Introduction to Modelling

5. Linear Discrete Dynamical Systems

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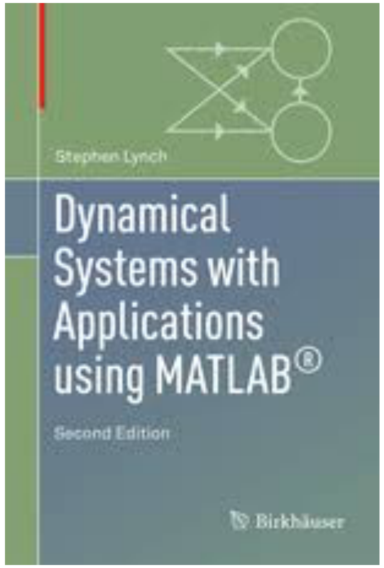
https://github.com/JimDuggan/CT248

First Order Difference Equation

- A recurrence relation can be defined by a difference equation of the form
- The next value is a function of the previous value
- This is known as a first-order difference equation as the suffices differ by one
- Compound interest example

$$x_{n+1} = f(x_n)$$



$$x_{n+1} = \left(1 + \frac{3}{100}\right) x_n$$

Find general solution, and balance after 5 years, with initial deposit = 10000

$$x_{n+1} = \left(1 + \frac{3}{100}\right) x_n$$

$$x_1 = \left(1 + \frac{3}{100}\right) \times 10{,}000 = \left(1 + \frac{3}{100}\right) \times x_0$$

$$x_2 = \left(1 + \frac{3}{100}\right) \times x_1 = \left(1 + \frac{3}{100}\right)^2 \times x_0$$

$$\boldsymbol{x_n = \left(1 + \frac{3}{100}\right)^n \times x_0} \qquad x_5 = \left(1 + \frac{3}{100}\right)^5 \times x_0 = 11{,}592.74$$

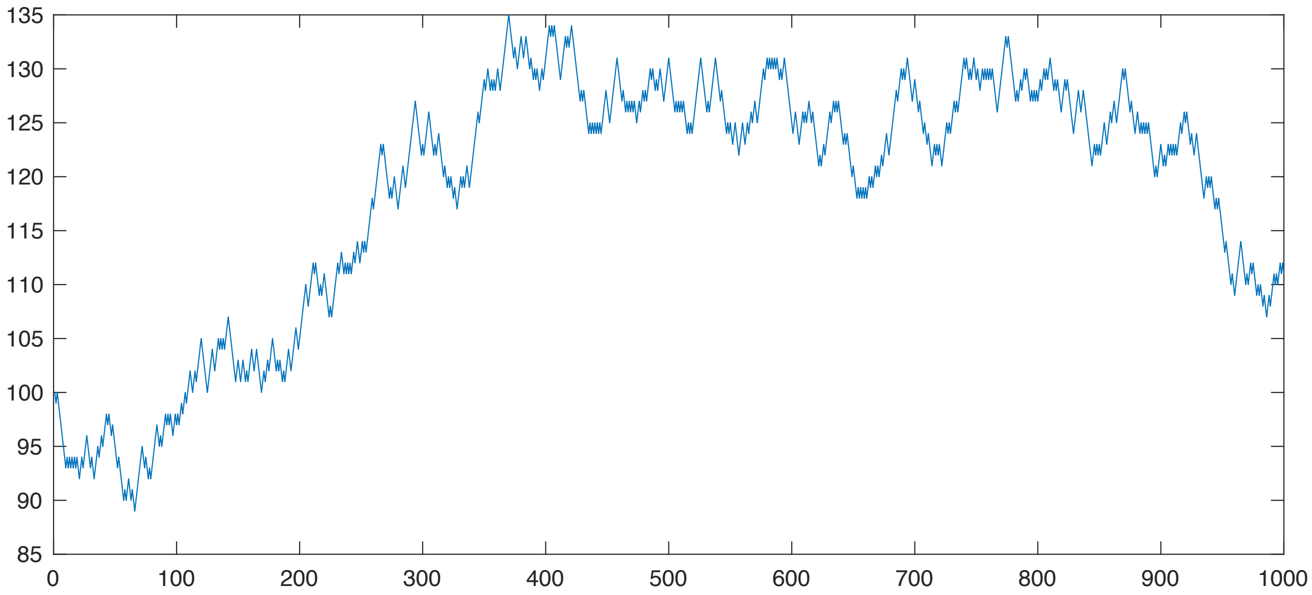
Challenge 5.1

- Write a MATLAB script to model the increase in savings over 30 years, assuming an interest rate of 5%.
- The initial value is 1000

Random Walks (Difference equation)

- Different to a list of random numbers
- The next value in a sequence is a modification of the previous value in the sequence
- Example stock price:
  - Starts at 100
  - Each time unit, a random draw (0,1)
    - <.5 declines by 1
    - >=.5 increases by 1
- $S_{t+1} = S_t + (-1 \text{ or } +1)$

Behaviour over time

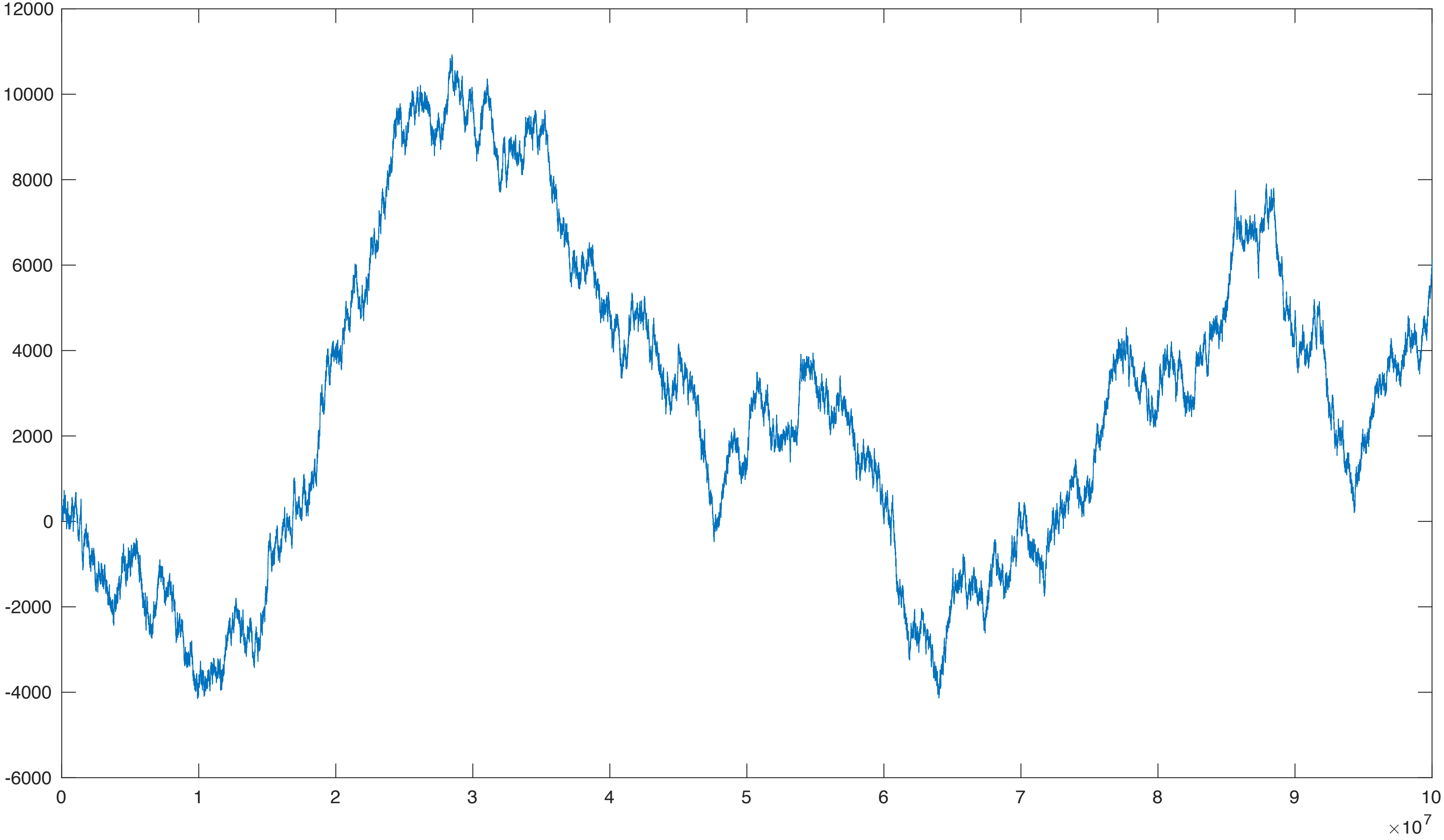


The MATLAB Code

```
clear;
rng(1);


N = 1000;
walk = zeros(N,1);

walk(1) = 100;

for i = 2:N
    walk(i) = walk(i-1) + roll();

end

plot(walk);
```

```
function state = roll()

r = rand();

if r < 0.5
    state = -1;
else
    state = 1;

end
```

One hundred million trials – rng(1)

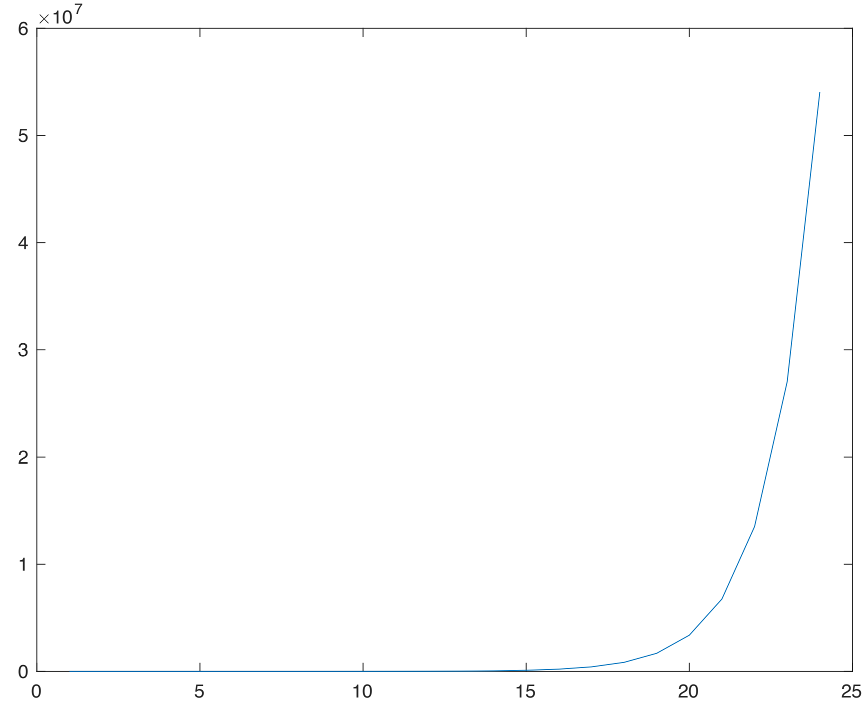


Challenge 5.2

- Modify the random walk code
- Allow the roll() function take a threshold parameter p Run for 100000 trials
- For the first 5000 trials, let p = 0.49
- For the second 5000 trials let p = 0.51
- Plot the result

Application Area: Population Modeling

- Difference Equations
- General idea:
  - Population of interest (different age cohorts)
  - Rules for aging and fertility are specified
  - Matrix multiplication used to solve



Difference Equations & Leslie Matrices

- Suppose we want to model a population of (female) rabbits, in that given their current number, what might the population be in a few years time
- One approach is to divide the population (X_t) into a number of equal-length age classes, e.g young (X_1) , middle-aged (X_2) and old-aged (X_3)
- The population level of these classes can be represented in a 3 x 1 column vector (a state vector)
- Key additional ideas are *survival factor* and *mean fertility*

$$X_t = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

Survival Factor (P_i)

- The proportion of the i^{th} class that survive to the $(i + 1)^{th}$ class (i.e. the proportion that "graduate").
- Y→M = 1/3; M→O = 1/2

From/ To	Young	Middle	Old
Young	0	0	0
Middle	1/3	0	0
Old	0	1/2	0

Mean Fertility (F)

- The mean number of newborn individuals expected to be produced during one time interval by each member of the i^{th} class at the beginning of the interval
- M→Y = 9, O→Y = 12

From/ To	Young	Middle	Old
Young	0	9	12
Middle	0	0	0
Old	0	0	0

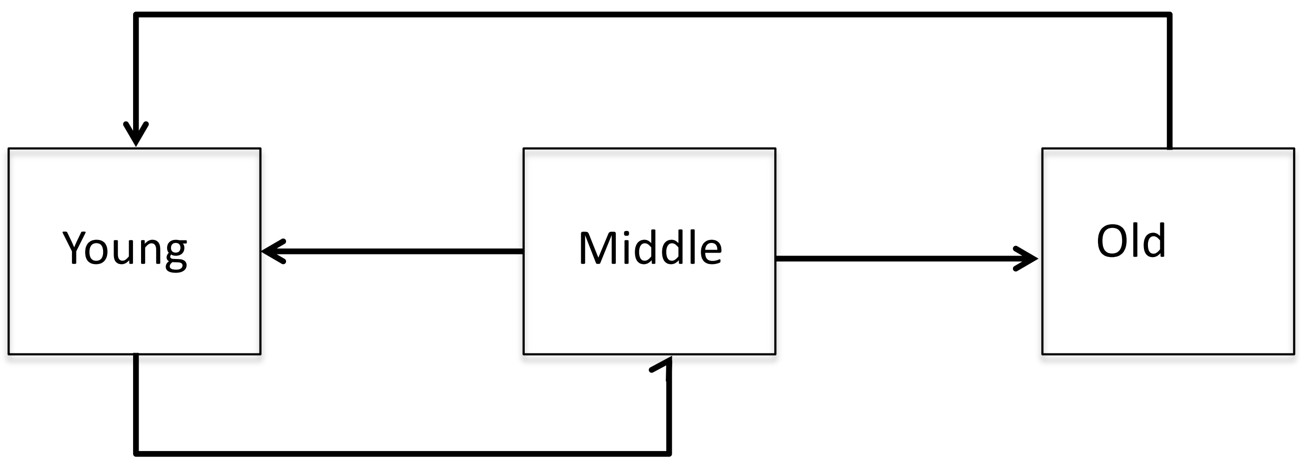
Leslie Matrix: *Combining Survival and Fertility*

From/ To	Young	Middle	Old
Young	0	F_2	F_3
Middle	P_2	0	0
Old	0	P_2	0

From/ To	Young	Middle	Old
Young	0	**9**	**12**
Middle	**1/3**	0	0
Old	0	**1/2**	0

Visualise Leslie Matrix

From/ To	Young	Middle	Old
Young	0	**9**	**12**
Middle	**1/3**	0	0
Old	0	**1/2**	0



$X_{t+1} = L\ X_t$, *Matrix multiplication*

From/ To	Young	Middle	Old
Young	0	F_2	F_3
Middle	P_2	0	0
Old	0	P_2	0

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}(t+1) = \begin{pmatrix} 0 & F_2 & F_3 \\ P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}(t)$$

$$X_1\ (t+1) =\ F_2 X_2\ (t) + F_3 X_3\ (t)$$

$$X_2\ (t+1) =\ P_1 X_1\ (t)$$

$$X_3\ (t+1) =\ P_2 X_2\ (t)$$

Code

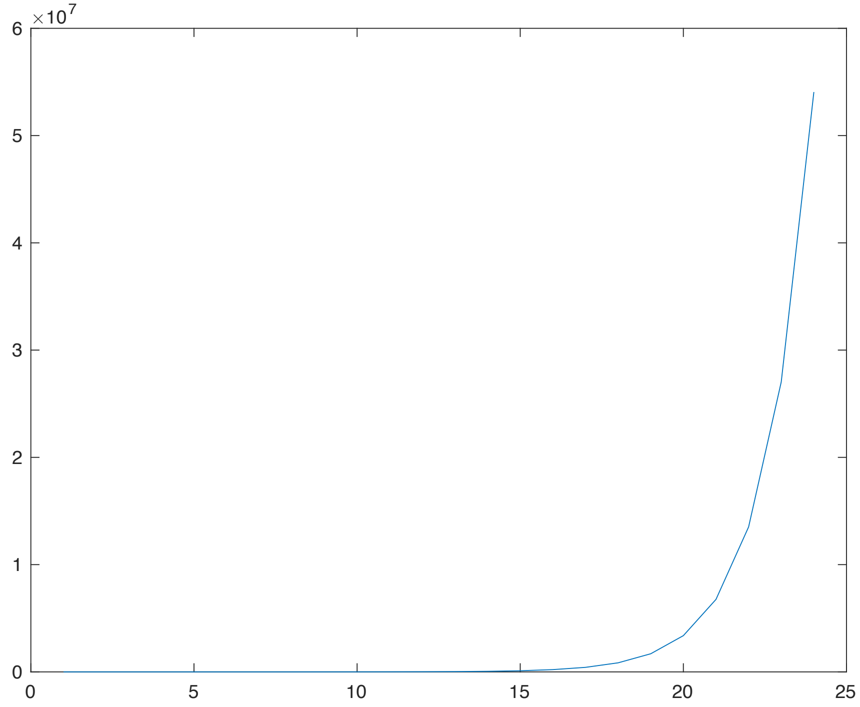
```
L = [0 9 12; 1/3 0 0; 0 1/2 0];

X = [0 0 1]';

for t = 1:24
    X = L * X;
    disp([t X' sum(X)]);
end
```

```
1   12    0    0   12

2    0    4    0    4

3   36    0    2   38

4   24   12    0   36

5  108    8    6  122
```

```
22   11184720   1864164    466020   13514904

23   22369716   3728240    932082   27030038

24   44739144   7456572   1864120   54059836
```

Plotting Results

```
L = [0 9 12; 1/3 0 0; 0 1/2 0];

X = [0 0 1]';

res=zeros(24,2);

for t = 1:24
    X = L * X;
    disp([t X' sum(X)]);
    res(t,:) = [t sum(X)];
end

plot(res(:,1),res(:,2))
```



Challenge 5.1

Consider a species of bird that can be split into three age groupings: 0-1, 1-2 and 2-3. The population is observed once a year.

$$L = \begin{pmatrix} 0 & 3 & 1 \\ 0.3 & 0 & 0 \\ 0 & 0.5 & 0 \end{pmatrix}$$

Given the Leslie matrix (L) and the initial population distribution of females, calculate the number of females in each group after (10), (20) and (30) years.

$$X = \begin{pmatrix} 1000 \\ 2000 \\ 3000 \end{pmatrix}$$

Recap $X_{t+1} = L\ X_t$

From/ To	Young	Middle	Old
Young	0	F_2	F_3
Middle	P_2	0	0
Old	0	P_2	0

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}(t+1) = \begin{pmatrix} 0 & F_2 & F_3 \\ P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}(t)$$

Non-loop solution

$$X_1 = L\,X_0$$

$$X_2 = L\,X_1 = L^2 X_0$$

$$X_k = LX_{k-1} = L^k X_0$$

Given the initial age distribution and the Leslie matrix L, it is possible to determine the female age distribution at any later time interval.

Implementation

```
y  = input ('Please enter the year: ');

L = [0 9 12; 1/3 0 0; 0 1/2 0];

X0 = [0 0 1]';

X = L^y * X0;

disp(['The prediction for year ' num2str(y) '
is...']);
disp(X);
```

```
>> LeslieEx2
Please enter the year: 10
The prediction for year 10 is...
        2688
         468
         108

>> LeslieEx2
Please enter the year: 11
The prediction for year 11 is...
        5508
         896
         234
```

Challenge 5.2

- Consider a human population that is divided into three ages classes: 0-15, 15-30 and 30-45
- Given the Leslie Matrix shown
  - Visualise the state transitions
  - Given the initial population, compute the number of females in each groupings after
    - 5 years
    - 10 years
    - 20 years
  - Estimate the steady-state growth rate of the population

$$L = \begin{pmatrix} 0 & 1 & 0.5 \\ 0.9 & 0 & 0 \\ 0 & 0.8 & 0 \end{pmatrix}$$

$$X = \begin{pmatrix} 10000 \\ 15000 \\ 8000 \end{pmatrix}$$

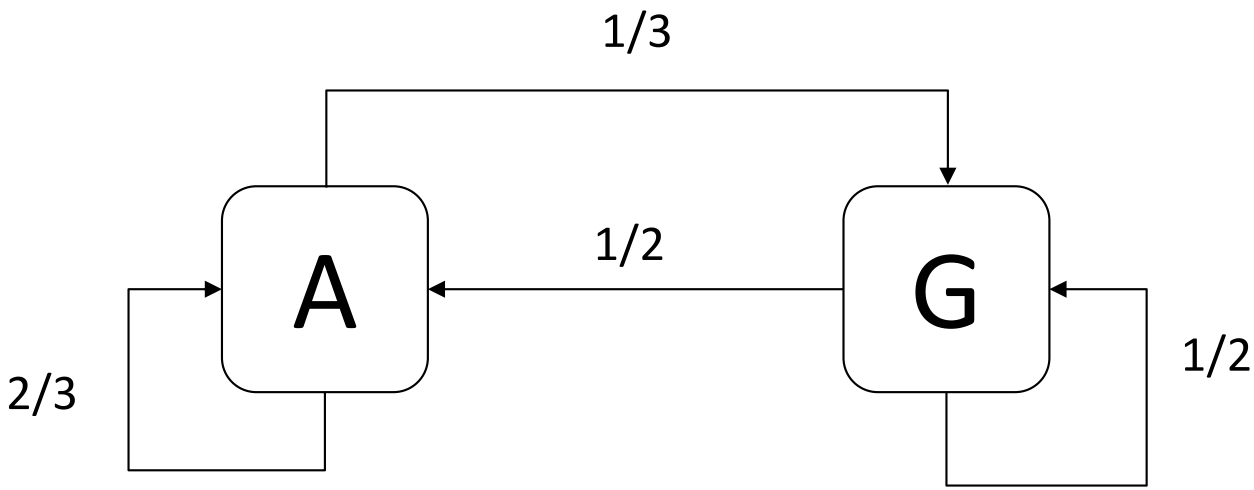
Apple IOS v Android

- Assume there are 50 users for each product (N = 100)
- Assume that after each time cycle:
  - 1/3 of IOS Users switch to Android, and the rest stay
  - 1/2 of Android users switch to Apple IOS, and the rest stay
- Formulate:
  - The state transitions
  - Code to simulate 10 time cycles
  - Find the steady state values
- Represent the customers numbers as proportions of market share
- Discuss the limitations of the model

See http://www.math.harvard.edu/~knill/teaching/math19b_2011/handouts/lecture27.pdf

Overall Equation and Transitions

$$\begin{pmatrix} A \\ G \end{pmatrix}_{t+1} = \begin{pmatrix} 2/3 & 1/2 \\ 1/3 & 1/2 \end{pmatrix} \begin{pmatrix} A \\ G \end{pmatrix}_{t}$$



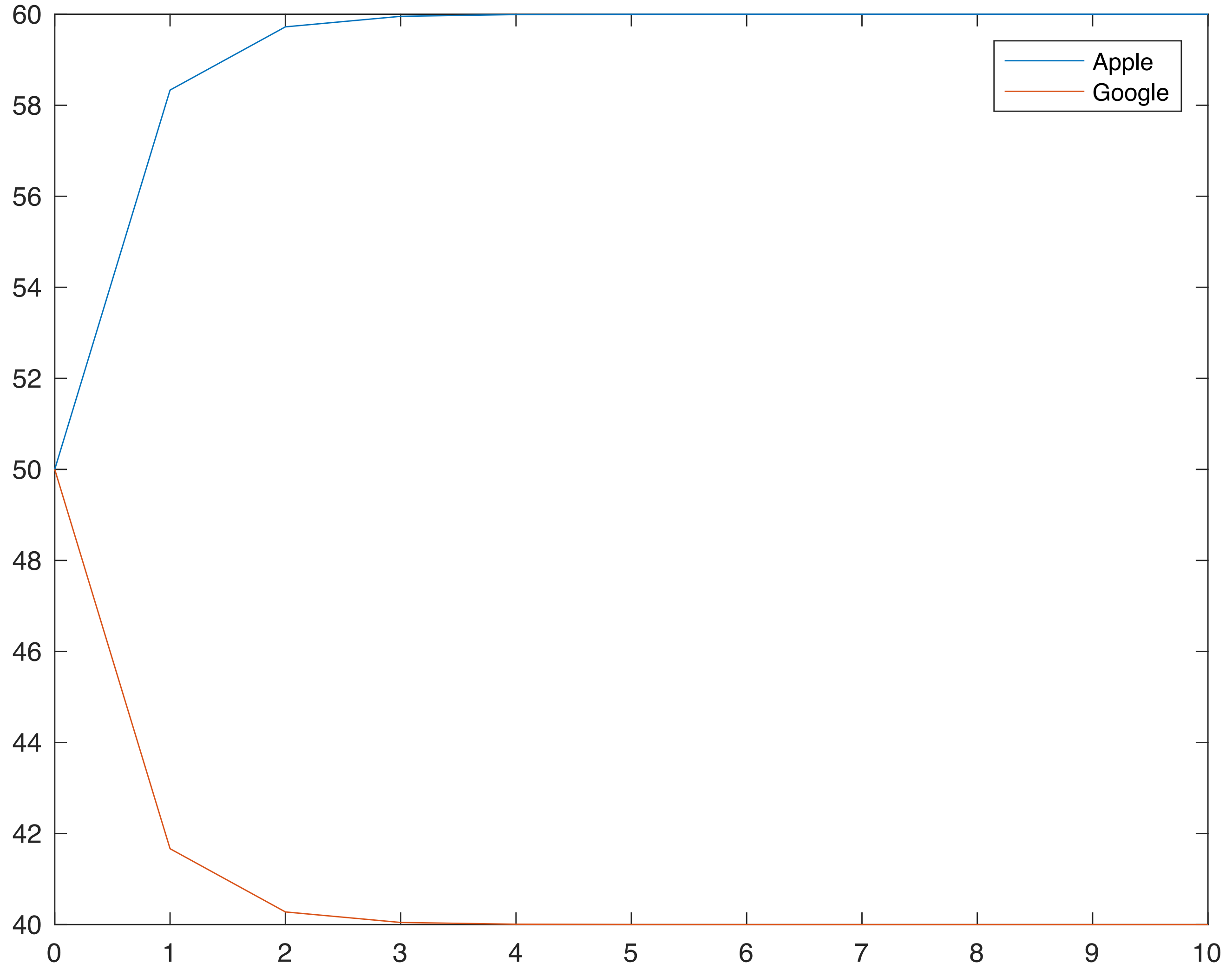
MATLAB Code

$$\begin{pmatrix} A \\ G \end{pmatrix}_{t+1} = \begin{pmatrix} 2/3 & 1/2 \\ 1/3 & 1/2 \end{pmatrix} \begin{pmatrix} A \\ G \end{pmatrix}_t$$

```
C = [50 50]';

A = [2/3 1/2; 1/3 1/2];

N = 10;
out = zeros(N+1,3);
out(:,1) = 0:N;
out(1,2:3) = C;


for i = 2:N+1
    C = A * C;
    out(i,2:3) = C';
end

plot(out(:,1),out(:,2),out(:,1),out(:,3));
legend('Apple','Google')
```

```
out =

        0   50.0000   50.0000
   1.0000   58.3333   41.6667
   2.0000   59.7222   40.2778
   3.0000   59.9537   40.0463
   4.0000   59.9923   40.0077
   5.0000   59.9987   40.0013
   6.0000   59.9998   40.0002
   7.0000   60.0000   40.0000
   8.0000   60.0000   40.0000
   9.0000   60.0000   40.0000
  10.0000   60.0000   40.0000
```

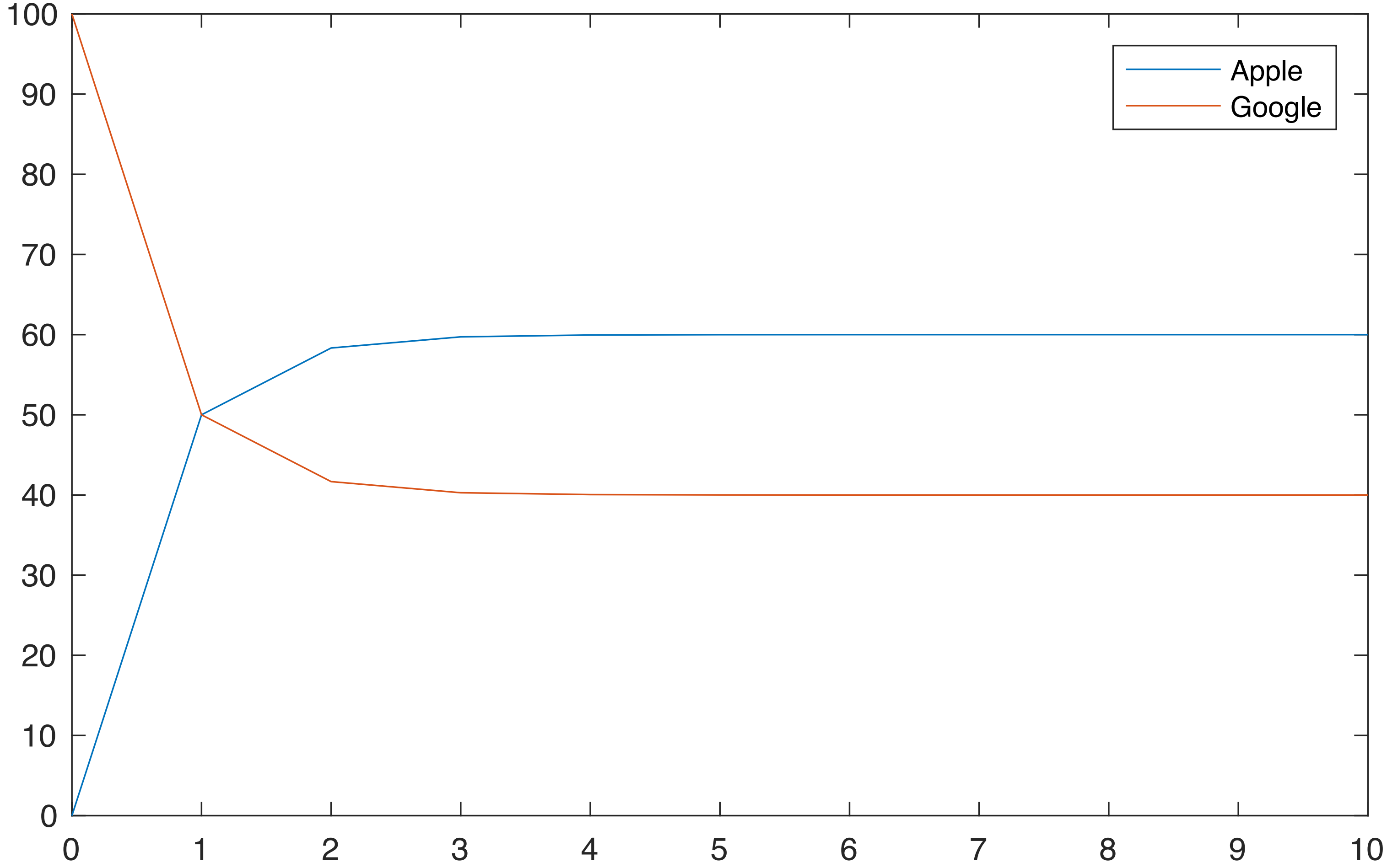
Output



Further scenario

- If everyone stayed at Google (change transition matrix), and 100 started at Apple (initial condition), use the difference equation model to explore what the final steady state would be.

When A = 0 and G = 100 (Initially)



Challenge 5.3

- Extend the model to Windows phones
- Draw the new state transition diagram
- Implement the algorithm to simulate the new marketplace

Migration Scenario

Incl. Footnotes

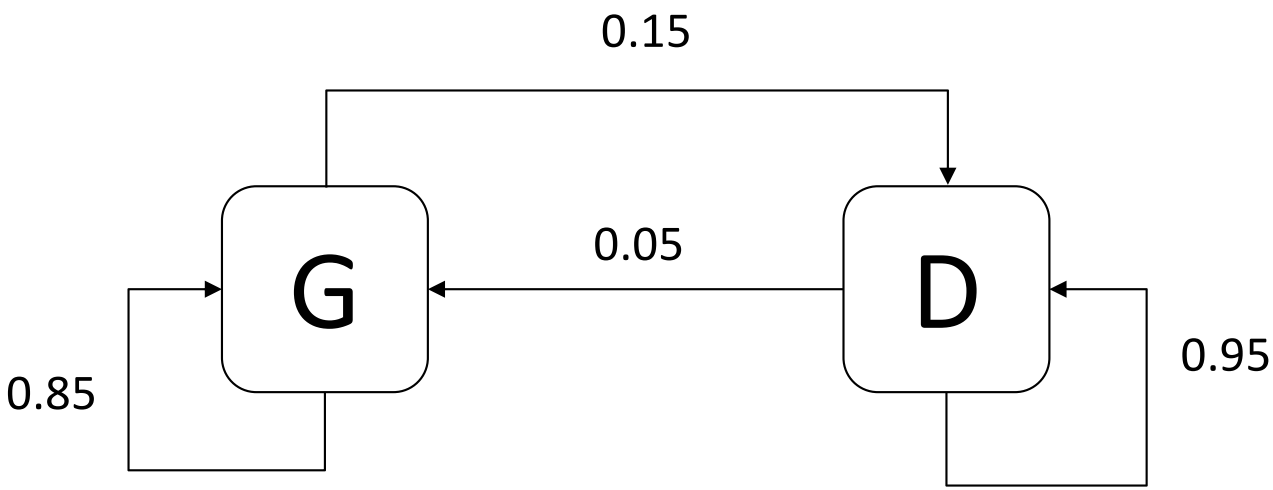
Population 2016 (Number) by Sex, Province County or City and Census Year

	2016
Both sexes	
Dublin	1,345,402
Cork County	416,574
Limerick County	136,856
Galway County	179,048

- Build a population migration model for Dublin and Galway
- Show the state transitions
- Use the model to estimate the population of each city in 20 years (starting from 2016 official numbers)
- Assume that 95% of people in Dublin stay in Dublin, and 85% of people in Galway stay in Galway
- The remainder migrate to the other city.

Overall Equation and Transitions

$$\begin{pmatrix} G \\ D \end{pmatrix}_{t+1} = \begin{pmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{pmatrix} \begin{pmatrix} G \\ D \end{pmatrix}_{t}$$

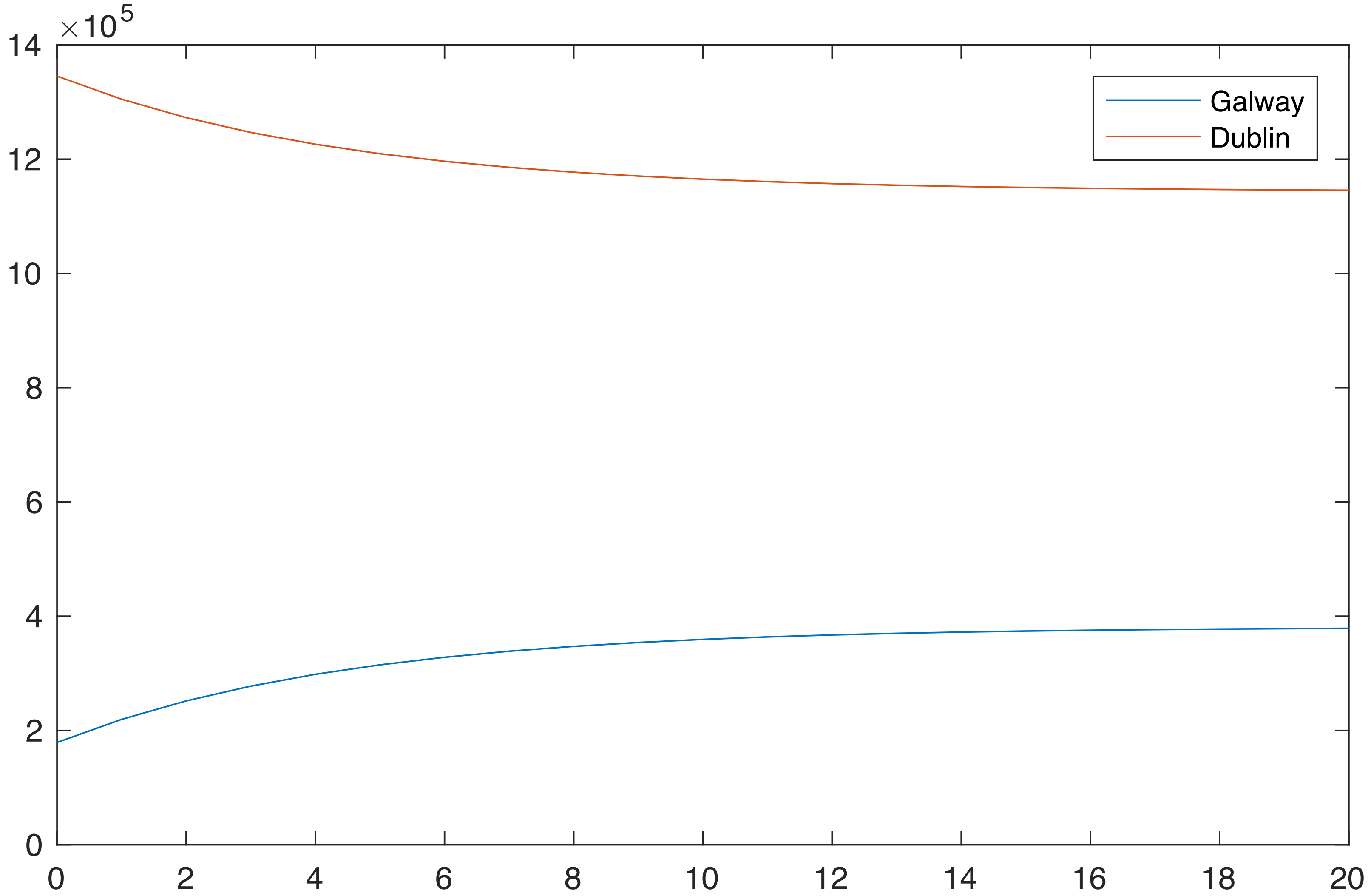


$$\begin{pmatrix} G \\ D \end{pmatrix}_{t+1} = \begin{pmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{pmatrix} \begin{pmatrix} G \\ D \end{pmatrix}_t$$

```
C = [179048 1345402]';

A = [0.85 0.05; 0.15 0.95];

N = 20;
out = zeros(N+1,3);
out(:,1) = 0:N;
out(1,2:3) = C;


for i = 2:N+1
   C = A * C;
   out(i,2:3) = C';
end

plot(out(:,1),out(:,2),out(:,1),out(:,3));
legend('Galway','Dublin')
```

```
>> out
         0    179048.00    1345402.00
      1.00    219460.90    1304989.10
      2.00    251791.22    1272658.78
      3.00    277655.48    1246794.52
      4.00    298346.88    1226103.12
      5.00    314900.00    1209550.00
      6.00    328142.50    1196307.50
      7.00    338736.50    1185713.50
      8.00    347211.70    1177238.30
      9.00    353991.86    1170458.14
     10.00    359415.99    1165034.01
     11.00    363755.29    1160694.71
     12.00    367226.73    1157223.27
     13.00    370003.89    1154446.11
     14.00    372225.61    1152224.39
     15.00    374002.99    1150447.01
     16.00    375424.89    1149025.11
     17.00    376562.41    1147887.59
     18.00    377472.43    1146977.57
     19.00    378200.44    1146249.56
     20.00    378782.85    1145667.15
```

Output



Challenge 5.4

- Modify the transition matrix so that for each cycle the matrix values are drawn randomly from a uniform [0,1] distribution, and ensure that the columns still sum to 1.
- How might this change the model output?