SUPPLEMENTARY MATERIAL. NOT TO HAND IN.

MATH 170A SUMMER 2018

Other important continuous random variables. In this class we met several important continuous random variables, such as the normal and the exponential. There are many more. These are not part of our syllabus but they may appear in other probability courses.

Doing these problems will also be further practice for you in computing expectation.

We say that f is a *probability density function* if f is nonnegative and

$$\int_{-\infty}^{\infty} f(x)\, dx = 1.$$

(1) Let $f_1(x)$ and $f_2(x)$ be probability density functions and let $\gamma \in (0,1)$. Define f_3 by,

$$f_3(x) = \gamma f_1(x) + (1-\gamma) f_2(x).$$

(a) Show that f_3 is also a probability density function.

(b) Suppose, for $i = 1, 2, 3$, X_i is a random variable with probability density f_i. Find the expectation of X_3 in terms of the expectations of X_1 and X_2.

We call f_3 a *mixture* of f_1 and f_2.

(a) Find the constant C such that the function

$$f(x) = \frac{C}{1+x^2}$$

is a probability density function.

(b) Find the constant C such that the function

$$f(x) = \begin{cases} \frac{C}{1+x^2} \text{ for } -1 < x < 1, \\ 0 \text{ otherwise,} \end{cases}$$

is a probability density function.

(c) If X is a random variable with probability density function given by f in part (b), find $E[X]$.

(d) If X is a random variable with probability density function given by f in part (a), show that the integral defining $E[X]$ does not converge.

Hint for parts (a), (b): recall

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}.$$

A random variable with probability density function f given in part (b) is said to be a *Cauchy* random variable.

(2) (a) Let $\lambda > 0$. Use integration by parts to show that, for any positive integer α,

$$f(x) = \begin{cases} \frac{1}{(\alpha-1)!}\lambda^\alpha \exp^{-\lambda x} & \text{for } x > 0, \\ 0 \text{ otherwise} \end{cases}$$

is a probability density function. A random variable with this probability density function is said to be a *Gamma* random variable.

(b) If X is a random variable with probability density function f, find $E[X]$.

(3) Chapter 3 end of chapter problem 2 introduces the *Laplace random variable.*

(4) Chapter 3 end of chapter problem 28 introduces the *two-sided exponential random variable.*

(5) Chapter 3 end of chapter problem 30 introduces the *beta random variable.*

Classical problems and paradoxes.

- **Bertrand's paradox.** This paradox concerns picking cords in a circle "at random". Two ways of doing this are explained on page 16 of the textbook.
  - Read this page and make sure you understand the resolution of the paradox. Can you think of a third way of picking chords at random from a circle? Answer the question posed in this paradox for the way that you chose.
  - Problem 22 at the end of chapter 3 shows another situation in which the colloquial words "random" or "arbitrary" can have many meanings.
- **Gambler's ruin.** A gambler enters a casino and makes a sequence of independent bets. In each one, they win \$1 with probability p and lose \$1 with probability $1-p$. The gambler starts out with \$$k$. They stay in the casino and play until either they run out of money, or until they win \$$n$ total.
  - What is the probability that the gambler wins \$$n$? (This problem is worked out in problem 42 at the end of chapter 1).
  - Let X_1, X_2, ... be Bernoulli random variables with parameter p. Fix an integer k. Express the amount of money that the player in the previous problem has after j bets in terms of the X_i. This is related to a so-called *random walk* on the integers $0, 1, ..., n$. You may learn about this in more generality if you continue studying probability.
- **Bernoulli's problem of joint lives.** Problem 32 at the end of chapter 2.
- **A famous betting strategy: always double your bet.** You are tossing a coin; the probability of heads is $p < 1/2$. You bet \$1 on the first toss being H. If you win you quit; if you lose, you place a second bet of \$2 on H. You continue in this manner: if H comes up you quit; so long as you keep getting T, your n-th bet is \$$2^{n-1}$.
  - Show that you are certain to win \$1 every time you use this strategy.
  - Find the expected size of your winning bet.

  Now suppose you are playing this in a casino, where the house limit is \$$2^L$, where L is a fixed natural number. In other words, if you bet \$$2^L$ and lose, you must quit.
  - What is your expected gain when you stop?
  - Would you prefer to play this game at a casino with a large or small house limit?