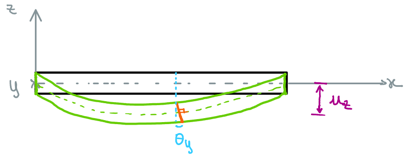
Project I – WP1 : Numerical simulation

1^{st} step: 2D Euler Bernoulli bending beam



* 2 DOF per node:
  → displacement in z u_z
  → rotation around y θ_y

* DOF mapping & boundary conditions:
  → DOF mapping:
  e.g. node 1 & node 2
  each beam connects 2 nodes with 2 DOF each ⇒ 4 DOF per beam
  vector of DOF : $\{u_{z_1}, \theta_{y_1}, u_{z_2}, \theta_{y_2}\}^T$
  → BC:
  $u_z = 0$ at both ends (node 1 and node n)
  ↳ $u_{z_1} = 0$ and $u_{z_n} = 0$
  ⇒ corresponds to indices 0 and 2(n-1) in DOF vector

* Element matrices:
  → Stiffness matrix Ke :

$$K_e = \frac{EI}{\ell^3} \begin{array}{c} \begin{array}{cccc} u_{z_1} & \theta_{y_1} & u_{z_2} & \theta_{y_2} \end{array} \\ \begin{bmatrix} 12 & 6\ell & -12^{*} & 6\ell \\ 6\ell & 4\ell^2 & 6\ell & 2\ell^2 \\ -12 & -6\ell & 12 & -6\ell \\ 6\ell & 2\ell^2 & -6\ell & 4\ell^2 \end{bmatrix} \end{array} \begin{array}{c} u_{z_1} \\ \theta_{y_1} \\ u_{z_2} \\ \theta_{y_2} \end{array}$$

with $I = \frac{\pi D^4}{64}$ moment of inertia for circular cross section

*: stiffness coupling btw. displacement in z at node 1 & displacement in z at node 2

  → Consistent mass matrix Me :

$$M_e = \frac{\rho A \ell}{420} \begin{bmatrix} 156 & 22\ell & 54 & -13\ell \\ 22\ell & 4\ell^2 & 13\ell & -3\ell^2 \\ 54 & 13\ell & 156 & -22\ell \\ -13\ell & -3\ell^2 & -22\ell & 4\ell^2 \end{bmatrix}$$

with $A = \frac{\pi D^2}{4}$ cross-sectional area

↳ ⚠ Note: different than lumped mass matrix !

* Matrices assembly to get $\underline{K}$ and $\underline{M}$
  superposition using the sum (for example node 2 will be in the 1^{st} & 2^{nd} element)

* Solve generalized eigenvalue problem
  $(\underline{K} - \omega^2 \underline{M})\vec{\varphi} = \vec{0}$

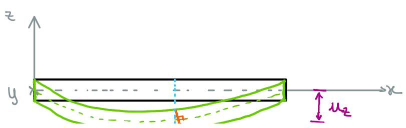
* Comparison to analytical method
  → Natural frequency ω:

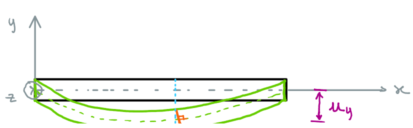
$$\omega_n = \frac{n^2\pi^2}{\ell^2}\sqrt{\frac{EI}{\rho A}}$$

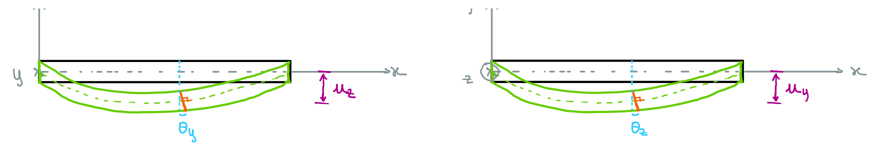
  → Mode shape:

$$\varphi_n(x) = \sin\left(\frac{n\pi x}{\ell}\right) \qquad x \in [0, \ell]$$

2^{nd} step: bending beam in 3D







* 4 DOF per node:
  - → displacement in z u_z
  - → rotation around y θ_y
  - → displacement in y u_y
  - → rotation around z θ_z

* DOF mapping & boundary conditions:
  - → DOF mapping:
    e.g. node 1 & node 2
    each beam connects 2 nodes with 4 DOF each ⇒ 8 DOF per beam
    vector of DOF : $\{u_{y_1}, \theta_{z_1}, u_{z_1}, \theta_{y_1}, u_{y_2}, \theta_{z_2}, u_{z_2}, \theta_{y_2}\}^T$
  - → BC:
    $u_y = 0$ and $u_z = 0$ at both ends (node 1 and node n)
    ↳ $u_{y_1} = 0$, $u_{z_1} = 0$, $u_{y_n} = 0$ and $u_{z_n} = 0$
    ⇒ corresponds to indices 0, 2, 4(n-1) and 4(n-1)+2 in DOF vector

* Element matrices:
  - → Stiffness matrix Ke:

$$K_{local} = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{cccc} u_{y_1} & \theta_{z_1} & u_{z_1} & \theta_{y_1} \end{array}$$

$$K_{local} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}\begin{array}{c} u_{y_1} \\ \theta_{z_1} \\ u_{z_1} \\ \theta_{y_1} \end{array}$$

(see code

K_{local} = same as in 2D

$$\begin{array}{cccccccc} u_{y_1} & \theta_{z_1} & u_{y_2} & \theta_{z_2} & u_{z_1} & \theta_{y_1} & u_{z_2} & \theta_{y_2} \end{array}$$

$$\rightarrow \underset{[8\times 8]}{K_e} = \begin{bmatrix} a & b & 0 & 0 & c & d & 0 & 0 \\ e & f & 0 & 0 & g & h & 0 & 0 \\ 0 & 0 & \cdots & & 0 & 0 & \cdots & \\ 0 & 0 & \cdots & & 0 & 0 & \cdots & \\ i & j & 0 & 0 & k & l & 0 & 0 \\ m & n & 0 & 0 & o & p & 0 & 0 \\ 0 & 0 & \cdots & & 0 & 0 & \cdots & \\ 0 & 0 & \cdots & & 0 & 0 & \cdots & \end{bmatrix}\begin{array}{c} u_{y_1} \\ \theta_{z_1} \\ u_{y_2} \\ \theta_{z_2} \\ u_{z_1} \\ \theta_{y_1} \\ u_{z_2} \\ \theta_{y_2} \end{array}$$

→ explains how 8 x 8 matrices gets filled up

  - → mass matrix Me:
    same process

* Matrices assembly to get $\underline{K}$ and $\underline{M}$
  superposition using the sum (for example node 2 will be in the 1st & 2nd element)

* Solve generalized eigenvalue problem
  $(\underline{K} - \omega^2 \underline{M})\vec{\varphi} = \vec{0}$

⇒ Results:

Each natural frequency is repeated twice but they have different mode shapes

→ beam that can bend around the z axis & y axis independently
⇒ 2 uncoupled bending systems:
  a) bending around Z-axis → deformation in Y direction
  b) bending around Y-axis → deformation in Z direction
↳ but with the same properties (length ℓ, young modulus E, ..)
⇒ same natural frequency

Mode shapes are different because they happen in orthogonal planes
↳ same frequency but different deformation pattern (vertical wave / horizontal wave)