Modern Analysis 1 Lecture 3

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Contents

1 Minimal

- $\mathbb{D}$ B is an ordered set, say $\alpha \in B$ is minimal iff $\forall x \in B, (x \leq \alpha \Rightarrow x = \alpha)$.
- Claim: Let B be the set of all upper bds for $A \subseteq S$. For $\alpha \in B$, the following are equivalent.
  1. α is minimal;
  2. $\forall x(x < \alpha \Rightarrow x \notin B)$ This is Rudins $\mathbb{D}$ of least UB.
- Proof of the Claim:
  We want to show that $x < \alpha \Rightarrow x \notin B$ is equivalent to α is minimal, i.e. $x < \alpha \Rightarrow x \notin B \equiv x \in B \Rightarrow (x \leq \alpha \Rightarrow x = \alpha)$
  From right to left:
  $x \in B \Rightarrow (x \leq \alpha \Rightarrow x = \alpha) \equiv (x \in B \bigwedge x \leq \alpha) \Rightarrow x = \alpha$
  $\equiv (x \leq \alpha \bigwedge x \in B) \Rightarrow x = \alpha$
  $\equiv x \leq \alpha \Rightarrow (x \in B \Rightarrow x = \alpha)$
  $\equiv x \leq \alpha \Rightarrow (x \neq \alpha \Rightarrow x \notin B)$

$\equiv x \leq \alpha \bigwedge x \neq \alpha \Rightarrow x \notin B$
$\equiv x < \alpha \Rightarrow x \notin B$

2 Least vs Minimal

- THM: Least $\Rightarrow$ Minimal. In a total ordered set, Least $\Leftrightarrow$ Minimal.
  - $\mathbb{M}$ For partially ordered set S, if $s \in S$ is the least element, then it is also a minimal element. In a total order set S, if $s \in S$ is a minimal element, then it is also a least element and vice versa.
  - $\mathbb{M}$ For a poset, if there exists a least element, then it is unique. Set $\{0, 0, 1\}$ is not a counterexample becaues according to the $\mathbb{D}$ of set, "In mathematics, a set is defined as a collection of **distinct**, well-defined objects forming a group.", $\{0, 0, 1\} \equiv \{0, 1\}$. Hence, the least element is still unique!
  - $\mathbb{M}$ in a poset, minimal element may not unique. For example:

```
C1 C2 C3 C4 C5 C6
  \ | /    \ | /
   \|/      \|/
    B1       B2
     \      /
      \    /
       \ /
        A
```

    In this case, if we define set S s.t. S contains all sets above A. Then B1 and B2 are the minimal elements of set S. Here, $A \leq B = A \subseteq B$.

3 Compatibility

- $\mathbb{D}$ Let F be a field. The total strict order $>$ on F is compatible iff $\forall a, b, c \in F$,
  1. $a > b, c \Rightarrow a + c > b + c$
  2. $a > 0, b > 0 \Rightarrow ab > 0$
- Remarks

1. $a > 0 \Rightarrow -a < 0$
   Proof:
   $a + (-a) > 0 + (-a) \Rightarrow 0 > -a \equiv -a < 0$
2. $a \neq 0 \Rightarrow a^2 > 0$
   Proof:
   $a \neq 0 \Rightarrow a > 0 \bigvee a < 0$

   - case 1: $a > 0$. Replace b = a of the 2 in $\mathbb{D}$, i.e. $a > 0, a > 0 \Rightarrow aa > 0 \equiv a^2 > 0$

   - case 2: $a < 0$. It implies that $-a > 0$. Again, by replacing $b = -a$ of the 2 in $\mathbb{D}$, then 2 becomes $-a > 0, -a > 0 \Rightarrow (-a)(-a) > 0$.
     But, $(-a)(-a) = a \cdot a$
     Proof this "But":

     * Option 1:
       First, $a(-a) + (-a)(-a) = (a + (-a))(-a) = 0(-a) = 0$. This implies that $(-a)(-a)$ is the additive inverse of $a(-a)$.
       Second, $a(-a) + aa = a(-a + a) = a \cdot 0 = 0 \Rightarrow a \cdot a$ is also the additive inverse of $a(-a)$.
       But the additive inverse is unique.
       Thus, $(-a)(-a) = a \cdot a = a^2$
     * Option 2:
       First, prove $(-1)a = -a$.

       · $\mathbb{E}$ Prove the above statement.
       Then, $(-a)(-a) = (-1)a(-1)a = a^2$.

4 Negative Elements

- $\mathbb{D}$ $P = \{x \in F | x > 0\}$ the positive elements of F. Then, $-P = \{-x | x \in P\} = \{x \in F | x < 0\}$.

- Claim: $F = P \cup \{0\} \cup -P$. In other words, F can be written as a disjoint union and P, the set of all positive elements of F, is closed under $+$ and $\cdot$.
  Proof:

1. P is closed under addition, i.e. $a > 0, b > 0 \Rightarrow a + b > 0$
   $a > 0 \Rightarrow a + b > 0 + b$
   and $b > 0 \Rightarrow 0 + b > 0 + 0$
   $\Rightarrow a + b > 0$
   Thus, P is closed under addition.
2. P is closed under multiplication. According to the second property of the $\mathbb{D}$ of compatibility, it is closed.

- THM: (No Order) Let $F = P \cup \{0\} \cup -P$ be a disjoint union in which P is closed under "+" and "·".
  Then, the rule, $a > b := a - b \in P$, defines a compatible total (strict) order on F.
  - Proof: We must show $>$ satisfies trichotomy, transitivity, and the compatibility. Note: trichotomy + transitivity $\equiv$ totality!
    1. Trichotomy:
       let $a, b \in F$, then $a - b \in P \cup \{0\} \cup -P$
       $a - b \in P \Rightarrow a > b$
       $a - b \in \{0\} \Rightarrow a = b$
       $a - b \in -P \Rightarrow a < b$
       Thus, trichotomy is satisfied.
    2. Transitivity:
       $a > b, b > c \Rightarrow a - b, b - c \in P$
       Then, the summation $(a - b) + (b - c) = a - c \in P$ since P is closed under addition. Thus $a - c \in P \Rightarrow a > c$, i.e. transitivity is satisfied.
    3. Compatibility:
       Coming next lecture!

5 Disjoint Union

- $\mathbb{D}$ Formally, let $\{A_i : i \in I\}$ be a family of sets indexed by I. The **disjoint union** is of this family is the set $\bigsqcup_{i \in I} A_i = \bigcup_{i \in I} \{(x, i) : x \in A_i\}$.
- $\mathbb{M}$ In this disjoint set, all elements have the format as (x, i). Therefore, this union is generated to be disjoint since even if A_i and A_j are not disjoint, the index element will make the resultant element disjoint.