Mathematical Modeling of System (2)

Lecture Outlines

1. Review on Complex Variable
2. Review on The Laplace Transform
3. The Transfer Function of Linear Systems

1. Review on Complex Number

Complex Number. A complex number has a real part and an imaginary part, both of which are constant:

$$3 + j4 \qquad\qquad -5 + j2$$

Complex Variable. If the real part and/or imaginary part are variables, a complex quantity is called as a complex variable. In the Laplace transform we use the notation s as complex variable:

$$s = \sigma + j\omega$$

Complex Function. A complex function G(s) is a function of s such as

$$G(s) = \frac{1}{s+1}$$

2. Review on The Laplace Transform

- It is necessary to solve the mathematical model of the physical systems.
- The solution of the mathematical model can be obtained by using the Laplace transform method

Definition

- The Laplace transformation for a function of time, f(t), is

$$F(s) = \int_{0}^{\infty} f(t) e^{-st} dt$$

- The Inverse Laplace transformation is written as:

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st} ds$$

- Alternatively, the Laplace variable s can be considered to be differential operator so that

$$s \equiv \frac{d}{dt} \qquad \frac{1}{s} \equiv \int_{0-}^{t} dt$$

Laplace Transform Table

	Item no.	f(t)	F(s)
Impulse function	1.	$\delta(t)$	1
Step function	2.	$u(t)$	$\frac{1}{s}$
Ramp function	3.	$tu(t)$	$\frac{1}{s^2}$
	4.	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
	5.	$e^{-at}u(t)$	$\frac{1}{s+a}$
	6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
	7.	$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

Laplace Transform Properties

Item no.	Theorem		Name
1.	$\mathcal{L}[f(t)] = F(s)$	$= \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathcal{L}[kf(t)]$	$= kF(s)$	Linearity theorem
3.	$\mathcal{L}[f_1(t) + f_2(t)]$	$= F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathcal{L}[e^{-at}f(t)]$	$= F(s+a)$	Frequency shift theorem
5.	$\mathcal{L}[f(t-T)]$	$= e^{-sT}F(s)$	Time shift theorem
6.	$\mathcal{L}[f(at)]$	$= \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathcal{L}\left[\frac{df}{dt}\right]$	$= sF(s) - f(0-)$	Differentiation theorem
8.	$\mathcal{L}\left[\frac{d^2f}{dt^2}\right]$	$= s^2F(s) - sf(0-) - \dot{f}(0-)$	Differentiation theorem
9.	$\mathcal{L}\left[\frac{d^nf}{dt^n}\right]$	$= s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorem
10.	$\mathcal{L}\left[\int_{0-}^{t} f(\tau)\,d\tau\right]$	$= \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty)$	$= \lim_{s\to 0} sF(s)$	Final value theorem[1]
12.	$f(0+)$	$= \lim_{s\to\infty} sF(s)$	Initial value theorem[2]

Example 2a:

$$\frac{dy(t)}{dt} + 3y(t) = 0 \qquad y(0) = 3$$

$$\Downarrow$$

$$\left[sY(s) - y(0)\right] + 3Y(s) = 0$$

$$sY(s) + 3Y(s) = 3$$

Example 2b:

$$\frac{d^2 y(t)}{dt^2} + 12\frac{dy(t)}{dt} + 32y(t) = 32u(t) \qquad y(0) = \dot{y}(0) = 0$$

$$\Downarrow$$

$$s^2Y(s) + 12sY(s) + 32Y(s) = \frac{32}{s}$$

Final Value Theorem (FVT)

$$f(\infty) = \lim_{s \to 0} sF(s)$$

This theorem applies, if and only if, all poles of sF(s) lies in the left half s plane (poles on the imaginary axis and in the right half plane are excluded).

Example 2c:

Find f(∞) for the following system

$$F(s) = \frac{1}{s(s+1)}$$

Example 2d:

Find f(∞) for the following system

$$F(s) = \frac{5}{s(s^2 + 49)}$$

Inverse Laplace Transform

The inverse Laplace transform can be obtained by use of the inversion integral as

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st} ds$$

However, the inversion integral is complicated.

A convenient method for obtaining inverse Laplace transform is to use a table of Laplace transform.

Example 2e:

Find the inverse Laplace transform of

$$F(s) = \frac{3}{s+4}$$

$$f(t) = ?$$

Example 2f:

Find the inverse Laplace transform of

$$F(s) = \frac{s+6}{s^2+2s+10}$$

Partial Fraction Expansion Method

Partial fraction expansion method can be used to find the inverse Laplace transform of a complicated function, we can convert the function to a sum of simpler terms for which we know the inverse Laplace transform.

$$F(s) = \frac{B(s)}{A(s)} \Rightarrow F(s) = F_1(s) + F_2(s) + \cdots + F_n(s)$$

$$f(t) = \ell^{-1}\left[F_1(s)\right] + \ell^{-1}\left[F_2(s)\right] + \cdots + \ell^{-1}\left[F_n(s)\right]$$

$$= f_1(t) + f_2(t) + \cdots + f_n(t)$$

We will consider three cases and show an F(s) can be expanded into partial fractions:

- Case 1: Roots of denominator A(s) are real and distinct,
- Case 2: Roots of denominator A(s) are real and repeated,
- Case 3: Roots of denominator A(s) are complex conjugates

Case 1: Roots of denominator A(s) are real and distinct.

Consider F(s) is written as.

$$F(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} \qquad for \quad m<n$$

If F(s) involves distinct poles only, then can be written as

$$F(s) = \frac{B(s)}{A(s)} = \frac{a_1}{(s+p_1)} + \frac{a_2}{(s+p_2)} + \cdots + \frac{a_k}{(s+p_k)} + \cdots + \frac{a_n}{(s+p_n)}$$

Example 2g:

Find the inverse Laplace transform of

$$F(s) = \frac{32}{s\left(s^2 + 12s + 32\right)}$$

Case 2: Roots of denominator A(s) are real and repeated.

Consider F(s) written as

$$F(s) = \frac{B(s)}{A(s)} = \frac{B(s)}{(s+p)^n}$$

F(s) can be written as

$$\mathrm{F(s)} = \frac{\mathrm{B(s)}}{(\mathrm{s}+\mathrm{p})^{\mathrm{n}}} = \frac{\mathrm{b}_1}{(\mathrm{s}+\mathrm{p})^{\mathrm{n}}} + \frac{\mathrm{b}_2}{(\mathrm{s}+\mathrm{p})^{\mathrm{n}-1}} + \cdots + \frac{\mathrm{b}_{\mathrm{n}}}{(\mathrm{s}+\mathrm{p})}$$

Example 2h:

Find the inverse Laplace transform of

$$F(s) = \frac{2}{(s+1)(s+2)^2}$$

Case 3: Roots of denominator A(s) are complex conjugates.

Partial fraction method of F(s) with real roots in the denominator can be used for complex and imaginary roots. However the residues are complex conjugates. After taking the inverse Laplace, the result can be simplified by using the following Euler equations:

$$\frac{e^{j\theta} + e^{-j\theta}}{2} = \cos\theta \qquad \frac{e^{j\theta} - e^{-j\theta}}{2j} = \sin\theta$$

Example 2i:

Find the inverse Laplace transform of

$$F(s) = \frac{3}{s\left(s^2 + 2s + 5\right)}$$

Solving Differential Equation

- Obtain the differential equations
- Obtain the Laplace transform of the differential equations
  - Use the Laplace transform table and properties of Laplace transform
- Solving the resulting algebraic transform of the variable of interest
  - Refer to the Laplace transform table to find the inverse Laplace transform
  - Use partial fraction expansion method if necessary

Exercise 2a:

Find the inverse Laplace transform of

1) $$F_1(s) = \frac{6s+3}{s^2}$$

2) $$F_2(s) = \frac{6s}{(s+1)(s+2)^2}$$

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Exercise 2b:

What is the solution of the following differential equation?

1) $2\frac{d^2x(t)}{dt^2}+7\frac{dx(t)}{dt}+3x(t)=0, \quad x(0)=3, \dot{x}(0)=0$

2) $\frac{d^2x(t)}{dt^2}+2\frac{dx(t)}{dt}+10x(t)=e^{-t}, \quad x(0)=0, \dot{x}(0)=0$

3. Transfer Function (TF)

Definition

- Transfer function G(s) of a linear system is defined as the ratio of the Laplace transform of the output variable to the Laplace transform of the input variable, with all initial conditions assumed to be zero.

$$G(s) = \frac{Output(s)}{Input(s)}$$

TF G(s) can be expressed by $G(s) = \dfrac{B(s)}{A(s)}$

where

$$A(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_0$$

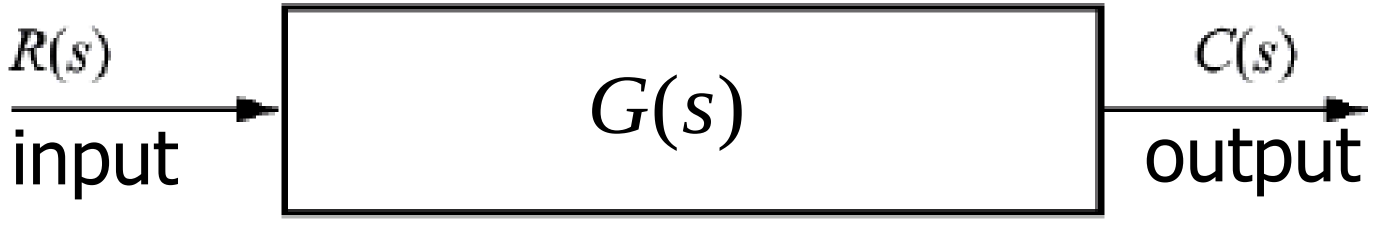
$$B(s) = b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \ldots + b_0$$

A(s)=0 is the **characteristic equation**, and *n* is the order of the system.

System poles are the roots of A(s).
System zeros are the roots of B(s).

Block Diagram

- The system transfer function is also represented by block diagram.
- Block diagram consist of unidirectional, operational block that represents system transfer function

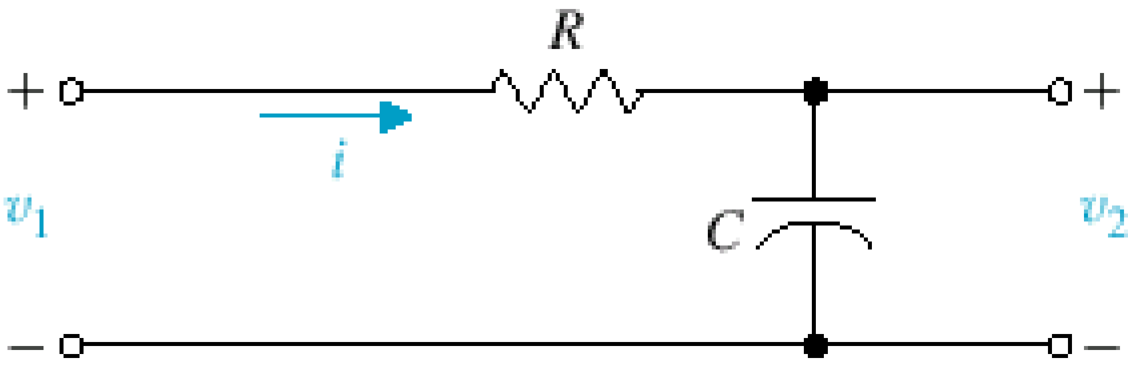


Electrical Network TF

TABLE 2.3 Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)\, d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

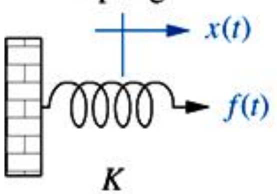
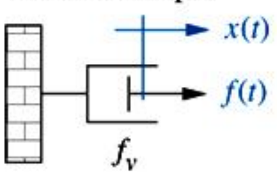
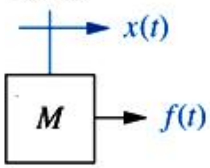
Example 3a:

input →  ← output

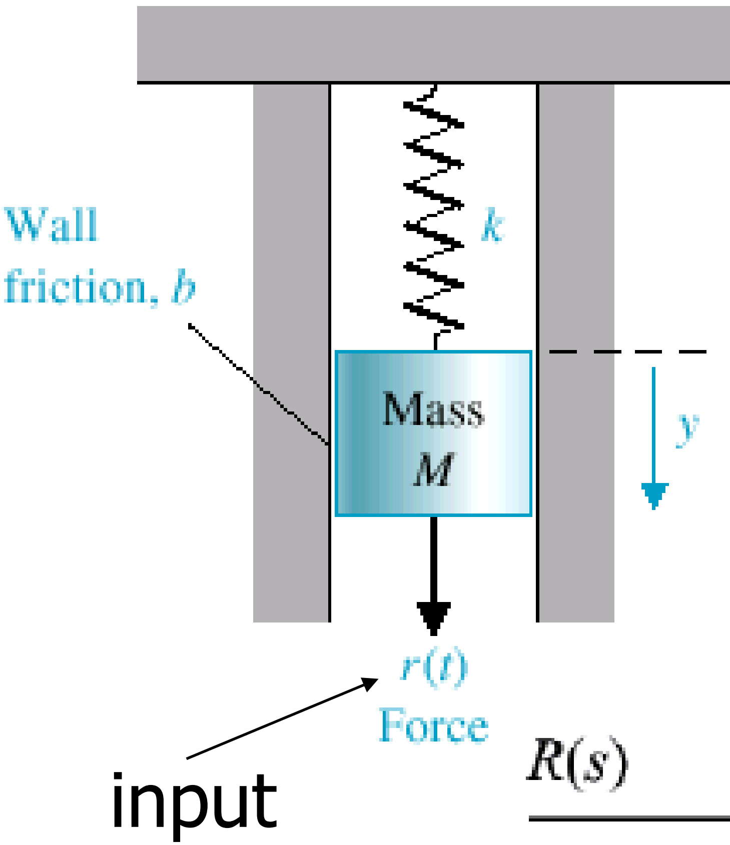
$$G(s) = \frac{V_2(s)}{V_1(s)} = ?$$

Translational mechanical system TF

TABLE 2.4 **Force-velocity, force-displacement, and impedance translational relationships for springs, viscous dampers, and mass**

Component	Force-velocity	Force-displacement	Impedence $Z_M(s) = F(s)/X(s)$
Spring 	$f(t) = K \int_0^t v(\tau)d\tau$	$f(t) = Kx(t)$	K
Viscous damper 	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
Mass 	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

Example 3b:



$$Ms^2Y(s) + bsY(s) + kY(s) = R(s)$$

$$\Downarrow$$

$$G(s) = \frac{Y(s)}{R(s)} = \frac{1}{Ms^2 + bs + k}$$

$$R(s) \longrightarrow \boxed{\frac{1}{Ms^2 + bs + k}} \longrightarrow Y(s)$$

Exercise 3a:

For each of the following Transfer Function, write the corresponding differential equation

(a) $$\frac{X(s)}{F(s)} = \frac{1}{s^2 + 2s + 7}$$

(c) $$\frac{X(s)}{F(s)} = \frac{1}{s^3 + 8s^2 + 9s + 15}$$

(b) $$\frac{X(s)}{F(s)} = \frac{10}{(s+7)(s+8)}$$

Exercise 3b:

Find the transfer function $G(s) = \dfrac{V_L(s)}{V(s)}$