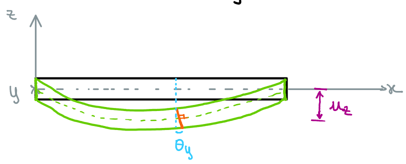
Project I – WP1 : Numerical simulation

1st step: 2D Euler Bernoulli bending beam



* 2 DOF per node:
  - → displacement in z u_z
  - → rotation around y θ_y

* DOF mapping & boundary conditions:
  - → DOF mapping:
    e.g. node 1 & node 2
    each beam connects 2 nodes with 2 DOF each ⇒ 4 DOF per beam
    vector of DOF : $\{u_{z_1}, \theta_{y_1}, u_{z_2}, \theta_{y_2}\}^T$
  - → BC:
    $u_z = 0$ at both ends (node 1 and node n)
    ↳ $u_{z_1} = 0$ and $u_{z_n} = 0$
    ⇒ corresponds to indices 0 and 2(n-1) in DOF vector

* Element matrices:
  - → Stiffness matrix Ke:

$$K_e = \frac{EI}{l^3} \begin{array}{c} \begin{array}{cccc} u_{z_1} & \theta_{y_1} & u_{z_2} & \theta_{y_2} \end{array} \\ \begin{bmatrix} 12 & 6l & -12^{*} & 6l \\ 6l & 4l^2 & 6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \end{array} \begin{array}{c} u_{z_1} \\ \theta_{y_1} \\ u_{z_2} \\ \theta_{y_2} \end{array}$$

with $I = \frac{\pi D^4}{64}$ moment of inertia for circular cross section

*: stiffness coupling btw. displacement in z at node 1 & displacement in z at node 2

  - → Consistent mass matrix Me:

$$M_e = \frac{\rho A l}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}$$

with $A = \frac{\pi D^2}{4}$ cross-sectional area

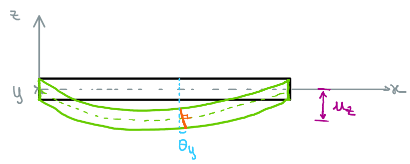
↳ ⚠ Note: different than lumped mass matrix!

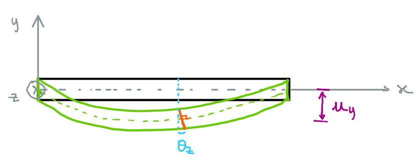
* Matrices assembly to get $\underset{\sim}{K}$ and $\underset{\sim}{M}$
  superposition using the sum (for example node 2 will be in the 1st & 2nd element)

* Solve generalized eigenvalue problem
  $(\underset{\sim}{K} - \omega^2 \underset{\sim}{M})\vec{\varphi} = \vec{0}$

* Comparison to analytical method
  - → Natural frequency ω:
    $$\omega_n = \frac{n^2\pi^2}{l^2}\sqrt{\frac{EI}{\rho A}}$$
  - → Mode shape:
    $$\varphi_n(x) = \sin\left(\frac{n\pi x}{l}\right) \quad x \in [0, l]$$

2^{nd} step: bending beam in 3D





* 4 DOF per node:
  - → displacement in z u_z
  - → rotation around y θ_y
  - → displacement in y u_y
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* DOF mapping & boundary conditions:
  - → DOF mapping:

    e.g. node 1 & node 2

    each beam connects 2 nodes with 4 DOF each ⇒ 8 DOF per beam

    vector of DOF : $\{u_{y_1}, \theta_{z_1}, u_{z_1}, \theta_{y_1}, u_{y_2}, \theta_{z_2}, u_{z_2}, \theta_{y_2}\}^T$
  - → BC:

    $u_y = 0$ and $u_z = 0$ at both ends (node 1 and node n)

    ↳ $u_{y_1} = 0$, $u_{z_1} = 0$, $u_{y_n} = 0$ and $u_{z_n} = 0$

    ⇒ corresponds to indices 0, 2, $4(n-1)$ and $4(n-1)+2$ in DOF vector

* Element matrices:
  - → Stiffness matrix Ke :

$$K_{local} = \begin{array}{c c} \begin{array}{cccc} u_{y_1} & \theta_{z_1} & u_{z_1} & \theta_{y_1} \end{array} & \\ \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix} & \begin{array}{c} u_{y_1} \\ \theta_{z_1} \\ u_{z_1} \\ \theta_{y_1} \end{array} \end{array}$$

(see code

K_{local} = same as in 2D

$$\to K_e = \begin{array}{c c} \begin{array}{cccccccc} u_{y_1} & \theta_{z_1} & u_{y_2} & \theta_{z_2} & u_{z_1} & \theta_{y_1} & u_{z_2} & \theta_{y_2} \end{array} & \\ \begin{bmatrix} a & b & 0 & 0 & c & d & 0 & 0 \\ e & f & 0 & 0 & g & h & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots \\ i & j & 0 & 0 & k & l & 0 & 0 \\ m & n & 0 & 0 & o & p & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots \end{bmatrix} & \begin{array}{c} u_{y_1} \\ \theta_{z_1} \\ u_{y_2} \\ \theta_{z_2} \\ u_{z_1} \\ \theta_{y_1} \\ u_{z_2} \\ \theta_{y_2} \end{array} \end{array}$$

[8x8]

→ explains how 8 x 8 matrices gets filled up

  - → mass matrix Me :

    same process

* matrices assembly to get $\underset{\sim}{K}$ and $\underset{\sim}{M}$

  superposition using the sum (for example node 2 will be in the 1^{st} & 2^{nd} element)

* Solve generalized eigenvalue problem

$$(\underset{\sim}{K} - \omega^2 \underset{\sim}{M})\vec{\varphi} = \vec{0}$$

⇒ Results :

Each natural frequency is repeated twice but they have different mode shapes

→ beam that can bend around the z axis & y axis independently

⇒ 2 uncoupled bending systems :

a) bending around z-axis → deformation in y direction

b) bending around y-axis → deformation in z direction

↳ but with the same properties (length ℓ, young modulus E, ..)

→ same natural frequency

mode shapes are different because they happen in orthogonal planes
↳ same frequency but different deformation pattern (vertical wave / horizontal wave)