Linear Algebra

Module code: 501032

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1 Objective

In this Lab you will find out about orthogonality and least squares solution in MATLAB.

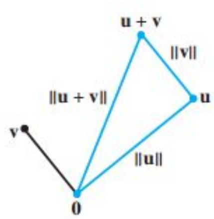
2 Orthogonality

2.1 Definition

Two vectors **u** and **v** in R^n are orthogonal (to each other) if $u \cdot v = 0$

2.2 Theorem 1:

Two vectors **u** and **v** are orthogonal if and only if $\| u + v \|^2 = \| u \|^2 + \| v \|^2$



2.3 Theorem 2: The Gram-Schmidt Process

Given a basis $\{x_1, ..., x_p\}$ for a nonzero subspace W of R^n, define

$$
\begin{gathered}
v_1 = x_1 \\
v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 v_1} v_1 \\
v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 v_2} v_2 \\
. \\
. \\
. \\
v_p = x_p - \frac{x_p \cdot v_1}{v_1 v_1} v_1 - \frac{x_p \cdot v_2}{v_2 v_2} v_2 - ... - \frac{x_p \cdot v_{p-1}}{v_{p-1} v_{p-1}} v_{p-1}
\end{gathered}
$$

Then, $\{v_1, .., v_p\}$ is an orthogonal basis for W. In addition

$Span\{v_1, ..., v_k\} = Span\{x_1, ..., x_k\}$ for $1 \leq k \leq p$

3 Least Squares solution

3.1 Definition.

Let $\mathbf{Ax} = \mathbf{b}$ be a linear system where of $\mathbf{A}$ is an $m \times n$ matrix. A vector $\mathbf{u} \in R^n$ is called the least squares solution to the linear system if $\|\mathbf{b}\text{-}\mathbf{Au}\| \leq \|\mathbf{b}\text{-}\mathbf{Av}\|$ for all $\mathbf{v} \in R^n$

3.2 Theorem.

Let $\mathbf{Ax} = \mathbf{b}$ be a linear system. Then $\mathbf{u}$ is a least square solution to $\mathbf{Ax} = \mathbf{b}$ if and only if $\mathbf{u}$ is a solution to $A^T Ax = A^T b$

3.3 Least square in MALAB

Example Find the least squares solution to the linear system $\mathbf{Ax} = \mathbf{b}$ with:

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 1 & 6 \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$$

Solution
We can check easily that A has rank 2 with rank(A) command.

```
>> A = [1  3;  2  4;  1  6];  b = [4  1  3]'
   rank(A)


ans
       2
```

And then we can compute $x = (A^T A)^{-1} A^T b$ and you get

$$\mathbf{x} = \begin{pmatrix} -0.3766 \\ 0.6623 \end{pmatrix}$$

```
>> x=inv((A'*A))*A'*b

   x =


   -0.3766
    0.6623
```

Let $\mathbf{Ax}$ and then you will obtain:

$$b_{ls} = \begin{pmatrix} 1.61 \\ 1.90 \\ 3.60 \end{pmatrix}$$

You can consider b_{ls} is not exactly b, but as close as we are going to get.
On the other hand, You can solve least squares problems with bounds or linear constrains in MALAB's command.

Functions	**Description**
lsqlin	Solve constrained linear least quares problems
lsqnonneg	Solve nonnegative least squares constrain problems
mldivide, \	Solve systems of linear equations $\mathbf{Ax} = \mathbf{b}$ for $\mathbf{x}$

4 Exercises

4.1 Exercise 1:

Write a function to show that $\{u_1, u_2, ..., u_n\}$ is an orthogonal set.
Hint: Consider each possible pairs of distinct vectors $u_i u_j = 0$ whenever $i \neq j$
For example $u_1 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$, $u_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$, $u_3 = \begin{pmatrix} \frac{-1}{2} \\ 2 \\ \frac{7}{2} \end{pmatrix}$

4.2 Exercise 2:

Let $\mathbf{y}$ and $\mathbf{u}$ vector. Write a function to find the orthogonal projection of $\mathbf{y}$ on $\mathbf{u}$.
Hint $proj_u y = \frac{y \cdot u}{u \cdot u} u$. For example, $y = \begin{pmatrix} 7 \\ 6 \end{pmatrix}$ and $u = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

4.3 Exercise 3:

Let a matrix $m \times n$, write a function to check that has orthonormal columns.
Hint: An $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I$

4.4 Exercise 4:

Use the Gram - Schmidt process to produce an orthogonal basis for column space of

$$\mathbf{A} = \begin{pmatrix} -10 & 13 & 7 & -11 \\ 2 & 1 & -5 & 3 \\ -6 & 3 & 13 & -3 \\ 16 & -16 & -2 & 5 \\ 2 & 1 & -5 & -7 \end{pmatrix}$$

4.5 Exercise 5:

To find the least square solution to $\mathbf{Ax} = \mathbf{b}$. For this case, the equation $A^T Ax = A^T b$ with

$$\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

4.6 Exercise 6:

Solve the linear system below and find the least squares solutions.

$$\begin{cases} e = 0.5 \\ d + e = 1.6 \\ c + 2d + e = 2.8 \\ c + e = 0.8 \\ 4c + d + e = 5.1 \\ 4c + 2d + e = 5.9 \end{cases} \Leftrightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \\ 4 & 1 & 1 \\ 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} c \\ d \\ e \end{pmatrix} = \begin{pmatrix} 0.5 \\ 1.6 \\ 2.8 \\ 0.8 \\ 5.1 \\ 5.9 \end{pmatrix} \Leftrightarrow \mathbf{Ax} = \mathbf{b}$$

4.7 Exercise 7

An engineer is tracking the friction index over mileage of a breaking system of a vehicle. She expects that the mileage-friction relationship is approximately linear and she collects five data points that are show in the table below.

Mileage	2000	6000	20000	30000	40000
Friction Index	20	18	10	6	2

Write a function to describe these points above and the approximately points by graph.
Hint: Consider $\mathbf{b}$ is the vector of friction index data values and $\mathbf{y}$ values when we plug in the mileage data for $\mathbf{x}$ and find $\mathbf{y}$ by the equation of the line $ax+b = y$. We want minimizes the distance between $\mathbf{b}$ and $\mathbf{y}$.

4.8 Exercise 8

A bioengineer is studying the growth of a genetically engineered bacteria culture and suspects that is it approximately follows a cubic model. He collects six data points that are show table below.

Time in Days	1	2	3	4	5	6
Grams	2.1	3.5	4.2	3.1	4.4	6.8

He assumes the equation has the form $ax^3 + bx^2 + cx + d = y$. Write a function to describe these points above by graph.

5 Reference

1. Linear Algebra Laboratory, National University of Singapore
2. Matlab Linear Algebra, Cesar Petez Lopez, Spinger, 2014.
3. Linear Algebra Laboratory, South California University.
4. $http://ltcconline.net/greenl/courses$