Linear Algebra

Module code: 501032

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1 Objective

In this Lab you will study to do the following tasks:

- Matrix operations
- The determinant of a matrix
- The inverse of a matrix
- The norm of a vector and matrix

2 Matrix Operations

2.1 Matrix Addition

2.2 Matrix Multiplication

3 Matrix Analysis

3.1 The determinant of a matrix

3.1.1 Definition

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j}\det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, ..., a_{1n}$ are from the first now of A. In symbols,

$$\begin{aligned}\det A &= a_{11}\det A_{11} - a_{12}\det A_{12} + ... + (-1)^{1+n} a_{1n}\det A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{i+j}\det A_{1j}\end{aligned}$$

In MATLAB, to calculate the determinant of the matrix, using **det** function.

For example, Let a matrix $A = \begin{pmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{pmatrix}$, using a MATLAB command to calculate the determinant of A.

```
>> det(A)
ans =
    -2
```

3.2 The inverse of a matrix

3.2.1 Theorem

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $ab - bc \neq 0$, the A is invertible and

$$A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

If $ad - bc = 0$, the A is not invertible. To find the inverse of A, MATLAB provide **inv** function or use A^{-1} to do it.

For example, Calculate the inverse of $A = \begin{pmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{pmatrix}$

```
>> inv(A)
ans =
        1       5.55111512312578e-17    2.5
        0       0                       -0.5
        2       -1                      3
>> A^-1
ans =
        1       5.55111512312578e-17    2.5
        0       0                       -0.5
        2       -1                      3
```

3.3 The norm of a vector and matrix

3.3.1 Definition (Vector Norm)

A vector norm $\parallel \mathbf{x} \parallel$ is any mapping from R^n to R with the following three properties.

1. $\parallel \mathbf{x} \parallel > 0$, if $\mathbf{x} \neq 0$
2. $\parallel \alpha\mathbf{x} \parallel = \mid \alpha \mid \parallel \mathbf{x} \parallel$, for any $\alpha \in R^n$
3. $\parallel \mathbf{x} + \mathbf{y} \parallel \leq \parallel \mathbf{x} \parallel + \parallel \mathbf{y} \parallel$

for any vector $\mathbf{x}, \mathbf{y} \in R^n$

3.3.2 Definition (Matrix Norm)

A matrix norm of a matrix $\| \mathbf{A} \|$ is any mapping from $R^{n \times n}$ to R with the following three properties

1. $\| \mathbf{A} \| > 0$, if $\mathbf{xA} \neq 0$
2. $\| \alpha \mathbf{A} \| = | \alpha | \| \mathbf{Ax} \|$, for any $\alpha \in R^n$
3. $\| \mathbf{A} + \mathbf{B} \| \leq \| \mathbf{A} \| + \| \mathbf{B} \|$

for any vector $\mathbf{A}, \mathbf{B} \in R^n$

3.3.3 The commonly vector and matrix norms

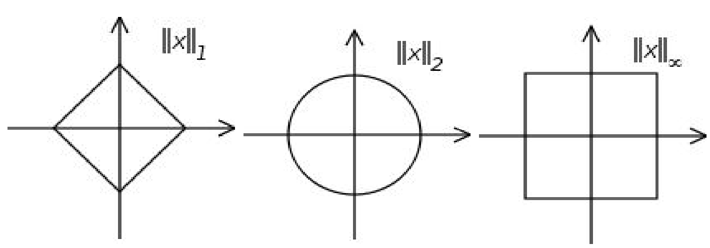
The commonly vector norms are:

1. l_1 norm: $\| \mathbf{x} \|_1 = \sum\limits_{i=1}^{n} | x_i |$
2. l_2 norm (the Euclidean norm): $\| \mathbf{x} \|_2 = (\sum\limits_{i=1}^{n} x_i^2)^{\frac{1}{2}} = \sqrt{x^T.x}$
3. $l_\propto$ norm: $\| \mathbf{x}_\propto \| = \max\limits_{1 \leq i \leq n} | x_i |$

The commonly matrix norms are:

1. The absolute-value norm: $\| \mathbf{A} \|_1 = \max\limits_{1 \leq j \leq n} (\sum\limits_{i=1}^{n} | a_{ij} |)$
2. The Euclidean norm: $\| \mathbf{A} \|_2 = \sqrt{\sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} (a_{ij}^2)^{\frac{1}{2}}}$
3. The infinity norm: $\| \mathbf{A} \|_\propto = \max\limits_{1 \leq i \leq n} (\sum\limits_{j=1}^{n} | a_{ij} |)$

Illustrations of unit circles in different norms.



3.3.4 Calculate vector and matrix norms in MALAB

Syntax: **norm(v, p)** or **norm(X, p)** where **v** is vector and **X** is matrix need to calculate norm. The valid values of p and what they return depend on whether the first input to norm is matrix or vector as show in the table below:

p	Matrix	Vector
1	The absolute-value norm	l_1-norm
2	The Euclidean norm	sum(abs(X) .^2)^(1/2)
Positive, real-valued numeric p	-	sum(abs(X) .^p)^(1/p)
Inf	max(sum(abs(X')))	max(abs(X))
-Inf	-	min(abs(X))

Example Calculate the 1-norm of the vector, 2-norm of vector (-2,3,-1) in 3-D space.

```
>> v = [-2 3 -1];
>> l1 = norm(v,1)
>> l2 = norm(v,2)
>> l_infinity = norm(v, Inf)
>> l_neinfinity = norm(v, -Inf)
>> l_p = norm(v, 3)
```

```
l1 =

    6.0000
l2 =

    3.7417
l_infinity =

     3
l_neinfinity =

     1
l_p =

    3.3019
```

4 Exercises

4.1 Exercise 1:

Let $A = \begin{pmatrix} -1 & 4 & 8 \\ -9 & 1 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 5 & 8 \\ 0 & -6 \\ 5 & 6 \end{pmatrix}$ $C = \begin{pmatrix} -4 & 1 \\ 6 & 5 \end{pmatrix}$ $D = \begin{pmatrix} -6 & 3 & 1 \\ 8 & 9 & -2 \\ 6 & -1 & 5 \end{pmatrix}$.

Computing the following, if possible

1. (AB^T)
2. (BC^T)
3. $(C - C^T)$
4. $(D - D^T)$
5. $((D^T)^T)$
6. $(2C^T)$
7. $(A^T + B)$
8. $((A^T + B)^T)$
9. $((2A^T - 5B)^T)$
10. $((-D)^T)$
11. $(-(D)^T)$
12. $((C^2)^T)$
13. $((C^T)^2)$

4.2 Exercise 2:

Write a MATLAB command to compute the determinants of the matrices below:

$$A = \begin{pmatrix} 6 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 2 & 0 & 0 & 0 \\ 8 & 3 & 1 & 8 \end{pmatrix}, B = \begin{pmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 5 & 0 & 4 & 4 \end{pmatrix},$$

$$C = \begin{pmatrix} 3 & 5 & -8 & 4 \\ 0 & -2 & 3 & -7 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 2 \end{pmatrix}, D = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 7 & -1 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ 5 & -8 & 3 & 0 \\ 5 & -8 & 4 & -3 \end{pmatrix},$$

$$E = \begin{pmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{pmatrix}, F = \begin{pmatrix} 6 & 3 & 2 & 4 & 0 \\ 9 & 0 & -4 & 1 & 0 \\ 8 & -5 & 6 & 7 & 1 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{pmatrix}$$

4.3 Exercise 3:

Is it true that $\det(A+B) = \det A + \det B$? To find out, generate random 5×5 matrices A and B, and compute $\det(A+B) - \det A - \det B$. Repeat the calculations for three other pairs of $n \times n$ matrices, for various values of n.

4.4 Exercise 4:

Is it true that $\det AB = (\det A)(\det B)$?. Repeat the calculation for four pairs of random matrices.

4.5 Exercise 5:

Compute $\det AA^T$ for several random 4×5 matrices and several random 5×6 matrices. What can you obtain to $A^T A$ and AA^T when A has more columns than rows.

4.6 Exercise 6:

Let A and B be the following 3×3 matrices $\begin{pmatrix} 2 & 4 & \frac{5}{2} \\ \frac{-3}{4} & 2 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & \frac{-1}{2} & \frac{3}{4} \\ \frac{3}{2} & \frac{1}{2} & -2 \\ \frac{1}{4} & 1 & \frac{1}{2} \end{pmatrix}$

Using MATLAB command to

1. Calculate $A^{-1}B^{-1}$, $(AB)^{-1}$, and $(BA)^{-1}$
2. Find $(A^{-1})^T$ and $(A^T)^{-1}$

4.7 Exercise 7:

Calculate the norms of the matrices below:

1. Calculate the 1-norm

   (a)
   $$A_1 = \begin{pmatrix} 1 & -7 \\ -2 & -3 \end{pmatrix}$$

   (b)
   $$A_2 = \begin{pmatrix} -2 & 8 \\ 3 & 1 \end{pmatrix}$$

   (c)
   $$A_3 = \begin{pmatrix} 2 & -8 \\ 3 & 1 \end{pmatrix}$$

   (d)
   $$A_4 = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$$

(e)

$$A_5 = \begin{pmatrix} 5 & -4 & 2 \\ -1 & 2 & 3 \\ -2 & 1 & 0 \end{pmatrix}$$

2. Calculate the infinity-norm

(a)

$$B_1 = \begin{pmatrix} 1 & -7 \\ -2 & -3 \end{pmatrix}$$

(b)

$$B_2 = \begin{pmatrix} 3 & 6 \\ 1 & 0 \end{pmatrix}$$

(c)

$$B_3 = \begin{pmatrix} 5 & -4 & 2 \\ -1 & 2 & 3 \\ -2 & 1 & 0 \end{pmatrix}$$

(d)

$$B_4 = \begin{pmatrix} 3 & 6 & -1 \\ 3 & 1 & 0 \\ 2 & 4 & -7 \end{pmatrix}$$

(e)

$$B_5 = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

3. Calculate the Euclidean-norm

(a)

$$C_1 = \begin{pmatrix} 5 & -4 & 2 \\ -1 & 2 & 3 \\ -2 & 1 & 0 \end{pmatrix}$$

(b)

$$C_2 = \begin{pmatrix} 1 & 7 & 3 \\ 4 & -2 & -2 \\ -2 & -1 & 1 \end{pmatrix}$$

(c)

$$C_3 = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$$

4.8 Exercise 8:

Write a MATLAB function to decode the encoded following message
$E = \begin{pmatrix} 80 & 98 & 99 & 85 & 106 & 94 \\ 71 & 92 & 76 & 95 & 100 & 92 \\ 124 & 163 & 140 & 160 & 176 & 161 \end{pmatrix}$. Given $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 4 \end{pmatrix}$
and lookup table

Alphabet	A	B	C	D	...	W	X	Y	Z	
Position	1	2	3	4	...	23	24	25	26	27
Position +3	4	5	6	7	...	26	27	28	29	30

Hint: In order to decode the message D, you need to know the inverse matrix A^{-1} and then you calculate $A^{-1}E$ to get the matrix D. Remember that the matrix D contains the message. Finally, using the above lookup table to obtain the message which can read.

4.9 Exercise 9:

Using lookup table in the Exercise 8, write a MATLAB function to encode the following message

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using the matrix $A = \begin{pmatrix} 3 & 4 & 5 \\ 1 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix}$
Hint: In order to encode the message, you need to create the matrix E with lookup table and then calculate AE.

4.10 Exercise 10:

Write a MATLAB function to calculate the similarities among documents. Consider the following document-term matrix

	T1	T2	T3	T4	T5	T6	T7	T8
Doc1	0	4	0	0	0	2	1	3
Doc2	3	1	4	3	1	2	0	1
Doc3	3	0	0	0	3	0	3	0
Doc4	0	1	0	3	0	0	2	0
Doc5	2	2	2	3	1	4	0	2

Using the Cosine Similarity $= \frac{\|Doc_i \cdot Doc_j\|_1}{\|Doc_i\|_2 \|Doc_j\|_2}$. Cosine Similarity is used to measure the angle between two unit length.

4.11 Exercise 11:

Write a MATLAB function reuse the Cosine Similarity measure to retrieve the documents which is the nearest with vector $\mathbf{q} = \begin{pmatrix} 0 & 0 & 0.7 & 0.5 & 0 & 0.3 \end{pmatrix}$. Given the documents are represented as vectors.

	nova	**galaxy**	**heat**	**actor**	**film**	**role**
D1	1.0	0.5	0.3	0	0	0
D2	0.5	1.0	0	0	0	0
D3	0	1.0	0.8	0.7	0	0
D4	0	0.9	1.0	0.5	0	0
D5	0	0	0	1.0	0	1.0
D6	0	0	0	0	0.7	0
D7	0.5	0	0.7	0	0	0.9
D8	0	0.6	0	1.0	0.3	0.2

5 Reference

1. Linear Algebra Laboratory, National University of Singapore
2. Matlab Linear Algebra, Cesar Petez Lopez, Spinger, 2014.
3. Linear Algebra Laboratory, South California University.