Modern Analysis 1 L5

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Contents

1 Superposition of Supremum

- THM: If $A, B \subseteq \mathbb{R}$ are nonempty and bounded above, then $A + B := \{a + b : a \in A, b \in B\}$ is bounded above and $sup(A + B) = sup(A) + sup(B)$.
  - Proof:
    Let $\alpha = sup(A), \beta = sup(B)$
    For each $a \in A$ and $b \in B$, we have $a \leq \alpha, b \leq \beta$
    So, $a + b \leq \alpha + \beta$ $(\mathbb{E})$
    Therefore, $a + b$ is an upper bound for $A + B$.
    Therefore, $sup(A + B) \leq sup(A) + sup(B)$ coz $sup(A) + sup(B)$ is "an" upper bound, and $sup(A + B)$ is the least upper bound.
    To prove the reverse equality: two approaches.
    * Option 1: show $\alpha + \beta \leq u$ for each upper bound u of $A + B$
      Let u be any upper bound for $A + B$
      Fix $a \in A$ and let $b \in B$ be arbitrary,
      Then, $a + b \leq u$
      $\therefore b \leq u - a$ for any $b \in B$
      $\therefore u - a$ is an upper bound for B $\Rightarrow \beta \leq u - a$
      $\therefore a \leq u - \beta$
      Now, unfix a, then $u - \beta$ is an upper bound for A.

So, $\alpha \leq u - \beta$
$\therefore \alpha + \beta \leq u$
$\therefore \alpha + \beta = sup(A + B)$.

* Option 2:
Spse $sup(A+B) < \alpha + \beta$ and then aim at $\Rightarrow\Leftarrow$
Write $\epsilon = \alpha + \beta - sup(A+B) > 0$
Then, $\alpha - \frac{1}{2}\epsilon < \alpha \Rightarrow \alpha - \frac{1}{2}$ is not an upper bound for A coz $\alpha = lub(A)$
Thus, $\exists a \in A : \alpha - \frac{1}{2}\epsilon < a$
Similarly, $\exists b \in B : \beta - \frac{1}{2}\epsilon < b$
But now, $a + b > \alpha + \beta - \epsilon = sup(A+B)$
$\Rightarrow\Leftarrow$

– $\mathbb{E}$ $AB = \{ab : a \in A, b \in B\}$, $sup(AB) = sup(A)sup(B)$. This works if $A, B \subseteq \mathbb{R}^+$ and A, B are nonempty and bounded above.

2 Square Root

- THM: Each positive real numebr has a unique positive square root.
  - Proof:
    Let $a > 0$
    $\exists! \alpha \in \mathbb{R}^+$ s.t. $\alpha\alpha = a$

    1. Uniqueness:

       * Option 1:
         Spse $\alpha\alpha = a = \alpha'\alpha'$ $(\alpha, \alpha' \in \mathbb{R}^+)$
         Then $0 = \alpha'\alpha' - \alpha\alpha = (\alpha' + \alpha)(\alpha' - \alpha)$
         Since $\alpha' + \alpha > 0$, $\alpha' - \alpha = 0$
         So, $0 = \alpha' - \alpha \Rightarrow \alpha' = \alpha$. Hence, uniqueness is proved.
       * Option 2:
         Spse $\alpha' \neq \alpha$ and then use trichotomy and aim for $\Rightarrow\Leftarrow$
         The above assumption leads to either $\alpha' < \alpha \bigvee \alpha' > \alpha$. This implies that $\alpha'\alpha' < \alpha\alpha \bigvee \alpha'\alpha' > \alpha\alpha$, both of which lead to $\Rightarrow\Leftarrow$.
    2. Existence:
       Define $A = \{x \in \mathbb{R}^+ : x^2 < a\}$
       Then, $A \neq \varnothing$ and bounded above. Here, we need to show **nonempty** and **Bounded above**.

- Nonempty:
  If $a < 1, x = a$ coz $a^2 < a$
  If $a = 1, x = \frac{1}{2}a$
  If $a > 1, x = 1$.
  Therefore, A contains any real x satisfying $0 < x < a \wedge 1$
  (Here, $\wedge$ means take the minimum value.)
  $x < a \wedge 1 \Rightarrow x^2 < x < a$
- Bounded above by $a \vee 1$ (Here, $\vee$ means take the maximum value.)
  Coming Next Lecture...