WHAT IS ALGEBRAIC ABOUT ALGEBRAIC EFFECTS AND HANDLERS?

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This note recapitulates and expands the contents of a tutorial on the mathematical theory of algebraic effects and handlers which I gave at the Dagstuhl seminar *"Algebraic effect handlers go mainstream"* [4]. It is targeted roughly at the level of a doctoral student with some amount of mathematical training, or at anyone already familiar with algebraic effects and handlers as programming concepts who would like to know what they have to do with algebra.

Our goal is to draw an uninterrupted line of thought between algebra and computational effects. We begin on the mathematical side of things, by reviewing the classic notions of universal algebra: signatures, algebraic theories, and their models. We then generalize and adapt the theory so that it applies to computational effects. In the last step we replace traditional mathematical notation with one that is closer to programming languages.

1. ALGEBRAIC THEORIES

In algebra we study mathematical structures that are equipped with operations satisfying equational laws. For example, a group is a structure $(G, \mathsf{u}, \cdot, {}^{-1})$, where u is a constant, $\cdot$ is a binary operation, and ${}^{-1}$ is a unary operation, satisfying the familiar group identities:

$$(x \cdot y) \cdot z = x \cdot (y \cdot z),$$
$$\mathsf{u} \cdot x = x = x \cdot \mathsf{u},$$
$$x \cdot x^{-1} = \mathsf{u} = x^{-1} \cdot x.$$

There are alternative axiomatizations, for instance: a group is a monoid $(G, \mathsf{u}, \cdot)$ in which every element is invertible, i.e., $\forall x \in G . \exists y \in G . x \cdot y = \mathsf{u} = y \cdot x$. However, a formulation all of whose axioms are equations is preferred, because its simple logical form grants its models good structural properties.

It is important to distinguish the theory of an algebraic structure from the algebraic structures it describes. In this section we shall study the descriptions, which are known as *algebraic* or *equational theories*.

1.1. **Signatures, terms and equations.** A *signature* Σ is a collection of *operation symbols* with *arities* $\{(\mathsf{op}_i, \mathsf{ar}_i)\}_i$. The operation symbols op_i may be any anything, but are usually thought of as syntactic entities, while arities ar_i are non-negative integers. An operation symbol whose arity is 0 is called a *constant* or a *nullary* symbol. Operation symbols with arities 1, 2 and 3 are referred to as *unary*, *binary*, and *ternary*, respectively.

A (possibly empty) list of distinct variables $x_1, \ldots, x_k$ is called a *context*. The Σ-*terms in context* $x_1, \ldots, x_k$ are built inductively using the following rules:

(1) each variable x_i is a Σ-term in context $x_1, \ldots, x_k$,
(2) if $t_1, \ldots, t_{\mathsf{ar}_i}$ are Σ-terms in context $x_1, \ldots, x_k$ then $\mathsf{op}_i(t_1, \ldots, t_{\mathsf{ar}_i})$ is a Σ-term in context $x_1, \ldots, x_k$.

We write

$$x_1, \ldots, x_k \mid t$$

to indicate that t is a Σ-term in the given context. A *closed Σ-term* is a Σ-term in the empty context. No variables occur in a closed term.

A Σ-*equation* is a pair of Σ-terms ℓ and r in a context $x_1, \ldots, x_k$. We write

$$x_1, \ldots, x_k \mid \ell = r$$

to indicate an equation in a context. We shall often elide the context and write simply $\ell = r$, but it should be understood that there is an ambient context which contains at least all the variables mentioned by ℓ and r.

A Σ-equation really is just a list of variables and a pair of terms, and *not* a logical statement. The context variables are *not* universally quantified, and we are not talking about first-order logic. Of course, a Σ-equation is suggestively written as an equation because we do eventually want to *interpret* it as an assertion of equality, but until such time (and even afterwards) it is better to think of contexts, terms, and equations as ordinary mathematical objects, devoid of any imagined or special meta-mathematical status. This remark will hopefully become clearer in Section 1.9.

When no confusion can arise we drop the prefix "Σ-" and simply speak about terms and equations instead of Σ-terms and Σ-equations.

Example 1.1. The signature for the theory of a monoid has a nullary symbol u and a binary symbol m. There are infinitely many expressions in context x, y, such as

$$\mathsf{u}(), \quad x, \quad y, \quad \mathsf{m}(\mathsf{u}(), \mathsf{u}()), \quad \mathsf{m}(\mathsf{u}(), x), \quad \mathsf{m}(y, \mathsf{u}()), \quad \mathsf{m}(x, x), \quad \mathsf{m}(y, x), \ldots$$

An equation in context x, y is

$$x, y \mid \mathsf{m}(y, x) = \mathsf{m}(\mathsf{m}(\mathsf{u}(), x), y).$$

It is customary to write a nullary symbol $\mathsf{u}()$ simply as u, and to use the infix operator $\cdot$ in place of m. With such notation the above equation would be written as

$$x, y \mid y \cdot x = (\mathsf{u} \cdot x) \cdot y.$$

One might even omit $\cdot$ and the context, in which case the equation is written simply as $y\,x = (\mathsf{u}\,x)\,y$. If we agree that $\cdot$ associates to the left then $(\mathsf{u}\,x)\,y$ may be written as $\mathsf{u}\,x\,y$, and we are left with $y\,x = \mathsf{u}\,x\,y$, which is what your algebra professor might write down. Note that we are *not* discussing validity of equations but only ways of displaying them.

1.2. **Algebraic theories.** An *algebraic theory* $\mathsf{T} = (\Sigma_\mathsf{T}, \mathcal{E}_\mathsf{T})$, also called an *equational theory*, is given by a signature Σ_T and a collection $\mathcal{E}_\mathsf{T}$ of Σ_T-equations. We impose no restrictions on the number of operation symbols or equations, but at least in classical treatments of the subject certain complications are avoided by insisting that arities be nonnegative integers.

Example 1.2. The theory Group of a group is algebraic. In order to follow closely the definitions we eschew the traditional notation $\cdot$ and $^{-1}$, and explicitly display the contexts.

We abide by such formalistic requirements once to demonstrate them, but shall take notational liberties subsequently. The signature Σ_{Group} is given by operation symbols u, m, and i whose arities are 0, 2, and 1, respectively. The equations $\mathcal{E}_{\mathsf{Group}}$ are:

$$x, y, z \mid \mathsf{m}(\mathsf{m}(x, y), z) = \mathsf{m}(x, \mathsf{m}(y, z)),$$
$$x \mid \mathsf{m}(\mathsf{u}(), x) = x$$
$$x \mid \mathsf{m}(x, \mathsf{u}()) = x,$$
$$x \mid \mathsf{m}(x, \mathsf{i}(x)) = \mathsf{u}()$$
$$x \mid \mathsf{m}(\mathsf{i}(x), x) = \mathsf{u}().$$

Example 1.3. The theory Semilattice of a semilattice is algebraic. It is given by a nullary symbol $\bot$ and a binary symbol $\vee$, satisfying the equations

$$x \vee (y \vee z) = (x \vee y) \vee z,$$
$$x \vee y = y \vee x,$$
$$x \vee x = x,$$
$$x \vee \bot = x.$$

It should be clear that the first equation has context x, y, z, the second one in x, y, and the last two in x.

Example 1.4. The theory of a field, as usually given, is not algebraic because the inverse 0^{-1} is undefined, whereas the operations of an algebraic theory are always taken to be total. However, a proof is required to show that there is no equivalent algebraic theory.

Example 1.5. The theory $\mathsf{Set}_\bullet$ of a *pointed set* has a constant $\bullet$ and no equations.

Example 1.6. The *empty theory* Empty has no operation symbols and no equations.

Example 1.7. The theory of a *singleton* Singleton has a constant $\star$ and the equation $x = y$.

Example 1.8. A bounded lattice is a partial order with finite infima and suprema. Such a formulation is not algebraic because the infimum and supremum operators do not have fixed arities, but we can reformulate it in terms of nullary and binary operations. Thus, the theory Lattice of a bounded lattice has constants $\bot$ and $\top$, and two binary operation symbols $\vee$ and $\wedge$, satisfying the equations:

$$
\begin{aligned}
x \vee (y \vee z) = (x \vee y) \vee z, && x \wedge (y \wedge z) = (x \wedge y) \wedge z, \\
x \vee y = y \vee x, && x \wedge y = y \wedge x, \\
x \vee x = x, && x \wedge x = x, \\
x \vee \bot = x, && x \wedge \top = x.
\end{aligned}
$$

In addition we need the *absorbtion* laws:

$$x \vee (x \wedge y) = x, \qquad\qquad x \wedge (x \vee y) = x.$$

Notice that the theory of a bounded lattice is the juxtaposition of two copies of the theory of a semi-lattice from Example 1.3, augmented with laws that relate them. The partial order is recovered because $x \leq y$ is equivalent to $x \vee y = y$ and to $x \wedge y = x$.

Example 1.9. A *finitely generated group* is a group which contains a finite collection of elements, called the *generators*, such that every element of the group is obtained by multiplications and inverses of the generators. It is not clear how to express this condition

using only equations, but a proof is required to show that there is no equivalent algebraic theory.

Example 1.10. An example of an algebraic theory with many operations and equations is the theory of a $\mathcal{C}^\infty$-ring. Let $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^m)$ be the set of all smooth maps from $\mathbb{R}^n$ to $\mathbb{R}^m$. The signature for the theory of a $\mathcal{C}^\infty$-ring contains an n-ary operation symbol op_f for each $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$. For all $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$, $h \in \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{R})$, and $g_1, \ldots, g_n \in \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{R})$ such that

$$f \circ (g_1, \ldots, g_n) = h,$$

the theory has the equation

$$x_1, \ldots, x_m \mid \mathsf{op}_f(\mathsf{op}_{g_1}(x_1, \ldots, x_m), \ldots, \mathsf{op}_{g_n}(x_1, \ldots, x_m)) = \mathsf{op}_h(x_1, \ldots, x_m).$$

The theory contains the theory of a commutative unital ring as a subtheory. Indeed, the ring operations on $\mathbb{R}$ are smooth maps, and so they appear as op_+, $\mathsf{op}_\times$, op_- in the signature, and so do constants op_0 and op_1, because all maps $\mathbb{R}^0 \to \mathbb{R}$ are smooth. The commutative ring equations are present as well because the real numbers form a commutative ring.

1.3. **Interpretations of signatures.** Let a signature Σ be given. An *interpretation I* of Σ is given by the following data:

   (1) a set $|I|$, called the *carrier*,
   (2) for each operation symbol op_i a map

$$[\![\mathsf{op}_i]\!]_I : \underbrace{|I| \times \cdots \times |I|}_{\mathsf{ar}_i} \to |I|,$$

   called an *operation*.

The double bracket $[\![\ ]\!]_I$ is called the *semantic bracket* and is typically used when syntactic entities (operation symbols, terms, equations) are mapped to their mathematical counterparts. When no confusion can arise, we omit the subscript I and write just $[\![\ ]\!]$.

   We abbreviate an n-ary product $|I| \times \cdots \times |I|$ as $|I|^n$. A nullary product $|I|^0$ contains a single element, namely the empty tuple (), so it makes sense to write $|I|^0 = 1 = \{()\}$. Thus a nullary operation symbol is interpreted by an map $1 \to |I|$, and such maps are in bijective correspondence with the elements of $|I|$, which would be the constants.

   An interpretation I may be extended to Σ-terms. A Σ-term in context

$$x_1, \ldots, x_k \mid t$$

is interpreted by a map

$$[\![x_1, \ldots, x_k \mid t]\!]_I : |I|^k \to |I|,$$

as follows:

   (1) the variable x_i is interpreted as the i-th projection,

$$[\![x_1, \ldots, x_k \mid x_i]\!]_I = \pi_i : |I|^k \to |I|,$$

   (2) a compound term in context

$$x_1, \ldots, x_k \mid \mathsf{op}_i(t_1, \ldots, t_{\mathsf{ar}_i})$$

   is interpreted as the composition of maps

$$|I|^k \xrightarrow{\ ([\![t_1]\!]_I, \ldots, [\![t_{\mathsf{ar}_i}]\!]_I)\ } |I|^{\mathsf{ar}_i} \xrightarrow{\ [\![\mathsf{op}_i]\!]_I\ } |I|$$

   where we elided the contexts $x_1, \ldots, x_k$ for the sake of brevity.

Example 1.11. One interpretation of the signature from Example 1.1 is given by the carrier set $\mathbb{R}$ and the interpretations of operation symbols

$$[\![\mathsf{u}]\!]() = 1 + \sqrt{5},$$
$$[\![\mathsf{m}]\!](a, b) = a^2 + b^3.$$

The term in context $x, y \mid \mathsf{m}(\mathsf{u}, \mathsf{m}(x, x))$ is interpreted as the map $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$, given by the rule

$$(a, b) \mapsto (a + 1)^3 a^6 + 2(3 + \sqrt{5}).$$

The same term in a context y, x, z is interpreted as the map $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, given by the rule

$$(a, b, c) \mapsto (b + 1)^3 b^6 + 2(3 + \sqrt{5}).$$

These are not the same map, as they do not even have the same domains!

The previous examples shows why contexts should not be ignored. In mathematical practice contexts are often relegated to guesswork for the reader, or are handled implicitly. For example, in real algebraic geometry the solution set of the equation $x^2 + y^2 = 1$ is either a unit circle in the plane or an infinitely extending cylinder of unit radius in the space, depending on whether the context might be x, y or x, y, z. Which context is meant is indicated one way or another by the author of the mathematical text.

1.4. **Models of algebraic theories.** A *model* M of an algebraic theory T is an interpretation of the signature Σ_T which validates all the equations $\mathcal{E}_\mathsf{T}$. That is, for every equation

$$x_1, \ldots, x_k \mid \ell = r$$

in $\mathcal{E}_\mathsf{T}$, the maps

$$[\![x_1, \ldots, x_k \mid \ell]\!]_M : |M|^k \to |M| \qquad \text{and} \qquad [\![x_1, \ldots, x_k \mid r]\!]_M : |M|^k \to |M|$$

are equal. We refer to a model of T as a T-*model* or a T-*algebra*.

Example 1.12. A model G of Group, cf. Example 1.2, is given by a carrier set $|G|$ and maps

$$[\![u]\!]_G : 1 \to |G|, \qquad [\![m]\!]_G : |G| \times |G| \to |G|, \qquad [\![i]\!]_G : |G| \to |G|,$$

interpreting the operation symbols u, m, and i, respectively, such that the equations $\mathcal{E}_{\mathsf{Group}}$. This amounts precisely to $(|G|, [\![u]\!]_G, [\![m]\!]_G, [\![i]\!]_G)$ being a group, except that the unit is viewed as a map $1 \to |G|$ instead of an element of $|G|$.

Example 1.13. Every algebraic theory has the *trivial model*, whose carrier is the singleton 1, and whose operations are interpreted by the unique maps $1^k \to 1$. All equations are satisfied because any two maps $1^k \to 1$ are equal.

The previous example explains why one should *not* require $0 \neq 1$ in a ring, as that prevents the theory of a ring from being algebraic.

Example 1.14. The empty set is a model of a theory T if, and only if, every operation symbol of T has non-zero arity.

Example 1.15. A model of the theory $\mathsf{Set}_\bullet$ of a pointed set, cf. Example 1.5, is a set S together with an element $s \in S$ which interprets the constant $\bullet$.

Example 1.16. A model of the theory Empty, cf. Example 1.6, is the same thing as a set.

Example 1.17. A model of the theory Singleton, cf. Example 1.7, is any set with precisely one element.

Suppose L and M are models of a theory T. Then we may form the *product of models* $L \times M$ by taking the cartesian product as the carrier,

$$|L \times M| = |L| \times |M|,$$

and pointwise operations,

$$[\![\mathsf{op}_i]\!]_{M \times L}(a, b) = ([\![\mathsf{op}_i]\!]_M(a), [\![\mathsf{op}_i]\!]_L(b)).$$

The equations $\mathcal{E}_\mathsf{T}$ are valid in $L \times M$ because they are valid on each coordinate separately. This construction can be extended to a product of any number of models, including an infinite one.

Example 1.18. We may now prove that the theory of a field from Example 1.4 is not equivalent to an algebraic theory. There are fields of size 2 and 3, namely $\mathbb{Z}_2$ and $\mathbb{Z}_3$. If there were an algebraic theory of a field, then $\mathbb{Z}_2 \times \mathbb{Z}_3$ would be a field too, but it is not, and in fact there is no field of size 6.

Example 1.19. Similarly, the theory of a finitely generated group from Example 1.9 cannot be formulated as an algebraic theory, because an infinite product of non-trivial finitely generated groups is not finitely generated.

Example 1.20. Let us give a model of the theory of a $\mathcal{C}^\infty$-ring from Example 1.10. Pick a smooth manifold M, and let the carrier be the set $\mathcal{C}^\infty(M, \mathbb{R})$ of all smooth scalar fields on M. Given $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$, interpret the operation op_f as composition with f,

$$[\![\mathsf{op}_f]\!] : \mathcal{C}^\infty(M, \mathbb{R})^n \to \mathcal{C}^\infty(M, \mathbb{R})$$
$$[\![\mathsf{op}_f]\!] : (u_1, \ldots, u_n) \mapsto f \circ (u_1, \ldots, u_n).$$

We leave it as an exercise to verify that all equations are validated by this interpretation.

1.5. **Homomorphisms and the category of models.** Suppose L and M are models of a theory T. A T-*homomorphism* from L to M is a map $\phi : |L| \to |M|$ between the carriers which commutes with operations: for every operation symbol op_i of T, we have

$$\phi \circ [\![\mathsf{op}_i]\!]_L = [\![\mathsf{op}_i]\!]_M \circ \underbrace{(\phi, \ldots, \phi)}_{\mathsf{ar}_i}.$$

Example 1.21. A homomorphism between groups G and H is a map $\phi : |G| \to |H|$ between the carriers such that, for all $a, b \in |G|$,

$$\phi([\![\mathsf{u}()]\!]_G) = [\![\mathsf{u}()]\!]_H,$$
$$\phi([\![\mathsf{m}]\!]_G(a, b)) = [\![\mathsf{m}]\!]_H(\phi(a), \phi(b)),$$
$$\phi([\![\mathsf{i}]\!]_G(a)) = [\![\mathsf{i}]\!]_H(\phi(a)).$$

This is a convoluted way of saying that the unit maps to the unit, and that ϕ commutes with the group operation and the inverses. Algebra textbooks usually require only that a group homomorphism commute with the group operation, which then implies that it also preserves the unit and commutes with the inverse.

We may organize the models of an algebraic theory T into a category $\mathbf{Mod}(\mathsf{T})$ whose objects are the models of the theory, and whose morphisms are homomorphisms of the theory.

Example 1.22. The category of models of theory Group, cf. Example 1.2, is the usual category of groups and group homomorphisms.

Example 1.23. The category of models of the theory Set$_\bullet$, cf. Example 1.5, has as its objects the pointed sets, which are pairs (S, s) with S a set and $s \in S$ its *point*, and as homomorphisms the point-preserving functions between sets.

Example 1.24. The category of models of the empty theory Empty, cf. Example 1.6, is just the category **Set** of sets and functions.

Example 1.25. The category of models of the theory of a singleton Singleton, cf. Example 1.7, is the category whose objects are all the singleton sets. There is precisely one morphisms between any two of them. This category is equivalent to the trivial category which has just one object and one morphism.

1.6. **Models in a category.** So far we have taken the models of an algebraic theory to be sets. More generally, we may consider models in any category **C** with finite products. Indeed, the definitions of an interpretation and a model from Sections 1.3 and 1.4 may be directly transcribed so that they apply to **C**. An interpretation I in **C** is given by

(1) an object $|I|$ in **C**, called the *carrier*,
(2) for each operation symbol op_i a morphism in **C**

$$\llbracket \mathsf{op}_i \rrbracket_I : \underbrace{|I| \times \cdots \times |I|}_{\mathsf{ar}_i} \to |I|.$$

Once again, we abbreviate the k-fold product of $|I|$ as $|I|^k$. Notice that a nullary symbol is interpreted as a morphism $|I|^0 \to |I|$, which is a morphisms from the terminal object $1 \to |I|$ in **C**.

An interpretation I is extended to Σ-terms in contexts as follows:

(1) the variable $x_1, \ldots, x_k \mid x_i$ is interpreted as the i-th projection,

$$\llbracket x_1, \ldots, x_k \mid x_i \rrbracket_I = \pi_i : |I|^k \to |I|,$$

(2) a compound term in context

$$x_1, \ldots, x_k \mid \mathsf{op}_i(t_1, \ldots, t_{\mathsf{ar}_i})$$

is interpreted as the composition of morphisms

$$|I|^k \xrightarrow{(\llbracket t_1 \rrbracket_I, \ldots, \llbracket t_{\mathsf{ar}_i} \rrbracket_I)} |I|^{\mathsf{ar}_i} \xrightarrow{\llbracket \mathsf{op}_i \rrbracket_I} |I|$$

A model of an algebraic theory T in **C** is an interpretation M of its signature Σ_T which validates all the equations. That is, for every equation

$$x_1, \ldots, x_k \mid \ell = r$$

in $\mathcal{E}_\mathsf{T}$, the morphisms

$$\llbracket x_1, \ldots, x_k \mid \ell \rrbracket_M : |M|^k \to |M| \qquad \text{and} \qquad \llbracket x_1, \ldots, x_k \mid r \rrbracket_M : |M|^k \to |M|$$

are equal.

The definition of a homomorphism carries over to the general setting as well. A T-*homomorphism* between T-models L and M in a category **C** is a morphism $\phi : |L| \to |M|$ in **C** such that, for every operation symbol op_i in T, ϕ commutes with the interpretation of op_i,

$$\phi \circ \llbracket \mathsf{op}_i \rrbracket_L = \llbracket \mathsf{op}_i \rrbracket_M \circ \underbrace{(\phi, \ldots, \phi)}_{\mathsf{ar}_i}.$$

The T-models and T-homomorphisms in a category **C** form a category $\mathbf{Mod_C}(\mathsf{T})$.

Example 1.26. A model of the theory Group in the category **Top** of topological spaces and continuous maps is a topological group.

Example 1.27. What is a model of the theory Group in the category of groups **Grp**? Its carrier is a group (G, u, m, i) together with group homomorphisms $\upsilon : 1 \to G$, $\mu : G \times G \to G$, and $\iota : G \to G$ which satisfy the group laws. Because υ is a group homomorphism, it maps the unit of the trivial group 1 to u, so the units u and υ agree. The operations m and μ agree too, because

$$\mu(x, y) = \mu(m(x, u), m(u, y)) = m(\mu(x, u), \mu(u, y)) = m(x, y),$$

where in the middle step we used the fact that μ is a group homomorphism. It is now clear that the inverses i and ι agree as well. Furthermore, taking into account that m and μ agree, we also obtain

$$m(x, y) = m(m(u, x), m(y, u)) = m(m(u, y), m(x, u)) = m(y, x).$$

The conclusion is that a group in the category of groups is an abelian group. The category $\mathbf{Mod_{Grp}}(\mathsf{Group})$ is therefore equivalent to the category of abelian groups.

Example 1.28. A model of the theory of a pointed set, cf. Example 1.5, in the category of groups **Grp** is a group (G, u, m, i) together with a homomorphism $1 \to G$ from the trivial group 1 to G. However, there is precisely one such homomorphism which therefore need not be mentioned at all. Thus a pointed set in groups amounts to a group.

1.7. **Free models.** Of special interest are the free models of an algebraic theory. Given an algebraic theory T and a set X, the *free* T-*model*, also called the *free* T-*algebra*, generated by X is a model M together with a map $\eta : X \to |M|$ such that, for every T-model L and every map $f : X \to |L|$ there is a unique T-homomorphism $\overline{f} : M \to L$ for which the following diagram commutes:

$$
\begin{array}{ccc}
X & \xrightarrow{\ \eta\ } & |M| \\
& {\scriptstyle f}\searrow & \downarrow{\scriptstyle \overline{f}} \\
& & |L|
\end{array}
$$

The definition is a bit of a mouthful, but it can be understood as follows: the free T-model generated by X is the "most economical" way of making a T-model out of the set X.

Example 1.29. The free group generated by the empty set is the trivial group 1 with just one element. The map $\eta : \emptyset \to 1$ is the unique one, and given any (unique) map $f : \emptyset \to |G|$ to a carrier of another group G, there is a unique group homomorphism $\overline{f} : 1 \to G$. The relevant triangle commutes automatically because it originates at $\emptyset$.

Example 1.30. The free group generated by the singleton set 1 is the group of integers $(\mathbb{Z}, 0, +, -)$. The map $\eta : \{\star\} \to \mathbb{Z}$ takes the generator $()$ to 0. As an exercise you should verify that the integers have the required universal property.

Example 1.31. Let $\mathcal{P}_{<\omega}(X)$ be the set of all finite subsets of a set X. We show that $(\mathcal{P}_{<\omega}(X), \emptyset, \cup)$ is the free semilattice generated by X, cf. Example 1.3. The map $\eta : X \to \mathcal{P}_{<\omega}(X)$ takes $x \in X$ to the singleton set $\eta(x) = \{x\}$. Given any semilattice $(L, \perp, \vee)$ and a map $f : X \to |L|$, define the homomorphism $\overline{f} : \mathcal{P}_{<\omega}(X) \to |L|$ by

$$\overline{f}(\{x_1, \ldots, x_n\}) = f(x_1) \wedge \cdots \wedge f(x_n).$$

Clearly, the required diagram commutes because

$$\overline{f}(\eta(x)) = \overline{f}(\{x\}) = f(x).$$

If $g : \mathcal{P}_{<\omega}(X) \to |L|$ is another homomorphism satisfying $g \circ \eta = f$ then

$$g(\{x_1, \ldots, x_n\}) = g(\eta(x_1) \cup \cdots \cup \eta(x_n)) = g(\eta(x_1)) \wedge \cdots \wedge g(\eta(x_n))$$
$$= f(x_1) \wedge \cdots \wedge f(x_n) = \overline{f}(\{x_1, \ldots, x_n\}),$$

hence $\overline{f}$ is indeed unique.

Example 1.32. The free model generated by X of the theory of a pointed set, cf. Example 1.5, is the disjoint union $X + 1$ whose elements are of the form $\iota_1(x)$ for $x \in X$ and $\iota_2(y)$ for $y \in 1$. The point is the element $\iota_2(())$. The map $\eta : X \to X + 1$ is the canonical inclusion ι_1.

Example 1.33. The free model generated by X of the empty theory, cf. Example 1.6, is X itself, with $\eta : X \to X$ the identity map.

Example 1.34. The free model generated by X of the theory of a singleton, cf. Example 1.7, is the singleton set 1, with $\eta : X \to 1$ the only map it could be. This example shows that η need not be injective.

Every algebraic theory T has a free model. Let us sketch its construction. Given a signature Σ and a set X, define $\mathsf{Tree}_\Sigma(X)$ to be the set of well-founded trees built inductively as follows:

(1) for each $x \in X$, there is a tree return $x \in \mathsf{Tree}_\Sigma(X)$,
(2) for each operation symbol op_i and trees $t_1, \ldots, t_{\mathsf{ar}_i} \in \mathsf{Tree}_\Sigma(X)$, there is a tree, denoted by $\mathsf{op}_i(t_1, \ldots, t_n) \in \mathsf{Tree}_\Sigma(X)$, whose root is labeled by op_i and whose subtrees are $t_1, \ldots, t_{\mathsf{ar}_i}$.

By labeling the tree leaves with the keyword "return" we are anticipating their role in effectful computations, as will become clear later on. From a purely formal point of view the choice of the label is immaterial.

The Σ-terms in context $x_1, \ldots, x_n$ are precisely the trees in $\mathsf{Tree}_\Sigma(\{x_1, \ldots, x_n\})$, except that a variable x_i is labeled as return x_i when construed as a tree.

Suppose $x_1, \ldots, x_n \mid t$ is a Σ-term in context, and we are given an assignment $\sigma : \{x_1, \ldots, x_n\} \to \mathsf{Tree}_\Sigma(X)$ of trees to variables. Then we may build the tree $\sigma(t)$ inductively as follows:

(1) $\sigma(t) = \sigma(x_i)$ if $t = x_i$,
(2) $\sigma(t) = \mathsf{op}_i(\sigma(t_1), \ldots, \sigma(t_n))$ if $t = \mathsf{op}_i(t_1, \ldots, t_n)$.

In words, the tree $\sigma(t)$ is obtained by replacing each variable x_i in t with the corresponding tree $\sigma(x_i)$.

Given a theory T, let $\approx_\mathsf{T}$ be the least equivalence relation on $\mathsf{Tree}_{\Sigma_\mathsf{T}}(X)$ such that:

(1) for every equation $x_1, \ldots, x_n \mid \ell = r$ in $\mathcal{E}_\mathsf{T}$ and for every assignment $\sigma : \{x_1, \ldots, x_n\} \to \mathsf{Tree}_{\Sigma_\mathsf{T}}(X)$, we have

$$\sigma(\ell) \approx_\mathsf{T} \sigma(r).$$

(2) $\approx_\mathsf{T}$ is a Σ_T-*congruence*: for every operation symbol op_i in Σ_T, and for all trees $s_1, \ldots, s_{\mathsf{ar}_i}$ and $t_1, \ldots, t_{\mathsf{ar}_i}$, if

$$s_1 \approx_\mathsf{T} t_1, \quad \ldots \quad, s_{\mathsf{ar}_i} \approx_\mathsf{T} t_{\mathsf{ar}_i}$$

then
$$\mathsf{op}_i(s_1, \ldots, s_{\mathsf{ar}_i}) \approx_{\mathsf{T}} \mathsf{op}_i(t_1, \ldots, t_{\mathsf{ar}_i}).$$

Define the carrier of the free model $F_{\mathsf{T}}(X)$ to be the quotient set

$$|F_{\mathsf{T}}(X)| = \mathsf{Tree}_{\Sigma_{\mathsf{T}}}(X)/\approx_{\mathsf{T}}.$$

Let $[t]$ be the $\approx_{\mathsf{T}}$-equivalence class of $t \in \mathsf{Tree}_{\Sigma_{\mathsf{T}}}(X)$. The interpretation of the operation symbol op_i in $F_{\mathsf{T}}(X)$ is the map $[\![\mathsf{op}_i]\!]_{F_{\mathsf{T}}(X)}$ defined by

$$[\![\mathsf{op}_i]\!]_{F_{\mathsf{T}}(X)}([t_1], \ldots, [t_{\mathsf{ar}_i}]) = [\mathsf{op}_i(t_1, \ldots, t_{\mathsf{ar}_i})].$$

The map $\eta_X : X \to F_{\mathsf{T}}(X)$ is defined by

$$\eta_X(x) = [\mathsf{return}\ x].$$

To see that we successfully defined a T-model, and that it is freely generated by X, one has to verify a number of mostly straightforward technical details, which we omit.

When a theory T has no equations the free models generated by X is just the set of trees $\mathsf{Tree}_T(X)$ because the relation $\approx_{\mathsf{T}}$ is equality.

1.8. **Operations with general arities and parameters.** We have so far followed the classic mathematical presentation of algebraic theories. To get a better fit with computational effects, we need to generalize operations in two ways.

1.8.1. *General arities.* We shall require operations that accept an arbitrary, but fixed collection of arguments. One might expect that the correct way to do so is to allow arities to be ordinal or cardinal numbers, as these generalize natural numbers, but that would be a thoroughly non-computational idea. Instead, let us observe that an n-ary cartesian product

$$\underbrace{X \times \cdots \times X}_{n}$$

is isomorphic to the exponential $X^{[n]}$, where $[n] = \{0, 1, \ldots, n-1\}$. Recall that an exponential B^A is the set of all functions $A \to B$, and in fact we shall use the notations B^A and $A \to B$ interchangeably. If we replace $[n]$ by an arbitrary set A, then we can think of a map

$$X^A \to X$$

as taking A-many arguments. We need reasonable notation for writing down an operation symbol applied to A-many arguments, where A is an arbitrary set. One might be tempted to adapt the tuple notation and write something silly, such as

$$\mathsf{op}_i(\cdots t_a \cdots)_{a \in A},$$

but as computer scientists we know better than that. Let us use the notation that is already provided to us by the exponentials, namely the λ-calculus. To have A-many elements of a set X is to have a map $\kappa : A \to X$, and thus to apply the operation symbol op_i to A-many arguments κ we simply write $\mathsf{op}_i(\kappa)$.

Example 1.35. Let us rewrite the group operations in the new notation. The empty set $\emptyset$, the singleton 1, and the set of boolean values

$$\mathsf{bool} = \{\mathsf{false}, \mathsf{true}\}$$

serve as arities. We use the conditional statement

$$\mathsf{if}\ b\ \mathsf{then}\ x\ \mathsf{else}\ y$$

as a synonym for what is usually written as definition by cases,

$$\begin{cases} x & \text{if } b = \text{true}, \\ y & \text{if } b = \text{false}. \end{cases}$$

Now a group is given by a carrier set G together with maps

$$\mathsf{u} : G^{\emptyset} \to G,$$
$$\mathsf{m} : G^{\mathsf{bool}} \to G,$$
$$\mathsf{i} : G^{1} \to G,$$

satisfying the usual group laws, which we ought to write down using the λ-notation. The associativity law is written like this:

$\mathsf{m}(\lambda b \,.\, \text{if } b \text{ then } \mathsf{m}(\lambda c \,.\, \text{if } c \text{ then } x \text{ else } y) \text{ else } z) =$

$$\mathsf{m}(\lambda b \,.\, \text{if } b \text{ then } x \text{ else } \mathsf{m}(\lambda c \,.\, \text{if } c \text{ then } y \text{ else } z)).$$

Here is the right inverse law, where $\mathsf{O}_X : \emptyset \to X$ is the unique map from $\emptyset$ to X:

$$\mathsf{m}(\lambda b \,.\, \text{if } b \text{ then } x \text{ else } \mathsf{i}(\lambda\_\,.\, x)) = \mathsf{u}(\mathsf{O}_G).$$

The symbol $\_$ indicates that the argument of the λ-abstraction is ignored, i.e., that the function defined by the abstraction is constant. One more example might help: x squared may be written as $\mathsf{m}(\lambda b \,.\, \text{if } b \text{ then } x \text{ else } x)$ as well as $\mathsf{m}(\lambda\_\,.\, x)$.

Such notation is not appropriate for performing algebraic manipulations, but is bringing us closer to the syntax of a programming language.

1.8.2. *Operations with parameters.* To motivate our second generalization, consider the theory of a module M over a ring R (if you are not familiar with modules, think of the elements of M as vectors and the elements of R as scalars). For it to be an algebraic theory, we need to deal with scalar multiplication $\cdot : R \times M \to M$, because it does not fit the established pattern. There are three possibilities:

(1) We could introduce *multi-sorted* algebraic theories whose operations take arguments from several carrier sets. The theory of a module would have two sorts, say R and M, and scalar multiplication would be a binary operation of arity $(\mathsf{R}, \mathsf{M}; \mathsf{M})$. (We hesitate to write $\mathsf{R} \times \mathsf{M} \to \mathsf{M}$ lest the type theorists get useful ideas.)

(2) Instead of having a single binary operation taking a scalar and a vector, we could have many unary operations taking a vector, one for each scalar.

(3) We could view the scalar as an additional *parameter* of a unary operation on vectors.

The second and the third options are superficially similar, but they differ in their treatment of parameters. In one case the parameters are part of the indexing of the signature, while in the other they are properly part of the algebraic theory. We shall adopt operations with parameters because they naturally model algebraic operations that arise as computational effects.

Example 1.36. The theory of a module over a ring $(R, 0, +, -, \cdot)$ has several operations. One of them is scalar multiplication, which is a *unary* operation mul parameterized by elements of R. That is, for every $r \in R$ and term t, we may form the term

$$\mathsf{mul}(r; t),$$

which we think of as t multiplied with r. The remaining operations seem not to be parameterized, but we can force them to be parameterized by fiat. Addition is a binary operation add parameterized by the singleton set 1: the sum of t_1 and t_2 is written as

$$\mathsf{add}((); t_1, t_2).$$

We can use this trick in general: an operation without parameters is an operation taking parameters from the singleton set.

Note that in the previous example we mixed theories and models. We spoke about the *theory* of a module with respect to a *specific ring R*.

Example 1.37. The theory of a $\mathcal{C}^\infty$-ring, cf. Example 1.10, may be reformulated using parameters. For every $n \in \mathbb{N}$ there is an n-ary operation symbol app_n whose parameter set is $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$. What was written as

$$\mathsf{op}_f(t_1, \ldots, t_n)$$

in Example 1.10 is now written as

$$\mathsf{app}_n(f; t_1, \ldots, t_n).$$

If you insist on the λ-notation, replace the tuple $(t_1, \ldots, t_n)$ of terms with a single function t mapping from $[n]$ to terms, and write $\mathsf{app}_n(f; t)$.

The operations app_n tell us what $\mathcal{C}^\infty$-rings are about: they are structures whose elements can feature as arguments to smooth functions. In contrast, an ordinary (commutative unital) ring is one whose elements can feature as arguments to "finite degree" smooth maps, i.e., the polynomials.

1.9. **Algebraic theories with parameterized operations and general arities.** Let us restate the definitions of signatures and algebraic operations, with the generalizations incorporated. For simplicity we work with sets and functions, and leave consideration of other categories for another occasion.

A *signature* Σ is given by a collection of *operation symbols* op_i with associated *parameter sets* P_i and *arities* A_i. For reasons that will become clear later, we write

$$\mathsf{op}_i : P_i \rightsquigarrow A_i$$

to display an operation symbol op_i with parameter set P_i and arity A_i. The symbols may be anything, although we think of them as syntactic entities, while P_i's and A_i's are sets.

Arbitrary arities require an arbitrary number of variables in context. We therefore generalize terms in contexts to *well-founded trees* over Σ generated by a set X. These form a set $\mathsf{Tree}_\Sigma(X)$ whose elements are generated inductively as follows:

(1) for every generator $x \in S$ there is a tree return x,
(2) if $p \in P_i$ and $\kappa : A_i \to \mathsf{Tree}_\Sigma(X)$ then $\mathsf{op}_i(p, \kappa)$ is a tree whose root is labeled with op_i and whose A_i-many subtrees are given by κ.

The usual Σ-terms in context $x_1, \ldots, x_k$ correspond to $\mathsf{Tree}_\Sigma(\{x_1, \ldots, x_k\})$. Or to put it differently, the elements of $\mathsf{Tree}_\Sigma(X)$ may be thought of as terms with variables X. In fact, we shall customarily refer to them as terms.

An *interpretation I of a signature* Σ is given by:

(1) a carrier set $|I|$,
(2) for each operation symbol op_i with parameter set P_i and arity A_i, a map

$$[\![\mathsf{op}_i]\!]_I : P_i \times |I|^{A_i} \longrightarrow |I|.$$

The interpretation I may be extended to trees. A tree $t \in \mathsf{Tree}_\Sigma(X)$ is interpreted as a map

$$\llbracket t \rrbracket_I : |I|^X \to |I|$$

as follows:

(1) the tree return x is interpreted as the x-th projection,

$$\llbracket \mathsf{return}\ x \rrbracket_I : |I|^X \to |I|,$$
$$\llbracket \mathsf{return}\ x \rrbracket_I : \eta \mapsto \eta(x),$$

(2) the tree $\mathsf{op}_i(p, \kappa)$ is interpreted as the map

$$\llbracket \mathsf{op}_i(p, \kappa) \rrbracket_I : |I|^X \longrightarrow |I|$$
$$\llbracket \mathsf{op}_i(p, \kappa) \rrbracket_I : \eta \mapsto \llbracket \mathsf{op}_i \rrbracket_I(p, \lambda a\,.\,\llbracket \kappa(a) \rrbracket_I(\eta)),$$

A Σ-*equation* is a set X and a pair of Σ-terms $\ell, r \in \mathsf{Tree}_\Sigma(X)$, written

$$X \mid \ell = r.$$

We usually leave out X. Given an interpretation I of signature Σ, we say that such an equation is *valid* for I when the interpretations of ℓ and r give the same map.

An *algebraic theory* $\mathsf{T} = (\Sigma_\mathsf{T}, \mathcal{E}_\mathsf{T})$ is given by a signature Σ_T and a collection of Σ-equations $\mathcal{E}_\mathsf{T}$. A T-*model* is an interpretation for Σ_T which validates all the equations $\mathcal{E}_\mathsf{T}$.

The notions of T-morphisms and the category $\mathbf{Mod}(\mathsf{T})$ of T-models and T-morphisms may be similarly generalized. We do not repeat the definitions here, as they are almost the same. You should convince yourself that every algebraic theory has a free model, which is still built as a quotient of the set of well-founded trees.

2. Computational effects as algebraic operations

It is high time we provide some examples from programming. The original insight by Gordon Plotkin and John Power [8, 9] was that many computational effects are naturally described by algebraic theories. What precisely does this mean?

When a program runs on a computer, it interacts with the environment by performing *operations*, such as printing on the screen, reading from the keyboard, inspecting and modifying external memory store, launching missiles, etc. We may model these phenomena mathematically as operations on an algebra whose elements are *computations*. Leaving the exact nature of computations aside momentarily, we note that a computation may be

- pure, in which case it terminates and returns a value, or
- effectful, in which case it performs an operation.

(We are ignoring a third possibility, non-termination.) Let us write

$$\mathsf{return}\ v$$

for a pure computation that returns the value v. Think of a value as an inert datum that needs no further computation, such as a boolean constant, a numeral, or a λ-abstraction. An operation takes a *parameter* p, for instance the memory location to be read, or the string to be printed, and a *continuation* κ, which is a suspended computation expecting the result of the operation, for instance the contents of the memory location that has been read. Thus it makes sense to write

$$\mathsf{op}(p, \kappa)$$

for the computation that performs the operation op, with parameter p and continuation κ. The similarity with algebraic operations from Section 1.9 is not incidental!

Example 2.1. The computation which increases the contents of memory location ℓ by 1 and returns the original contents is written as

$$\mathsf{lookup}(\ell, \lambda x \,.\, \mathsf{update}((\ell, x + 1), \lambda\_ \,.\, \mathsf{return}\; x)).$$

In some venerable programming languages we would write this as $\ell{+}{+}$. Note that the operations happen from outside in: first the memory location ℓ is read, its value is bound to x, then $x + 1$ is written to memory location ℓ, the result of writing is ignored, and finally the value of x is returned.

So far we have a notation that looks like algebraic operations, but to do things properly we need signatures and equations. These depend on the computational effects under consideration.

Example 2.2. The algebraic theory of *state* with *locations* L and *states* S has operations

$$\mathsf{lookup} : L \rightsquigarrow S \qquad \text{and} \qquad \mathsf{update} : L \times S \rightsquigarrow 1.$$

First we have equations which state what happens on successive lookups and updates to the same memory location. For all $\ell \in L$, $s \in S$ and all continuations κ:

$$\mathsf{lookup}(\ell, \lambda s \,.\, \mathsf{lookup}(\ell, \lambda t \,.\, \kappa\, s\, t)) = \mathsf{lookup}(\ell, \lambda s \,.\, \kappa\, s\, s)$$
$$\mathsf{lookup}(\ell, \lambda s \,.\, \mathsf{update}((\ell, s), \kappa)) = \kappa\,()$$
$$\mathsf{update}((\ell, s), \lambda\_ \,.\, \mathsf{lookup}(\ell, \kappa)) = \mathsf{update}((\ell, s), \lambda\_ \,.\, \kappa\, s)$$
$$\mathsf{update}((\ell, s), \lambda\_ \,.\, \mathsf{update}((\ell, t), \kappa)) = \mathsf{update}((\ell, t), \kappa)$$

For example, the first equations says that two consecutive lookups from a memory location give equal results. We ought to explain the precise nature of κ in the above equations. If we translate the earlier examples into the present notation, we see that κ corresponds to variables, which leads to the idea that we should use a *generic* κ. Thus we let κ take some arguments and just return them as a tuple. In the first equation we take $\kappa = \lambda s\, t \,.\, \mathsf{return}\; (s, t)$; in the second and fourth equations are $\kappa = \lambda\_ \,.\, \mathsf{return}\;()$; and in the third equation $\kappa = \lambda s \,.\, \mathsf{return}\; s$. All equations have empty contexts, as no free variables occur in them. Unless specified otherwise, we shall always take κ to be such a generic continuation.

There is a second set of equations stating that lookups and updates from *different* locations $\ell \neq \ell'$ distribute over each other:

$$\mathsf{lookup}(\ell, \lambda s \,.\, \mathsf{lookup}(\ell', \lambda s' \,.\, \kappa\, s\, s')) = \mathsf{lookup}(\ell', \lambda s' \,.\, \mathsf{lookup}(\ell, \lambda s \,.\, \kappa\, s\, s'))$$
$$\mathsf{update}((\ell, s), \lambda\_ \,.\, \mathsf{lookup}(\ell', \kappa)) = \mathsf{lookup}(\ell', \lambda t \,.\, \mathsf{update}((\ell, s), \lambda\_ \,.\, \kappa\, t))$$
$$\mathsf{update}((\ell, s), \lambda\_ \,.\, \mathsf{update}((\ell', s'), \kappa)) = \mathsf{update}((\ell', s'), \lambda\_ \,.\, \mathsf{update}((\ell, s), \kappa)).$$

Have we forgotten any equations? It turns out that the theory is Hilbert-Post complete: if we add any equation that does not already follow from these, the theory trivializes in the sense that all equations become derivable.

Example 2.3. The theory of *input and output* (I/O) has operations

$$\mathsf{print} : S \rightsquigarrow 1 \qquad \text{and} \qquad \mathsf{read} : 1 \rightsquigarrow S,$$

where S is the set of entities that are read or written, for example bytes, or strings. There are no equations. We may now write the obligatory hello world:

$$\mathsf{print}(\text{'Hello world!'}, \lambda\_ \,.\, \mathsf{return}\;()).$$

Example 2.4. The theory of a point set, cf. Example 1.5, is the theory of an *exception*. The point • is a constant, which we rename to a nullary operation

$$\mathsf{abort} : 1 \rightsquigarrow \emptyset.$$

There are no equations. For example, the computation

$$\mathsf{read}((), \lambda x \,.\, \mathsf{if}\ x < 0\ \mathsf{then}\ \mathsf{abort}((), \mathsf{O}_{\mathbb{Z}})\ \mathsf{else}\ \mathsf{return}\ (x + 1))$$

reads an integer x from standard input, raises an exception if x is negative, otherwise it returns its successor.

Example 2.5. Let us take the theory of semilattice, cf. Example 1.3, but without the unit. It has a binary operation $\vee$ satisfying

$$\begin{aligned}
x \vee x &= x, \\
x \vee y &= y \vee x, \\
(x \vee y) \vee z &= x \vee (y \vee z).
\end{aligned}$$

This is the algebraic theory of (one variant of) *non-determinism*. Indeed, the binary operation $\vee$ corresponds to a choice operation

$$\mathsf{choose} : 1 \rightsquigarrow \mathsf{bool}$$

which (non-deterministically) returns a bit, or chooses a computation, depending on how we look at it. Written in continuation notation, it chooses a bit b and passes it to the continuation κ,

$$\mathsf{choose}((), \lambda b \,.\, \kappa\, b),$$

whereas with the traditional notation it chooses between two computations κ_1 and κ_2,

$$\mathsf{choose}(\kappa_1, \kappa_2).$$

Example 2.6. Algebraic theories may be combined. For example, if we want a theory describing state and I/O we may simply adjoin the signatures and equations of both theories to obtain their combination.

Sometimes we want to combine theories so that the operations between them interact. To demonstrate this, let us consider the theory of a single stateful memory location holding elements of a set S. The operations are

$$\mathsf{get} : 1 \rightsquigarrow S \qquad \text{and} \qquad \mathsf{put} : S \rightsquigarrow 1.$$

The equations are

(1) $$\mathsf{get}((), \lambda s \,.\, \mathsf{get}((), \lambda t \,.\, \kappa\, s\, t)) = \mathsf{get}((), \lambda s \,.\, \kappa\, s\, s)$$

(2) $$\mathsf{get}((), \lambda s \,.\, \mathsf{put}(s, \kappa)) = \kappa\,()$$

(3) $$\mathsf{put}(s, \lambda\_ \,.\, \mathsf{get}((), \kappa)) = \mathsf{put}(s, \lambda\_ \,.\, \kappa\, s)$$

(4) $$\mathsf{put}(s, \lambda\_ \,.\, \mathsf{put}(t, \kappa)) = \mathsf{put}(t, \kappa)$$

This is just the first group of equations from Example 2.2, except that we need not specify which memory location to read from.

Can the theory of states with many locations from Example 2.2 be obtained by a combination of many instances of the theory of a single state? That is, to model I-many states, we combine I-many copies of the theory of a single state, so that for every $\iota \in I$ we have operations

$$\mathsf{get}_\iota : 1 \rightsquigarrow S_\iota \qquad \text{and} \qquad \mathsf{put}_\iota : S_\iota \rightsquigarrow 1,$$

with the above equations. We also need to postulate *distributivity* laws expressing the fact that operations from instance ι distribute over those of instance ι', so long as $\iota \neq \iota'$:

$$\mathsf{get}_\iota((), \lambda s \,.\, \mathsf{get}_{\iota'}((), \lambda s' \,.\, \kappa\, s\, s')) = \mathsf{get}_{\iota'}((), \lambda s' \,.\, \mathsf{get}_\iota((), \lambda s \,.\, \kappa\, s\, s'))$$

$$\mathsf{put}_\iota(s, \lambda\_ \,.\, \mathsf{get}_{\iota'}((), \kappa)) = \mathsf{get}_{\iota'}((), \lambda t \,.\, \mathsf{put}_\iota(s, \lambda\_ \,.\, \kappa\, t))$$

$$\mathsf{put}_\iota(s, \lambda\_ \,.\, \mathsf{update}_{\iota'}(s', \kappa)) = \mathsf{update}_{\iota'}(s', \lambda\_ \,.\, \mathsf{put}_\iota(s, \kappa)).$$

The theory so obtained is similar to that of Example 2.2, with two important differences. First, the locations $\ell \in L$ are parameters of operations in Example 2.2, whereas in the present case the instances $\iota \in I$ index the operations themselves. Second, all memory locations in Example 2.2 share the same set of states S, whereas the combination of I-many separate states allows a different set of states S_ι for every instance $\iota \in I$.

2.1. **Computations are free models.** Among all the models of an algebraic theory of computational effects, which one best described the actual computational effects? If a theory of computational effects truly is adequately described by its signature and equations, then the free model ought to be the desired one.

Example 2.7. Consider the theory State of a state storing elements of S from Example 2.6. Let us verify whether the free model $F_{\mathsf{State}}(V)$ adequately describes stateful computations returning values from V. As we saw in Section 1.7, the free model is a quotient of the set of trees $\mathsf{Tree}_{\Sigma_{\mathsf{State}}}(V)$ by a congruence relation $\approx_{\mathsf{State}}$. Every tree is congruent to one of the form

(5) $$\mathsf{get}((), \lambda s \,.\, \mathsf{put}(f(s), \lambda\_ \,.\, \mathsf{return}\, g(s)))$$

for some maps $f : S \to S$ and $g : S \to V$. Indeed, by applying the equations from Example 2.6, we may contract any two consecutive get's to a single one, and similarly for consecutive put's, we may disregard a get after a put, and cancel a get followed by a put. We are left with four forms of trees,

$$\mathsf{return}\, v,$$
$$\mathsf{get}((), \lambda s \,.\, \mathsf{return}\, g(s)),$$
$$\mathsf{put}(t, \lambda\_ \,.\, \mathsf{return}\, v),$$
$$\mathsf{get}((), \lambda s \,.\, \mathsf{put}(f(s), \lambda\_ \,.\, \mathsf{return}\, g(s))),$$

but the first three may be brought into the form of the fourth one:

$$\mathsf{return}\, v = \mathsf{get}((), \lambda s \,.\, \mathsf{put}(s, \lambda\_ \,.\, \mathsf{return}\, v)),$$
$$\mathsf{get}((), \lambda s \,.\, \mathsf{return}\, g(s)) = \mathsf{get}((), \lambda s \,.\, \mathsf{put}(s, \lambda\_ \,.\, \mathsf{return}\, g(s))),$$
$$\mathsf{put}(t, \lambda\_ \,.\, \mathsf{return}\, v) = \mathsf{get}((), \lambda\_ \,.\, \mathsf{put}(t, \lambda\_ \,.\, \mathsf{return}\, v)).$$

Therefore, the free model $F_{\mathsf{Sate}}(()V)$ is isomorphic to the set of functions

$$S \to S \times V.$$

The isomorphism takes the element represented by (5) to the function $\lambda s \,.\, (f(s), g(s))$. (It takes extra effort to show that each element is represented by unique f and g.) The inverse takes a function $h : S \to S \times V$ to the computation represented by the tree

$$\mathsf{get}((), \lambda s \,.\, \mathsf{put}(\pi_1(h(s)), \lambda\_ \,.\, \mathsf{return}\, \pi_2(h(s)))).$$

Functional programers will surely recognize the genesis of the state monad.

Let us expand on the last thought of the previous example and show, at the risk of wading a bit deeper into category theory, that free models of an algebraic theory T form a monad. We describe the monad structure in the form of a Kleisli triple, because it is familiar to functional programmers. First, we have an endofunctor F_{T} on the category of sets which takes a set X to the free model $F_{\mathsf{T}}(X)$ and a map $f : X \to Y$ to the unique T-homomorphism $\overline{f}$ for which the following diagram commutes:

$$
\begin{array}{ccc}
X & \xrightarrow{\ \eta_X\ } & F_{\mathsf{T}}(X) \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \overline{f}} \\
Y & \xrightarrow[\ \eta_Y\ ]{} & F_{\mathsf{T}}(Y)
\end{array}
$$

Second, the unit of the monad is the map $\eta_X : X \to F_{\mathsf{T}}(X)$ taking x to return x. Third, a map $\phi : X \to F_{\mathsf{T}}(Y)$ is lifted to the unique map $\phi^{\dagger} : F_{\mathsf{T}}(X) \to F_{\mathsf{T}}(Y)$ for which the following diagram commutes:

$$
\begin{array}{ccc}
X & \xrightarrow{\ \eta_X\ } & F_{\mathsf{T}}(X) \\
 & {\scriptstyle \phi}\searrow & \downarrow{\scriptstyle \phi^{\dagger}} \\
 & & F_{\mathsf{T}}(Y)
\end{array}
$$

Concretely, $\phi^{\dagger}$ is defined by recursion on ($\approx_{\mathsf{T}}$-equivalence classes of) trees by

$$
\phi^{\dagger}([\mathsf{return}\ x]) = \phi(x),
$$
$$
\phi^{\dagger}([\mathsf{op}(p, \kappa)]) = [\mathsf{op}(p, \phi^{\dagger} \circ \kappa)],
$$

where op ranges over the operations of T. The first equation holds because the above diagram commutes, and the second because $\phi^{\dagger}$ is a T-homomorphism. We leave the verification of the monad laws as an exercise.

Example 2.8. Let us resume the previous example. If there is any beauty in mathematics, the monad for F_{State} should be isomorphic to the usual state monad $(T, \theta, {}^{*})$, given by

$$
T(X) = (S \to S \times X),
$$
$$
\theta_X(x) = (\lambda s \,.\, (s, x)),
$$
$$
\psi^{*}(h) = (\lambda s \,.\, \psi(\pi_2(h(s)))(\pi_1(h(s)))),
$$

where $x \in X$, $\psi : X \to T(Y)$, and $h : S \to S \times X$. In the previous example we already verified that $F_{\mathsf{State}}(X) \cong T(X)$ by the isomorphism

$$
\Xi : [\mathsf{get}((), \lambda s \,.\, \mathsf{put}(f(s), \lambda\_ \,.\, \mathsf{return}\ g(s)))] \mapsto (\lambda s \,.\, (f(s), g(s))).
$$

Checking that Ξ transfers η to θ and † to * requires a tedious but straightforward calculation which is best done in the privacy of one's notebook. Nevertheless, here it is. Note that

$$
\eta_X(x) = [\mathsf{return}\ x] = [\mathsf{get}((), \lambda s \,.\, \mathsf{put}(s, \lambda\_ \,.\, \mathsf{return}\ x))]
$$

hence $\eta_X(x)$ is isomorphic to the map $x \mapsto (\lambda s \,.\, (s, x))$, which is just $\theta_X(x)$, as required. For lifting, consider any $\phi : X \to F_{\mathsf{State}}(Y)$. There corresponds to it a unique map $\psi : X \to (S \to S \times Y)$ satisfying

$$
\phi(x) = [\mathsf{get}((), \lambda t \,.\, \mathsf{put}(\pi_1(\psi(x)(t)), \lambda\_ \,.\, \mathsf{return}\ \pi_2(\psi(x)(t))))].
$$

First we compute $\phi^\dagger$ applied to an arbitrary element of the free model:

$\phi^\dagger([\mathsf{get}((), \lambda s . \mathsf{put}(f(s), \lambda\_ . \mathsf{return}\ g(s)))]) =$
$\quad [\mathsf{get}((), \lambda s . \mathsf{put}(f(s), \lambda\_ . \phi(g(s))))] =$
$\quad [\mathsf{get}((), \lambda s . \mathsf{put}(f(s), \lambda\_ . \mathsf{get}((), \lambda t . \mathsf{put}(\pi_1(\psi(g(s))(t)), \lambda\_ . \mathsf{return}\ \pi_2(\psi(g(s))(t))))))] =$
$\quad [\mathsf{get}((), \lambda s . \mathsf{put}(f(s), \lambda\_ . \mathsf{put}(\pi_1(\psi(g(s))(f(s))), \lambda\_ . \mathsf{return}\ \pi_2(\psi(g(s))(f(s)))))))] =$
$\quad [\mathsf{get}((), \lambda s . \mathsf{put}(\pi_1(\psi(g(s))(f(s))), \lambda\_ . \mathsf{return}\ \pi_2(\psi(g(s))(f(s)))))].$

Then we compute ψ^* applied to the corresponding element of the state monad:

$$\psi^*(\lambda s . (f(s), g(s))) = (\lambda s . \psi(g(s))(f(s)))$$
$$= (\lambda s . (\pi_1(\psi(g(s))(f(s))), \pi_2(\psi(g(s))(f(s))))),$$

And we have a match with respect to Ξ.

2.2. **Sequencing and generic operations.** We seem to have a good theory of computations, but our notation is an abomination which neither mathematicians nor programmers would ever want to use. Let us provide a better syntax that will make half of them happy.

Consider an algebraic theory T. For an operation $\mathsf{op} : P \rightsquigarrow A$ in Σ_T, define the corresponding *generic operation*

$$\overline{\mathsf{op}}(p) := \mathsf{op}(p, \lambda x . \mathsf{return}\ x).$$

In words, the generic version performs the operation and returns its result. When the parameter is the unit we write $\overline{\mathsf{op}}()$ instead of the silly looking $\overline{\mathsf{op}}(())$. After a while one also grows tired of the over-line and simplifies the notation to just $\mathsf{op}(p)$, but we shall not do so here.

Next, we give ourselves a better notation for the monad lifting. Suppose $t \in F_\mathsf{T}(X)$ and $h : X \to F_\mathsf{T}(Y)$. Define the *sequencing*

$$\mathsf{do}\ x \leftarrow t\ \mathsf{in}\ h(x),$$

to be an abbreviation for $h^\dagger(t)$, with the proviso that x is bound in $h(x)$. Generic operations and sequencing allow us to replace the awkward looking

$$\mathsf{op}(p, \lambda x . t(x))$$

with

$$\mathsf{do}\ x \leftarrow \overline{\mathsf{op}}(p)\ \mathsf{in}\ t(x).$$

Even better, nested operations

$$\mathsf{op}_1(p_1, \lambda x_1 . \mathsf{op}_2(p_2, \lambda x_2 . \mathsf{op}_3(p_2, \lambda x_3 . \cdots)))$$

may be written in Haskell-like notation

$$\mathsf{do}\ x_1 \leftarrow \overline{\mathsf{op}_1}(p_1)\ \mathsf{in}$$
$$\mathsf{do}\ x_2 \leftarrow \overline{\mathsf{op}_3}(p_3)\ \mathsf{in}$$
$$\mathsf{do}\ x_2 \leftarrow \overline{\mathsf{op}_3}(p_3)\ \mathsf{in}\ \cdots$$

The syntax of a typical programming language only ever exposes the generic operations. The generic operation $\overline{\mathsf{op}}$ with parameter set P and arity A looks to a programmer like a function of type $P \to A$, which is why we use the notation $\mathsf{op} : P \rightsquigarrow A$ to specify signatures.

Because sequencing is just lifting in disguise, it is governed by the same equations as lifting:

$$(\text{do } x \leftarrow \text{return } v \text{ in } h(x)) = h(v),$$
$$(\text{do } x \leftarrow \text{op}(p, \kappa) \text{ in } h(x)) = \text{op}(p, \lambda y \,.\, \text{do } x \leftarrow \kappa(y) \text{ in } h(x)).$$

These allow us to eliminate sequencing from any expression. When we rewrite the second equation with generic operations we get an associativity law for sequencing:

$$(\text{do } x \leftarrow (\text{do } y \leftarrow \overline{\text{op}}(p) \text{ in } \kappa(y)) \text{ in } h(x)) = (\text{do } y \leftarrow \overline{\text{op}}(p) \text{ in do } x \leftarrow \kappa(y) \text{ in } h(x)).$$

The ML aficionados may be pleased to learn that the sequencing notation in an ML-style language is none other than let-binding,

$$\text{let } x = t \text{ in } h(x).$$

3. HANDLERS

So far the main take-away is that computations returning values from V and performing operations of a theory T are the elements of the free model $F_\mathsf{T}(V)$. What about *transformations* between computations, what are they? An easy but useless answer is that they are just maps between the carriers of free models,

$$|F_\mathsf{T}(X)| \longrightarrow |F_{\mathsf{T}'}(X')|,$$

whereas a better answer should take into account the algebraic structure. Having put so much faith in algebra, let us continue to do so and postulate that a transformation between computations be a homomorphism. Should it be a homomorphism with respect to T or T'? We could weasel out of the question by considering only homomorphisms of the form $F_\mathsf{T}(X) \to F_\mathsf{T}(X')$, but such homomorphisms are rather uninteresting, because they amount to maps $X \to F_\mathsf{T}(X')$. We want transformation between computations that transform the operations as well as values.

To get a reasonable notion of transformation, let us recall that the universal property of free models speaks about maps *from* a free model. Thus, a transformation between computations should be a T-homomorphism

$$H : |F_\mathsf{T}(X)| \longrightarrow |F_{\mathsf{T}'}(X')|.$$

For this to make any sense, the carrier $|F_{\mathsf{T}'}(X')|$ must carry the structure of a T-model, i.e., in addition to H we must also provide a T-model on $|F_{\mathsf{T}'}(X')|$. If we take into account the fact that H is uniquely determined by its action on the generators, we arrive at the following notion. A *handler* from computations $F_\mathsf{T}(X)$ to computations $F_{\mathsf{T}'}(X')$ is given by the following data:

(1) a map $f : X \to |F_{\mathsf{T}'}(X')|$,
(2) for every operation $\text{op}_i : P_i \rightsquigarrow A_i$ in Σ_T, a map

$$h_i : P_i \times |F_{\mathsf{T}'}(X')|^{A_i} \to |F_{\mathsf{T}'}(X')|$$

such that
(3) the maps h_i form a T-model on $|F_{\mathsf{T}'}(X')|$, i.e., they validate the equations $\mathcal{E}_\mathsf{T}$.

The map $H : |F_\mathsf{T}(X)| \longrightarrow |F_{\mathsf{T}'}(X')|$ induced by these data is the unique one satisfying

$$H([\text{return } x]) = f(x),$$
$$H([\text{op}(p, \kappa)]) = h_i(p, H \circ \kappa).$$

When H is a handler from $F_\mathsf{T}(X)$ to $F_{\mathsf{T}'}(X')$ we write

$$H : F_\mathsf{T}(X) \Rightarrow F_{\mathsf{T}'}(X').$$

From a mathematical point of view handlers are just a curious combination of algebraic notions, but they are much more interesting from a programming point of view, as practice has shown.

We need a notation for handlers that neatly collects its defining data. Let us write

(6) $\mathsf{handler}\ \{\mathsf{return}\ x \mapsto f(x),\ \big(\mathsf{op}_i(y; \kappa) \mapsto h_i(y, \kappa)\big)_{\mathsf{op}_i \in \Sigma_\mathsf{T}}\}$

for the handler H determined by the maps f and h_i, as above, and

$$\mathsf{with}\ H\ \mathsf{handle}\ C$$

for the application of H to a computation C. The defining equations for handlers written in the new notation are, where H stands for the handler (6):

$$(\mathsf{with}\ H\ \mathsf{handle}\ \mathsf{return}\ v) = f(v),$$

$$(\mathsf{with}\ H\ \mathsf{handle}\ \mathsf{do}\ x \leftarrow \overline{\mathsf{op}_i}(p)\ \mathsf{in}\ \kappa(x)) = h_i(p, \lambda x\,.\,\mathsf{with}\ H\ \mathsf{handle}\ \kappa(x))$$

Example 3.1. Let us consider the theory Exn of an exception, cf. Example 2.4. A handler

$$H : F_{\mathsf{Exn}}(X) \Rightarrow F_\mathsf{T}(Y)$$

is given by a return clause and an abort clause,

$$\mathsf{handler}\ \{\mathsf{return}\ x \mapsto f(x), \mathsf{abort}(y; \kappa) \mapsto c\},$$

where $f : X \to F_\mathsf{T}(Y)$ and $c \in F_\mathsf{T}(Y)$. Note that c does not depend on $y \in 1$ and $\kappa : \emptyset \to F_\mathsf{T}(Y)$ because they are both useless. The theory of an exception has no equations, so the handler is automatically well defined. Such a handler is quite similar to exception handlers from mainstream programming languages, except that it handles both the exception and the return value.

4. What is coalgebraic about algebraic effects and handlers?

Handlers are a form of flow control (like loops, conditional statements, exceptions, coroutines, and the dreaded "goto"), to be used by programmers in programs. In other words, they can be used to *simulate* computational effects, a bit like monads can simulate computational effects in a purely functional language. What we still lack is a mathematical model of computational effects at the level of the external environment in which the program runs. There is always a barrier between the program and its external environment, be it a virtual machine, the operating system, or the underlying hardware. The actual computational effects cross the barrier, and cannot be modeled as handlers. A handler gets access to the continuation, but when a real computational effects happens, the continuation is not available. If it were, then after having launched missiles, the program could change its mind, restart the continuation, and establish world peace.

4.1. **Comodels of algebraic theories.** We shall model the top-level computational effects with comodels, which were proposed by Gordon Plotkin and John Power [10]. A *comodel* of a theory T in a category $\mathbf{C}$ is a model in the opposite category $\mathbf{C}^{\mathsf{op}}$. Comodels form a category

$$\mathbf{Comod}_\mathbf{C}(\mathsf{T}) := (\mathbf{Mod}_{\mathbf{C}^{\mathsf{op}}}(\mathsf{T}))^{\mathsf{op}}.$$

We steer away from category-theory and just spell out what a comodel is in the category of sets. When we pass to the dual category all morphisms turn around, and concepts are replaced with their duals.

Recall that the interpretation of an operation $\mathsf{op} : P \rightsquigarrow A$ in a model M is a map

$$[\![\mathsf{op}]\!]_M : P \times |M|^A \longrightarrow |M|,$$

which in the curried form is

$$|M|^A \longrightarrow |M|^P.$$

In the opposite category the map turns its direction, and the exponentials become products:

$$A \times |M| \longleftarrow P \times |M|.$$

Thus, in a comodel W an operation $\mathsf{op} : P \rightsquigarrow A$ is interpreted as a map

$$[\![\mathsf{op}]\!]^W : P \times |W| \to A \times |W|,$$

which we call a *cooperation*.

Example 4.1. Non-deterministic choice $\mathsf{choose} : 1 \rightsquigarrow \mathsf{bool}$, cf. Example 2.5, is interpreted as a cooperation

$$|W| \to \mathsf{bool} \times |W|,$$

where on the left we replaced $1 \times |W|$ with the isomorphic set $|W|$. If we think of $|W|$ as the set of all possible worlds, the cooperation choose is the action by which the world produces a boolean value and the next state of the world. Thus an external source of binary non-determinism is a stream of booleans.

Example 4.2. Printing to standard output $\mathsf{print} : S \rightsquigarrow 1$ is interpreted as a cooperation

$$S \times |W| \to |W|.$$

It is the action by which the world is modified according to the printed message (for example, the implants on your retina might induce your visual center to see the message).

Example 4.3. Reading from standard input $\mathsf{read} : 1 \rightsquigarrow S$ is interpreted as a cooperation

$$|W| \to S \times |W|.$$

This is quite similar to non-deterministic choice, except that the world provides an element of S rather than a boolean value. The world might accomplish such a task by inducing the user (who is considered as part of the world) to press buttons on the keyboard.

Example 4.4. An exception $\mathsf{abort} : 1 \rightsquigarrow \emptyset$ is interpreted as a cooperation

$$1 \times |W| \to \emptyset \times |W|.$$

Unless $|W|$ is the empty set, there is no such map. An exception cannot propagate to the outer world. The universe is safe from segmentation fault!

The examples are encouraging, so let us backtrack and spell out the basic definitions properly. A *cointerpretation* I of a signature Σ is given by a carrier set $|I|$, and for each operation symbol $\mathsf{op} : P \rightsquigarrow A$ a map

$$[\![\mathsf{op}]\!]^I : P \times |I| \to A \times |I|,$$

called a *cooperation*. The cointerpretation I may be extended to well-founded trees. A tree $t \in \mathsf{Tree}_\Sigma(X)$ is interpreted as a map

$$[\![t]\!]^I : |I| \to X \times |I|$$

as follows:

(1) the tree return x is interpreted as the x-th injection,

$$[\![X \mid x]\!]^I : |I| \to X \times |I|,$$

$$[\![X \mid x]\!]^I : \omega \to (x, \omega).$$

(2) the tree $\mathsf{op}_i(p, \kappa)$ is interpreted as the map

$$[\![X \mid \mathsf{op}_i(p, \kappa)]\!]^I : |I| \to X \times |I|,$$

$$[\![X \mid \mathsf{op}_i(p, \kappa)]\!]^I : \omega \to [\![X \mid \kappa(a)]\!]^I(\varpi) \quad \text{where } (a, \varpi) = [\![\mathsf{op}_i]\!]^I(p, \omega).$$

A *comodel* W of a theory T is a Σ_{T}-cointerpretation which validates all the equations $\mathcal{E}_{\mathsf{T}}$. As before, an equation is valid when the interpretations of its left- and right-hand sides yield equal maps.

Example 4.5. Let us work out what constitutes a comodel W of the theory of state, cf. Example 2.6. The operations

$$\mathsf{get} : 1 \rightsquigarrow S \qquad \text{and} \qquad \mathsf{put} : S \rightsquigarrow 1$$

are respectively interpreted by cooperations

$$g : |W| \to S \times |W| \qquad \text{and} \qquad p : S \times |W| \to |W|,$$

where we replaced $1 \times |W|$ with the isomorphic set $|W|$ (and we shall continue doing so in the rest of the example). The cooperations p and g must satisfy the equations from Example 2.6. We first unravel the interpretation of equation (2). Recall that κ is the generic continuation $\kappa\,() = \mathsf{return}\,()$, and the context contains no variables so it is interpreted by 1. Thus the right- and left-hand sides are interpreted as maps $|W| \to |W|$, namely

$$[\![\kappa\,()]\!]^W : w \mapsto w,$$

$$[\![\mathsf{get}((), \lambda s\,.\,\mathsf{put}(s, \kappa))]\!]^W : w \mapsto p(g(w)).$$

These are equal precisely when, for all $w \in |W|$,

$$(7) \qquad\qquad p(g(w)) = w.$$

Keeping in mind that the dual nature of cooperations requires reading of expressions from inside out, so that in $p(g(w))$ the cooperation g happens before p, the intuitive meaning of (7) is clear: the external world does not change when we read the state and write it right back. Equations (1), (3), and (4) may be similarly treated to respectively give

$$(8) \qquad\qquad g(\pi_2(g(w))) = g(w),$$

$$g(p(s, w)) = (s, p(s, w)),$$

$$p(t, p(s, w)) = p(t, w).$$

From these equations various others can be derived. For instance, by (7), the cooperation g is a section of p, therefore we may cancel it on both sides of (8) to derive $\pi_2(g(w)) = w$, which says that reading the state does not alter the external world.

Example 4.6. A comodel W of the theory of non-determinism, cf. Example 2.5, is given by a cooperation

$$c : |W| \to \mathsf{bool} \times |W|.$$

The cooperation must satisfy (the interpretations of) associativity, idempotency, and commutativity. Commutativity is problematic because we get from it that if $c(w) = (b, w')$ then also $c(w) = (\mathsf{not}\,b, w')$, implying the nonsensical requirement $b = \mathsf{not}\,b$. It appears that comodels of non-determinism require fancier categories than the good old sets.

4.2. **Tensoring comodels and models.** If we construe the elements of a T-model $|M|$ as effectful computations and the elements of a T-comodel $|W|$ as external environments, it is natural to ask whether M and W interact to give an account of running effectful programs in effectful external environments. Let $\sim_\mathsf{T}$ be the least equivalence relation on $|M| \times |W|$ such that, for every operation symbol $\mathsf{op} : P \rightsquigarrow A$ in Σ_T, and for all $p \in P$, $a \in A$, $\kappa : A \to |M|$, and $w, w' \in |M|$ such that $[\![\mathsf{op}]\!]^W(p, w) = (a, w')$,

$$(9) \qquad ([\![\mathsf{op}]\!]_M(p, \kappa), w) \sim_\mathsf{T} (\kappa(a), w').$$

Define the *tensor* $M \otimes W$ to be the quotient set $(|M| \times |W|)/\sim_\mathsf{T}$.

The tensor represents the interaction of M and W. The equivalence $\sim_\mathsf{T}$ in (9) has an operational reading: to perform the operation $[\![\mathsf{op}]\!]_M(p, \kappa)$ in the external environment w, run the corresponding cooperation $[\![\mathsf{op}]\!]^W(p, w)$ to obtain $a \in A$ and a new environment w', then proceed by executing $\kappa(a)$ in environment w'.

Example 4.7. Let us compute the tensor of $M = F_{\mathsf{State}}(X)$, the free model of the theory of state generated by X, and the comodel W defined by

$$|W| := S, \qquad [\![\mathsf{get}]\!]^W := \lambda s . (s, s), \qquad [\![\mathsf{put}]\!]^W := \lambda(s, t) . s.$$

We may read the equivalences

$$([\![\mathsf{get}((), \kappa)]\!]_M, s) \sim_{\mathsf{State}} ([\![\kappa]\!]_M(s), s),$$
$$([\![\mathsf{put}(t, \kappa)]\!]_M, s) \sim_{\mathsf{State}} ([\![\kappa]\!]_M(), t),$$

from left to right as rewrite rules which allows us to "execute away" all the operations until we are left with a pair of the form $([\![\mathsf{return}\ x]\!]_M, s)$. Because $([\![\mathsf{return}\ x]\!]_M, s) \sim_{\mathsf{State}} ([\![\mathsf{return}\ y]\!]_M, t)$ implies $x = y$ and $s = t$ (the proof of which we skip), it follows that $M \otimes W$ is isomorphic to $X \times S$. In other words, the execution of a program in an initial state always leads to a return value paired with the final state.

The next time the subject of tensor products comes up, you may impress your mathematician friends by mentioning that you know how to tensor software with hardware.

5. Making a programming language

The mathematical theory of algebraic effects and handlers may be used in programming language design, both as a mathematical foundation and a source of inspiration for new programming concepts. This is a broad topic which far exceeds the purpose and scope of these notes, so we only touch on the main questions and issues, and provide references for further reading.

Figure 1 shows the outline of a core language based on algebraic theories, as presented so far. Apart from a couple of changes in terminology and notation there is nothing new. Instead of generators and generating sets we speak of *values* and *value types*, and instead of trees and free models we speak of *computations* and *computation types*. The computation type $A \mathbin{!} \{\mathsf{op}_1, \ldots, \mathsf{op}_k\}$ corresponds to the free model $F_\mathsf{T}(A)$ where T is the theory with operations $\mathsf{op}_1, \ldots, \mathsf{op}_k$ without any equations. The rest of the table should look familiar. And operational semantics and typing rules still have to be given. For these we refer to Matija Pretnar's tutorial [12], and to [2, 11] for a more thorough treatment of the language.

The programming language in Figure 1 can express only the terminating computations. To make it more realistic, we should add to it general recursion and allow non-terminating computations. Such modifications cannot be accommodated by the set-theoretic semantics, but they can be handled by domain theory, as was shown in [2]. Therein you can find an

$$
\begin{array}{rll}
\text{Value } v \; ::= & x & \text{variable} \\[2pt]
& \mid \; \text{false} \;\mid\; \text{true} & \text{boolean constant} \\[2pt]
& \mid \; \lambda x \,.\, c & \text{function} \\[2pt]
& \mid \; \text{handler} \{ \, \text{return } x \mapsto c_r, & \text{handler} \\[2pt]
& \qquad \ldots, \text{op}_i(x; k) \mapsto c_i, \ldots \} & \\[4pt]
\text{Computation } c \; ::= & \text{return } v & \text{pure computation} \\[2pt]
& \mid \; \overline{\text{op}}(v) & \text{operation} \\[2pt]
& \mid \; \text{do } x \leftarrow c_1 \text{ in } c_2 & \text{sequencing} \\[2pt]
& \mid \; \text{if } v \text{ then } c_1 \text{ else } c_2 & \text{conditional} \\[2pt]
& \mid \; v_1 \, v_2 & \text{application} \\[2pt]
& \mid \; \text{with } v \text{ handle } c & \text{handling} \\[4pt]
\text{Value type } A, B \; ::= & \text{bool} & \text{boolean type} \\[2pt]
& \mid \; A \times B & \text{product type} \\[2pt]
& \mid \; A \to \underline{C} & \text{function type} \\[2pt]
& \mid \; \underline{C} \Rightarrow \underline{D} & \text{handler type} \\[4pt]
\text{Computation type } \underline{C}, \underline{D} \; ::= & A \,!\, \{\text{op}_1, \ldots, \text{op}_k\} &
\end{array}
$$

FIGURE 1. A core language with algebraic effects and handlers

adequate domain-theoretic semantics for algebraic effects and handlers with support for general recursion.

Once a programming language is in place, the next task is to explore its possibilities. Are user-defined operations and handlers good for anything? Practice so far has shown that indeed they can be used for all sorts of things, but also that it is possible to overuse and misuse them, just like any programming concept. Handlers have turned out to be a versatile tool that unifies and generalizes a number of techniques: exception handlers, backtracking and other search strategies, I/O redirection, transactional memory, coroutines, cooperative multi-threading, delimited continuations, probabilistic programming, and many others. As this note is already getting quite long, we recommend existing material [12, 3, 7] for further reading. For experimenting with handlers in practice, you can try out one of the languages that implements handlers. The first such language was Eff [1], but there are by now others. The Effects Rosetta Stone [6] is a good starting point to learn about them and to see how they compare. The Effect bibliography [5] is a good source for finding out what has been published in the area of computational effects.

6. Exploring new territories

Lastly, we mention several aspects of algebraic effects and handlers that have largely remained unexplored so far.

Perhaps the most obvious one is that existing implementations of effects and handlers largely ignore equations. In a sense this is expected and understandable. For (6) to define a handler $F_{\mathsf{T}}(X) \Rightarrow F_{\mathsf{T}'}(Y)$, the operation clauses h_i must satisfy the equations of T. In general it is impossible to check algorithmically whether this is the case, and so a compiler

or a language interpreter should avoid trying to do so. Thus existing languages with handlers solve the problem by ignoring the equations. This is not as bad as it sounds, because in practice we often want handlers that break the equations. Moreover, dropping equations just means that we work with trees as representatives of their equivalence classes, which is a common implementation technique (for instance, when we represent finite sets by lists). Nevertheless, incorporating equations into programming languages would have many benefits.

The idea of tensoring comodels and models as a mathematical explanation of the interaction between a program and its external environment is very pleasing, but has largely not been taken advantage of. There should be a useful programming concept in there, especially if we can make tensoring a user-definable feature of a programming language. The only known (to me) attempt to do so were the *resources* in an early version of Eff [3], but those disappeared from later versions of the language. May they see the light of day again.

At the Dagstuhl seminar [4] the topic of dynamic creation of computational effects was recognized as important and mostly unsolved. The operations of an algebraic theory are fixed by the signature, but in real-world situations new instances of computational effects are created and destroyed dynamically, for example, when a program opens or closes a file, allocates or deallocates memory, spawns or terminates a new thread, etc. How should such phenomena be accounted for mathematically? A good answer would likely lead to new programming concepts for general resource management. Once again, the only known implementation of dynamically created instances of effects was provided in the original version of Eff [3], although some languages allow dynamic creation of new effects by indirect means.

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