$ is a function $S : \Omega \to \mathbb{C}$ holomorphic on Ω and continuous on its boundary, Γ, which satisfies

$$S(\zeta) = \overline{\zeta} \qquad \text{on } \Gamma. \tag{1.1}$$

In his Acta Mathematica paper [19], Sakai proved that Schwarz functions completely characterise the shape of Γ. One of the technical tools used was Phragmén-Lindenöf Principle in the form below, but it is far from being the key to his proof; his paper is full of very subtle tricks.

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Theorem 1.1. *Let Ω be an open set in $\mathbb{C}$ and let ζ_0 be a non-isolated boundary point of Ω. Let f be a holomorphic function on Ω and $D(\zeta_0, \delta)$ be a ball satisfying the following:*

(i) $\limsup |f(z)| \leq 1$ *while* $\Omega \ni z \to \zeta$ *for every* $\zeta \in \partial\Omega \cap D(\zeta_0, \delta) \setminus \{\zeta_0\}$ *and*

(ii) $|f(z)| \leq \alpha |z - \zeta_0|^{-\beta}$ *in* $\Omega \cap D(\zeta_0, \delta)$ *for some positive constants* α *and* β.

Then,

$$\limsup |f(z)| \leq 1$$

while $\Omega \ni z \to \zeta_0$.

In particular, Sakai proved the following [19, Theorem 5.2]:

Theorem 1.2. *Set $D = D(\zeta_0, r)$. Let $\Omega \subset D$ be an open set and ζ_0 an accumulation point of its boundary, $\Gamma = \partial\Omega \cap D$. Suppose S is a Schwarz function on $\Omega \cup \Gamma$, that is,*

(i) S *is holomorphic on* Ω,

(ii) *continuous on* $\Omega \cup \Gamma$, *and*

(iii) $S(\zeta) = \bar{\zeta}$ *on* Γ.

Then, for some small $0 < \delta \leq r$ one of the following must occur:

(1) $\Omega \cap D$ *is simply connected and* $\Gamma \cap D$ *is a regular real analytic simple arc through* ζ_0.

(2a) $\Gamma \cap D$ *determines uniquely a regular real analytic arc through* ζ_0. $\Gamma \cap D$ *is either an infinite proper subset of this arc with* ζ_0 *as an accumulation point or equal to it. Also,* $\Omega \cap D = D \setminus \Gamma$.

(2b) $\Omega \cap D = \Omega_1 \cup \Omega_2$ *where* Ω_1 *and* Ω_2 *are simply connected and* $\partial\Omega_1 \cap D$ *and* $\partial\Omega_2 \cap D$ *are regular real analytic simple arcs through* ζ_0 *and tangent at* ζ_0.

(2c) $\Omega \cap D$ *is simply connected and* $\Gamma \cap D$ *is a regular real analytic simple arc except for a cusp at* ζ_0. *The cusp points into* Ω.

Recall that *regular arc* means a differentiable arc whose derivative never vanishes and *simple* that it is parametrised by an injective continuous function.

Remarks 1.3. When $\zeta_0 = 0$ there is an easy to describe the cusp of (2c). In fact, there exists a holomorphic function $T : \bar{D}(0, \eta) \to \mathbb{C}$, for some $\eta > 0$, which has a zero of order 2 at 0, is univalent on closed upper half-disk $K_\eta \equiv \{|z| \leq \eta : \operatorname{Im}(z) \geq 0\}$, and satisfies $\Gamma \cap D \subset T(-\eta, \eta)$ and $T(K_\eta) \subset \Omega \cup \Gamma$.

The converse of this theorem also holds, in the sense that if any of the conditions (1), (2a), (2b) or (2c) is satisfied, then Ω admits a Schwarz function.

In order to differentiate between the cases, Sakai also showed an auxiliary result [19, Proposition 5.1], which we will also use here:

Theorem 1.4. *Set $D' = D(0, r)$. Let $\Omega' \subset D'$ be an open set and 0 an accumulation point of its boundary, $\Gamma' = \partial\Omega' \cap D'$. Then, (for some $r' \leq r$) either*

(1) *there exists a Schwarz function,* S_t, *of* $(\Omega' \cup \Gamma') \cap D(0, r')$ *at* 0 *if and only if there exists a function* Φ_1 *defined on* $(\Omega' \cup \Gamma') \cap D(0, \delta)$ *for some* $\delta > 0$ *such that*

  (i) Φ_1 *is holomorphic and univalent in* $\Omega' \cap D(0, \delta)$,

  (ii) Φ_1 *is continuous on* $(\Omega' \cup \Gamma') \cap D(0, \delta)$,

  (iii) $\Phi_1(\zeta) = |\zeta|^2$ *on* $\Gamma' \cap D(0, \delta)$

  or

(2) *there exists a Schwarz function,* S_t, *of* $(\Omega' \cup \Gamma') \cap D(0, r')$ *at* 0 *if and only if there exists a function* Φ_2 *defined on* $(\Omega' \cup \Gamma') \cap D(0, \delta)$ *for some* $\delta > 0$ *such that*

  (i') Φ_2 *is holomorphic and univalent in* $\Omega' \cap D(0, \delta)$,

  (ii') Φ_2^2 *is continuous on* $(\Omega' \cup \Gamma') \cap D(0, \delta)$,

(iii') $\Phi_2^2(\zeta) = |\zeta|^2$ *on* $\Gamma' \cap D(0,\delta)$
(iv') $\Phi_2(\Omega' \cap D(0,\delta)) \cup (-\varepsilon, \varepsilon)$ *contains a neighbourhood of* 0 *for* $\varepsilon > 0$.

In particular, the functions Φ_1, Φ_2 *are related to* S_t *by* $\Phi_1(z) = zS_t(z)$ *and* $\Phi_2(z) = \sqrt{zS_t(z)}$.

Unfortunately, Theorem 1.4 only holds around 0 in this form. Nevertheless, we can "translate" the setup of Theorem 1.2 by setting $\Omega' = \Omega - \zeta_0$, $\Gamma' = \Gamma - \zeta_0$ and $S_t(z) = S(z+\zeta_0) - \overline{\zeta}_0$ for $z \in \Omega'$. Then, S_t is a Schwarz function on $\Omega' \cup \Gamma'$ at 0. Cases (1) of the two theorems correspond with one another as do (2a), (2b) and (2c) with (2).

Sakai gives two applications of his results: the first one describes the local structure of the boundary of quadrature domains, while the second one deals with a free boundary problem of the classical type, namely, what is the boundary of the set of positivity of a smooth *non-negative* function in the disc such that $\Delta u = 1$ on the set $\{u > 0\}$.

It is natural to wonder how to extend this result to more a general form of eq. (1.1). In this text, we examine three different scenarios for a simply-connected domain Ω. In sections 2 to 4 equation (1.1) is replaced by

$$f_1(\zeta) = \overline{\zeta} f_2(\zeta) \qquad \text{for all } \zeta \in \partial\Omega \tag{1.2}$$

where f_1, f_2 are holomorphic functions continuous up to the boundary. This is closely connected with the model subspaces K_θ and Nevanlinna domains, which will be important here. It is shown that there are domains so that eq. (1.2) holds for which $\partial\Omega$ is C^∞ but not real analytic. Further, in section 5 we replace the quantity $\overline{\zeta} f_2(\zeta)$ with $\Phi(\zeta, \overline{\zeta})$, where Φ is a holomorphic function of two variables, to find that the boundary is locally composed of real analytic arcs. Finally, in section 6 we consider two positive harmonic functions $\mathcal{U}$ and $\mathcal{V}$, which are zero on a Jordan arc, Γ, of the boundary. If their ratio on Γ is equal to a real analytic function of the form $|A|^2$, where A is holomorphic, then Γ is real analytic itself with the possible exception of some cusps.

Our interests to the problems considered below also was spurred by an application, which originates from complex dynamics. Certain complex dynamics question naturally brought the second author to a another *free boundary problem* described in Section 6. After that it was very natural to ask related questions, where Sakai setup was generalized in yet two other ways. To our surprise the answers were quite different and required different techniques: from the use of Nevanlinna domains and pseudo-continuation to multi-valued analytic functions.

In the rest of this paper, $\mathbb{D}$ will stand for the unit disk and $\mathbb{T}$ for the unit circle.

2. Polynomials & Analytic Functions

Let Ω be an open domain, ζ_0 a non-isolated boundary point of Ω and $\Gamma = \partial\Omega \cap D(\zeta_0, r)$ for some $r > 0$. Suppose S is a holomorphic function on Ω continuous on $\Omega \cup \Gamma$. We start with a simple yet important case. Instead of eq. (1.1), we consider

$$S(\zeta) = \overline{\zeta} p(\zeta) \qquad \text{on } \Gamma, \tag{2.1}$$

where p is a polynomial. We will shortly show that $f(z) = \frac{S(z)}{p(z)}$ is, in fact, a Schwarz function.

Lemma 2.1. *Assume* $S : \Omega \to \mathbb{C}$ *is holomorphic on* $\Omega \subset D(\zeta_0, r)$*, continuous on* $\Omega \cup \Gamma$*, and that it satisfies*

$$S(\zeta) = \overline{\zeta}(\zeta - \zeta_0)^n \qquad \textit{on } \Gamma.$$

Then, the function $S_t(z) = S(z+\zeta_0) - \overline{\zeta_0} z^n$ *is holomorphic on* $\Omega - \zeta_0 \subset D(0,r)$*, continuous on* $(\Omega - \zeta_0) \cup (\Gamma - \zeta_0)$ *and it satisfies*

$$S_t(\zeta) = \overline{\zeta}\zeta^n \qquad \textit{on } \Gamma - \zeta_0.$$

Proposition 2.2. *Assume $0 \in \Gamma$ is a non-isolated boundary point of $\Omega \subset D(0,r)$ and suppose S is a holomorphic function of Ω continuous on $\Omega \cup \Gamma$ satisfying*

$$S(\zeta) = \overline{\zeta}\zeta^n \qquad \textit{on } \Gamma.$$

Then, for any positive $\delta < r$ the function $\frac{S(z)}{z^n}$ is holomorphic on $\Omega \cap D(0,\delta)$ and continuous on $(\Omega \cup \Gamma) \cap D(0,\delta) \setminus \{0\}$. Moreover, the following holds while $z \in \Omega \cup \Gamma \setminus \{0\}$:

$$\lim_{z \to 0} \frac{S(z)}{z^n} = 0.$$

Proof. The function $\frac{S(z)}{z^n}$ is clearly holomorphic on $\Omega \cap D(0,\delta)$ and continuous on $(\Omega \cup \Gamma) \cap D(0,\delta) \setminus \{0\}$ for any $\delta \in (0,r)$. It remains to see what happens at 0.

Fix $\delta \in (0,r)$. Since S is bounded on $\Omega \cap D(0,r)$, say by m, we get

$$\left| \frac{S(z)}{z^n} \right| \le m|z|^{-n} \qquad \text{on } \Omega \cap D(0,\delta)$$

and additionally for any $\zeta \in \Gamma \cap D(0,\delta) \setminus \{0\}$ it holds that

$$\lim \left| \frac{S(z)}{z^n} \right| = |\overline{\zeta}| \le \delta \qquad \text{while } \Omega \ni z \to \zeta.$$

Hence, by the Phragmén-Lindelöf Principle 1.1 we have that

$$\limsup \left| \frac{S(z)}{z^n} \right| \le \delta \qquad \text{while } \Omega \ni z \to 0.$$

This last inequality holds for any positive $\delta < r$ and therefore $\lim \frac{S(z)}{z^n} = 0$ as $z \to 0$. □

Corollary 2.3. *Let p be a complex polynomial. Assume $\zeta_0 \in \Gamma$ is a non-isolated boundary point of Ω other than zero and suppose S is a holomorphic function of $\Omega \subset D(\zeta_0,r)$ continuous on $\Omega \cup \Gamma$ satisfying*

$$S(\zeta) = \overline{\zeta}p(\zeta) \qquad \textit{on } \Gamma.$$

Set $f(z) = S(z)/p(z)$ on $\Omega \cup \Gamma \setminus \{\zeta_0\}$ and $f(\zeta_0) = \overline{\zeta_0}$. Then, f is a Schwarz function of $\Omega \cup \Gamma$ on $D(\zeta_0,r)$ for sufficiently small $r > 0$.

Proof. Take r small enough so that p has no zeroes on $\overline{D(\zeta_0,r)} \setminus \{\zeta_0\}$. If $p(\zeta_0) \neq 0$, the result is immediate.

If $p(\zeta_0) = 0$, we just need to show that f is continuous on $(\Omega \cup \Gamma) \cap D(\zeta_0,r)$. Denote by n the order of ζ_0 as a zero of p and consider the function

$$S_n(z) = S(z)\frac{(z-\zeta_0)^n}{p(z)}.$$

S_n is holomorphic on Ω, is continuous on $\Omega \cup \Gamma$ and satisfies

$$S_n(\zeta) = \overline{\zeta}(\zeta - \zeta_0)^n \qquad \text{on } \Gamma.$$

From Lemma 2.1 we get that

$$(S_n)_t(\zeta) = \overline{\zeta}\zeta^n \qquad \text{on } \Gamma - \zeta_0$$

and from Proposition 2.2 that while $z \in \Omega$

$$\lim_{z\to 0}\frac{(S_n)_t(z)}{z^n}=0 \implies \lim_{z\to 0}\frac{S_n(z+\zeta_0)-\overline{\zeta_0}z^n}{z^n}=0$$
$$\implies \lim_{z\to\zeta_0}\frac{S_n(z)-\overline{\zeta_0}(z-\zeta_0)^n}{(z-\zeta_0)^n}=0$$
$$\implies \lim_{z\to\zeta_0} f(z)=\lim_{z\to\zeta_0}\frac{S_n(z)}{(z-\zeta_0)^n}=\overline{\zeta_0}$$

and the conclusion follows. □

Notice that the same proof works with p replaced by any function F which is analytic in a neighbourhood of ζ_0. This along with Lemma 2.1 give us the following corollary:

Corollary 2.4. *Assume $\zeta_0 \in \Gamma$ is a non-isolated boundary point of $\Omega \subset D(\zeta_0, r)$. Suppose F is a function analytic around ζ_0 and S a holomorphic function of Ω continuous on $\Omega \cup \Gamma$ satisfying*

$$S(\zeta)=\overline{\zeta}F(\zeta) \qquad \text{on } \Gamma.$$

Set $f(z) = S(z)/F(z)$ on $(\Omega\cup\Gamma)\cap D(\zeta_0,\delta)\setminus\{\zeta_0\}$ for some small enough $\delta > 0$ and $f(\zeta_0) = \overline{\zeta_0}$. Then, f is a Schwarz function of $\Omega \cup \Gamma$ on $D(\zeta_0, \delta)$.

The converse of this corollary also holds in the sense that if Γ has certain shape, in particular, if it satisfies (1), (2a), (2b) or (2c) of 1.2, then there is a Schwarz function f of $\Omega \cup \Gamma$ at ζ_0 such that $S(\zeta) = \overline{\zeta}F(\zeta)$ on Γ where $S = Ff$.

In fact, we can slightly modify the same proof to get a little more, again through the Phragmén-Lindelöf principle 1.1:

Corollary 2.5. *Let p be a polynomial, F a function analytic in a neighbourhood of $\bar{\Omega}$ and S a function holomorphic on Ω and continuous on $\Omega \cup \Gamma$. Suppose that for all $\zeta \in \Gamma$ we have that*

$$S(\zeta)=p(\overline{\zeta})F(\zeta).$$

Then, around every non-isolated point ζ_0 of Γ for which $p'(\zeta_0) \neq 0$ there is some small $\delta > 0$ such that the function $p^{-1}(S/F)$ is a Schwarz function of $\Omega \cup \Gamma$ on $D(\zeta_0, \delta)$.

We wish to examine what happens in the more general case where p in (2.1) is replaced with any analytic function of Ω continuous on its boundary, but not necessarily analytic on that boundary. More specifically, suppose that f_1 and f_2 are functions analytic on Ω, continuous on $\Omega \cup \Gamma$ and which satisfy

$$f_1(\zeta)=\overline{\zeta}f_2(\zeta) \qquad \text{on } \Gamma. \tag{2.2}$$

As above, if $f_2(\zeta_0) \neq 0$, the function $f = f_1/f_2$ is a Schwarz function around $\zeta_0 \in \Gamma$ and no issues arise. However, if $f_2(\zeta_0) = 0$, the situation is very complicated in general.

We start with a lemma analogous to Lemma 2.1:

Lemma 2.6. *Assume $f_1, f_2 : \Omega \to \mathbb{C}$ are holomorphic on $\Omega \subset D(\zeta_0, r)$, continuous on $\Omega \cup \Gamma$, and that they satisfy*

$$f_1(\zeta)=\overline{\zeta}f_2(\zeta) \qquad \text{on } \Gamma.$$

Then, there exist functions $(f_1)_t$ and $(f_2)_t$ holomorphic on $\Omega - \zeta_0$, continuous on $(\Omega - \zeta_0) \cup (\Gamma - \zeta_0)$ and such that

$$(f_1)_t(\zeta)=\overline{\zeta}(f_2)_t(\zeta) \qquad \text{on } \Gamma-\zeta_0.$$

If additionally $f_2(\zeta_0) = 0$, then $(f_2)_t(0) = 0$.

Proof. Define $(f_1)_t$ by
$$(f_1)_t(z) = f_1(z+\zeta_0) - \overline{\zeta_0} f_2(z+\zeta_0).$$
Then for $\zeta \in \Gamma - \zeta_0$ we have
$$\begin{aligned}(f_1)_t(\zeta) &= f_1(\zeta+\zeta_0) - \overline{\zeta_0} f_2(\zeta+\zeta_0)\\ &= \overline{\zeta+\zeta_0} f_2(\zeta+\zeta_0) - \overline{\zeta_0} f_2(\zeta+\zeta_0)\\ &= \overline{\zeta} f_2(\zeta+\zeta_0)\end{aligned}$$
Setting $(f_2)_t(z) = f_2(z+\zeta_0)$, we have the desired equality.

Clearly, $(f_1)_t(0) = 0$ and also if $f_2(\zeta_0) = 0$, $(f_2)_t(0) = 0$. □

Abusing the notation, we denote these new functions again by f_1 and f_2.

It remains to show an analogous result to Corollary 2.3 with p replaced by f_2. In particular, we would want to show that the function $f = f_1/f_2$ is holomorphic on Ω, continuous on Γ, and that it satisfies
$$f(\zeta) = \frac{f_1(\zeta)}{f_2(\zeta)} = \overline{\zeta} \qquad \text{for all } \zeta \in \Gamma.$$
However, the limit of $f_1(z)/f_2(z)$ as $\Omega \ni z \to 0$ might not even exist when $f_2(0) = 0$ and we cannot apply the Phragmén-Lindelöf principle here. We will need to see this problem from a different scope.

3. Nevanlinna Domains and Inner Functions

Let Ω be a bounded simply-connected set with boundary $\Gamma = \partial\Omega$ and suppose there are holomorphic functions $f_1, f_2 : \Omega \to \mathbb{C}$ continuous up to the boundary which satisfy
$$f_1(\zeta) = \overline{\zeta} f_2(\zeta) \qquad \text{for } \zeta \in \Gamma. \tag{3.1}$$
In order to better understand the situation, we rewrite (3.1) as
$$\frac{f_1(\zeta)}{f_2(\zeta)} = \overline{\zeta} \tag{3.1'}$$
which now holds almost everywhere on Γ except for the measure-zero set $\Gamma \cap f_2^{-1}\{0\}$. This leads us to the notion of Nevanlinna domains.

Definition 3.1. A bounded simply connected open set $\Omega \subset \mathbb{C}$ is called a *Nevanlinna domain* if there exist bounded holomorphic functions $f, g \in H^\infty(\Omega)$ on Ω ($g \not\equiv 0$) such that
$$\overline{\zeta} = \frac{f(\zeta)}{g(\zeta)} \qquad \text{almost everywhere on } \partial\Omega$$
in the sense of conformal mappings. That is, $\overline{\psi(\xi)} = f(\psi(\xi))/g(\psi(\xi))$ for almost every $\xi \in \mathbb{T}$ where $\psi : \mathbb{D} \to \Omega$ is a conformal map onto Ω and $\psi(\xi)$ its angular boundary values.

This definition is independent of the choice of ψ and by the Lusin-Privalov uniqueness theorem the ratio f/g is uniquely defined on Ω.

We need to note that this definition does not imply any additional regularity for f or g on $\partial\Omega$ or $\bar{\Omega}$, not even continuity. Nevertheless, in (3.1) we do require the continuity of our functions. We will try to overcome this obstacle in the next section 4.

Now, let $\varphi : \mathbb{D} \to \Omega$ be a conformal map and consider the functions $F_1 = f_1 \circ \varphi$ and $F_2 = f_2 \circ \varphi$. Equalities (3.1) and (3.1') transform respectively to
$$F_1(\zeta) = \overline{\varphi(\zeta)} F_2(\zeta) \tag{3.2}$$

and

$$\frac{F_1(\zeta)}{F_2(\zeta)} = \overline{\varphi(\zeta)} \tag{3.2'}$$

both of which hold in angular boundary values almost everywhere on $\mathbb{T}$, since φ might not extend "nicely" on $\bar{\mathbb{D}}$. By the factorization Theorem,we can write F_1 and F_2 in $\mathbb{D}$ as

$$F_1 = \theta_1 \mathcal{F}_1 \quad \text{and} \quad F_2 = \theta_2 \mathcal{F}_2 \tag{3.3}$$

where $\mathcal{F}_i$ are the outer factors of F_i and θ_i their inner factors. Because $F_1, F_2 \in H^\infty$, also $\mathcal{F}_1, \mathcal{F}_2 \in H^\infty$ and from (3.2') we get

$$\frac{\theta_1(\zeta)}{\theta_2(\zeta)} \frac{\mathcal{F}_1(\zeta)}{\mathcal{F}_2(\zeta)} = \overline{\varphi(\zeta)}, \tag{3.4}$$

almost everywhere on $\mathbb{T}$ in angular boundary values. We distinguish between two cases: either θ_2 divides θ_1, that is, $\theta_1/\theta_2 \in H^\infty$, or it doesn't.

3.1. $\theta_2 \mid \theta_1$**.** Let $h = \theta_1/\theta_2 \in H^\infty$. Then, the function $(h\mathcal{F}_1)/\mathcal{F}_2$ belongs to the class N^+, defined as

$$N^+ = \left\{ \frac{f}{g} \ : \ f, g \in H^\infty, \ g \text{ is an outer function} \right\},$$

and its (angular) boundary values are equal almost everywhere on $\mathbb{T}$ to the (angular) boundary values of $\overline{\varphi}$. However, since Ω is bounded, $\varphi \in L^\infty(\mathbb{T}, m)$ where m is the normalised Lebesgue measure on $\mathbb{T}$. Smirnov's Theorem tells us that in fact $(h\mathcal{F}_1)/\mathcal{F}_2 \in H^\infty$. Therefore, we have a bounded holomorphic function on the disk which is equal to $\overline{\varphi}$ almost everywhere on $\mathbb{T}$. This is impossible whenever φ is a bounded holomorphic function on $\mathbb{D}$.

We are necessarily left with the other case.

3.2. $\theta_2 \nmid \theta_1$**.** We begin with some notations and definition which will be important for the rest of this text:

Let $\mathbb{D}_e = \widehat{\mathbb{C}} \setminus \bar{\mathbb{D}}$. For any function $h : \mathbb{D} \to \mathbb{C}$ we define $\tilde{h}$ as

$$\tilde{h}(z) = \overline{h(1/\overline{z})}.$$

The notation $\tilde{H}$ will stand for a function $\tilde{H} : \mathbb{D}_e \to \mathbb{C}$ and we will write H instead of $\tilde{\tilde{H}}$ for the function $\overline{\tilde{H}(1/\overline{z})}$. Observe that $h \in H^\infty$ if, and only if, $\tilde{h} \in H^\infty(\mathbb{D}_e)$ and $h(0) = 0$ if, and only if, $\tilde{h}(\infty) = 0$.

We will also consider the backward shift operator, $\mathcal{B} : H^p \to H^p$, for $p \in [1, \infty)$, that is

$$\mathcal{B} : f \mapsto \frac{f(z) - f(0)}{z}.$$

Definition 3.2. Let f be a meromorphic function on $\mathbb{D}$. We say f admits a *pseudo-continuation* (across $\mathbb{T}$) if there exists another meromorphic function g on $\mathbb{D}_e$ such that $f = g$ almost everywhere (on $\mathbb{T}$) in the sense of non-tangential limits.

The pseudo-continuation of f is called *of bounded type* or a *Nevanlinna-type pseudo-continuation* if g is of the form $g = h_1/h_2$ for some $h_1, h_2 \in H^\infty(\mathbb{D}_e)$.

Definition 3.3. A function $f \in H^p$ is called *a cyclic vector for* $\mathcal{B}$, or simply *cyclic for* $\mathcal{B}$ if the set $\{\mathcal{B}^n f\}_{n=0}^\infty$ spans the space H^p.

The following important result is due to Douglas, Shapiro and Shields:

Theorem 3.4. *Consider* $1 \le p < \infty$. *A function* $f \in H^p$ *is not cyclic for* $\mathcal{B}$ *if, and only if,* f *has a pseudo-continuation of bounded type.*

In the case when $p = 2$ it is known that any non-cyclic function of $\mathcal{B}$ belongs to a proper $\mathcal{B}$-invariant subspace.As a consequence of Beurling's Theorem, these spaces are of the form $(\theta H^2)^\perp$ and are known as *model spaces* and denoted by K_θ. Here we will need the fact that

$$K_\theta = (\theta H^2)^\perp = H^2(\mathbb{T}) \cap \theta \overline{H_0^2(\mathbb{T})},$$

where in the last equality we mean the boundary values of the corresponding functions and where $H_0^2 = \{f \in H^2 \, : \, f(0) = 0\}$.

We can now proceed with the case when $\theta_2 \nmid \theta_1$:

After dividing both θ_1 and θ_2 with their greatest common divisor we can assume that θ_1 and θ_2 have no common zeroes and that the Borel supports of their singular measures are disjoint. Similarly as above we see that the function $F = (\theta_1 \mathcal{F}_1)/\mathcal{F}_2 = F_1/\mathcal{F}_2$ belongs the class N^+ and thus $F \in H^\infty$, since $\theta_2 \overline{\varphi} \in L^\infty(\mathbb{T}, m)$. Then the following holds in angular boundary values for almost every $\zeta \in \mathbb{T}$:

$$\overline{\varphi(\zeta)} = \frac{\theta_1(\zeta)\mathcal{F}_1(\zeta)}{\theta_2(\zeta)\mathcal{F}_2(\zeta)} = \frac{F(\zeta)}{\theta_2(\zeta)}$$

$$\Longleftrightarrow \varphi(\zeta) = \theta_2(\zeta)\overline{F}(\zeta) \tag{3.5}$$

$$\Longleftrightarrow \varphi(\zeta) = \frac{\tilde{F}(\zeta)}{\tilde{\theta}_2(\zeta)} \tag{3.6}$$

Since $\tilde{F}, \tilde{\theta}_2 \in H^\infty(\mathbb{D}_e)$, $\varphi \in H^\infty \subset H^2$ has pseudo-continuation across $\mathbb{T}$ of bounded type and from Theorem 3.4 φ is not cyclic for $\mathcal{B}$. So, it has to belong to some model space K_θ. See [11, Theorem 1] for more details. In fact, from eq. (3.5) and because we "need" to have $F(0) = 0$ it follows that either

$$\varphi \in K_{\theta_2}, \text{ if } \theta_1(0) = 0, \quad \text{or} \quad \varphi \in K_{z\theta_2}, \text{ if } \theta_1(0) \neq 0.$$

4. Boundary behaviour of conformal maps in K_θ

In this section we show that Theorem 1.2 fails when condition (iii) is replaced by (3.1). To this end, we will find a simply-connected domain Ω and a conformal map $\varphi : \mathbb{D} \to \Omega$ continuous up the boundary, which has a pseudo-continuation of bounded type and is smooth but not real-analytic on $\mathbb{T}$. The functions participating into this pseudo-continuation will also be continuous on the boundary. First, we go one step back and work with Nevanlinna domains. Thanks to [11, Theorem 1] from Fedorovskiy, this is equivalent to studying the model subspaces, K_θ, for different inner functions θ. Also see [2, Theorems A and B].

If $\theta(z_0) = 0$ for some $z_0 \in \mathbb{D}$, the function

$$\varphi(z) = \frac{1}{1 - \overline{z_0} z} \in K_\theta \cap C^\infty(\mathbb{T})$$

is bounded type pseudo-continuation across $\mathbb{T}$ and thus $\varphi(\mathbb{D})$ is a Nevanlinna domain. In fact, φ can be analytically extended on the whole closed disk, $\bar{\mathbb{D}}$, and $\varphi(\mathbb{T})$ is real-analytic. On the other hand, in a series of papers, [15, 6, 11, 16, 4], it is shown that the boundary of a Nevanlinna domain can be "arbitrarily bad". In particular, it can be nowhere analytic [15], of class C^1 but not in any $C^{1,\alpha}$ for no $\alpha > 0$ [11], unrectifiable [16], or even have any possible dimension between 1 and 2 [4].

We need to mention that in all the above work the inner function θ is a Blaschke product (or has a Blaschke part). The exception is [4, Theorem 4] where for any $\beta \in [1, 2]$ the authors construct bounded univalent function, ψ, in the Paley-Wiener space $\mathcal{PW}^\infty_{[0,1]}$ for which the boundary of the Nevanlinna domain $\Omega = \psi(\mathbb{C}_+)$ has Hausdorff dimension β. This is the simplest example of a model space K_θ where θ is purely singular; in fact $\theta(z) = \exp\left(-\frac{1+z}{1-z}\right)$ in this case.

However, in order to compare with Sakai's theorem, we have to consider the case where the functions $\tilde{F_1}, \tilde{F_2} \in H^\infty(\mathbb{D}_e)$ for which $\varphi = \tilde{F_1}/\tilde{F_2}$ on $\mathbb{T}$ are continuous up to $\mathbb{T}$. This is not always possible when θ is not purely singular (see [6, Example 5.8]).

Therefore, in this section θ will be a singular inner function of the form

$$\theta(z) = \exp\left(-\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z} d\mu_\theta(\zeta)\right)$$

with μ_θ supported on a Carleson set, $E \subset \mathbb{T}$. We will show there is a conformal map $\varphi \in K_\theta$ continuous on $\bar{\mathbb{D}}$ which is in $C^\infty(\mathbb{T})$ but not real-analytic on $\mathbb{T}$.

In view of [10, Theorem 2.1], since $\operatorname{supp}(\mu_\theta)$ is Carleson, the space K_θ then contains a non-trivial function from some smoothness class, for example a function $g \in H^\infty \cap C^\infty(\mathbb{T})$, (or in a Bergman space, i.e. $g \in A^{p,1}$ for some $p > 1$). Since $g \in K_\theta$, it admits a bounded type pseudo-continuation of the form

$$g = \tilde{G}/\tilde{\theta} \qquad \text{almost everwhere on } \mathbb{T},$$

where $\tilde{G} \in H^\infty(\mathbb{D}_e)$ vanishes at infinity (see [7, Theorem 5.1.4]). Additionally, g has analytic continuation, say $\mathcal{G}$, on $\widehat{\mathbb{C}} \setminus \operatorname{supp}(\mu)$. Of course, $\mathcal{G} = \tilde{G}/\tilde{\theta}$ on $\mathbb{D}_e$ and observe that $\mathcal{G}$ cannot be bounded in $\mathbb{D}_e$; otherwise g would be constant, as $\mathcal{G}_{|\mathbb{D}} = g$ and $\mathcal{G}_{|\mathbb{D}_e}$ coincide almost everywhere on $\mathbb{T}$.

Now, consider $\alpha \in \mathbb{D}_e$ with $\theta(1/\overline{\alpha}) \neq 0$ and the following aggregate:

$$\varphi(z) = \frac{\mathcal{G}(z) - \mathcal{G}(\alpha)}{z - \alpha}.$$

We will show that $\varphi \in K_\theta \cap C^\infty(\mathbb{T})$ and is conformal in $\bar{\mathbb{D}}$.

Clearly, φ is inside $H^2(\mathbb{D})$ and also

$$\overline{\theta(\zeta)}\varphi(\zeta) = \frac{\overline{\theta(\zeta)}g(\zeta) - \overline{\theta(\zeta)}\mathcal{G}(\alpha)}{\zeta - \alpha} = \frac{\tilde{G}(\zeta) - \tilde{\theta}(\zeta)\mathcal{G}(\alpha)}{\zeta - \alpha}.$$

For $z \in \mathbb{D}_e$ the function

$$\frac{\tilde{G}(z) - \tilde{\theta}(z)\mathcal{G}(\alpha)}{z - \alpha} = \frac{1}{z - \alpha}\left(\tilde{G}(z) - \frac{\tilde{\theta}(z)}{\tilde{\theta}(\alpha)}\tilde{G}(\alpha)\right)$$

is analytic around α and vanishes at infinity. Hence, $\varphi \in K_\theta$.

Furthermore, φ is univalent in $\bar{\mathbb{D}}$. Indeed, suppose it's not. Then, there exist $z, w \in \bar{\mathbb{D}}$ with $z \neq w$ and $\varphi(z) = \varphi(w)$ or equivalently

$$\begin{aligned}
&\frac{g(z) - \mathcal{G}(\alpha)}{z - \alpha} = \frac{g(w) - \mathcal{G}(\alpha)}{w - \alpha} \\
\iff &\frac{g(z)}{z - \alpha} - \frac{g(w)}{w - \alpha} = \frac{\mathcal{G}(\alpha)}{z - \alpha} - \frac{\mathcal{G}(\alpha)}{w - \alpha} = \mathcal{G}(\alpha)\frac{z - w}{(z - \alpha)(w - \alpha)} \\
\iff &-\alpha\frac{g(z) - g(w)}{z - w} + w\frac{g(z) - g(w)}{z - w} - g(w) = \mathcal{G}(\alpha).
\end{aligned}$$

The left-hand side is bounded, since $g \in C^\infty(\bar{\mathbb{D}})$, whereas we can pick $1 < |\alpha| < 2$ so that $|\mathcal{G}(\alpha)|$ is arbitrarily large (recall $\mathcal{G}_{|\mathbb{D}_e}$ is not bounded), a contradiction, and therefore φ is univalent in $\bar{\mathbb{D}}$.

Consequently, if $g \in K_\theta \cap C^\infty(\mathbb{T})$ and θ is a singular inner function, φ is univalent in $\bar{\mathbb{D}}$ and $\varphi \in K_\theta \cap C^\infty(\mathbb{T})$. Also see [1, Section 4] or [3] for more details. At the same time, note that $\mathcal{G}$ cannot be analytically extended on the whole $\bar{\mathbb{D}}$, since it is unbounded near the unit circle, and thus neither can φ; this fails exactly on the Carleson set E.

Now, since $\varphi \in K_\theta$, we can write that

$$\varphi = \theta\overline{F} \iff \theta\overline{\varphi} = F \tag{4.1}$$

almost everywhere on $\mathbb{T}$ for some function $F \in H^2$ with $F(0) = 0$. In fact, $F \in H^\infty$ since $\varphi \in C^\infty(\bar{\mathbb{D}})$.

It is known there exists some analytic function, $\mathcal{H}$, with $\mathcal{H}_{|E} = 0$ such that both $\mathcal{H}$ and $\mathcal{H}\theta$ are Lipschitz on $\bar{\mathbb{D}}$. In fact, we can further consider $\mathcal{H}$ to be an outer function in $C^\infty(\mathbb{T})$.Multiplying $\mathcal{H}$ in (4.1), we get

$$(\mathcal{H}\theta)\overline{\varphi} = \mathcal{H}F \tag{4.2}$$

almost everywhere on $\mathbb{T}$. In particular, the left hand side is now smooth on the whole $\mathbb{T}$ and the same therefore holds for the right hand side. In a sense, $\mathcal{H}$ "annihilates" the singularities of θ as (4.1) fails exactly on the support, E, of μ_θ.

At this point, set $F_1 = \mathcal{H}F$, $F_2 = \mathcal{H}\theta$, and $f_j = F_j \circ \varphi^{-1}$ for $j = 1, 2$. Then, eq. (4.2) becomes

$$F_1 = \overline{\varphi}F_2,$$

which now holds on the whole boundary $\mathbb{T}$, and in turn

$$f_1(\zeta) = \overline{\zeta}f_2(\zeta) \qquad \text{for all } \zeta \in \Gamma.$$

This is exactly the setup we were looking for, albeit it contrasts with Sakai's result: Even though $\Gamma = \varphi(\mathbb{T})$ is C^∞-smooth, φ cannot be analytic on the Carleson set E and thus neither can Γ.

It is worth mentioning that as of now there no known examples of Nevanlinna domains which come from singular inner function with particularly irregular boundaries. It is unknown whether there exist conformal maps in K_θ which are only Lipschitz on the boundary.

5. Holomorphic Functions in $\mathbb{C}^2$

In this section we attempt to replace the function $\overline{\zeta}f_2(\zeta)$ in eq. (3.1) with a more general formula.

For some positive $r > 0$ let $\Omega \subset D(\zeta_0, r)$ be a simply-connected set, $\Gamma = \partial\Omega \cap D(\zeta_0, r)$, and $\zeta_0 \in \Gamma$. Here, we will also need the extra assumption that Γ is Jordan arc (or possibly a union of Jordan arcs).

Let Φ be a holomorphic function of two variables, that is, a function of the form

$$\Phi(z, w) = \sum_{n,m=0}^{+\infty} b_{nm} z^n w^m$$

where each of the functions $\Phi(z, \cdot\,)$ and $\Phi(\,\cdot\,, w)$ is itself holomorphic. In fact, we only need Φ to be holomorphic on a $\mathbb{C}^2$ neighbourhood of $(\zeta_0, \overline{\zeta}_0)$, but here we assume the whole $\mathbb{C}^2$ for simplicity.

Suppose there exists a function R which is

(i) holomorphic on Ω,
(ii) continuous on $\bar{\Omega}$, and
(iii) satisfies $R(\zeta) = \Phi(\zeta, \overline{\zeta})$ on Γ.

In view of Lemma 2.6 we can assume $\zeta_0 = 0$ and $b_{00} = 0$ so that $R(0) = \Phi(0,0) = 0$. Notice $R(z)$ and $\Phi(z, \overline{z})$ are bounded on $\bar{\Omega}$ and thanks to the Phragmén-Lindelöf Principle 1.1 we can assume without loss of generality that there exist some non-negative integer k for which

$$\Phi(0,0) = \frac{\partial}{\partial w}\Phi(0,0) = \cdots = \frac{\partial^{k-1}}{\partial w^{k-1}}\Phi(0,0) = 0 \quad \text{and} \quad \frac{\partial^k}{\partial w^k}\Phi(0,0) \neq 0 \tag{5.1}$$

otherwise Φ would be identically zero.

We would like to use Weierstraß Approximation Theorem for the function $\Phi(z,w) - R(z)$ around 0, but R is not holomorphic on the boundary. But since its continuous by (ii) and Γ is Jordan, we can use Mergelyan's Theorem to get a sequence of polynomials p_n that converge to R uniformly on $\bar{\Omega}$. And we can pick this sequence so that $p_n(0) = 0$ for every $n = 0, 1, \dots$

Next, we define the functions

$$\Psi(z,w) = \Phi(z,w) - R(z) \quad \text{and} \quad \Psi_n(z,w) = \Phi(z,w) - p_n(z).$$

Ψ_n are holomorphic on $\mathbb{C}^2$ and converge uniformly to Ψ on $\bar{\Omega} \times \mathbb{C}$. Observe that for all n we have $\Psi_n(0,0) = \Phi(0,0) - p_n(0) = 0$ and also

$$\frac{\partial^\kappa}{\partial w^\kappa}\Psi_n = \frac{\partial^\kappa}{\partial w^\kappa}\Phi \qquad \text{for all integers } \kappa \geq 1$$

and all points (z,w). Then, from (5.1) and from Weierstraß Approximation Theoremthere exist unique holomorphic functions $a_{0;n}, \dots, a_{k-1;n} : \mathbb{C} \to \mathbb{C}$ and $c_n : \mathbb{C}^2 \to \mathbb{C}$ with $a_{j;n}(0) = 0$ and $c_n(0,0) \neq 0$ such that

$$\Psi_n(z,w) = c_n(z,w)\left(w^k + a_{k-1;n}(z)w^{k-1} + \cdots + a_{0;n}(z)\right).$$

Following the proof of the Weierstraß theorem and since the convergence $\Psi_n \to \Psi$ is uniform on $\bar{\Omega} \times \mathbb{C}$, we can find sufficiently small δ and ρ with $\rho \geq \delta > 0$ so that $a_{0;n}, \dots, a_{k-1;n}$ and c_n converge uniformly on $\bar{\Omega} \cap D(0,\delta)$ and $\left(\bar{\Omega} \cap D(0,\delta)\right) \times D(0,\rho)$, respectively, to some functions $a_0, \dots, a_{k-1}$ and c with $a_j(0) = 0$ and $c(0,0) \neq 0$. Note that the functions a_j are holomorphic on $\Omega \cap D(0,\delta)$ and continuous on $\bar{\Omega} \cap D(0,\delta)$. Subsequently, we get

$$\Phi(z,w) - R(z) = c(z,w)\left(w^k + a_{k-1}(z)w^{k-1} + \cdots + a_0(z)\right). \tag{5.2}$$

Let us write

$$P(z,w) = w^k + a_{k-1}(z)w^{k-1} + \cdots + a_0(z)$$

for the polynomial factor. From (iii), eq. (5.2) and since $c(0,0) \neq 0$, we have that

$$P(\zeta, \overline{\zeta}) = \overline{\zeta}^k + a_{k-1}(\zeta)\overline{\zeta}^{k-1} + \cdots + a_0(\zeta) = 0 \qquad \text{for all } \zeta \in \Gamma \cap D(0,\delta). \tag{5.3}$$

Remark. Functions of the form

$$P(z, \overline{z}) = \overline{z}^k + a_{k-1}(z)\overline{z}^{k-1} + \cdots + a_0(z),$$

where a_j are polynomials, are called a poly-analytic polynomials. One can find more details on these at [12], [16] or [21].

We are interested in the roots of the polynomial $P(z, \cdot\,)$ when $z \in \bar{\Omega} \cap D(0,\delta)$. In other words, we will study the equation (in w)

$$P(z,w) = 0 \iff w^k + a_{k-1}(z)w^{k-1} + \cdots + a_0(z) = 0 \qquad \text{when } z \in \bar{\Omega} \cap D(0,\delta). \tag{5.4}$$

Let $\mathcal{D}(z)$ be the discriminant of $P(z, \cdot\,)$ (for any fixed z). Then, $\mathcal{D}(z)$ is a polynomial of the coefficients $a_0(z), \dots, a_{k-1}(z)$ and is equal to 0 if, and only if, $P(z,w)$ and $\frac{\partial}{\partial w}P(z,w)$ share a common factor. The roots of $P(z, \cdot\,)$ are given by a multi-valued holomorphic function, $\mathcal{W}$, depending on $a_1, \dots, a_{k-1}$, and the points where $\mathcal{W}$ changes branch inside $\Omega \cap D(0,\delta)$ are exactly the zeroes of $\mathcal{D}$ (in $\Omega \cap D(0,\delta)$).

We distinguish between two cases: when $\mathcal{D}$ is identically 0 and when it's not.

Before moving on let us note that the set $\mathcal{M}(\Omega,\Gamma,\delta)$ of all meromorphic functions on $\Omega \cap D(0,\delta)$ continuous up to $(\Omega \cup \Gamma) \cap D(0,\delta)$ except possibly a (closed) measure zero subset of Γ is a field with the usual operations of addition and multiplication.

5.1. $\mathcal{D} \neq 0$**.** Here $P(z,w)$ is irreducible over $\mathcal{M}(\Omega,\Gamma,\delta)$. Since $\mathcal{D}$ is continuous on $(\Omega \cup \Gamma) \cap D(0,\delta)$, the set $(\mathcal{D}^{-1}\{0\} \cap \Gamma) \cap D(0,\delta)$ is closed and of zero harmonic measure. Now, we decompose $(\Gamma \setminus \mathcal{D}^{-1}\{0\}) \cap D(0,\delta)$ into countably many open connected arcs.

Let γ be one of these arcs. Then, there exists a simply-connected set $D \subset \Omega \cap D(0,\delta)$ such that $\partial D \cap \partial\Omega = \gamma$. Because $\mathcal{D}$ has no zeroes on $D \cup \gamma$, by the monodromy theorem the multivalued function $\mathcal{W}$ "splits" into k many distinct holomorphic functions, W_j $(j = 1, \dots, k)$, and let $C_j = \{\zeta \in \gamma : W_j(\zeta) = \overline{\zeta}\}$. Notice that C_j's are closed (in γ), they cover γ, and any two of them intersect at a (closed) set of zero harmonic measure.

Unfortunately, C_j need not be connected, but we can further decompose each $\mathring{C}_j$ (whenever it's non-empty) into countably many open arcs as in $\mathring{C}_j = \cup_i \gamma_j^i$, for $j = 1, \dots, k$. Again, around each γ_j^i we consider a neighbourhood $D_j^i \subset D$ with $\partial D_j^i \cap \partial D = \gamma_j^i$ (these can, but need not be simply-connected) and let $W_j^i = W_{j|D_j^i \cup \gamma_j^i}$.

Then, for each $j = 1, \dots, k$ and $i = 1, 2 \dots$ the functions W_j^i are holomorphic on D_j^i, continuous on $D_j^i \cup \gamma_j^i$ and satisfy $W_j^i(\zeta) = \overline{\zeta}$ for all $\zeta \in \gamma_j^i$; in other words they are Schwarz functions on $D_j^i \cup \gamma_j^i$. Since Γ is Jordan, all γ_j^i are also Jordan and from Theorem 1.2 we conclude that each γ_j^i is, in fact, a regular real-analytic simple arc except possibly some cusps.

5.2. $\mathcal{D} = 0$**.** In this case the $P(z,w)$ has to be reducible over $\mathcal{M}(\Omega,\Gamma,\delta)$. In particular, we can write $P(z,w) = P_1(z,w) \cdots P_{\tilde{k}}(z,w)$ for some $\tilde{k} \leq k$ where each $P_\kappa(z,w)$ has now coefficients in $\mathcal{M}(\Omega,\Gamma,\delta)$ and is irreducible, i.e. $\mathcal{D}_\kappa \neq 0$ where $\mathcal{D}_\kappa$ is the discriminant of $P_\kappa(z,\,\cdot\,)$. Since $P(\zeta,\overline{\zeta}) = 0$ for all $\zeta \in \Gamma$, we can split $(\Gamma \setminus E) \cap D(0,\delta)$, where E is some closed zero-(harmonic)-measure set, into open sets O_κ for $\kappa = 1, \dots, \tilde{k}$ so that $P_\kappa(\zeta,\overline{\zeta}) = 0$ for all $\zeta \in O_\kappa$. Notice that $O_\kappa \cap O_{\kappa'} = \emptyset$ when P_κ and $P_{\kappa'}$ are different.

Observe that, because $P(z,w)$ factors into the polynomials $P_\kappa(z,w)$ (over $\mathcal{M}(\Omega,\Gamma,\delta)$) and the roots of $P(z,\,\cdot\,)$ are given by the multivalued holomorphic function $\mathcal{W}$, the roots of each $P_\kappa(z,\,\cdot\,)$ are also given by a multivalued holomorphic function $\mathcal{W}_\kappa$ whose branches are comprised of branches of $\mathcal{W}$.

Working as above for each $\kappa = 1, \dots, \tilde{k}$, we separate $O_\kappa \setminus \mathcal{D}_\kappa^{-1}\{0\}$ into countably many open arcs and for each such arc, γ, we find some simply-connected neighbourhood, $D \subset \Omega$, with $\partial D \cap \partial\Omega = \gamma$ so that $\mathcal{W}_\kappa$ "splits" into its different branches. Again following the above arguments, we can decompose γ — minus a zero-measure set — into countably many open arcs over which $W_j(\zeta) = \overline{\zeta}$ for some branch W_j of $\mathcal{W}_\kappa$. Constructing appropriate neighbourhoods, we conclude that except a zero-measure set γ is a countable union of regular real-analytic simple arcs except possibly some cusps.

In either case, the cusps (if they exist) point into Ω and may only accumulate on the endpoints of each open arc.

Let us now formulate the above results into a theorem:

Theorem 5.1. *Let $\Omega \subset D(\zeta_0, r)$ be a bounded simply connected domain such that the set $\Gamma = \partial\Omega \cap D(\zeta_0, r)$ is a (union of) Jordan arc(s) for some (small) $r > 0$. Also, let Φ be a (non-trivial) function of two variables holomorphic on a $\mathbb{C}^2$ neighbourhood of $(\zeta_0, \overline{\zeta_0})$, and suppose there exists a function R*

(i) holomorphic on Ω,
(ii) continuous on $\bar{\Omega}$, and such that
(iii) $R(\zeta) = \Phi(\zeta, \overline{\zeta})$ for all $\zeta \in \Gamma$.

Then, there exists a closed set, $E \subset \Gamma$, of zero harmonic measure so that $\Gamma \setminus E$ is a countable union of regular real-analytic simple arcs except possibly for some cusps. The cusps (if they exist) point into Ω and may only accumulate on E.

6. The $\mathcal{U}$-$\mathcal{V}$ Problem

In this section, we are interested in the following setup:

Let $\Omega \subset \mathbb{C}$ be a simply-connected domain and let $\zeta_0 \in \partial\Omega$ be a boundary point of Ω. Assume that for some $\rho > 0$ the connected component, Γ, of $\partial\Omega \cap D(\zeta_0, \rho)$ containing ζ_0 is a Jordan curve. Note that $\rho \geq \operatorname{dist}(\zeta_0, \partial\Omega \setminus \Gamma) > 0$. For convenience we will be writing simply Ω to denoted $\Omega \cap D(\zeta_0, \rho)$.

Let A be an analytic function in a neighbourhood, $D(\zeta_0, \varepsilon)$, of ζ_0 and suppose we have two functions $\mathcal{U}$ and $\mathcal{V}$ defined on Ω, which are not proportional, with the following properties:

I) $\mathcal{U}$ and $\mathcal{V}$ are positive and harmonic on Ω,
II) continuous on $\Omega \cup \Gamma$,
III) $\mathcal{U} = \mathcal{V} = 0$ on Γ and
IV) $\frac{\mathcal{U}(\zeta)}{\mathcal{V}(\zeta)} = |A(\zeta)|^2 \neq const.$ for $\zeta \in \Gamma$.

Notice that because $\mathcal{U} \neq c\mathcal{V}$ the function $|A|$ needs to be non-constant. Otherwise, we could have $\mathcal{U} = c\mathcal{V}$ and all our conditions work trivially for any Γ. Also, we may assume $\rho < \varepsilon$ without loss of generality so that A is defined over the whole Ω to avoid unnecessary technical difficulties.

Equality (IV) is to be understood in the sense of limits, i.e. the limit of $\mathcal{U}(z)/\mathcal{V}(z)$ as $\Omega \ni z \to \zeta \in \Gamma$ exists and is equal to $|A(\zeta)|^2$. In fact, this limit always exists when Ω is simply-connected and Γ Jordan (see Remark 6.1), so the only assumption here is the values it takes.

Consider a conformal map from the Poincaré plane to Ω, $\varphi : \mathbb{H} \to \Omega$. Since Γ is connected and Jordan, Carathéodory's Theorem implies that φ extends conformally to a function (abusing the notation) $\varphi : \mathbb{H} \cup \gamma \to \Omega \cup \Gamma$ which we can pick so that $\gamma \subset \mathbb{R}$ is some bounded open interval with $\varphi(\gamma) = \Gamma$ and $\varphi(0) = \zeta_0$. Utilising this φ, we can "transfer" the information about $\mathcal{U}$ and $\mathcal{V}$ over Ω to information over $\mathbb{H}$. Define

$$u \equiv \mathcal{U} \circ \varphi, \quad v \equiv \mathcal{V} \circ \varphi \quad \text{and} \quad a \equiv A \circ \varphi$$

and note that a is analytic on $\mathbb{H}$ and continuous on $\mathbb{H} \cup \gamma$. Analogously as above we have

i) u and v are positive and harmonic on $\mathbb{H}$,
ii) continuous on $\mathbb{H} \cup \gamma$,
iii) $u = v = 0$ on γ and
iv) $\frac{u}{v} = |a|^2$ on γ.

Again equality (iv) is to be understood in the sense of limits.

Now, harmonically extend u and v on $\mathbb{H}\cup\gamma\cup\mathbb{H}^-$ by

$$u^*(z)=\begin{cases}u(z), & z\in\mathbb{H}\\ 0, & z\in\gamma\\ -u(\overline{z}), & z\in\mathbb{H}^-\end{cases}\quad\text{and}\quad v^*(z)=\begin{cases}v(z), & z\in\mathbb{H}\\ 0, & z\in\gamma\\ -v(\overline{z}), & z\in\mathbb{H}^-\end{cases}$$

and let h be the function

$$h(z)=\begin{cases}\frac{u^*(z)}{v^*(z)}, & z\in\mathbb{H}\cup\mathbb{H}^-\\ \frac{u^*_y(z)}{v^*_y(z)}, & z\in\gamma\end{cases}$$

We claim that h is well-defined and, in fact, real analytic on $\mathbb{H}\cup\gamma\cup\mathbb{H}^-$. Indeed, using Harnack's inequality, for any $(x,0)\in\gamma$ there exists a constant, $c>0$, (dependent on v^*) such that

$$c\frac{y}{2-y}\le v^*(x,y)\le c\frac{2-y}{y}\qquad\text{for every } 0<y<1,\ \text{or}$$

$$c\frac{1}{2-y}\le\frac{v^*(x,y)}{y}\le c\frac{2-y}{y^2}\tag{6.1}$$

Recall that $v^*(x,0)=0$ and take limits as $y\to 0^+$. Since v^* is harmonic on $\mathbb{H}\cup\gamma\cup\mathbb{H}^-$, eq. (6.1) guarantees that $v^*_y>0$ on γ (the same holds for u^*) and therefore the limit

$$\lim_{y\to 0}\frac{u^*(x,y)}{v^*(x,y)}=\frac{u^*_y(x,0)}{v^*_y(x,0)}$$

exists and is finite. Hence, h is a well-defined continuous function on $\mathbb{H}\cup\gamma\cup\mathbb{H}^-$. In fact, because u^*_y and v^*_y are real analytic and non-zero around γ, h is also real analytic on $\mathbb{H}\cup\gamma\cup\mathbb{H}^-$. What is more is that

$$h(\xi)=\frac{u^*_y(\xi)}{v^*_y(\xi)}=\lim_{\mathbb{H}\ni z\to\xi}\frac{u(z)}{v(z)}=|a(\xi)|^2\qquad\text{for any } \xi\in\gamma\tag{6.2}$$

because of (iv) and therefore $|a|^2$ is also real analytic on γ.

Remark 6.1. The above conversation is the reason why the relation (IV) is meaningful. When we write $\frac{\mathcal{U}}{\mathcal{V}}$ on Γ it really means the limit of $h\circ\varphi^{-1}$ as we approach Γ from the inside of Ω. This limit always exist on a Jordan arc Γ when Ω is simply-connected thanks to Harnack's inequality.

It is worth mentioning the work of Jerison and Kenig who showed [14, Theorems 5.1 and 7.9] that equation (IV) makes sense whenever Ω is assumed to be a non-tangentially accessible (NTA) domain.

Next, consider $h_{|\gamma}$. Its power series around $0\in\gamma$ is given by

$$h_{|\gamma}(x)=\sum_{n=0}^{\infty}b_nx^n$$

for some real numbers $b_0,b_1,\ldots$ This readily extends to a complex analytic function, say r, on some open neighbourhood, $D(0,\varepsilon')$:

$$r(z)=\sum_{n=0}^{\infty}b_nz^n,$$

where we can choose φ and ε' so that $\gamma \subset D(0,\varepsilon')$. Of course, by construction and form eq. (6.2) we get $r_{|\gamma} = h_{|\gamma} = |a|^2$.

At this point, we want to revert back to Ω: We set $V \equiv \varphi(\mathbb{H} \cap D(0,\varepsilon')) \subset \Omega$ and observe that ∂V is a closed Jordan arc such that $\Gamma \subsetneq \partial V \cap D(\zeta_0,\rho)$. Define a new function

$$R \equiv r \circ (\varphi^{-1})_{|V\cup\Gamma} \tag{6.3}$$

which is holomorphic on V, continuous on $V \cup \Gamma$ and on Γ it satisfies $R(\zeta) = |A(\zeta)|^2$.

Now, consider the function $\Phi(z,w) = A(z)\overline{A(\overline{w})}$. Φ is holomorphic on $D(\zeta_0,\varepsilon) \times D(\overline{\zeta}_0,\varepsilon)$ and it satisfies that $\Phi(\zeta,\overline{\zeta}) = A(\zeta)\overline{A(\zeta)} = |A(\zeta)|^2$ when $z = \overline{w} = \zeta \in \Gamma$. As a corollary of Theorem 5.1 the next theorem follows:

Theorem 6.2. *Let $\Omega \subset D(\zeta_0,\rho)$ be a simply-connected domain of $\mathbb{C}$ and let Γ be an open Jordan arc of its boundary with $\zeta_0 \in \Gamma$. Suppose there are two positive non-proportional harmonic functions $\mathcal{U}$ and $\mathcal{V}$ on Ω, continuous on $\Omega \cup \Gamma$ which satisfy*

$$\mathcal{U}(\zeta) = \mathcal{V}(\zeta) = 0 \quad \text{and} \quad \frac{\mathcal{U}(\zeta)}{\mathcal{V}(\zeta)} = |A(\zeta)|^2 \qquad \text{for all } \zeta \in \Gamma$$

where A is a non-trivial analytic function on a neighbourhood of Ω.

Then, there exists some neighbourhood D of ζ_0 and a closed set $E \subset \Gamma$ of zero harmonic measure so that $(\Gamma \setminus E) \cap D$ is a countable union of regular real-analytic simple arcs except possibly for some cusps. The cusps (if they exist) point into Ω and may only accumulate on $E \cap D$.

Of course, Theorems 5.1 and 6.2 are somewhat far from Sakai's result. Nevertheless, because of the special form of the function $\Phi(z,w) = A(z)\overline{A(\overline{w})}$, we can actually say more in this case.

Proposition 6.3. *Let $\Omega \subset D(\zeta_0,\rho)$ be a simply-connected domain of $\mathbb{C}$ and let Γ be an open Jordan arc of its boundary with $\zeta_0 \in \Gamma$. Suppose there are two positive non-proportional harmonic functions $\mathcal{U}$ and $\mathcal{V}$ on Ω, continuous on $\Omega \cup \Gamma$ which satisfy*

$$\mathcal{U}(\zeta) = \mathcal{V}(\zeta) = 0 \quad \text{and} \quad \frac{\mathcal{U}(\zeta)}{\mathcal{V}(\zeta)} = |A(\zeta)|^2 \qquad \text{for all } \zeta \in \Gamma$$

where A is a non-trivial analytic function on a neighbourhood of Γ.

Then, there exists a neighbourhood D of ζ_0 and a function R satisfying the following:

(i) R is holomorphic on $\Omega \cap D$,
(ii) R is continuous on $(\Omega \cup \Gamma) \cap D$ and
(iii) $R(\zeta) = |A(\zeta)|^2$ for $\zeta \in \Gamma \cap D$.

Additionally, around any $\zeta_0 \in \Gamma$ with $A'(\zeta_0) \neq 0$ either

(1) there exist a function Ψ_1 holomorphic and univalent on $\Omega \cap D$ such that Ψ_1 is continuous on $(\Omega \cup \Gamma) \cap D$, and $\Psi_1(\zeta) = |A(\zeta) - A(\zeta_0)|^2$ for $\zeta \in \Gamma \cap D$, or
(2) there exist a function Ψ_2 holomorphic and univalent on $\Omega \cap D$ such that Ψ_2^2 is continuous on $(\Omega \cup \Gamma) \cap D$, and $\Psi_2^2(\zeta) = |A(\zeta) - A(\zeta_0)|^2$ for $\zeta \in \Gamma \cap D$.

Proof. We have already established the existence of such a function R in (6.3).

For the rest, $A'(\zeta_0) \neq 0$ and we assume without loss of generality that $\rho > 0$ is small enough so that A is conformal on a neighbourhood of $\Omega \cap D(\zeta_0,\rho)$. Recall that V from the definition of R in (6.3) is such that ∂V is Jordan and $\Gamma \subsetneq \partial V \cap D(\zeta_0,\rho)$. Because A is continuous and injective on $\bar{V}$, there exist some small $\delta > 0$ such that $\partial(A(V)) \cap D(\zeta_0',\delta) \subset A(\Gamma)$.

Now, let $A(\zeta_0) = \zeta_0'$, $\Omega' = A(V) \cap D(\zeta_0', \delta)$ and $\Gamma' = \partial\Omega' \cap D(\zeta_0', \delta)$. The function

$$S(z) \equiv \frac{1}{z} R \circ A^{-1}(z) \tag{6.4}$$

is a Schwarz function of $\Omega' \cup \Gamma'$ in $D(\zeta_0', \delta)$:

(i) S is holomorphic on Ω',
(ii) continuous on $\Omega' \cup \Gamma'$ and
(iii) $S(\zeta) = \frac{1}{\zeta} R(A^{-1}(\zeta)) = \overline{\zeta}$ on Γ'.

Notice that from eq. (6.1) the functions $a = A \circ \varphi$ and A are always non-zero and thus S is a well-defined holomorphic function, since 0 cannot be a point of $\bar{\Omega}'$.

Finally, consider the function $S_t(z) = S(z + \zeta_0') - \overline{\zeta_0'}$, which is a Schwarz function on $(\Omega' - \zeta_0') \cup (\Gamma' - \zeta_0')$ at 0. From Theorem 1.4, we know that one of the functions $\Phi_1(z) = zS_t(z)$ and $\Phi_2(z) = \sqrt{zS_t(z)}$ is univalent on $(\Omega' - \zeta_0') \cap D(0, \delta')$ for some $\delta' \le \delta$. Changing variables to get back to our initial domain Ω, we find that one of the following functions, Ψ_1 or Ψ_2, has to be univalent on $\Omega \cap D'$:

$$\Psi_1(z) = (A(z) - A(\zeta_0)) \left(\frac{R(z)}{A(z)} - \overline{A(\zeta_0)} \right) \quad \text{and} \quad \Psi_2(z) = \sqrt{(A(z) - A(\zeta_0)) \left(\frac{R(z)}{A(z)} - \overline{A(\zeta_0)} \right)}$$

for $z \in \Omega \cap D'$, where $D' = A^{-1}(D(\zeta_0', \delta'))$. The rest of the desired properties are obvious. □

In the above proof, Γ' is the image of a Jordan arc under the (conformal) map A. Therefore, the existence of a Schwarz function, S, along with Theorem 1.2 imply that Γ', and in turn Γ, satisfy (1) or (2c) of Theorem 1.2. Case (1) corresponds to (1) of Proposition 6.3 and (2c) to (2), that is, Γ' (respectively Γ), has a cusp if, and only if, the function $\sqrt{z\left(S(z + \zeta_0') - \overline{\zeta_0'}\right)}$ is univalent on $(\Omega' - \zeta_0') \cap D(0, \delta')$ (respectively Ψ_2 on $\Omega \cap D$).

As a consequence, we have the following theorem which is the main result of this section:

Theorem 6.4. *Let $\Omega \subset D(\zeta_0, \rho)$ be a simply-connected domain of $\mathbb{C}$ and let Γ be an open Jordan arc of its boundary with $\zeta_0 \in \Gamma$. Suppose there are two positive non-proportional harmonic functions $\mathcal{U}$ and $\mathcal{V}$ on Ω, continuous on $\Omega \cup \Gamma$ which satisfy*

$$\mathcal{U}(\zeta) = \mathcal{V}(\zeta) = 0 \quad \textit{and} \quad \frac{\mathcal{U}(\zeta)}{\mathcal{V}(\zeta)} = |A(\zeta)|^2 \qquad \textit{for all } \zeta \in \Gamma$$

where A is a non-trivial analytic function on a neighbourhood of Γ.

Then, for all but possibly finitely many points $\zeta_0 \in \Gamma$ there exist some small neighbourhood D of ζ_0 such that the following holds:

$$\Gamma \cap D \textit{ is a regular real-analytic simple arc through } \zeta_0 \textit{ except possibly a cusp at } \zeta_0. \tag{6.5}$$

The finitely many points around which (6.5) might fail are the points $\zeta \in \Gamma$ where $A'(\zeta) = 0$, i.e. where A might not to be invertible.

There is a cusp at ζ_0 if, and only if, (2) of Proposition 6.3 holds.

Of course, one can ask at this point whether it is possible to actually have a cusp. The answer is yes as the next example shows:

Example 6.5. Let Ω be open and $\Gamma = \partial\Omega \cap D(0, \rho)$ (with $\rho \ge 1$ small enough) be such that Γ has a cusp at 0 (i.e. $\zeta_0 = 0$). Then, from Remarks 1.3, for some $\eta > 0$, there is holomorphic function T defined on $\{|z| \le \eta\}$, which maps conformally the closed upper half-disk $K_\eta = \{|z| \le \eta : \operatorname{Im}(z) \ge 0\}$ into $\Omega \cup \Gamma$ and $\Gamma \cap D \subset T(-\eta, \eta)$ for some small neighbourhood of 0, D. Also, $T(0) = 0$ with order 2. By dilating appropriately, we may assume that everything happens in the unit circle, that is, $\eta = 1$, T is defined on $\bar{\mathbb{D}}$ and is univalent on $K_1 = \{|z| \leq 1 : \operatorname{Im}(z) \geq 0\}$, $T(K_1) \subset \Omega \cup \Gamma$, and $\Gamma \cap D(0,\rho) \subset T(-1,1)$.

Next, consider two positive harmonic functions, u and v, on the upper half-disk $\mathbb{D} \cup \mathbb{H}$ which are zero on $(-1,1)$. As we saw in the beginning of this section, u and v and be extended on the whole disk and the ratio u/v is a positive analytic function. Therefore, on $(-1,1)$ we can write that $u/v = |a|^2$ for some function a holomorphic on $\mathbb{D}$.

Finally, construct the functions

$$\mathcal{U} = u \circ T^{-1}, \quad \mathcal{V} = v \circ T^{-1} \quad \text{and} \quad A = a \circ T^{-1}.$$

Then, A is holomorphic around the cusp at 0, and $\mathcal{U}, \mathcal{V}$ are positive harmonic functions on $\Omega \cap D(0,\rho)$ and zero on the boundary Γ. Moreover, $\mathcal{U}$ and $\mathcal{V}$ satisfy $\mathcal{U}/\mathcal{V} = |A|^2$ on Γ.

7. Unsolved "free boundary" problems

All problems treated above are the examples of so-called free boundary problems (non-variational free boundary problems).

We would like to call the attention of the reader to the open question: what if the domain Ω for positive harmonic functions is not simply connected? Finitely connected situation presents no difficulties, but what if, for example, Γ is a Cantor set and $\Omega = \mathbb{D} \setminus \Gamma$? Suppose we know that the ratio of two positive harmonic functions in Ω vanishing on the Cantor set Γ has a well defined ratio on Γ (this happens for a wide class of Γ's, for example for all regular Cantor sets of positive Hausdorff dimension). Suppose this ratio is equal to $|A(\zeta)|^2 \neq const.$ for $\zeta \in \Gamma$, where A is a holomorphic function on $\mathbb{D}$. What we can say about the Cantor set Γ? The "desired" answer is that this is impossible to happen on a Cantor set. This type of problems — we may call them "one-phase free boundary problems" — appear naturally in certain problems of complex dynamics, see e.g. [23]. If we would know the aforementioned answer (we believe it is true), then a long standing problem of dimension of harmonic measure on Cantor repellers would be solved.

Another similar one-phase boundary problem concerns functions in $\mathbb{R}^n$ for $n > 2$. Let Ω be a bounded domain in $\mathbb{R}^n$, $n > 2$, and $\Gamma = \partial\Omega \cap D(x,r)$, where $x \in \partial\Omega$. Again, let $\mathcal{U}, \mathcal{V}$ be two positive (non-proportional) harmonic functions in Ω vanishing continuously on Γ. If Ω is assumed to be a Lipschitz domain, then [14] claims that $\mathcal{U}/\mathcal{V}$ makes sense on Γ and is additionally a Hölder function on Γ (boundary Harnack principle). Here is a question: Let R be a real analytic function on $D(x,r)$ and let $\mathcal{U}/\mathcal{V} = R$ on Γ. Is it true that Γ is real analytic, maybe with the exception of some lower dimensional singular set?

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