Determining cylindrical parameters - an alternative approach

This paper describes the mathematical basis of a program which can fit a cylinder to a series of measured 3D co-ordinates of points which are assumed to lie on a cylindrical surface. The approach is a simpler alternative to that recently described by Feltham (1990). Once the best fitting cylinder has been determined it is possible to examine departures from the geometric surface by unwrapping and projecting onto a plane. Application of the method to the 200-year old cylindrical ceiling of the ballroom in the Mansion House, London has revealed the quality of earlier methods of construction. **By *J H Chandler BSc PhD ARICS* and *M A R Cooper BSc FRICS FInstCES*, Engineering Photogrammetry Unit, Department of Civil Engineering, The City University**

1. An article published recently in the *Photogrammetric Record* by Feltham (1990) describes a method of fitting a cylindrical model to a set of three-dimensional coordinates of arbitrary points on the surface of a cylindrical object. This was read with considerable interest because the Engineering Photogrammetry Unit (EPU), City University, London had been approached by two potential clients who needed such a facility.

The widespread and growing use of CAD systems suggests that the fitting of geometric shapes to spatial data is becoming increasingly necessary. When CAD systems are used in a mapping context, differing geometric entities can be used to represent real world objects. For example, in the case of the representation of an oil refinery with numerous pipe racks and oil tanks, the cylinder represents one geometric figure which assumes great importance. The need to develop software to fit basic geometric structures to three dimensional data, obtained by either photogrammetry or field survey, is becoming increasingly important. Often the three-dimensional coordinates which can be obtained lie on the surface of such figures and these points cannot be assumed to lie in any pre-defined or assumed positions.

For these reasons it seemed prudent to begin finding solutions to the general problem of fitting specific geometric entities to three-dimensional coordinates. A logical starting point entailed developing a cylinder-fitting routine. The work that has been undertaken is a development of the method presented in the *Photogrammetric Record* (Feltham, 1990) and that journal would have provided the logical means of publication. With the growing use of CAD systems in survey, land surveyors will be increasingly presented with this type of problem and so the decision was made to publish in *Land and Hydrographic Survey*.

2.1 Mathematical theory

In three-dimensional analytical geometry a cylinder can be defined by seven parameters (Figure 1). Six of these refer to the equation of the line representing the central axis of the cylinder. They are the three-dimensional coordinates of a point on the centre line (X,Y,Z) and three direction ratios (l,m,n) defining orientation. The final parameter is simply the radius of the cylinder (R).

The requirement of any cylinder-fitting program is the determination of the best estimates for the seven parameters using the three-dimensional coordinates of a series of points measured anywhere on the cylindrical surface. Account should be taken of the statistical nature of these coordinates by including their standard deviations in the estimation procedure. The surface points cannot be assumed to lie in any pre-defined plane, such as on the circumference of two circles at opposite ends of the cylinder. The principle of least squares will yield the 'best fit' cylinder by minimising the sum of the squares of the measurement residuals. The problem is non-linear and an iterative procedure is required in which the unknowns to be estimated are corrections to initial values.

2.1.1 Estimation of cylindrical parameters

Originally it was decided to follow the approach discussed by Feltham (1990) and set out in the published paper. Feltham tackles the problem by establishing relationships between three-dimensional coordinates of measured surface points, cylindrical parameters and local cylindrical coordinates. For each point on the surface an observation equation is developed which functionally links the seven parameters with the two sets of coordinates. In addition to the seven cylindrical parameters which are required, each measured point contributes two extra unknowns, the cylindrical coordinates of that point. This introduces the problem that the number of unknowns increases linearly with the number of measurements and effectively limits the number of measurements that can be included. Feltham solves this particular problem by introducing a relationship which removes these additional unrequired parameters. The functional basis of this is obscure and does create further problems associated with developing certain components of the revised observation equations.

After examination of Feltham's (1990) method it was decided that a simpler and neater alternative could be developed. It was felt that the introduction of the local cylindrical coordinates in the estimation was making the problem unduly complicated. All that was required was to find or develop a function which could relate the seven cylindrical parameters with the three-dimensional coordinates of points on the surface. From Sommerville (1934) a suitable functional model is:

Equation (1)

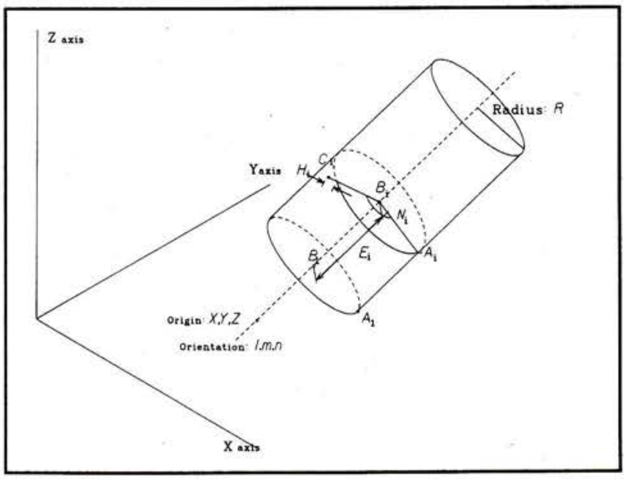
$$R^2 = \frac{[l(y_i-Y)-m(x_i-X)]^2+[m(z_i-Z)-n(y_i-Y)]^2+[n(x_i-X)-l(z_i-Z)]^2}{(l^2+m^2+n^2)} = 0$$

Where: X, Y, and Z are 3D coordinates of a point on the cylinder axis; l, m, and n are direction ratios of the cylinder axis; R is the radius of the cylinder, and x_i, y_i and z_i are the 3D coordinates of point i on the surface.

The function (l) is normally used to derive the distance between a point and a line in three-dimensional space. If the distance is kept constant the function describes the locus of a cylindrical surface where the distance R is the radius.

Using this functional model and the principle of least squares estimation a solution for the seven unknown parameters $(X, Y, Z, l, m, n$ and $R)$ can be obtained from measurements of coordinates (x_i, y_i, z_i)

Figure 1: Cylindrical parameters and mapping to a plane



of points on the surface. Unfortunately measurements and parameters cannot be separated in the functional model. The computationally simpler 'observation equation' approach cannot be used and the 'general solution' is required, (Cooper, 1987). The function is non-linear and so an iterative procedure is necessary, in which the estimated parameters are corrections to some initial assumed values of X, Y, Z, l, m, n and R. Using matrix notation the normal equations can be derived from:

$$\boldsymbol{Ax+Bv=b}$$

Where: $\boldsymbol{A}$ is the A or design matrix (dimensions 3n x 7); $\boldsymbol{B}$ is the B matrix (dimension 3n x 3n); $\boldsymbol{b}$ is the vector of observed minus computed terms (dimensions 3n x 1); $\boldsymbol{n}$ is the number of measured points; and $\boldsymbol{x}$ is the vector of the seven corrections to the initial values of the cylindrical parameters. The elements of the **A** matrix are composed of the partial differentials of the main function (Equation 1) with respect to each parameter. Similarly, the **B** matrix is composed of the partial differentials of the function with respect to each measured coordinate.

The least squares estimates of corrections to the cylindrical parameters are given by:

$$\hat{x} = (\boldsymbol{A}^t(\boldsymbol{B}\boldsymbol{W}^{-1}\boldsymbol{B}^t)^{-1}\boldsymbol{A})^{-1}\boldsymbol{A}^t(\boldsymbol{B}\boldsymbol{W}^{-1}\boldsymbol{B}^t)^{-1}\boldsymbol{b}$$

Where: $\boldsymbol{W}$ is the (3n x 3n) weight matrix of the measured coordinates, often assumed to be diagonal with elements equal to the reciprocals of the squares of the coordinate standard deviations.

The solution is iterative and must be repeated until the estimated corrections are no longer significantly greater than zero. Once this has been achieved the best estimates of the seven cylindrical parameters have been determined. These parameters define the best fit or 'design' cylinder and can be used to create a cylindrical object in a CAD package. The cylindrical parameters can also be used for further analyses.

2.1.2 Mapping of departures

One application specified by a client entailed quantitative assessment of damage to large bore cylindrical steel tubing. In this case the required information consists of the minimum distance between the three-dimensional coordinates of points measured on the damaged part of the pipe surface and the undamaged theoretical design cylinder. Departures from the design surface are required at many points distributed throughout the damaged area and must be presented to the engineer by a clear, concise, quantitative and unambiguous method. A graphical representation such as a contour plot of departures would be most suitable but how would the cylinder be represented? An orthogonal projection would be unsuitable as differing viewing directions would yield differing results. Unambiguous results would be obtained if the cylinder was 'unwrapped' by transforming the three-dimensional surface coordinates onto a planar surface.

Referring to Figure 1, a point such as C_i will possess three mapped coordinates: E_i the distance measured along the cylinder axis between an arbitrary reference point B_1 and point B_i; N_i the length of the circular arc, radius R, subtended by the angle C_i, B_i, A_i and H_i the discrepancy between radial distance $B_i C_i$ and the radius R of the design cylinder. The reference point A_1 is required for the transformation but selection is totally arbitrary and will not alter the discrepancy surface. For example, it can be assumed that A_1 refers to the first point in the data file, giving rise to the suffix of 1. Initially, the three-dimensional coordinates of point B_1 on the cylinder axis associated with the reference point A_1 must be derived. This procedure is then repeated for each surface point C_i giving rise to coordinates of B_i. Using analytical geometry the coordinates of B_i are defined by:

$$X_{Bi} = \mu\, l + X$$
$$Y_{Bi} = \mu\, m + Y$$
$$Z_{Bi} = \mu\, n + Z$$

Where:

$$\mu = \frac{(x_i-X)l+(y_i-Y)m+(z_i-Z)n}{l^2+m^2+n^2}$$

Once the three-dimensional coordinates of the point B_i have been obtained then the required mapped coordinates of the cylinder are:

$$E_i = [(x_{Bi}-x_{B1})^2+(y_{Bi}-y_{B1})^2+(z_{Bi}-z_{B1})^2]^{1/2}$$

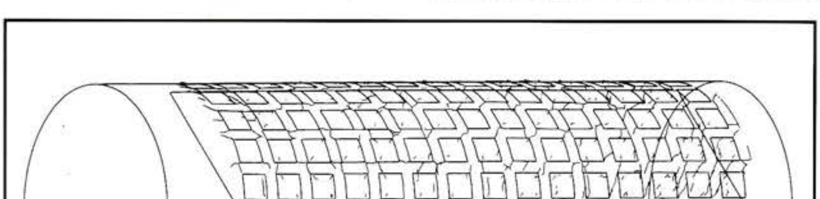
$$N_i=R.\cos^{-1}[(l_A l_i+m_A m_i+n_A n_i).(l^2_A+m^2_A+n^2_A)^{-1/2}.(l^2_i+m^2_i+n^2_i)^{-1/2}]$$

$$H_i = [(x_i-x_{Bi})^2+(y_i-y_{Bi})^2+(z_i-z_{Bi})^2]^{1/2}-R$$

Mapped coordinates can be determined for all points measured on the cylinder or pipe surface. Plan positions are defined by $E_i N_i$ while H_i describes the radial discrepancy between the theoretical cylinder surface and the coordinates of the measured point. These data can be used to form a digital terrain model, triangulated, contoured, and used to portray the extent and variability of the radial discrepancies.

3.1 The developed program

A computer program was developed which incorporated the mathematical theory indicated in section 2. The program initially determines a 'best fit' cylinder from a series of three-dimensional coordinates measured on the cylinder surface. The number of parameters requiring estimation can be reduced to six in the case when the radius of the cylinder or pipe is known. A full stochastic model is included in which the statistical properties of all measurements are allowed to propagate through the functional model. This enables the quality of the final estimation to be assessed by analysing the precision of the final estimated parameters, the residuals and the *a posteriori* variance factor. The six or seven parameters are then used to generate a cylindrical element in a 3D Intergraph design file. A secondary component of the program maps the cylinder onto a plane surface and determines radial discrepancies between the design surface and a file of three dimensional coordinates. Consistent with most programming solutions involved with least squares estimation and three dimensional analytical geometry there were certain practical problems to resolve. The main function (1) is non-linear and so an iterative approach is required, (Section 2. 1). This creates the classic problem of deriving adequate starting values. If these initial values are not sufficiently close to the final estimates the linearisation may not be valid and the solution may fail to converge. Deriving good starting values for the three direction ratios (l,m,n) is particularly critical. Starting values for these are obtained from the first two points



Right:Figure 2 Cylinder fitted to ceiling cracks

in the file of coordinates which can be assumed to have been measured on the cylinder surface in the direction of the cylinder axis. Starting values for the three positional parameters (X,Y,Z) are given by the mean coordinates of all measured surface points.

If all seven parameters are to be estimated, an initial value for the cylinder radius is determined from the three positional starting values and one of the other points. In tests carried out so far these methods have proved adequate and convergence has been achieved.

The other main practical problem occurs when the axis of the cylinder is aligned almost along, or orthogonal to, one of the axes of the coordinate system. Inaccuracies involved with representing very large and small numbers in the computer memory become critical and a solution may not be obtainable. This problem is overcome by checking the initial estimates of the direction ratios. If one of these values is close to either zero or unity then the coordinates of all points are rotated about the other two axes by say 30 degrees, using a three-dimensional similarity transformation (Albertz and Kreiling, 1975). The estimation of the cylindrical parameters is then carried out in the new rotated coordinate system. A reverse rotation is applied once the parameters have been finally determined. Such a rotation is frequently required in practice because pipes and cylinders are often aligned in horizontal and vertical planes.

It should be apparent that the number and distribution of surface coordinates can affect significantly the quality of the final estimated parameters. If fewer than three surface points are included then no solution can be obtained. The quality of the estimation is revealed in part by the precision of the determined cylindrical parameters and this alone accounts for the necessity of a full stochastic model. Points located on the circumference of two imaginary circles located at opposite ends of a cylinder will yield the most precise estimates for the seven parameters.

4.1 An Application

The program was originally developed using test data derived from a cylinder drawn within an Intergraph design file. Such perfect data is ideal for initial software development but real survey data is required for more rigorous testing.

A contract the Engineering Photogrammetry Unit (EPU) has recently carried out involved mapping cracks in the ceiling of the ballroom at the Mansion House, London. This ceiling (Photo 1) is approximately 12m above the floor, about 28m long and 11m wide. For the initial contract 18 small targets had been placed adjacent to the ceiling. Horizontal and vertical angles were measured to these from four survey stations. Scale was defined by taped distances between all survey stations. All measurements were used in a three-dimensional variation of coordinates estimation program to determine the best estimates for the three-dimensional coordinates of the targets. The rms standard deviations of these coordinates was 0.0015m. Terrestrial photographs were acquired using a Zeiss UMK metric camera placed on the floor using both panchromatic and holographic emulsions.

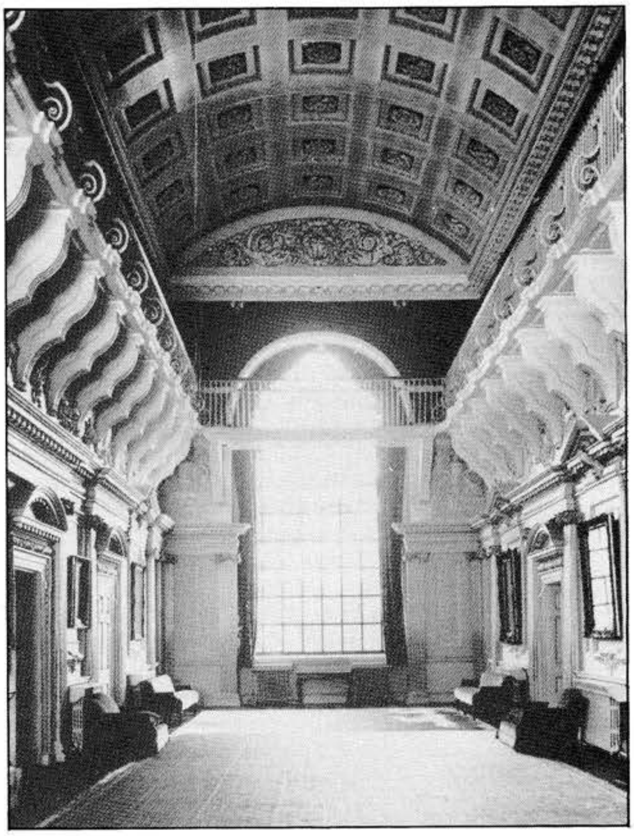
The Engineering Photogrammetry Unit's Intergraph InterMap Analytic analytical plotter was used to derive a three-dimensional digital representation of the extent and distribution of the cylindrical ceiling cracks. An analysis of the quality of the coordinates of points on the cracks, shows that these have been determined to a precision of 0.0025m.

The curved nature of the ceiling suggested that a subset of these crack coordinates could be used to test the cylinder-fitting program. Approximately 30 coordinates were derived from points distributed over the whole ceiling and used as input to the program. A solution was obtained and the estimated parameters used to derive the best fit cylinder (Figure 2). Interestingly the radial residuals for all coordinates used in the solution were small, less than 0.065m.

Further analysis included deriving a discrepancy surface using coordinates of all measured points on the cracks including those not used for the original estimation of cylindrical parameters (Figure 3). Again the radial departures from the design surface were all less than 0.092m and most were substantially less. This would appear quite surprising considering that the ceiling was originally constructed in 1750, has suffered the ravages of two world wars and tunnelling activities associated with Bank underground station underneath the Mansion House.

5.1 Conclusion

With the widespread and increasing use of three-dimensional CAD systems in which real objects are represented by geometric figures, the importance of computer programs to fit real measured data to geometric entities is increasing. It has been shown that suitable programs can be developed with comparative ease. Many graduate land surveyors possess the necessary skills: a basic understanding of the principles of least squares, computer programming and a little three-dimensional analytical geometry.



Above: Photo 1. The Mansion House ballroom ceiling.

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Below: Figure 3. Discrepancy surface contour int. 0.005m Mansion House ceiling