Modern Analysis 1 L6

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1 Nomenclature

$\vee$ means max and $\wedge$ means min

2 Squares

Given $a > 0$, $A = \{x \in \mathbb{R}^+ : x^2 < a\}$ is nonempty $A \supseteq (0, 1 \wedge a)$ and is bounded above by $1 \vee a$.

Proof. If $y > 1 \vee a \Rightarrow y^2 > y > a \Rightarrow y \notin A$.
$\therefore y \in A \Rightarrow y \leq 1 \vee a$ $\square$

Let $\alpha = sup(A)$

Claim. $\alpha^2 = a$

Proof. According to the Trichotomy, we want to rule out $\alpha^2 < a$ and $\alpha^2 > a$.

- Case 1: Suppose $\alpha^2 < a$
  Then, $\exists\ t > 0$ s.t. $(\alpha + t)^2 < a \Leftrightarrow \alpha^2 + 2\alpha t + t^2 < a \equiv 2\alpha t + t^2 < a - \alpha^2$.
  Choose t to be small enough to make

  1. $2\alpha t < \frac{a-\alpha^2}{2}$
     This implies that $t < \frac{a-\alpha^2}{4\alpha}$
  2. $t^2 < \frac{a-\alpha^2}{2}$
     This implies that $t < 1 \wedge \frac{a-\alpha^2}{2}$

  Therefore, if $0 < t < 1 \wedge \frac{a-\alpha^2}{4\alpha} \wedge \frac{a-\alpha^2}{2}$, then $(\alpha + t)^2 < a \Rightarrow (\alpha + t) \in A$.
  But, α is the least upper bound for A, $\Rightarrow\Leftarrow$
  Therefore, $\alpha^2 \not< a$.

- Case 2: Suppose $\alpha^2 > a$
  We can find $t < \alpha$ s.t. $(\alpha - t)^2 > a \Leftrightarrow \alpha^2 - 2\alpha t + t^2 > a \Leftarrow \alpha^2 - 2\alpha t > a \Leftarrow \alpha^2 - a > 2\alpha t$
  So, let $t < \frac{\alpha^2-a}{2\alpha} \wedge \alpha$ to get $\alpha - t > 0$ s.t. $(\alpha - t)^2 > a$.
  Now, $\alpha - t$ is an upper bound for A. But, α is the least upper bound for A, $\Rightarrow\Leftarrow$.

Therefore, $\alpha^2 = a$ $\square$

Remark. *In $\mathbb{R}$, the order structure can be recovered from the field structure because $a > 0 \Leftrightarrow a$ is a non-zero square.*

Theorem 2.1. *Let $f : [0,1] \to [0,1]$ be increasing. Then, f has a fixed point (i.e. $\exists\ x \in [0,1]$ s.t. $f(x) = x$)*

Proof. Consider $A = \{t \in [0,1] : t \leq f(t)\}$.
$A \neq \varnothing$ because $0 \in A$, and A is bounded above by 1.
Let $\alpha = supA$.
Claim: $f(\alpha) = \alpha$
Suppose we can show $\alpha \leq f(\alpha)$
Then $f \uparrow \Rightarrow f(\alpha) \leq f(f(\alpha))$
$\therefore\ f(\alpha) \in A$
$\therefore\ f(\alpha) \leq \alpha$ because α is an upper bound for A.
Now, all that remains is to show $f(\alpha)$ is an upper bound for A, because then $\alpha \leq f(\alpha)$ follows.
Let $t \in A$, then $t \leq f(t)$.
α is an upper bound for $A \Rightarrow t \leq \alpha$.
$f \uparrow \bigvee t \leq \alpha \Rightarrow f(t) \leq f(\alpha)$
$t \leq f(t) \bigvee f(t) \leq f(\alpha) \Rightarrow t \leq f(\alpha)$.
Thus, $f(\alpha)$ is an upper bound for $A \Rightarrow \alpha \leq f(\alpha)$.
We have showed that $f(\alpha) \leq \alpha$ and $\alpha \leq f(\alpha)$.
Therefore, $f(\alpha) = \alpha$. □

Remark. *This is a special case of a fixed-point theorem (due to Knaster and Taski) for complete lattices. The general gheorem leads to a nice proof of Schroder-Bernstein theorem.*

Theorem 2.2 (Schroder-Bernstein Theorem)**.** *If there exist injective functions $f : A \to B$ and $g : B \to A$ between the sets A, B, then there exists a bijective function $h : A \to B$.*