Introduction to Modelling

7. Introduction to Integration

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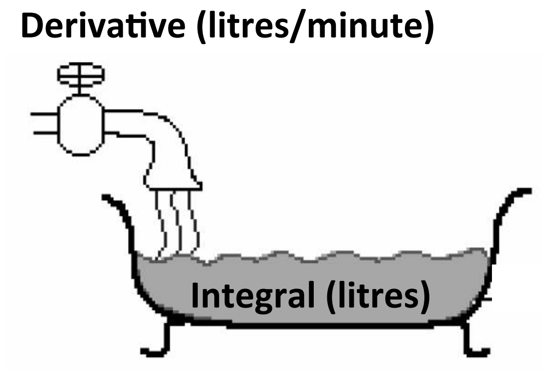
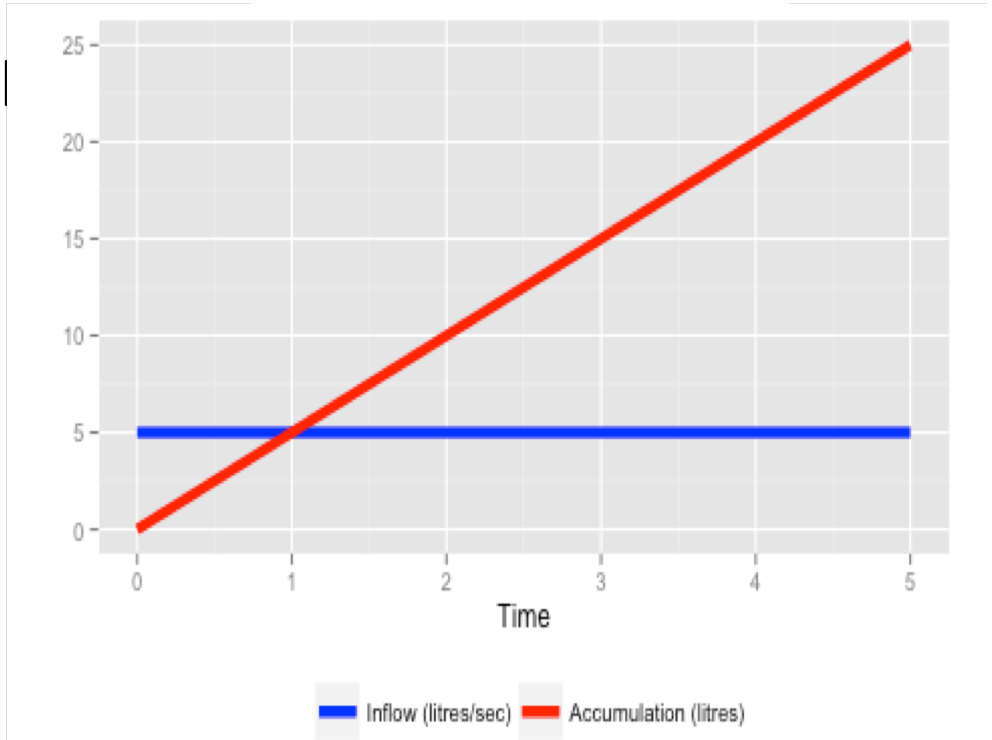
https://github.com/JimDuggan/CT248

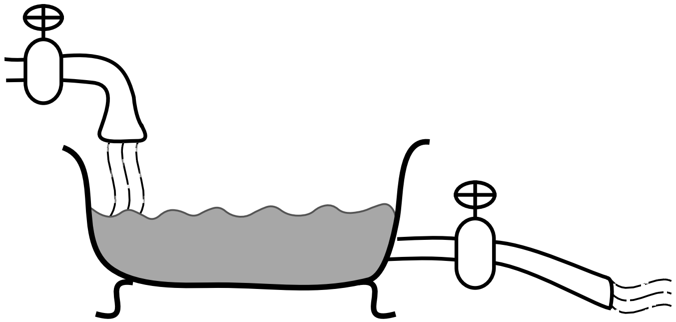
Integration

- Calculus is the study of how things **change over time**, and is described by Strogratz (2009) as "*perhaps the greatest idea that humanity has ever had*."
- Integration is the mathematical process of calculating the area under the net flow curve, between initial and final times.
- Given the flows and the initial condition, integration provides the stock values (how much is in the bathtub at any future time t).





Hydraulic Metaphor:



Stock and Flow Diagram:

Integral Equation:

$$\mathbf{Stock}(\mathbf{t}) = \int_{t_0}^{t} [\mathbf{Inflow}(\mathbf{s}) - \mathbf{Outflow}(\mathbf{s})]\mathbf{ds} + \mathbf{Stock}(\mathbf{t}_0)$$

Differential Equation:

$$\mathbf{d(Stock)/dt = Net\ Change\ in\ Stock = Inflow\ (t) - Outflow(t)}$$

Figure 6-2 Four equivalent representations of stock and flow structure.

Integrals (Stocks)

- **Stocks** are the elements of the system that you can see, feel, count, or measure at any given time.
- A **system stock** is, an accumulation of material or information that has built up over time
- Dimensions are units (litres, people, lines of code)

Meadows, Donella H. *Thinking in systems: A primer*. Chelsea Green Publishing, 2008.

Derivatives (Flows)

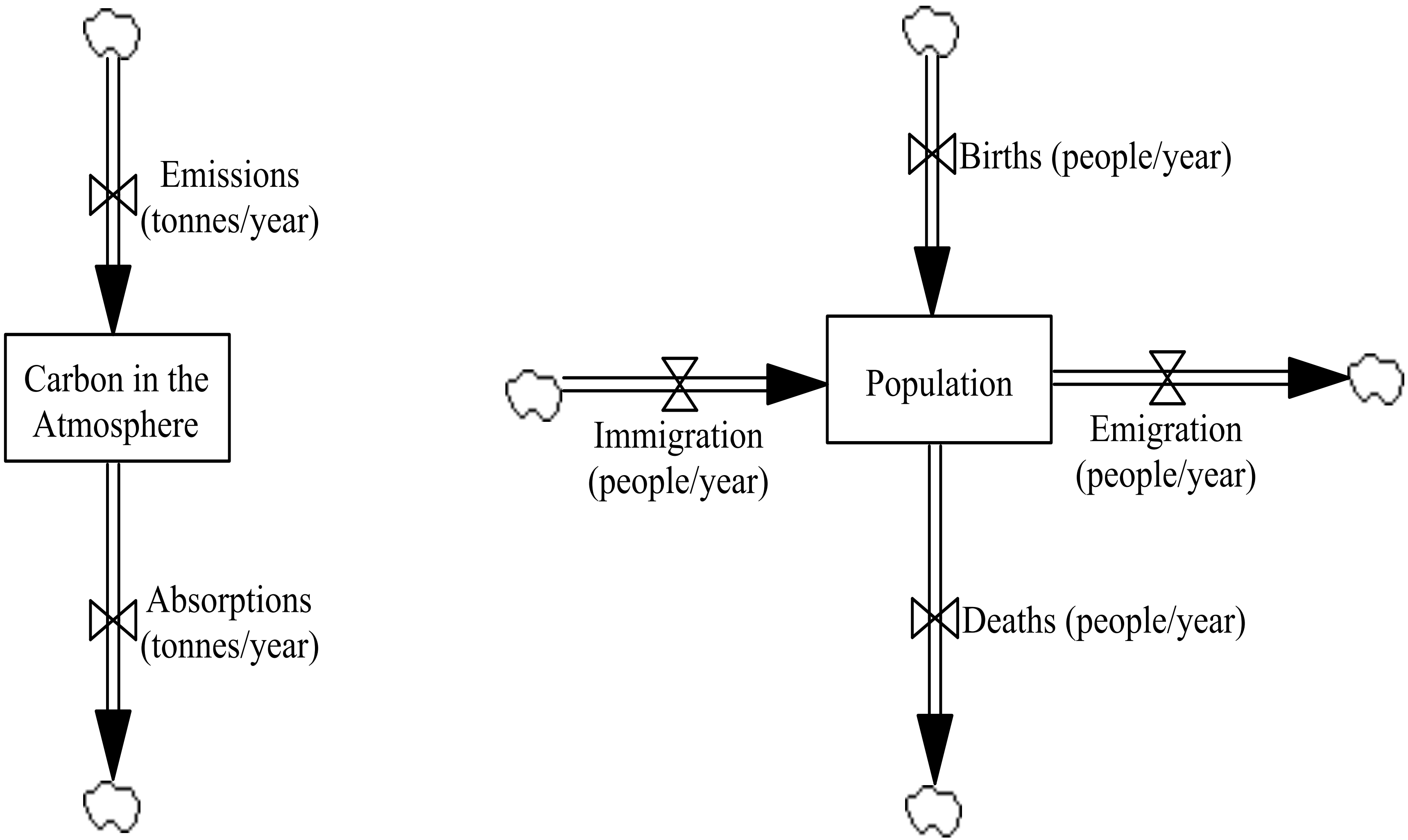
- Stocks change over time through the actions of a **flow**.
- Flows are:
  - filling and draining,
  - births and deaths,
  - purchases and sales,
  - deposits and withdrawals
  - enrolments and graduations
- Dimensions are units/time period (litres/day, people/year)

Field	Stocks	Flows
Mathematics, physics and engineering	Integrals, states, state variables, stocks	Derivatives, rates of change, flows
Chemistry	Reactants and reaction products	Reaction rates
Manufacturing	Buffers, inventories	Throughput
Economics	Levels	Rates
Accounting	Stocks, balance sheet items	Flows, cash flow or income statement items
Biology, physiology	Compartments	Diffusion rates, flows
Medicine, epidemiology	Prevalence, reservoirs	Incidence, infection, morbidity and mortality rates

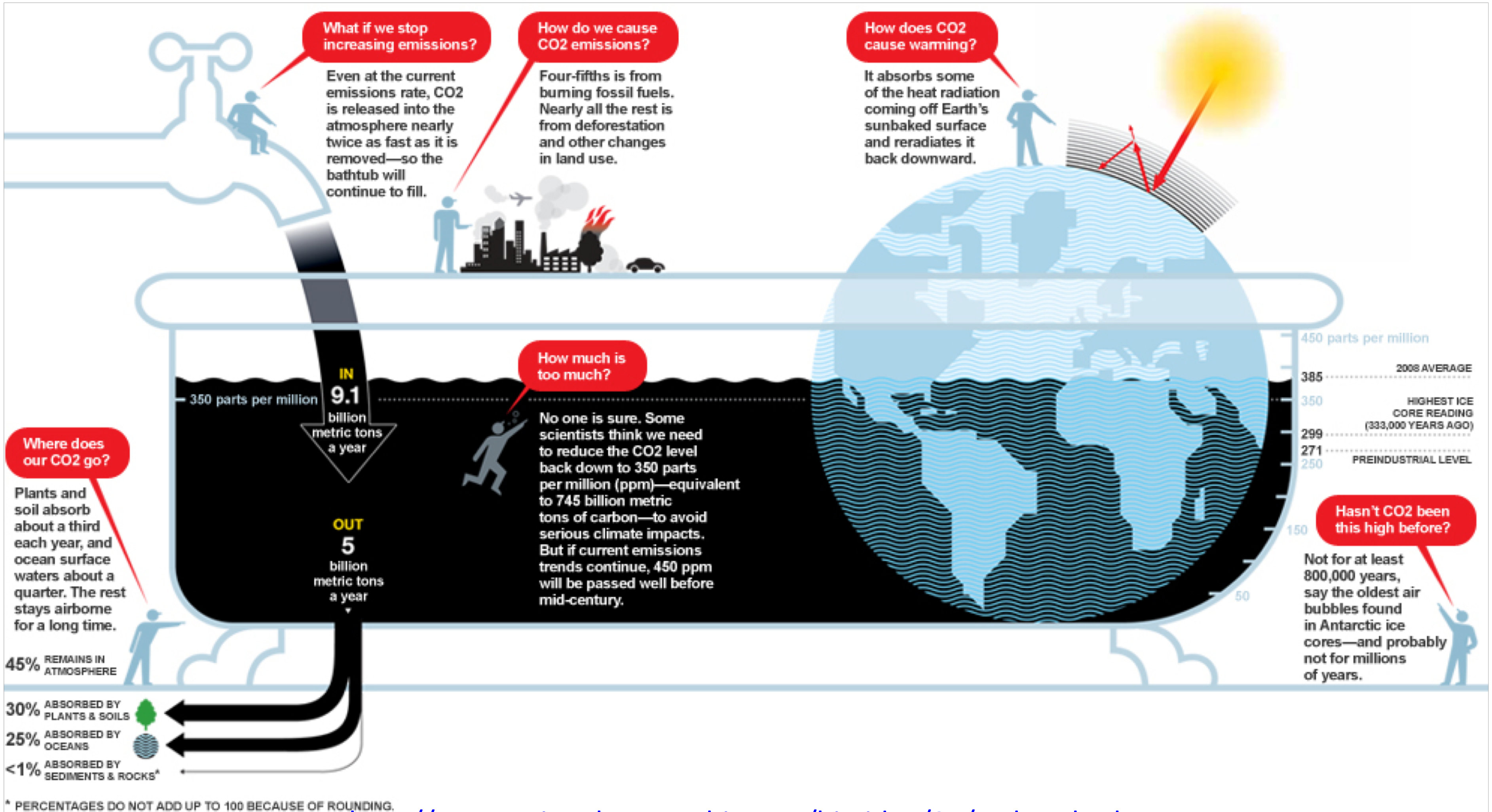
Table 6-1 Terminology used to distinguish between stocks and flows in different disciplines

Business Dynamics

Mathematical Models:
Stocks and Flows



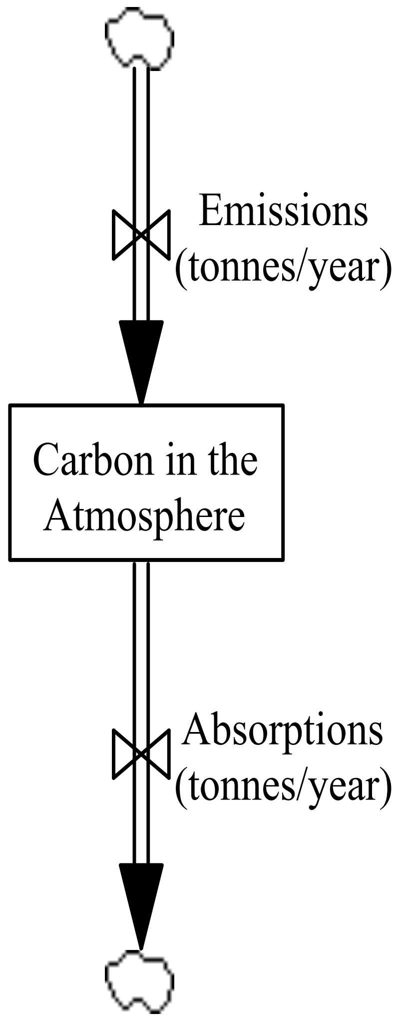
A Stock and Flow Model of Carbon in the Atmosphere



http://ngm.nationalgeographic.com/big-idea/05/carbon-bath

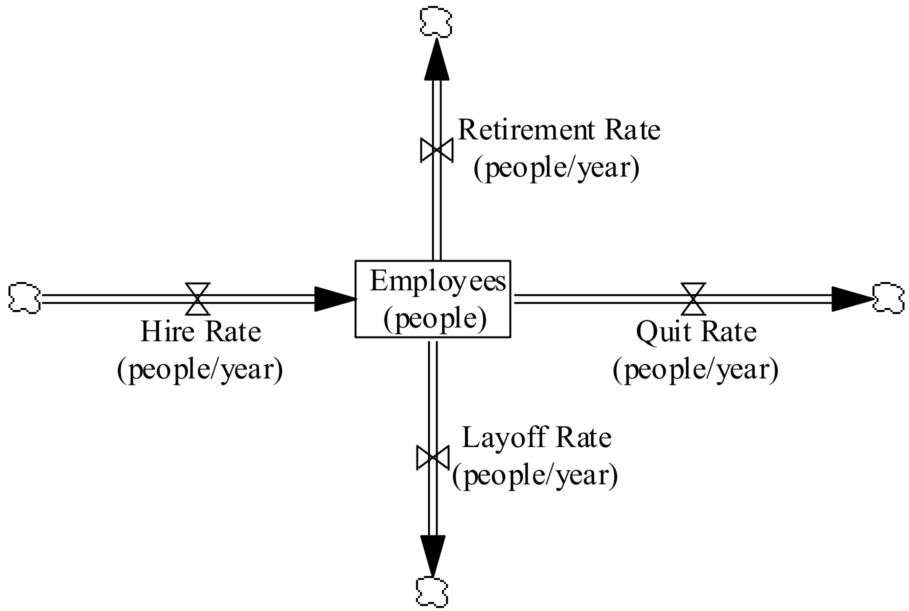
Challenge 7.1

- Represent the stock and flow model as a differential equation

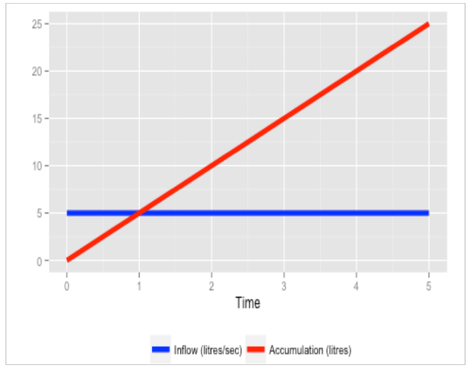


Calculus

- Given the dynamic of the flows, what is the behaviour of the stock?
  – Integration
- From the dynamics of the stock, can you infer the behaviour of the flows
  – Differentiation
- Calculus
  – "quite intuitive… it is the use of unfamiliar notation and a focus on analytic solutions that deters many people from the study of calculus" (Sterman 2000)



Analytical Solution



$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + c$$

$$f(x) = 5\, x^0$$

$$\int 5\, x^0 \, dx = 5 \int x^0 \, dx = \boxed{5x^1 + c}$$

$$\int_0^5 5x^0 dx = 5(5) - 5(0) = 25$$

$$\int_0^{1000} 5x^0 dx = 1000(5) - 5(0) = 5{,}000$$

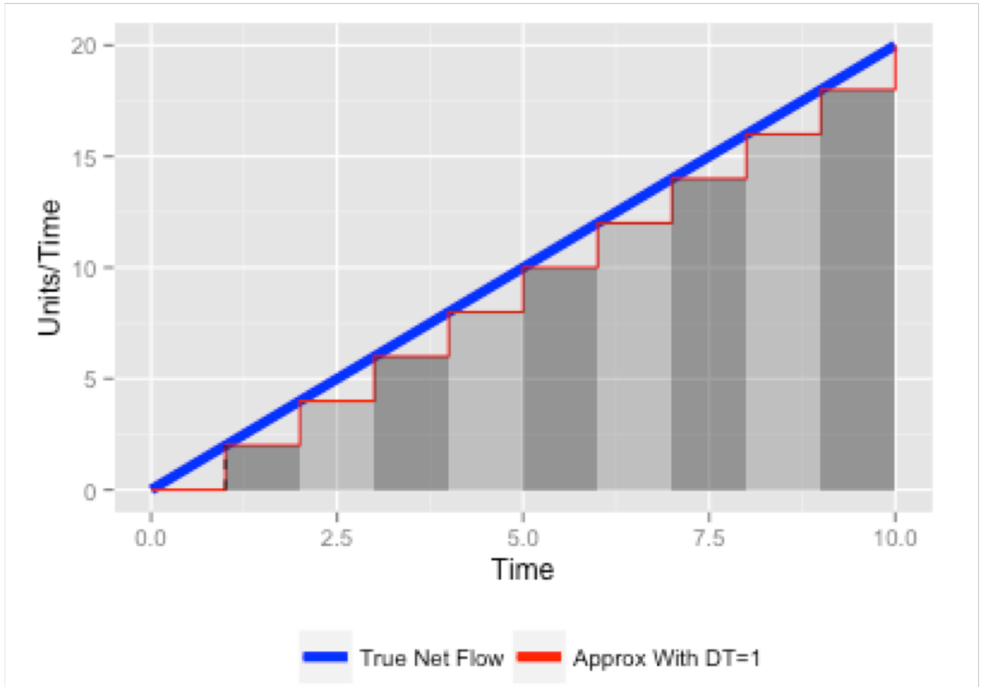
Challenge 7.2

$$\frac{dy}{dt} = 2t$$

- Given this net flow, solve for the value in the stock (integral) after 10 time units (analytical solution).
- Assume the initial stock value is 0

Numerical Integration

- Euler's Method
- Approximate area under the net flow curve as a summation of rectangles, of width DT
- The smaller DT, the more accurate the result



$$S_t = S_{t-dt} + NF_{t-dt} \times DT$$

Challenge 7.3

$$\frac{dy}{dt} = 2t$$

- Given this net flow, solve for the value in the stock (integral) after 10 time units using Euler’s method.
- Use a spreadsheet for the solution
- Assume the initial stock value is 0

MATLAB Provides Functions

[t,y] = ode23(odefun,tspan,y0), where tspan = [t0 tf]

integrates the system of differential equations y'=f(t,y) from t0 to tf with initial conditions y0.

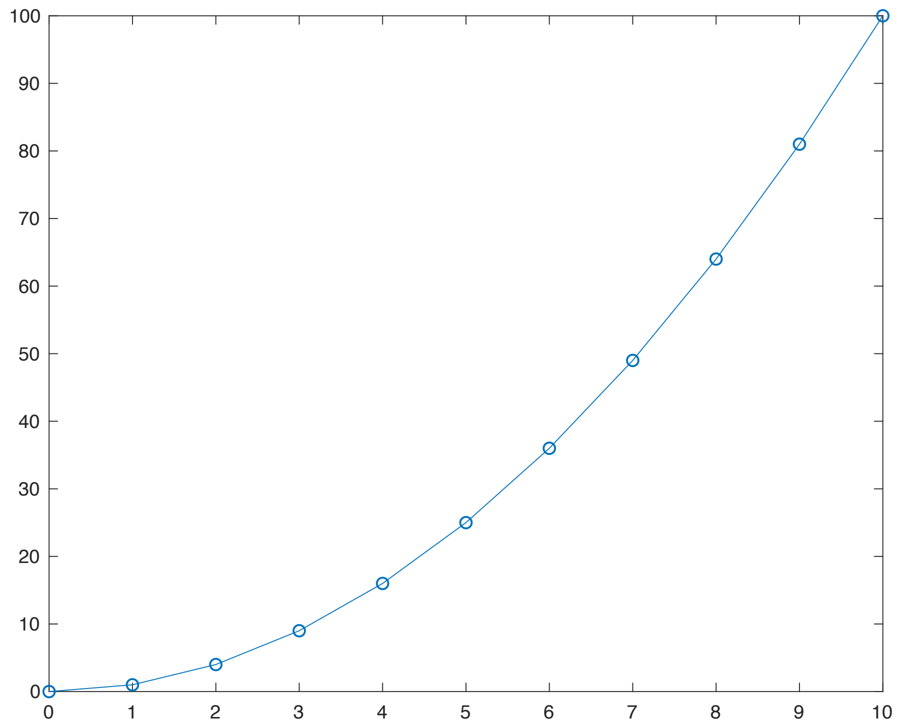
Each row in the solution array y corresponds to a value returned in column vector t.

An anonymous function can be used for odefun. It should return a column vector of solutions

Example

$$\frac{dy}{dt} = 2t$$

```
f = @(t, x) 2*t;

[t,y] = ode23(f, [0 10], 0);

plot(t, y,'-o');
```



Code Summary

Anonymous Function definition

MATLAB's ode solver

Time vector

```
f = @(t, x) 2*t;

[t,y] = ode23(f, [0 10], 0);

plot(t, y,'-o');
```

Initial conditions

Function that solves equations

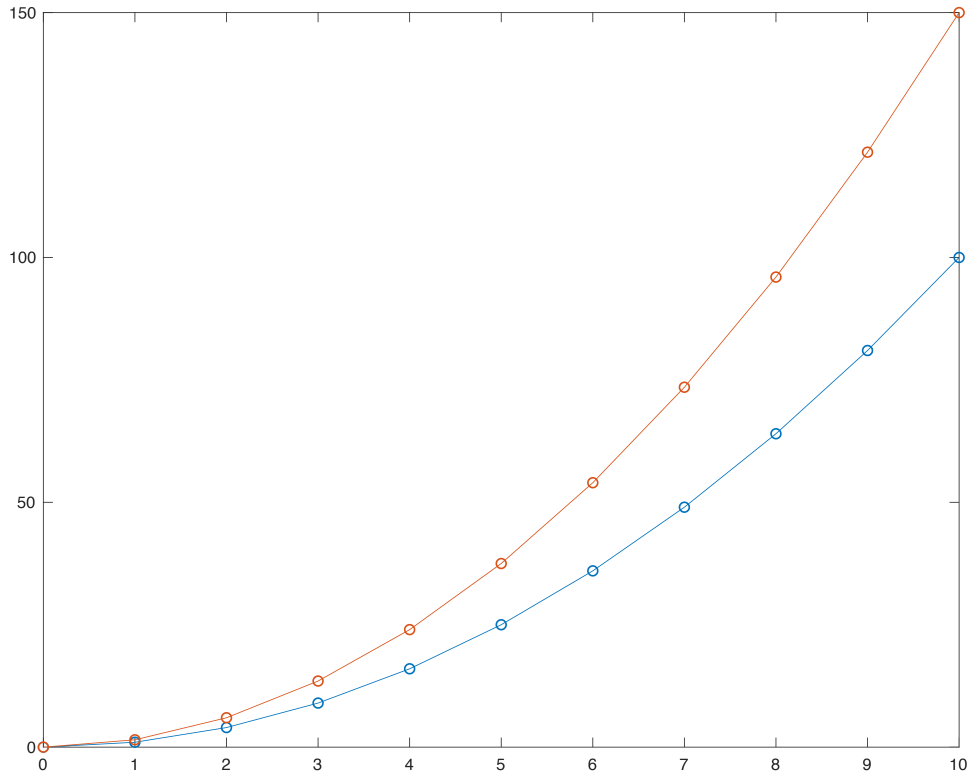
Returns time vector and solution vector

Solving two equations in one function

$$\frac{dy}{dt} = 2t$$

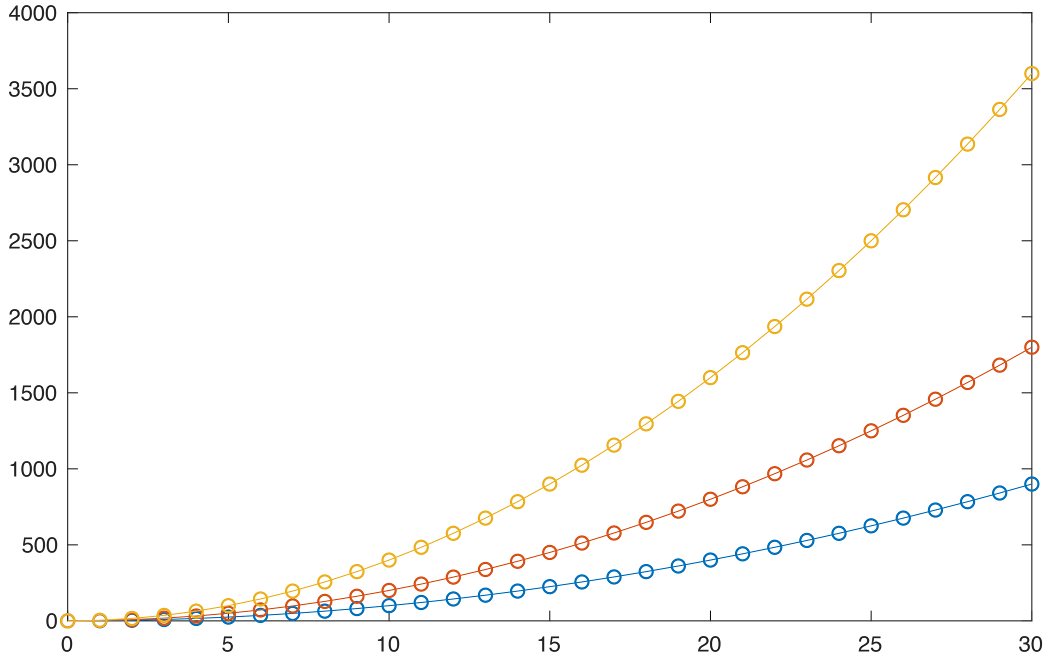
$$\frac{dz}{dt} = 3t$$

```
% must use column vector
% for multiple outputs
f = @(t,x) [2*t; 3*t];

[t,y] = ode23(f, [0 10], [0 0]);

plot(t, y,'-o');
```



Creating a flexible function

```
f = @(t,x,a,b,c) [a*t; b*t; c*t];

[t, y] = ode23(f, [0:30], [0,0,0],odeset(), 2,4,8);

plot(t, y,'-o');
```



Challenge 7.4

- Solve the following equation in MATLAB
- Assume a growth rate of 0.015
- Set the start time = 1960
- Set the finish time = 2010
- Set the initial value = 3 Billion (3e9)

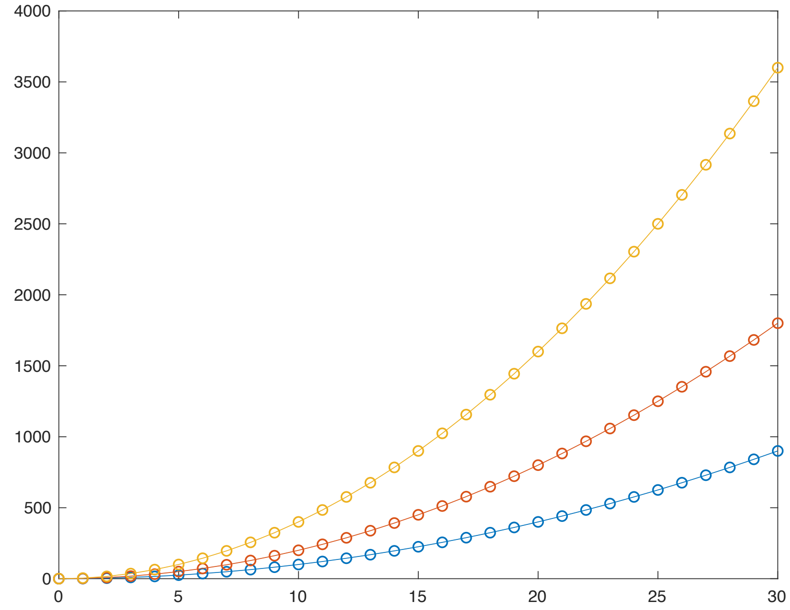
$$\frac{dP}{dt} = rP$$

Note

- Functions can also be implemented as m files
- Useful for a more complex ode model

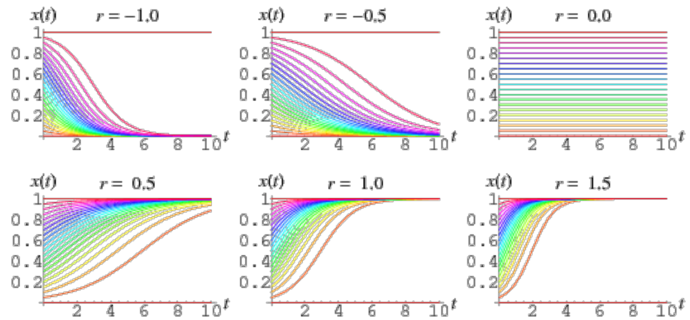
```
clear;

[t, y] = ode23(@ode_test, [0:30], ...
                     [0,0,0], odeset(), 2,4,8);

plot(t, y,'-o');
```

```
function [y] = ode_test(t,x,a,b,c)
   y = [0;0;0];
   y(1) = a*t;
   y(2) = b*t;
   y(3) = c* t;
end
```



Next Steps – Exploring ode Models

- Exponential Growth
- Limits to growth (Verhulst Model)
- Newton's Law of Cooling
- SIR Model



http://mathworld.wolfram.com/LogisticEquation.html

Mathematical models of malaria - a review

Sandip Mandal, Ram Rup Sarkar and Somdatta Sinha*

Newton's Law of Cooling states that the rate of change of the temperature of an object is proportional to the difference between its own temperature and the ambient temperature (i.e. the temperature of its surroundings).