PDE Assignment

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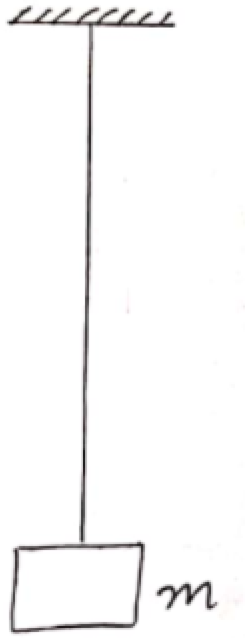
1) Vibrating wire (10 marks)

A steel wire is supporting an unknown static load under gravity. The wire diameter is 1mm and it is 2m long. You pluck the wire and it vibrates in its fundamental mode at a frequency of 2 Hz. What is the load in kg?
Assume that the (volume) density of steel is 7850kg/m^3.



To find linear density ρ:

Wire diameter $= \ 1mm \ = \ 0.001m$

$\therefore$ Wire area $= \ \pi r^2 = 2.5\pi \times 10^{-7}$

For a one-meter increase in length, volume density increases by $7850 \times 2.5\pi \times 10^{-7} = 6.165 \times 10^{-3}$

$\therefore \rho(linear\ density) = 0.0062\frac{kg}{m}$

Boundary conditions:

$u(0,t) = 0$ (assuming fixed upper end)

Using the general string equation from tutorial 1:

$$f = \frac{1}{2L}\sqrt{\frac{T}{\rho}}$$

$$2Hz = \frac{1}{2 \times 2}\sqrt{\frac{T}{0.0062}}$$

Rearranging:

$$(2 \times 4)^2 \times 0.0062 = T$$

$$T = 0.3946N$$

Using $F = mg = T$,

$$m = \frac{0.3946}{9.81} = 0.0402$$

Therefore, the load has a mass of 0.0402 kg.

2) Design a Pan Flute (40 marks)

The pan flute is an end blown flute. Sound is produced by the vibration of an air-stream blowing across an open hole at the end of a resonating tube that is closed at the other end. The length of the tube determines the fundamental frequency.

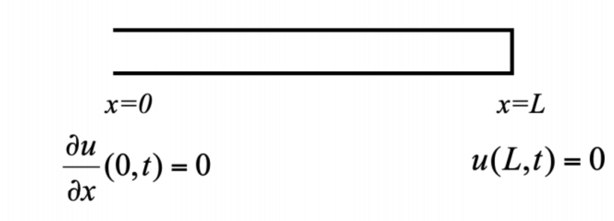
Design a 4 tube pan flute. The 4 tubes are tuned in a perfect 5th relation to each other: the higher pitched tube has a fundamental frequency that is 1.5 times greater than its neighbour.

Give the third tube in the sequence a fundamental of 440 Hz.

For your answer write down the length and frequencies of the other three tubes.

Hints and notes:
You want to find an expression for the frequency of the first mode of vibration. You go through the steps to do this:

1. Model the problem:



Air is free to move in and out at x=0. If u(x,t) represents the displacement of air molecules at x and t then with the end open $\frac{\partial u}{\partial x}(0,t) = 0$. At x=L the air can't move in the x direction so $u(L,t)=0$.

2. Find an equation:
   Use the 1 dim wave equation that you have a solution for.

3. Fit the boundary conditions to the solution.

4. Calculate tube lengths assuming that the wavespeed c= 331 m/s.

The 4 tubes must have frequencies $G_3(196\ Hz),\ D_4(293.7\ Hz), A_4,\ E_5(659.3\ Hz)$

Using the 1-D wave equation as given:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Substituting $u(x,t) = X(x) \times T(t)$:

$$X\frac{d^2T}{dt^2} = c^2 \frac{d^2X}{dx^2}T$$

Leads to the solution as given in the notes (2.6):

$$u(x,t) = \big(Asin(px) + Bcos(px)\big)\big(Csin(cpt) + Dcos(cpt)\big)$$

$$\therefore \frac{\partial u}{\partial x} = \big(Csin(cpt) + Dcos(cpt)\big)\big(pAcos(px) - pBsin(px)\big)$$

Using the condition $\frac{\partial u}{\partial x}(0,t) = 0$:

$$0 = \big(Csin(cpt) + Dcos(cpt)\big)\big(pA\overbrace{cos(0)}^{1} - pB\overbrace{sin(0)}^{0}\big)$$

$$0 = pA\big(Csin(cpt) + Dcos(cpt)\big)$$

Therefore $A = 0$,

$$u(x,t) = Bcos(px)\big(Csin(cpt) + Dcos(cpt)\big)$$

Applying $u(L,t) = 0$:

$$0 = Bcos(pL)\big(Csin(cpt) + Dcos(cpt)\big)$$

Discarding $B = 0$, we get:

$$\cos(pL) = 0$$

$$pL = \frac{(2n-1)\pi}{2}$$

$$p = \frac{(2n-1)\pi}{2L}$$

Giving us our solution:

$$u_n(x,t) = B_n cos\left(\frac{(2n-1)\pi}{2L}x\right)\left(C_n sin\left(\frac{(2n-1)\pi}{2L}ct\right) + D_n cos\left(\frac{(2n-1)\pi}{2L}ct\right)\right)$$

And giving frequencies of:

$$\omega_n = \frac{(2n-1)\pi}{2L}c;\ for\ n = 1,2,3\ldots$$

$$f_n = \frac{(2n-1)}{4L}c; (Hz)$$

c = 331m/s:

$$L = \frac{331}{4f_1}$$

	1st	2nd	3rd	4th
Frequency f_1 (Hz)	196	293.7	440	659.3
Length (M)	0.422	0.282	0.188	0.126

3) Heat distribution in a wall (50 marks)

Find an expression for the temperatures within a steel wall of thickness 10cm. One side of the wall (x=0) is fixed at temperature 20ºC. The other side (x=0.1m) is fixed at 30ºC. The initial temperature across the wall is 100 ºC. Plot the temperature at t = 10 sec, 100 sec and 1,000 sec.

For your answer:

1) Write expressions for the boundary and initial conditions.
2) Calculate the diffusivity for steel.
3) Derive and display the solution.
4) Provide a plot for temperature for the three times given above. Provide evidence of your Excel or Matlab calculations.
   Use the following properties for steel :

Specific heat	σ = 460 J/(kgC°)
Density	ρ = 8.01 g/cm^3
Thermal conductivity	K = 16.27 W/m*C°.

Boundary conditions:

$$u(0,t) = 20$$

$$u(0.1,t) = 30$$

Initial conditions:

$$u(x,t_0) = 100$$

Diffusivity $\alpha^2 = \frac{K}{\sigma\rho}$. By substituting,

$$\rho = 8.01\frac{g}{cm^3} = 8.01 \times 10^3\frac{kg}{m^3}$$

$$\alpha^2 = \frac{16.27}{460 \times 8.01 \times 10^3}$$

Diffusivity $= 4.416 \times 10^{-6}\frac{m^2}{s}$

In general, given the general 1-D heat equation $\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}$, we can use the substitution $u(x,t) = X(x) \times T(t)$ to derive (by separation of variables):

$$X\frac{dT}{dt} = \alpha^2\frac{d^2X}{dx^2}T$$

$$\frac{\frac{dT}{dt}}{\alpha^2 T} = \frac{\frac{d^2X}{dx^2}}{X}(= k)$$

$$\frac{dT}{dt} - k\alpha^2 T = 0; \frac{d^2X}{dx^2} - kX = 0$$

Defining k to be negative $(= -p^2)$:

$$\frac{dT}{dt} + p^2\alpha^2 T = 0$$

$$\frac{d^2X}{dx^2} + p^2 X = 0$$

Using the characteristic equation:

$$\lambda_T = -p^2\alpha^2$$

$$T = Ae^{-p^2\alpha^2 t}$$

$$\lambda_X^2 = -p^2;\ \lambda_X = 0 \pm pi$$

$$X = Ae^0\big(Bsin(px) + Ccos(px)\big)$$

$$u(x,t) = Ae^{-p^2\alpha^2 t}\big(Bsin(px) + Ccos(px)\big)$$

$$\therefore u(x,t) = e^{-p^2\alpha^2 t}\big(Dsin(px) + Ecos(px)\big)$$

In our case because our boundary conditions are non-homogenous, we have a final solution with a steady state and a transient part, i.e.:

$$u(x,t) = u_s(x) + U(x,t)$$

At steady state, $\frac{d^2u}{dx^2} = 0$ so the steady state solution to the 1-D heat equation is:

$$u_s(x) = Fx + G$$

$$u(0,t) = 20$$

$$u(0.1,t) = 30$$

$$\therefore G = 20, F = 100$$

$$u_s = 100x + 20$$

For the transient part, we can substitute in the steady state solution to solve for new boundary conditions:

$$u(0,t) = 20 = (20) + U(0,t)$$

$$\therefore U(0,t) = 0$$

$$u(0.1,t) = 30 = (100 \times 0.1 + 20) + U(0.1,t)$$

$$\therefore U(0.1,t) = 0$$

The transient solution can be derived from the general heat equation – we already did so above, and therefore by substituting these new boundary conditions:

$$U(x,t) = e^{-p^2\alpha^2 t}\big(Dsin(px) + Ecos(px)\big)$$

$$u(0,t) = 0$$

$$0 = e^{-p^2\alpha^2 t}(\cancel{Dsin(px)} \rightarrow \mathbf{0} + \cancel{Ecos(px)} \rightarrow \mathbf{1}E)$$

$$0 = Ee^{-p^2\alpha^2 t}$$

$$\therefore E = 0$$

$$U(x,t) = e^{-p^2\alpha^2 t}Dsin(px)$$

$$u(0.1,t) = 0$$

$$0 = e^{-p^2\alpha^2 t}Dsin(0.1p)$$

Discarding $D = 0$,

$$0.1p = n\pi$$

$$p = 10n\pi$$

Our transient solution is the sum of:

$$U_n = D_n e^{-(10n\pi)^2\alpha^2 t}\sin(10n\pi x)$$

Substituting the initial condition $u(x,0) = 100$,

$$U(x,t) = \sum_{n=1}^{\infty} D_n e^{-(10n\pi)^2\alpha^2 t}\sin(10n\pi x)$$

$$u_s = 100x + 20$$

$$u(x,0) = u_s + U(x,0) = 100$$

$$U(x,0) = 80 - 100x$$

We know from the notes that $\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \frac{L}{2}$ if $n = m$.

$$U(x,0) = \sum_{n=1}^{\infty} D_n \sin(10n\pi x)$$

$$\int_0^L U(x,0)\sin(10m\pi x)\,dx = \int_0^L \sum_{n=1}^{\infty} D_n \sin(10n\pi x)\sin(10m\pi x)\,dx$$

$$\int_0^L U(x,0)\sin(10m\pi x)\,dx = D_m\frac{L}{2}$$

$$D_m = \frac{2}{L}\left[\int_0^L 80\sin(10m\pi x)\,dx - \int_0^L 100x\sin(10m\pi x)\,dx\right]$$

$$D_m = \frac{2}{L}\left[80\left(\frac{-\cos(10m\pi L) + 1}{10\pi m}\right) - \left(\frac{-10\pi Lm\cos(10m\pi L) + \sin(10m\pi L)}{\pi^2 m^2}\right)\right]$$

$$D_m = \frac{2}{L}\left[\frac{8}{m\pi} - \frac{\sin(10m\pi L) + (8m\pi - 10m\pi L)\cos(10m\pi L)}{\pi^2 m^2}\right]$$

From the model, $L = 0.1$:

$$D_m = \frac{2}{0.1}\left[\frac{8}{m\pi} - \frac{\sin(m\pi) + (8m\pi - m\pi)\cos(m\pi)}{m^2\pi^2}\right]$$

Therefore, our transient solution is:

$$U(x,t) = \sum_{n=1}^{\infty} 20\left[\frac{8 - 7\cos(n\pi)}{n\pi} - \frac{\sin(n\pi)}{n^2\pi^2}\right] e^{-(10n\pi)^2\alpha^2 t}\sin(10n\pi x)$$

And our overall solution is:

$$u(x,t) = 100x + 20 + \sum_{n=1}^{\infty} 20\left[\frac{8 - 7\cos(n\pi)}{n\pi}\right] e^{-(10n\pi)^2 \times 4.416\times10^{-6} t}\sin(10n\pi x)$$

Plotting the solution:

```
function [u] = heat(x,t,diff)

u = (100*x + 20);
syms n
u_1 = symsum((20)*(exp(-(10*n*(pi))^2*diff*t))*(sin(10*n*(pi)*x))*((8-7*cos(n*(pi)))/(n*(pi))+sin(n*(pi))/((n*(pi))^2)),n,[1,50]);
u = u+u_1;
end
```

```
diff = (16.27/(460*8.01*1000))
x = linspace(0,0.1,50);

format short

figure

u_10 = heat(x,10,diff);

u_100 =heat(x,100,diff);

u_1000=heat(x,1000,diff);

subplot(3,1,1)
plot(x,u_10)
grid on
title("Heat distribution for t = 10")
xlabel("Position(m)")
ylabel("Temperature (Celsius)")

subplot(3,1,2)
plot(x,u_100)
grid on
xlabel("Position(m)")
ylabel("Temperature (Celsius)")
title("Heat distribution for t = 100")

subplot(3,1,3)
plot(x,u_1000)
grid on
xlabel("Position(m)")
ylabel("Temperature (Celsius)")
title("Heat distribution for t = 1000")
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```
t = 10
diff = 4.4157e-06
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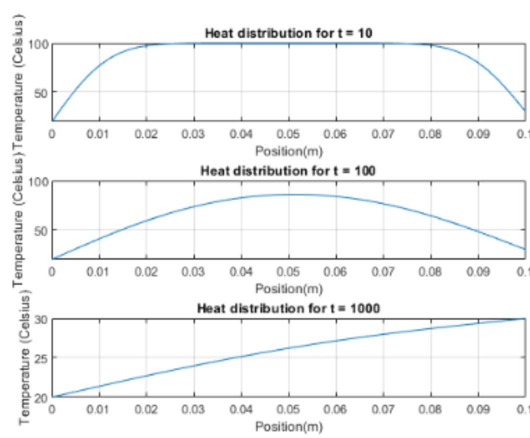
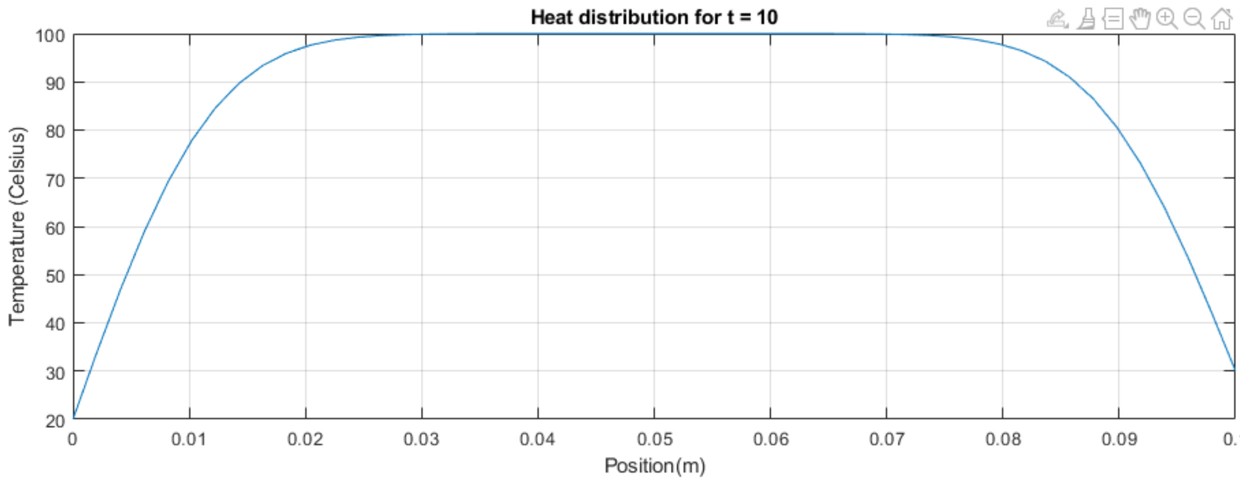
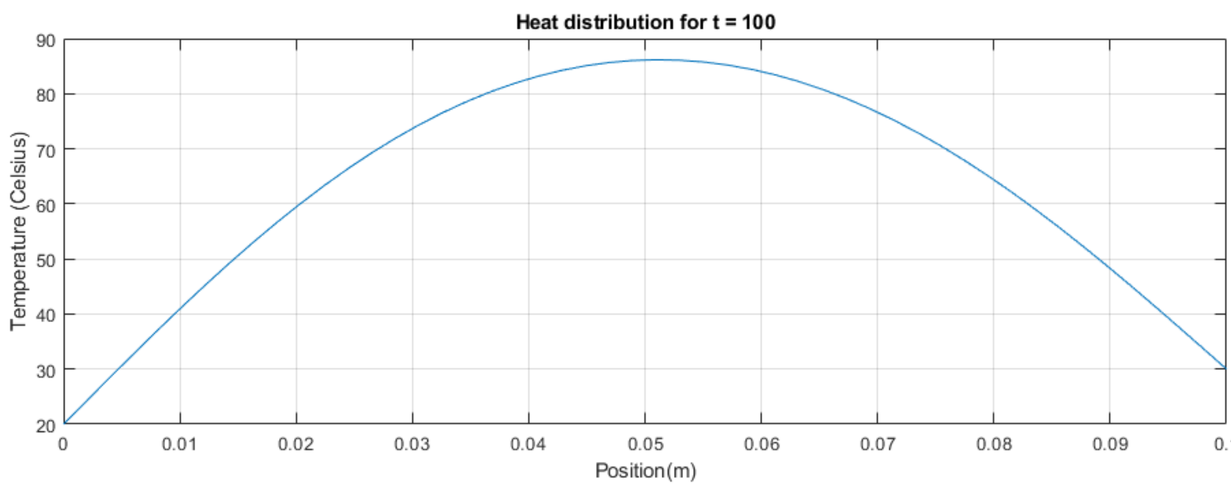
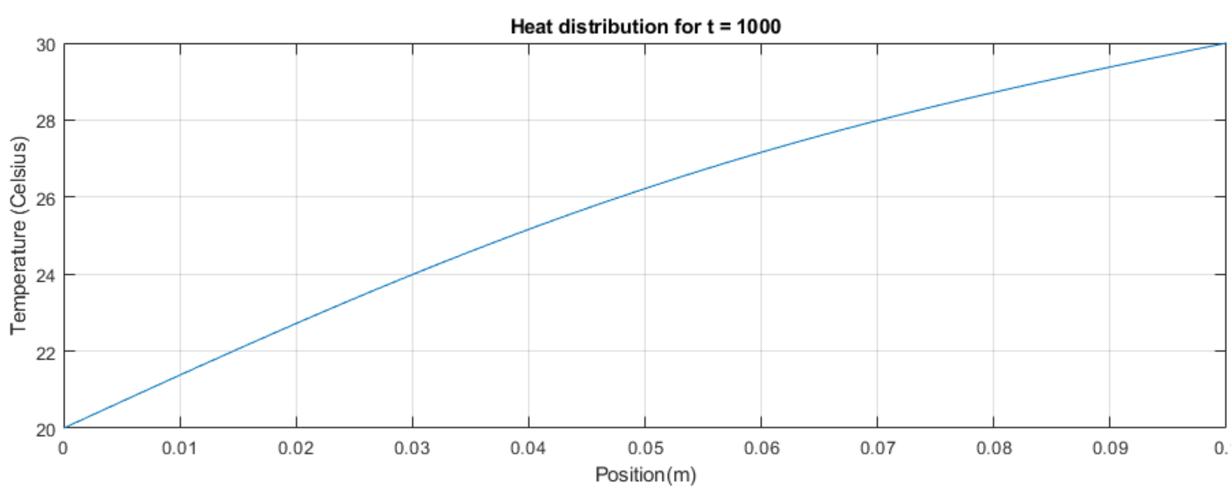



Heat distribution for t = 10
Temperature (Celsius)
Position(m)


Heat distribution for t = 100
Temperature (Celsius)
Position(m)


Heat distribution for t = 1000
Temperature (Celsius)
Position(m)