Modern Analysis 1 L8 and L9

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1 Recall

- **AMGM** If $a_1, \cdots, a_n \geq 0$, then $\frac{a_1+\cdots+a_n}{n} \geq (a_1 \cdot \cdots \cdot a_n)^{\frac{1}{n}}$ with equality $\Leftrightarrow a_1 = \cdots = a_n$.

2 Example 1

$\mathbb{E}$ Maximize $x(1-x)^n$ on $(0,1)$.

Solution:

Consider the following numbers $a_0 = x, a_1 = 1-x, ..., a_n = 1-x$, according to AMGM we have

$(a_0 a_1 \cdots a_n)^{\frac{1}{n+1}} \leq \frac{a_0+a_1+\cdots+a_n}{n+1}$

$\Rightarrow (nx(1-x)^n)^{\frac{1}{n+1}} \leq \frac{n}{n+1}$

$\Rightarrow nx(1-x)^n \leq \left(\frac{n}{n+1}\right)^{n+1}$

$\Rightarrow x(1-x)^n \leq \frac{n^n}{(n+1)^{n+1}}$

Equality holds $\Leftrightarrow a_0 = a_1 = \cdots = a_n$, that is $x = \frac{1}{n+1}$.

3 Example 2

Given $a_1, \cdots, a_N > 0$**, show that if** $\pi \in S_N$**,** $\sum_{n=1}^{N} \frac{a_{\pi(n)}}{a_n} \geq N$

For example, $a, b, c > 0$, $\frac{b}{c} + \frac{c}{a} + \frac{a}{b} \geq 3$

Proof. Apply AMGM to $\frac{a_{\pi_q}}{a_1}, \cdots, \frac{a_{\pi_N}}{a_N}$

$\frac{\frac{a_{\pi_1}}{a_1}+\cdots+\frac{a_{\pi_N}}{a_N}}{N} \geq (\frac{a_{\pi_1}}{a_1} \cdot \cdots \cdot \frac{a_{\pi_N}}{a_N})^{\frac{1}{N}}$

$\because \pi$ is permutation of N

$\therefore$ RHS's numerator is exactly same as the denominator with different order

$\therefore$ RHS $= 1$

$\therefore \frac{\frac{a_{\pi_1}}{a_1}+\cdots+\frac{a_{\pi_N}}{a_N}}{N} \geq 1 \therefore \frac{a_{\pi_1}}{a_1} + \cdots + \frac{a_{\pi_N}}{a_N} \geq N$ with equality holds if $\pi =$ identity, i.e. $a_{\pi_i} = a_i$ $\square$

4 Uniqueness of $\mathbb{R}$

Theorem 4.1. *If F and F' are fields with compatible total orders having the least upper bound property, then there exists a unique* ***field isomorphism*** *from $F \to F'$.*

Remark. *The smallest (or prime) subfield of F is an isomorphic copy of the rationals, say $\mathbb{Q}$. Similarly, $F' \supseteq \mathbb{Q}'$.*

Fact: There is a unique field isomorphism $\phi_0 : \mathbb{Q} \to \mathbb{Q}'$

Fact: If $a \in F$, and $\mathbb{Q}_a := \{q \in \mathbb{Q} : q < a\}$, then $sup\mathbb{Q}_a = a$

Proof. $\mathbb{Q}_a \neq$ due to Archimedies.

Thus, $\mathbb{Q}_a$ is nonempty and bounded above.

The $\mathbb{D}$ of $\mathbb{Q}_a \Rightarrow a$ is an upper bound for $\mathbb{Q}_a$

Now, we need to show a is the least upper bound by showing if there exists a $t < a$, then t is not an upper bound.

Let $t < a$

Archimedies gives us $q \in \mathbb{Q} s.t. t < q < a$

$a < a \bigwedge q \in \mathbb{Q} \Rightarrow q \in \mathbb{Q}_a$

Therefore, $t < q$ implies t is not an upper bound for $\mathbb{Q}_a$. Thus, $sup\mathbb{Q}_a = a$. $\square$

Let's define a function $\phi : F \to F'$ as $\phi(a) = sup[\phi_0(\mathbb{Q}_a)]$, where $\phi_0 : \mathbb{Q} \to \mathbb{Q}_a$. Then, we have the following claims:

- $\phi_0(\mathbb{Q}_a) \neq \varnothing$ because ϕ_0 is isomorphism, i.e. bijective mapping, and $\mathbb{Q}_a \neq \varnothing$.

- $\phi_0(\mathbb{Q}_a)$ is bounded above by Archimedies.

  Proof. Choose $n > a, n \in \mathbb{Z}$, because integers are not bounded above.

  Now, $q \in \mathbb{Q}_a \Rightarrow n > a > q$

$n > q \Rightarrow \phi_0(n) > \phi_0(q)$ because the isomorphism ϕ_0 preserves order ($\mathbb{E}$).

Therefore, $\phi_{0(}n)$ is an upper bound for $\phi_0(\mathbb{Q}_a)$ □

- $\phi : F \to F'$ is an isomorphism.

Proof. We need to show:

1. $\phi(a+b) = \phi(a) + \phi(b)$

   First, $\mathbb{Q}_{a+b} = \mathbb{Q}_a + \mathbb{Q}_b$.

   If so, (assumping this is correct), then $\phi_o(\mathbb{Q}_{a+b}) = \phi_0(\mathbb{Q}_a) + \phi_0(\mathbb{Q}_b)$, because ϕ_0 is isomorphism.

$$\begin{aligned} \therefore \phi(a+b) &= sup[\phi_0(\mathbb{Q}_{a+b})] \\ &= sup[\phi_0(\mathbb{Q}_a) + \phi_0(\mathbb{Q}_b)] \\ &= sup[\phi_0(\mathbb{Q}_a)] + sup[\phi_0(\mathbb{Q}_b)] \\ &= \phi(a) + \phi(b) \end{aligned}$$

   Now, to prove $\mathbb{Q}_{a+b} = \mathbb{Q}_a + \mathbb{Q}_b$, we need to show $\mathbb{Q}_{a+b} \subseteq \mathbb{Q}_a + \mathbb{Q}_b$ and $\mathbb{Q}_{a+b} \supseteq \mathbb{Q}_a + \mathbb{Q}_b$

   "$\subseteq$" is easy: if $x \in \mathbb{Q}_a$ and $y \in \mathbb{Q}_b$, then $x + y < a + b$ and $x + y \in \mathbb{Q}_{a+b}$

   "$\supseteq$" is not easy because the set $\mathbb{Q}_a$ is defined as q strictly less ($<$) than a.

   Let $z \in \mathbb{Q}_{a+b}$, then $\mathbb{Q} \ni z < a + b$.

   We want $z = x + y$ for some $x \in \mathbb{Q}_a$ and $y \in \mathbb{Q}_b$.

   Since $z - a < b$, according to Archimedes: $\exists y \in \mathbb{Q}\ s.t.\ z - a < y < b \Rightarrow y \in \mathbb{Q}_b$

   Let $x = z - y$, certainly $x \in \mathbb{Q}$.

   Since $z - b < a$, again by Archimedes, $\exists x \in \mathbb{Q}$

   Check $x < a$ **HOW?????**

2. $\phi(ab) = \phi(a)\phi(b)$

□

5 Isomorphism

Definition 5.1. *Given two groups $(G, *)$ and $(H, \odot)$, a* ***group isomorphism*** *from $(G, *)$ to $(H, \odot)$ is a bijective group homomorphism from G to H. This means that a group isomorphism is a bijective function $f : G \to H$such that $\forall u, v \in G$, it holds that*

$$f(u * v) = f(u) \odot f(v)$$

The two groups G and H are isomorphic if there exists an isomorphism from one to the other, denoted as $(G, *) \cong (H, \odot)$

6 Homomorphism

Definition 6.1. *Let $(G, *)$ and $(H, \odot)$ be groups. A homomorphism $f : G \to H$ is a function s.t. $\forall g_1, g_2 \in G$,*

$$f(g_1 * g_2) = f(g_1) \odot f(g_2)$$

Definition 6.2. *Equivalently, the function $f : G \to H$ is a group homomorphism if whenever $a, b, c \in G$*

$$a * b = c \Rightarrow f(a) \odot f(b) = f(c)$$

$\mathbb{M}$ The purpose of defining a group homomorphism is to create functions that **preserve** the algebraic structure. According to the alternative definition, the group H in some sense has a similar algebraic structure as G and the homomorphism h preserves that.

Proposition 6.0.1. *Let G and H be groups, written multiplicatively and let $f : G \to H$ be a homomorphism. Then,*

1. *$f(1) = 1$, where the 1 on the left is the identity in G and the 1 one the right is the identity in H. In other words, f taks the identity in G to the identity in H.*
2. *$\forall\ g \in G$, $f(g^{-1}) = \big(f(g)\big)^{-1}$*