The Application of GUP-deformed Quantum Mechanics to Quantum Entangled States

MS Thesis

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Final Approval

It is certified that this thesis titled ***"The Application of GUP-deformed Quantum Mechanics to Quantum Entangled States"*** submitted by Syed Masood Ahmad Shah Bukhari (Reg.No.330-FBAS/MSPHY/F14) is fully adequate to qualify, in scope and quality, for a dissertation of MS Physics.

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To my parents and family who taught me how the clock of patience ticks

<u>Declaration</u>

I hereby certify that this MS thesis entitled ***"The Application of GUP-deformed Quantum Mechanics to Quantum Entangled States"*** is my own work carried out at Department of Physics, International Islamic University, Islamabad, Pakistan. Due credit has been given to the sources thereof. Further, no part of it, in any version, has been submitted at any Institute/University elsewhere.

Syed Masood Ahmad Shah Bukhari

Forwarding Sheet by Research Supervisor

The thesis titled ***"The Application of GUP-deformed Quantum Mechanics to Quantum Entangled States"*** submitted by Syed Masood Ahmad Shah Bukhari in partial fulfillment of MS degree in physics has been completed under my guidance and supervision. I am satisfied with the quality of this thesis work and allow him to submit it to the Department of Physics, International Islamic University, Islamabad, Pakistan

Date:--------------------

Dr. Jamil Raza

Adjunct Professor,

Department of Physics, IIUI

[Hallowed be] He who has created seven heavens in full harmony with one another: no fault will thou see in the creation of the Most Gracious. And turn thy vision [upon it] once more: canst thou see any flaw? Yea, turn thy vision [upon it] again and yet again: [and every time] thy vision will fall back upon thee, dazzled and truly defeated.
——Al-Mulk (The Dominion), Quran, 67:3-4

Acknowledgments

........And when (Solomon) saw it truly before him, he exclaimed: "This is [an outcome] of my Sustainer's bounty, to test me as to whether I am grateful or ungrateful! However, he who is grateful [to God] is but grateful for his own good; and he who is ungrateful [should know that], verily, my Sustainer is self-sufficient, most generous in giving!"
——*From the story of the Queen of Saba, Quran, 27:40*

This master thesis work marks the end of my education at *International Islamic University, Islamabad, Pakistan*. My academic journey so far has been filled with memories, memories linked with so many people and incidents. As tradition of morality calls for me to pay my sincere gratitude to all those who have been very intrumental so far in my academic and personal life, here goes it.

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Abstract

The existence of a minimal measurable length in quantum gravity modifies the Heisenberg uncertainty principle to a new generalized uncertainty principle (GUP) along with a modified algebra. This also gives rise to a new class of spaces called non-commutaive spaces. In the present work, we have investigated the effects of quantum gravity through this modified Heisenberg algebra on some basic quantum mechanical systems like harmonic oscillator, Landau levels, Lamb shift etc. We have demonstrated that under a form of GUP, the microscopic structure of spacetime becomes dependent upon probing energy which resembles very closely that of rainbow gravity case.Finally, we have discussed quantum entanglement in the framework of GUP and non-commutative spaces. In particular, the analysis of entangled squeezed states of harmonic oscillator on noncomutative spaces indicate that these modified spaces impart extra degrees of freedom by which systems show a higher degree of entanglement as compared to ordinary spaces at low energy. Also, we have demonstrated that a straightforward application of modified Heisenberg algebra to the entangled state of a bipartite system also shows an increasing entanglement. As a conclusive remark, we have shown that gravity seems to enhance the quantum mechanical nature of systems, possibly shedding some light on the interface between quantum mechanics and gravity.

Contents

Chapter 1

Introduction

"But instinct is something which transcends knowledge. We have, undoubtedly, certain finer fibers that enable us to perceive truths when logical deduction, or any other willful effort of the brain, is futile." —*-Nikola Tesla*

Scientific exploration has ever led humans to unveil the deepest mysteries of universe, driving ahead the spirit of curiosity, pushing out the radius of this knowledge sphere like an expanding balloon from a known configuration to an unknown. The cornerstones of this odyssey comprise theory and experimention (or observation). An experimental finding may need a theoretical setup to map its structural organization. Or a theorist may put forward a testable prediction leaving the matter to an experimentalist to certify or to falsify his hypothesis. Among all other scientific disciplines, physics has been credited with being crown of this human breakthrough. In this thesis work, we will shed light on a particular ongoing struggle lying at frontiers of our knowledge: The quest for a viable theory of quantum gravity.

Einstein's theory of general relativity and quantum physics have been independently playing a leading role in our description of universe, working precisely in providing remarkable maps and guides of nature. But gravity has so far refused to follow the guidelines of quantum mechanics unlike other interactions, signalling an annoyance in scientific community regarding the understanding of two beautiful theories. Quantization of gravity in principle is not impossible, however, the theory holding both of them together suffers from the problem of a mathematical inconsistency: the emergence of infinities. This goes modelled in a well known statement that gravity is not renormalizable. So what is the remedy?

In scientific methodology, if a theory is not guided firmly by experimental data, there is no harm in seeking mathematical models based on the extracted knowledge from the whole setup of the original theory and formulate a subdomain of the original framework where predictions can be made testable. Such a formulation is called a phenomenogical model. For the case of quantum gravity, these efforts have been pressed in various directions with a varied emphasis. Here, we discuss a particular type of phenomenogical model of quantum gravity here that as been pursued for decades now. Minimal length physics and Generalized Uncerataintу Principle (GUP) along with Noncommutative spaces forms a considerable volume of such work. We start by giving introductory remarks about the problem of quantum gravity itself, followed by a introductroy discussion of GUP and noncommutative spacetimes.

1.0.1 Quantum Gravity: A Bird's-eye View

At present, there is no consistant and complete physical model that would essentially describe gravity by the principles of quantum mechanics. On one hand, Quantum Mechanics describes the physical behaviour of microscopic world and on other hand, General Relativity describes large scale structure of universe. Both of these theories work well in their respective regimes of application.However, certain difficulties arise when one tries to reconcile the principes of these two theories. A genuine question arises here; what lies so elegant with such a unification scheme? The answer is very loaded: It would help us to unveil the underlying physics of the early stages of our universe; its origin and evolution. It would also open a new way for unification of all fundamental interactions in the universe and the counting goes on. As Big Bang theory clearly indicated that our universe must have started of from an infinite dense, hot structure; it seems quite natural to study this microscopic structure as quantum effects start dominating. Moreover, an essential property of such a formulation would have a be the existence of fundamental units derived from Newtons constant G,Planck constant$\hbar$ and velocity of light$c$ viz. Planck length ($L_p = \sqrt{\frac{G\hbar}{c^3}} \approx 10^{-35}m$),Planck mass ($M_p = \frac{\hbar}{cL_p} \approx 10^{-5}gms$), Planck energy ($E_p = M_p c^2 \approx 10^{18}GeV$), and Planck time ($\frac{L_p}{c} \approx 10^{-42}s$) [1]. Unfortunately, it is very difficult to get a firm picture of how a quantized fabric of spacetime would look like. A number of models have been proposed to achieve such a unified framework of gravity and quantum mechanics,but we are still far from having a complete one. This is mainly due to the fact that quantum gravity is a very high energy phenomenon; so currently, we lack any experimental data to build the theory. Future labs might address this difficulty. It is worth mentioning here that with a full theory of quantum gravity, it will also pave a way for unification of gravity with all other interactions. In this section, we will have a brief overview of few approaches to quantum gravity.

One of the most succesful candidates for quantum gravity is string theory. It is a framework that comes up with remarkable physical insights. Ordinary quantum field theories describe the other three fundamental interactions very well at both high and low energies. Despite being succesful in describing gravity as an effective field theory at low energies [2], it fails to describe it at very high energies. In other words, we say gravity is non-renormalizable. Ordinary QFT's treat fundamental objects as point objects but string theory replaces the idea of these point objects by strings which are one dimensional extended objects whose each vibrational mode in a 10-dimensional(three ordinary space dimensions, one time and six extra space dimensions)space corresponds to a particle with different electric and other charges [3]. Proceeding with these ideas, we can easily appreciate that string theory is able to unify all particles and interactions into a single framework [4] where graviton emerges as a quantum of gravitational field. Much abstract mathematical rigor comes up when one combines string theory with that of supergravity (a unified model of GR and supersymmetry) into a hypothesisd eleven dimensional model called M-theory,which would hopefully provide a consistent theory of quantum gravity [5]. But all of these approaches still need to overcome certain theoretical anamolies.

Second most promising candidate theory is that of Loop Quantum Gravity(LQG), where one treats spacetime a dynamical entity as indicated by Einstein's General Relativity. This theory dictates that spacetime has granular structure at length scales close to planck length,a consequence of quantised nature of spacetime. So this is a theory of quantum spacetime. Here we assume that space comprising intervoven network(fabric) of loops called spin loops which during the course of time evolution evolve as a structure called spin foam. Thus space has a fundamental structure like matter [6]. This is an active field of research that involves large number of reserach groups worlwide [7]. The progress is going in two directions:traditional canonical loop quantum gravity and other with newer covariant(Lagrangian) approaches, commonly known as spin foam theory. Though not a complete theory of quantum grvity, it is still considered as a mature theory owing to its beutiful mathematical framework. One of the ramarkable features of this idea is that when incorporated in cosmology(an area known as loop quantum cosmology), it turns Big Bang into a Big Bounce [8]. There are other various other approaches for QG that include noncommutative geometry, causal set theory, Twister theory, to name a few; but these find a small number of reserach groups worldwide.

1.0.2 A Minimal Length Scale

One of the prominant features of special relativity and quantum mechanics is the introduction of two new constants in physics: the ubiquitous speed of light(c) and the smallest possible action in nature, the Planck constant($\hbar$)respectively. As such, no fundamental unit scale for either length or mass can be constructed from these two constants only. Planck noticed that if one tries to combine Newtons constant(G) with these constants in some way, a fundamental measurement scale for length, mass and time would probably emerge.

The very notion of a fundamental length scale dates to back to earlier works by Robert Levi regarding the "smallest unit of time" [9] in which he tried to develop the concept of discrete nature of time. This idea was soon followed by the idea of a fundamental length. Initially, people would doubt the idea of divisibilty of time and space. But, later, with the merger of quantum mechanics and special relativity into Quantum field Theory(QFT), the idea of a minimum length was taken up sriously by scientific community. In order to get rid of some divergent quantities in QFT, it was felt necessary to consider an idea of fundamental length. This was later taken by Heisenberg who conjectured that this fundamental length scale would be around a classical electron radius ($\approx 100fm$). He later applied this idea to Fermi's theory of β decay [10]. The work was rigourously later persued by Mead in which he argued that it is impossible to make a distance measurement with error less than Planck length [11]. The importance of Planck length seems to be relevant when system under consideration is quantum mechanical and gravity (G) or the structural geometry of spacetime has a role to play in describing the dynamics of the system. Also, it should possess very high energy and high velocity(close to velocity of light c). The systems in view that possibly possesses these features is the early universe, quark colisions at around the Planck energy. But achieving such a high energy for quark collisions in a laboratory would require an accelerator of the size of the order of a galaxy. Another example of such systems is real particle in the vicinity of virtual particles as these virtual particles can possess arbitrarily very high energies. This concept of minimal length can be motivated from various Gedanken experiments and almost all of the approaches to quantum gravity.

We know that compton wavelength of a particle is the length scale at which QFT becomes crucial for its description. Technically, it is equal to the wavelength of the photon whose energy is same as that of rest energy of particle. The essence of this idea is that if one tries to localize a particle within less than its compton wavelength, it makes momentum so uncertain that there is enough energy to create an extra particle. So this spoils the very concept of a particle. Next comes the gravity. As implied by GR, spacetime is not like a fixed background, it is dynamical. Since space in which we put our particle suffers flauctuations due to the gravity-energy interaction, this introduces additional uncertainty in the position of particle. So we have two uncertainities for particle:one, that is standard Heisenberg uncertainty and second that comes from these tiny fluctuations from the background spacetime which is its response to quantum uncertainty [12]. Suppose, in order to localize a particle within a spherical region of radius$l$, we need a radiation of wavelength smaller than l, which will have energy greater than$\frac{1}{l}$, which will correspond to puttting an energy density greater than $\frac{1}{l^4}$ in that region of space.Now coming to Einstein's equations that say:

$$\partial^2 g \approx L_p^2 \rho \geq \frac{L_p^2}{l^4} \tag{1.0.1}$$

So the gravitational potential generated by this photon is

$$g \geq \frac{L_p^2}{l^2} \tag{1.0.2}$$

where g is the gravitaional potential(spacetime metric) and L_p is the Planck length.This introduces an uncertainty in length as

$$\sqrt{gl^2} \geq L_p \tag{1.0.3}$$

So the fixed lower bound for uncertainty for any distance measurement automatically implies a fundamental(minimal) length [13].

This emergence of minimal length creates important physical scenarios that has a direct impact on physical systems. However, its effects are more pronounced at the energy scales close to Planck scales i.e the early universe and black holes; at lower energy scales, the changes occuring in physical systems are very small to be detected. In the wake of this minimal length, we arrive at an uncertainty principle that is a generalization of usual Heisenberg uncertainty principle(HUP). . Also, the structure of spacetime around Planck scale radically diverges from the standard one, giving birth to so called noncommutative spaces. We briefly discuss here this Generalized Uncertainty Principle(GUP) and noncommutative spaces.

1.0.3 Generalized Uncertainty Principle and Noncommutative Spaces

One of the major challenges of incorporating gravity into QFT's is that gravity renders them non-renormalizable, which indicates slim chances for achieving a full theory of quantum gravity(QG). This problem is mainly due to lack of any experimental quantum gravity signal. This has led many people to employ effective theories to describe effects of quantum gravity and to test these proposals in the lab for any possible hint. These formalisms are collectively known as quantum gravity phenomenology. In this field, some standard QFT or general relativity is subject to few predictions arising from QG theories. Some of these phenomenogical models include the famous Generalized Uncertainty Principle(GUP), Doubly Special Relativity(DSR), Lorentz-violating theories (for a review see, [40]). In addition, Noncommutative Spaces (Noncommutative algebra) may also possibly provide a potential roadmap for constructing QG phenomenogical models. Some of these frameworks can be seen as immediate consequences of the appearance of a minimal length in physics.

In order to get rid of the divergences due to merger of quantum mechanics with gravity, one thinks of to introduce the concept of minimum length as discussed in the previous section. It becomes clear that the very notion of spacetime breaks down beyond Planck scale. In other words, it puts a fundamental limit to our abilities to resolve distances that are of the order of lengths below Planck scale. In quantum mechanics, measurements of distances are governed by Heisenberg uncertainty principle and it gaurantees an utmost precision with which we can measure the position of a particle as far as the uncertainty product is satisfied. But here, this minimal uncertainty in position measurement implies to relearn standard HUP. To reconcile HUP with this minimal scale, it has been extended to a new Generalized Unceratinty Principle. So this correction to HUP is due to Quantum gravity; it is for this reason that GUP is sometimes called Gravitational Uncertainty Principle. The essence of this framework is that apart from providing a clue for experimental verification for QG proposals, it helps us to compute the spacetime curvature effects on the quantum systems with a well defined hamiltonian. We have different heuristic methods and certain Gedanken(thought) experiments to understand this modified uncertainty principle. We will invoke here Newtonian gravity and dimensional estimate to arrive at the said relation.

In the usual Heisenberg uncertainty, we never take into consideration the gravitational interaction between photon and electron. We assume here that a photon with which we want to localise an electron behaves like a classical particle with mass E/c^2. The region of interest has a characteristic length L. The gravitational interaction between photon and electron imparts an acceleration to the electron given by

$$\ddot{r} = \frac{G(E/c^2)}{r^2} \tag{1.0.4}$$

where r is the distance between electron and photon. The characteristic time of interaction between electron and photon is L/c. During this time,electron acquires a velocity

$$\Delta v \approx \frac{G(E/c^2)}{r^2}(L/c) \tag{1.0.5}$$

and moves a distance of

$$\Delta x_G \approx \frac{G(E/c^2)}{r^2}(L/c)^2 \tag{1.0.6}$$

Due to position uncertainty,electron can be found anywhere in the region as the photon scatters off electromagnetically at some indeterminate time. The distance between electron and photon would be of the order of $r \approx L$. Asthe momentum of the photon is related to energy by $E = pc$, equation(6) gives

$$\Delta x_G \approx \frac{Gp}{c^3} \tag{1.0.7}$$

Since the electron momentum uncertainty must be of the order of momentum uncertainty of photon; using $L_p^2 = G\hbar/c^3$ equation (7) leads to

$$\Delta x_G \approx \frac{Gp}{c^3} = \frac{G\Delta p}{c^3} = \left(\frac{G\hbar}{c^3}\right)\frac{\Delta p}{\hbar} = L_p^2\frac{\Delta p}{\hbar} \tag{1.0.8}$$

We add this uncertainty in the position of electron to the standard Heisenberg uncertainty $\Delta x \Delta p \geq \frac{\hbar}{2}$ we get,

$$\Delta x \geq \frac{\hbar}{\Delta p} + L_p^2\frac{\Delta p}{\hbar} \tag{1.0.9}$$

The initial form of GUP in course of string theory and various thought experiments can be found in Maggiore's work [14] and is given by

$$\Delta x \geq \frac{\hbar}{\Delta p} + const.G\Delta p \tag{1.0.10}$$

where G is Newton's constant. The extra term on right hand side of Eq.(1.0.9) or Eq.(1.0.10) becomes irrelevant much below the Planck energy. At such low energy scales, we recover the traditional

Heisenberg uncertainty principle. The extra term on the R.H.S of (4) can be ignored at the energy scales much lower than the Planck scale. At those scales, we recover the traditional Heisenberg relation [15]. Some people usually express GUP as [16]

$$\Delta x \Delta p \geq \frac{\hbar}{2}\left(1+\beta(\Delta p)^2\right) \tag{1.0.11}$$

β is constant that denotes the energy scale at which the measurement is carried out. Its value is usually given by

$$\beta = \frac{\beta_0 L_p}{\hbar} = \frac{\beta_0}{M_p^2 c^2} \tag{1.0.12}$$

In usual quantum mechanics, we can have any arbitrary value for position uncertainty i.e, we can minimize Δx as much as we can provided the usual Heisenberg unceratinty product is satisfied. But here in Eq.(1.0.11), as we try to minimize Δx by growing Δp, the RHS of grows much faster than LHS thereby increasing Δx. So this lower limit to the position uncertainty is a direct implication of minimal length. We represent the above situation graphically as follows:

It should be noticed from the above figure that a minimal uncertainty in position is $L_p\sqrt{\beta_0}$ and minimal uncertainty in momentum corresponds to $\frac{M_p c}{\sqrt{\beta_0}}$. Kempf et.al have further generalized the above modified uncertainty relation to [16]

$$\Delta x \Delta p \geq \frac{\hbar}{2}\left(1+\alpha(\Delta x)^2+\beta(\Delta p)^2+\gamma\right) \tag{1.0.13}$$

where γ generally depends upon $\langle p \rangle$ and not on Δp. We can easily see that such a generalized relation guarantees a minimal unceratinty in both position and momentum. The immediate consequence of this modification of Heisenberg uncertainty principle is that the associated Heisenberg algebra is also changed by the modified coordinate representation of position and momemtum operators. With the above GUP in Eq.(1.0.11) taken into consideration, our commutation relations between x and p change to

$$[x,p] = i\hbar(1+\beta p^2) \tag{1.0.14}$$

Likewise, our coordinate representation for momentum and position in position space are transformed in the following manner:

$$x \rightarrow x \tag{1.0.15}$$

and

$$p \rightarrow p(1+\beta p^2) \tag{1.0.16}$$

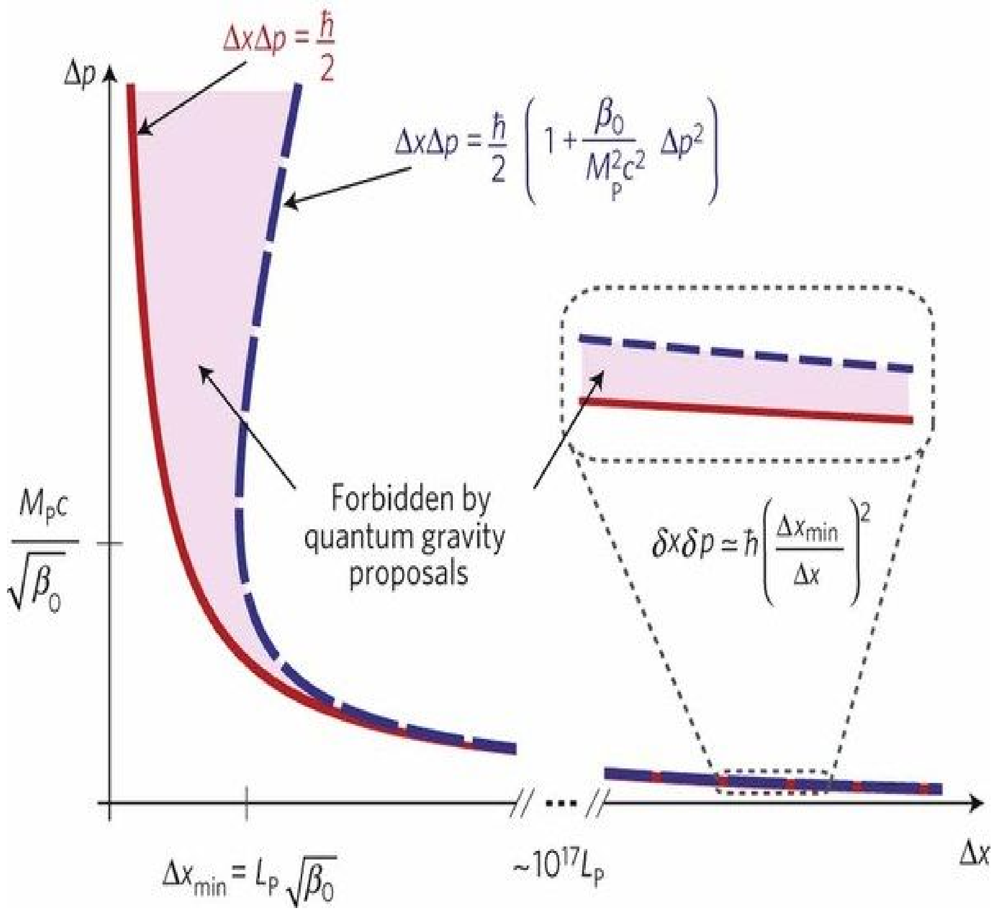

Figure 1.1: Position-momentum uncertainty relation in standard and modified framework (*Source: Igor Pikovski et.al. Nature Physics 8, 393-397 (2012)*)

This relation for momentum can be regarded as momentum at higher energies and at low energies, our momentum would be as usual p. The modification of Heisenberg algebra and the coordinate representaion of momentum operator modifies all quantum hamiltonians. For example, the modified hamiltonian corresponding to the unceratinty relation $\Delta x \Delta p \geq \hbar/2\left(1+\beta(\Delta p)^2\right)$ is given by

$$H = \frac{p^2}{2m} + \frac{\beta p^4}{m} + V \tag{1.0.17}$$

Since the value of constant β[1] is usually less than 1 with dimensions of inverse powers of momentum square, we use perturbative techniques to calculate the corrections to quantum systems due to these quantum gravity effects. The idea of this modified algebra has been applied to diverse fields of high energy physics and some interesting results have been found. It is found that significant effect on thermal properties of Quark-Gluon Plasma occur with these effects where it produces large amount of bag pressure [17]. It has also been applied to blackholes and their evaporation [18]. Some other

[1]Throughout the rest of this work, we stick to this notation β for the deformation parameter to avoid any ambiguity as there is a varied choice for it available in the literature

areas of application are quantum optical experiments [19], extra dimensions [20], thermal effects of a photon gas [21], precession of perihelion [22], discreteness of space [23], Relativistic Ramsauer-Townsend effect [24], inflationary universe [25], a few to mention. When applied to some quantum systems like harmonic oscillator, Landau levels, potential barrier and STM etc., it has been found that the corrections rendered by gravity are too weak to be detected at present energy scale. The effects are more pronounced at the energy scales close to Planck scale. However, certain bounds on the chracterising parameter β have been fixed which indicates that we can bring down these effects to much lower energy scales and using very precise instruments, we might possibly be able to detect these changes [15]. This way, we may be able seek some testable aspects of quantum gravity. This will also provide a way to check whether the approaches to quantum gravity provide a plausible solution for it.

There is another class of models based on the minimal length scenarios that arises in string theory which eliminates the use of point like structure in spacetime. In fact, the signatures for such a framework can be appreciated from the the analysis of quantum phase space by John Von Neumann with Heisenberg uncertainty principle taken into account [39]. It was called 'pointless geometry' and later paved the way for a new algebric framework identified as Noncommutative spaces (and hence noncommutative algebra). Here, the minimal length indicates a space-space and space-time uncertainities. The simplest of these noncommutative structure of spaces is represented by

$$[x^{\mu}, x^{\nu}] = i\theta^{\mu\nu}, \;\; [x^{\mu}, p_{\nu}] = i\hbar\delta^{\mu}_{\nu}, \text{ and } [p_{\mu}, p_{\nu}] = 0 \tag{1.0.18}$$

where $\theta^{\mu\nu}$ is a constant rank-two antisymmetric tensor having entries of dimension area, which behaves like a deformation parameter, with the indices $\mu, \nu = 1,2,3,4$. Thus, the standard commuting coordinates are replaced by Hermitian generator x^{μ} of C^{*} algebra in noncommutative spaces. Here the spacetime coordinates do not commute(even those which commute in ordinary standard quantum algebra), the underlying background space ceases and is taken up by noncommutative space. This is a particular type of noncommutative spaces called flat noncommutative spaces. In spite of some theorectical problems like breaking of Lorentz-Poincare symmetry due to nonvanishing $\theta^{\mu\nu}$ aassociated with noncommutative algebra; however, all other important properties like causality [31], unitarity [32] and UV/IR divergences [33] are believed to be describable in Lorentz invariant manner. The essence of noncommutativity being effective around Planck scales makes it a perturbative ingredient in the low energy physics especially the field theory where its first order correction in θ yields NC QFT. Well before the considerations of noncommutative spaces by A.Connes [34], we have similar formalism due to Snyder in 1947 [35] which possess the Lorentz invariance in their mathematical construction

$$[x^{\mu}, x^{\nu}] = i\theta(x^{\mu}p^{\nu} - x^{\nu}p^{\mu}) \tag{1.0.19}$$

$$[x^{\mu}, p_{\nu}] = i\hbar(\delta^{\mu}_{\nu} + \theta p^{\mu} p_{\nu}) \tag{1.0.20}$$

$$[p^{\mu}, p_{\nu}] = 0, \tag{1.0.21}$$

where θ is again the deformation parameter. It is found that Snyder algebra is Lorentz symmetric but they manifest a translational symmetry violation thereby indicating a Poincaré violating theory. However, some people have deduced a relation between deformed Poincaré algebra and Snyder algebra which can help us to easily formulate a scheme for noncommutative algebra. Even some people have argued that Poincaré symmetries could be subjected to quantm deformation in which case the deformed Poincaré generators preserve the original Poincaré algebra [128]. Recent studies have shown that these noncommutative spaces possess interesting physical properties unlike ordinary spaces which motivates us to study some crucial aspects of quantum entangled systems on these spaces. This could be an another working scheme to test the minimal length effects muc like the GUP framework and could constitute an promising QG phenomenogical model. The outline of the thesis is as follows.

In the next chapter, we will discuss the low energy effects of GUP or Planck scale corrections to various quantum systems like harmonic oscillator, Landau levels, Lamb shift etc. We will do it by considering different models of GUP. The most important parameter is the lower bound on the parameter β that characterizes the scale or regime of application of the this gravitational deformation and the corresponding possibility of detecting these effects experimentally. Also field theories and nonlocality is discussed based on GUP. In chapter 3, we will study some the crucial aspects of quantum entanglement like quantifying entanglement-entanglement entropy etc. We study the behaviour of entangled systems minimal length GUP and noncommutative spaces. We will also study the entanglement and the holography as a relevent ingredient to our work. Chapter 4 would be devoted to results and discussions. Finally, conclusions are drawn in chapter 5.

Chapter 2

Generalized Uncertainty Principle and its Implications

"Imagination is more important than knowledge."———Albert Einstein

With the minimal length scenarios taken into account, it becomes evident that this length scale is in an obvious contradiction with the standard Heisenberg unceratinty principle. The essential idea behind this is the emergence of divergences in physical theories due point like structural considerations which eventually necessitates the introduction of a minimal length that is motivated from various QG proposals. This inevitably asks for a reformulation of quantum mechanics and other theories to incorporate gravity effects in them, much pronounced around Planck scale. As mentioned in the preeeding section, the immediate implication of this minimal length scale is the modification of Heisenberg algebra and hence all well defined quantum hamiltonians. In this section, we show that it is possible to consider this deformation of quantum algebra(GUP-deformed or modified algebra) and predict some experimental techniques to measure these deformation efects. We will analyse the lower bounds on this deformation parameterβ by invoking some known observational and experimental data. Once gain consider the relations $(1.0.11)$-$(1.0.17)$, which show a working mathematical scheme to start for computing these effects. Looking at Eq(1.0.11), one may wonder why to choose a particular type of deformation like this one which is quadratic in momentum uncertainty. Well, in principle, nothing stops us atleast mathematically to comprehend any power in momentum uncertainty and hence the coordinate representation of momentum operator , but one needs to consider the physical implications of such algebra which potentially includes parity, unitarity and other fundamental considerations(including various invariance laws like translational invariance) manifested through the the resulting hamiltonian and algebra. In fact, the one proposal that is most general deformation so far will be extensively be dealt in the last section of this chapter. As a matter of fact, quadratic deformation is the most and well studied model of GUP which is straightaway implied by candidate

QG theories. In addition, some people have studied a quadratic-linear type of GUP, while others have extended this idea of generalization to time-energy uncertainty principle. Lets study these models in detail.

2.1 Linear and Quadratic Generalized Uncertainty Principle

We begin here by treating this gravitaional deformation as a reverse methodolgy to the standard formalism of modern phyiscs where every fundamental interaction is subjected to the rules of quantum mechanics which generically gave birth to quantum field theory, including quantum electrodynamics(QED) and quantum chromodynamics(QCD). In GUP, we subject quantum mechanics to the gravity effects. Hence, it could be very justified to call this gravitaional quantum mechanics. A quadratic and linear type of modified momentum representaion, discussed in this section, is achieved by combining a quadratic momentum deformation and Doubly Special Relativity(DSR) [37,38]. Lets first study the algebra associated with this deformed quantum mechanics and later analyse the real physical systems by incorporating this modified algebra and discuss the resulting configuration of these systems.

2.1.1 Modified Quantum Algebra

We study the correction(extension) to standard HUP, $\Delta x \Delta p \geq \hbar/2$ due to quantum gravity considerations. We will use a detailed notation in a cautious manner (as the subject is flooded with literature) while writing commutators and other relations taking into consideration the space dimensionality as well.

In view of the (1.0.11)-(1.0.17), we inroduce the the general commutator algebra for linear and quadratic GUP deformation using Jacobi identity as

$$[x_i, p_j] = i\hbar\Big[\delta_{ij} - \beta(p\delta_{ij} + \frac{p_i p_j}{p} + \beta^2\big(p^2\delta_{ij} + 3p_i p_j\big)\Big] \tag{2.1.1}$$

where $p^2 = \sum_{j=1}^{3} p_j p_j$ and $\beta = \beta_0/M_p c$. Planck energy $M_p c^2 \approx 10^{19} GeV$. The constant β must take care of dimensions in each term on R.H.S of (2.1.1) as discussed before. [15] (2.1.1) produces the correction terms to usual uncertainty principle upto $\mathscr{O}(\beta^2)$ as

$$\Delta x \Delta p \geq \frac{\hbar}{2}\Big[1 - 2\beta\langle p\rangle + 4\beta^2\langle p^2\rangle\Big] \tag{2.1.2}$$

or

$$\Delta x \Delta p \geq \frac{\hbar}{2}\Big[1 + \big(\frac{\beta}{\sqrt{\langle p^2\rangle}} + 4\beta^2\big)(\Delta p)^2 + 4\beta^2\langle p\rangle^2 - 2\beta\sqrt{\langle p^2\rangle}\Big] \tag{2.1.3}$$

The above commutation relation and hence the uncertainty principle imply a minimum uncertainty in position and a maximum uncertainty in the momentum upto a certain scale. These minimal position and maximal momentum are given as

$$\Delta x \geq \Delta x_{min} \approx \beta_0 L_p \tag{2.1.4}$$

and

$$\Delta p \leq \Delta p_{max} \approx \frac{M_p c}{\beta_0} \tag{2.1.5}$$

From these results, we see the immediate implications is the change in the coordinate representaion of both momentum as well as position in the representation of our choice(position or momentum). Thus, we write for position and momentum

$$x_i = x_{0i} \tag{2.1.6}$$

and

$$p_i = p_{0i}(1 - \beta p_0 + 2\beta^2 p_0^2) \tag{2.1.7}$$

This is the coordinate representation which satisfies the above modified commutator algebra. Note that in position representation, there is no modification to coordinate representation of x but only p is changed. The situation would be just reverse in momentum representation.

There is yet another form of GUP that is synonymous to the above formulation. Starting from commutation algebra [41],

$$[x_i, p_j] = i\hbar\left[\delta_{ij} + \beta\delta_{ij}p^2 + 2\beta p_i p_j\right] \tag{2.1.8}$$

we obtain

$$\Delta x_i \Delta p_j \geq \frac{\hbar}{2}\left[1 + \beta\left((\Delta p)^2 + \langle p\rangle^2\right) + 2\beta\left((\Delta p_i)^2 + \langle p_i\rangle^2\right)\right] \tag{2.1.9}$$

In this formulation of GUP, our coordinate representation reads as

$$x_i = x_{0i} \tag{2.1.10}$$

and

$$p_i = p_{0i}(1 + \beta p_0^2) \tag{2.1.11}$$

In other words, these can be treated represtations of position and momentum at high energies and Here x_{0i} and p_{0j} are representations at low energy and satisfy the usual undeformed commutation relation

$$[x_{0i}, p_{0j}] = -i\hbar\delta_{ij} \tag{2.1.12}$$

Note that at low energies, p_{0i} has the standard representation in position space $p_{0i} = -i\hbar\partial/\partial x_{0i}$ Since we would be mostly working in one dimension, the over-emphasizing subscripts depicting the space dimensions become redundant and hence we omit them. Moreover, to keep notation as simple and straightforward for the sake of computations, we further drop this 0 subscript and just write x and p as our new position and momenta and the corresponding relations become same as in (1.0.15) and (1.0.16). Like wise, the Hamiltonian due to these modified algebraic relations for both GUP's reads as the following respectively,

$$H = \frac{p^2}{2m} + V - \frac{\beta}{m}p^3 + \frac{5\beta^2}{2m}p^4 + \mathscr{O}(\beta^3) \tag{2.1.13}$$

and

$$H = \frac{p^2}{2m} + V + \frac{\beta}{m}p^4 + \mathscr{O}(\beta^2) \tag{2.1.14}$$

It is worth mentioning here that these modified commutation relations do not break the rotational symmetry and thus generators of rotation can still be expressed through position and momentum operators [16]

$$L_{ij} = \frac{1}{(1+\beta p^2)}(x_i p_j - x_j p_i) \tag{2.1.15}$$

With these basic algebraic tools of GUP, we are now in a position to analyze the systems with a well defined hamiltonian. In addition, we may also be able handle systems that conform to standard quantum algebra and hence get affected due to this deformation.

2.1.2 Constraining GUP with Physical Systems

Lets now turn our attention to the implications of this modified algebra and GUP. Since, there are lot of possible physical systems worth looking for with reference to the above algebra. However, we will analyze the most crucial systems that have some profound implications for our understanding of quantum gravity and cosmology and other areas of physics like atomic physics. Since the aim of this QG phenomenogical model of GUP is to look for some possible experimental signatures or predictions for quantum gravity, we will study this feature specially for these physical systems. Hence, the bounds obtained for the characterizing parameter β will have to be taken into consideration.

I) Hydrogen Atom and Atomic Physics

The atomic physics and other basic quantum mechanical systems include the systems like particle in a box, harmonic oscillator , Hydrogen atom and related physics, a few to mention. As these systems have a well defined quantum hamiltonian, it becomes easy to incorporate GUP perturbative energy to the original hamiltonian of these systems. Here, we study in detail the effect on the hydrogen atom.

Before that let's revisit the algebra and the corresponding Schrödinger equation [42]. As discussed earlier in (2.1.10),(2.1.11),(2.1.14), the modified algebra for a quadratic deformation in general yields the following hamiltonian for a quantum system.

$$H = \frac{p^2}{2m} + V + \frac{\beta}{m}p^4 + \mathscr{O}(\beta^2) \tag{2.1.16}$$

The higher powers of β do not contribute, hence can be neglected. Our calculations exist upto first order in β. Thus the hamiltonian reads

$$H = \frac{p^2}{2m} + V + \frac{\beta}{m}p^4 \tag{2.1.17}$$

We fisrt write down the Schrödinger equation for a system

$$\left[\frac{p^2}{2m} + V(x)\right]\psi(x) = E\psi(x) \tag{2.1.18}$$

With the modified hamiltonian in consideration, the modified or deformed Schrödinger equation reads

$$\left[\frac{p^2}{2m} + \frac{\beta}{m}p^4 + V(x)\right]\psi(x) = E\psi(x) \tag{2.1.19}$$

This is just the Schrödinger equation with an extra term in p^4 which acts like a purturbation and only first order corrections are siginificant in quantum systems. The energy spectrum of the system thus changes as

$$E_n = E_n^0 + \Delta E_n \tag{2.1.20}$$

where n is a quantum number that denotes the particular energy level, E_n^0 is the unperturbed energy level and ΔE_n are the matrix elements of perturbative energy $\frac{\beta p^4}{m}$ for any nth level. It is given by

$$\begin{aligned}\Delta E_n &= \frac{\beta}{m}\int_{-\infty}^{+\infty} \psi_n^{*0}(x)\psi_m^0(x)dx \\ &= \frac{\beta}{m}\langle\psi_n^0(x)|p^4|\psi_m(x)\rangle \\ &= \frac{\beta}{m}\langle n|p^4|m\rangle\end{aligned} \tag{2.1.21}$$

where ψ_n^0 and ψ_m^0 are unpurturbed states of the system that would correspond to the case $\beta = 0$. Since each state E_n^0 could have some multiplicity prior to perturbation, this deformation can sometimes remove the degeneracy of a particular energy level. As the behaviour of p^2 is very clear from Schrödinger equation, we can write the above matrix elements for a central potential V_r as follows

$$\Delta E_n = 4\beta m\left[(E_{n,l}^0)^2\delta_{nn'} - (E_{n,l}^0 + E_{n',l}^0)\langle nlm|V(r)|n'lm\rangle + \langle nlm|V(r)^2|n'lm\rangle\right]\delta_{ll'}\delta_{mm'} \tag{2.1.22}$$

which can be further simplified to

$$\Delta E_n = 4\beta m\left[(E_{n,l}^0)^2 - 2E_{n,l}^0\langle nlm|V(r)|nlm\rangle + \langle nlm|V(r)^2|nlm\rangle\right] \tag{2.1.23}$$

We consider a power law potential here $V(r) \sim r^p$ which greatly simplifies the above equation. Virial theorem furnishes

$$\langle nlm|V(r)|nlm\rangle = \left(\frac{2}{p+2}\right)E^0_{nl} \tag{2.1.24}$$

This gives a correction to the energy levels to a first order in β as shown below

$$\Delta E_n = 4\beta m\left[(E^0_{n,l})^2\left(\frac{p-2}{p+2}\right) + \langle nlm|V(r)^2|nlm\rangle\right] \tag{2.1.25}$$

Using the above expression, we can easily now compute the corrections to hydrogen atom spectrum or any other system with a well defined central potential like harmonic oscillator. However, we will only study the hydrogen atom case which is a relevant for atomic physics here, leaving the harmonic oscillator problem to be dealt in the section on non-local GUP.

For the case of hydrogen atom, the normalized unperturbed wavefunctions read as

$$\psi^0_{nlm}(r) = (2\gamma_n)^{3/2}\sqrt{\frac{(n-l-1)!}{2n(n+l)!}}\left(2\gamma_n r\right)^l e^{-\gamma_n r}L^{2l+1}_{n-l-1}(2\gamma_n r)Y_{lm}(\theta,\phi) \tag{2.1.26}$$

where $\gamma_n = \frac{m\alpha}{n}$ with α being fine structure constant. n is the principle quantum number, l the angular momentum quantum number and m the magnetic quantum number. $L(2\gamma_n r)$ are the Laguerre polynomials [43]. Letting $2\gamma_n r = x$, the energy shift reads as

$$\Delta E_n = -12\beta m\left(E^0_{n,l}\right)^2 + 8\beta m\gamma_n^2\alpha^2\frac{(n-l-1)!}{n(n+l)!}\int_0^\infty e^{2l}e^{-x}\left[L^{2l+1}_{n-l-1}(x)\right]^2 dx \tag{2.1.27}$$

Using the following relations for Laguerre polynomials

$$\sum_{m=0}^{n} L^\alpha_m(x) = L^{\alpha+1}_n(x) \tag{2.1.28}$$

and

$$\int_0^\infty e^{-x}x^\alpha L^\alpha_n(x)L^\alpha_m(x)dx = \frac{\Gamma(\alpha+n+1)}{n!}\delta_{nm} \tag{2.1.29}$$

with the summation formula

$$\sum_{p=0}^{b}\frac{(p+a)!}{p!} = \frac{a+b+1}{(1+a)b!} \tag{2.1.30}$$

our expression for coorections to energy levels take the following form

$$E_n = \frac{-m\alpha^2}{2n^2} + (\Delta x_0)^2\frac{m^3\alpha^4}{5}\frac{\left[4n-3(l+1/2)\right]}{n^4(l+1/2)} \tag{2.1.31}$$

where Δx_0 is the minimum uncertainty in length given by $\Delta x_0 = \hbar\sqrt{5\beta}$. Here, the energy corrections are poisitive and maximum for ground state $n=0$. Among a given l values of a particular n, it is maximum for $l = 0$. Also the degeneracy for energy levels is removed by the energy correction. This means that the extended structure of electrons breaks the degeneracy of energy levels of hydrogen atom unlike the ordinary situation where we assume electrons to be point particles with no internal structural degrees of freedom.

Now, given the measurement accuracy for frequency of the radiation emiited by a transition from $1s$ to $2s$ is about 1kHz [44], the precision of measurement is around $10^{-12}eV$, then the upper bound on minimum position uncertainty Δx_0 comes out to be

$$\Delta x_0 \leq 0.01 \text{ fm or } 10^{-17}\text{m} \tag{2.1.32}$$

This upper bound on the minimal position uncertainty Δx_0 is very interesting in the sense that it yields the non-point like or extended or short distance structure in nature. In particular, here for hydrogen atom, it demonstrates the extended structure of electrons by proposing an upper bound for experimental determination of size of an electron. The lower bound for electron mass in excited state comes out to be around 85 GeV [100] which means that a particular photon of 85GeV energy cannot excite an electron or is unable to resolve the size or dimensions of electron. This particular photon wavelength would be an upper bound for electron size given by

$$\Delta x_0 \leq \lambda \sim 0.015 \text{ fm} \tag{2.1.33}$$

Possibly, future instruments with a great precision might be able to lower down the upper bound for Δx_0 or we will be able to detect the minimal uncertainty in position measurement.

II) Black Hole Physics

Black holes are the most astonishing and interesting objects in cosmology and astrophysics. Generally speaking, it is a region of spacetime which does not allow the passage of any material or energy to escape from its gravitational influence. Even the photons can't escape from its pull. Hence the name 'black hole'. According to general relativity, which considers gravity as geometrical aspect of spacetime structure, a large suffficient amount of matter or energy can produce a enormous curvature in spacetime and eventually form a balck hole. Further, the surface or the boundary around the black-hole where from it becomes impossible for something to escape is called event horizon [46, 47]. In a mathematical language, we briefly describe it as follows. In a Newtonian sense, given a spherical body with mass M and radius r, the escape velocity for it is $v_{esc} = \sqrt{\frac{2GM}{r}}$ which shows that it is independent of the mass of the escaping object and only depends on the mass of this massive object. Escape velocity would exceed speed of light c if $r < r_s$, where r_s is the Schwarzschild radius for mass M. Black hole comes in various types with deferent Schwarzschild radii. Typical ones include a collapsed star with $r_s = 3$km, collapsed star cluster with $r_s = 20$AU, primordial black hole with

$r_s = 10^{-13}$cm! [48]. Since the chracteristic feature of a blackhole is that it does not radiate anything except the hypothesized Hawking radiation, which makes it very hard to be directly observed. That is why people have always relied on some indirect obseravtional techniques like observing its gravitaional ineteraction with its surroundings. Some evidences of existing magnetic fields around this blackhole have revealed some interesting physical properties of blackholes [49]. One of the groundbreaking discoveries in this line was the announcement of first ever gravitational waves detection by LIGO which are believed to have been emiitted by merger of two massive blackholes: one of around 36 solar masses and another with 29 solar masses. It is the most concrete evidence of blackholes in general and binary blackholes with mass greater than 25 solar masses in particular [50].
A rigorous study of blackholes physics during the past couple of decades has depicted an interesting and fundamental relationship between gravity, thermodynamics and quantum physics. The ordinary laws that govern the classical thermodynamics applied to blackholes have shown a considerable understanding of their quantum counterparts in a region of strong gravitational fields [51]. Before starting the thermodynamic aspect of blackholes, we first consider the blackhole as one of the solution to Einstein's General Relativity. The simplest solution to General Relativity is the vacuum solution with four spacetime coordinates; this particular metric is called Schwarzschild metric where the line element(in units $c = \hbar = 1$) is given by

$$ds^2 = -\left(1 - \frac{2MG}{r}\right)dt^2 + \left(1 - \frac{2MG}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{2.1.34}$$

or in temporal sense, the metric is given by

$$d\tau^2 = \left(1 - \frac{2MG}{r}\right) - \left(1 - \frac{2MG}{r}\right)^{-1} - r^2(d\theta^2 + sin^2\theta d\phi^2) = g_{\mu\nu}dx^\mu dx^\nu \tag{2.1.35}$$

where t is Shcwarzschild time that is recorded by a clock at spatial infinity at rest, r is Schwarzschild radial coordinate that is choosen such that area of 2-sphere is $4\pi r^2$. θ and ϕ are azimuthal coordinates. It might appear from the above relation that the equation has a singularity at $r = 2M$ which is actually due choice of a bad coordinate system and can be easily removed by choosing a proper system for the description. The different coordinate systems for blackhole description, each with different advantages, may include tortoise coordinates, New horizon coordinates and Rindler space, Kruskal-Szekeres Coordinates. The geometry compactification and causal structure of spacetimes for blackholes with spherical geometry is achieved through Penrose diagrams [46]. Since blackhole studies are vast and enormous, it is impossible to discuss all of them. What interests us here is the thermodynamics of blackholes which psssibly provides us the most appropriate platform to analyze the merger of quantum principles with gravity.

Among the most remarkable works in the thermodynamic realm of blackholes is the relationship for blackhole entropy by Bekenstein [52] and the temperature by Hawking [53]. In fact, the temperature of a blackhole automatically hints at its entropy. The Hawking temerature is given by(not working here in Planck units)

$$T_H = \frac{\hbar c^3}{8\pi GM} = \frac{M_p^2 c^2}{8\pi M},\ M_p = \sqrt{\frac{\hbar c}{G}} \tag{2.1.36}$$

and the Bkenstein entropy(also called Bekenstein-Hawking entropy) reads as

$$S_B = \frac{4\pi GM^2}{\hbar c} = \frac{4\pi M^2}{M_p^2} \tag{2.1.37}$$

which is sometimes written as

$$S_B = \frac{\text{Area}}{4G} \tag{2.1.38}$$

It means that the entropy of a blackhole is proportional to area of the horizon which is gratifying. This also includes the entropy that emerges from the matter fields just outside the horizon.

In view of the GUP algebra, we seek corrections to these two relations and we would analyze the effect of the extended structure considerations on the fate of blackholes. It can argued that the blackhole evaporation could continue till its characteristic size reaches the Planck scale which marks the end of spacetime notion and the post-Planckian regime is a subject matter of discussion. Arguments against the existence of remnant structure beyond Planck scale see no evident quantum conservation principle that could guarantee the halt of evaporation beyond Planck scale which means that the whole blackhole should evaporate to photons and other quantum particles till complete elimination. Though a variety of alternative proposals regarding Hawking radiation can be found in the literature(see for example [54, 55]), but the situation is still hazy. The reason for this obscurity is the absence of a complete quantum gravity framework. Now the modified relation for blackhole temperature in GUP framework is given by

$$T_{GUP} = \frac{Mc^2}{4\pi}\left[1 \mp \sqrt{1 - \left(\frac{M_p^2}{M^2}\right)}\right] \tag{2.1.39}$$

with the momentum uncertainty

$$\Delta p = \frac{\hbar \Delta x}{2L_p^2}\left[1 \mp \sqrt{1 - \left(\frac{4L_p^2}{\Delta x^3}\right)}\right] \tag{2.1.40}$$

which gives the standard result of Eq(2.1.36) for temperature with a very large mass M, if we choose the negative sign within the bracket. Positive sign has as such no physical implications. It is quite evident from the Eq(2.1.39), that temperature becomes physically meaningless and complex as soon as the mass of the body becomes less than Planck mass M_p. Also, as the Schwarzschild radius approaches a value less than $2L_p$, the temperature again turns unphysical. We plot the temperature and

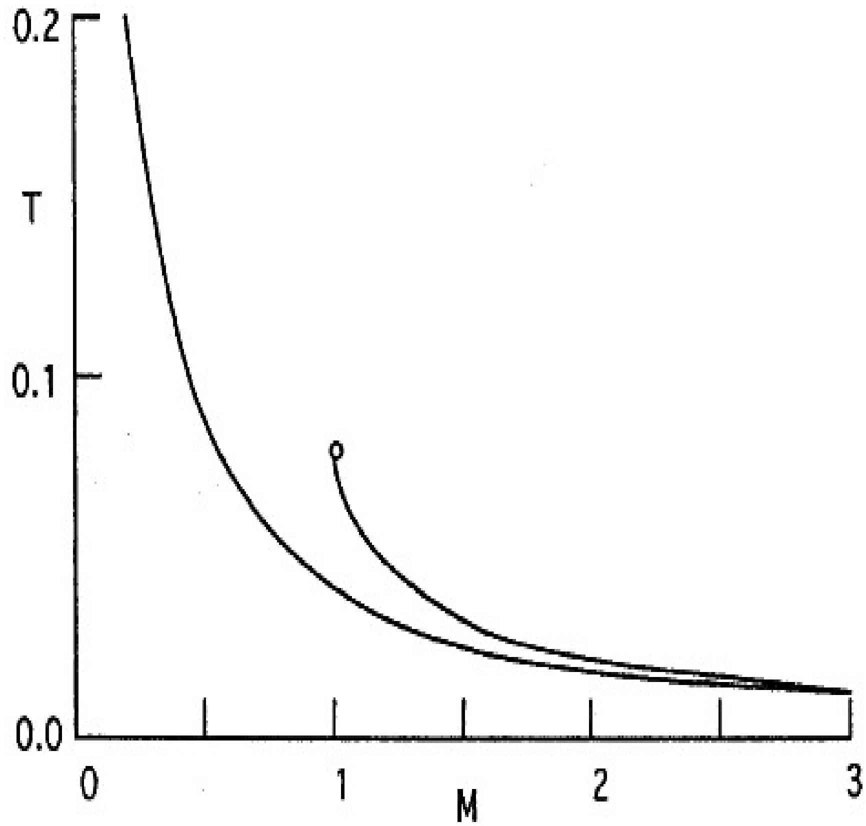

Figure 2.1: Temperature-Mass plot for a black hole in Planck units indicating GUP-corrected (upper curve) and standard Hawking(lower curve) $\big[$*Source: Adler et.al. Gen.Rel.Grav. 33 (2001) 2101-2108*$\big]$

mass relation graphically as shown in Figure 2.1.

The result is very interesting. The graphical relation shows that it is impossible for blackhole evaporation to continue beyond Planck mass M_p provided one takes GUP into the consideration. Further, the Schwarzschild radius of $2L_p$ marks the end of blackhole size, where the evaporation process in principle should cease. During this process, the temperature of the blackhole goes on increasing. Next, we compute the GUP corrections to Bekenstein-Hawking entropy. As usual the entropy is gotten by integrating $dS = c^2 T dM$. Taking GUP corrected relation for temperature into account, we get the modified relation for entropy as follows.

$$S_{GUP} = 2\pi \left[\frac{M^2}{M_p^2} \left(1 - \frac{M_p^2}{M^2} + \sqrt{1 - \frac{M_p^2}{M^2}} \right) - \log \left(\frac{M + \sqrt{M^2 - M_p^2}}{M_p} \right) \right] \tag{2.1.41}$$

The situation is graphically shown in Figure 2.2.

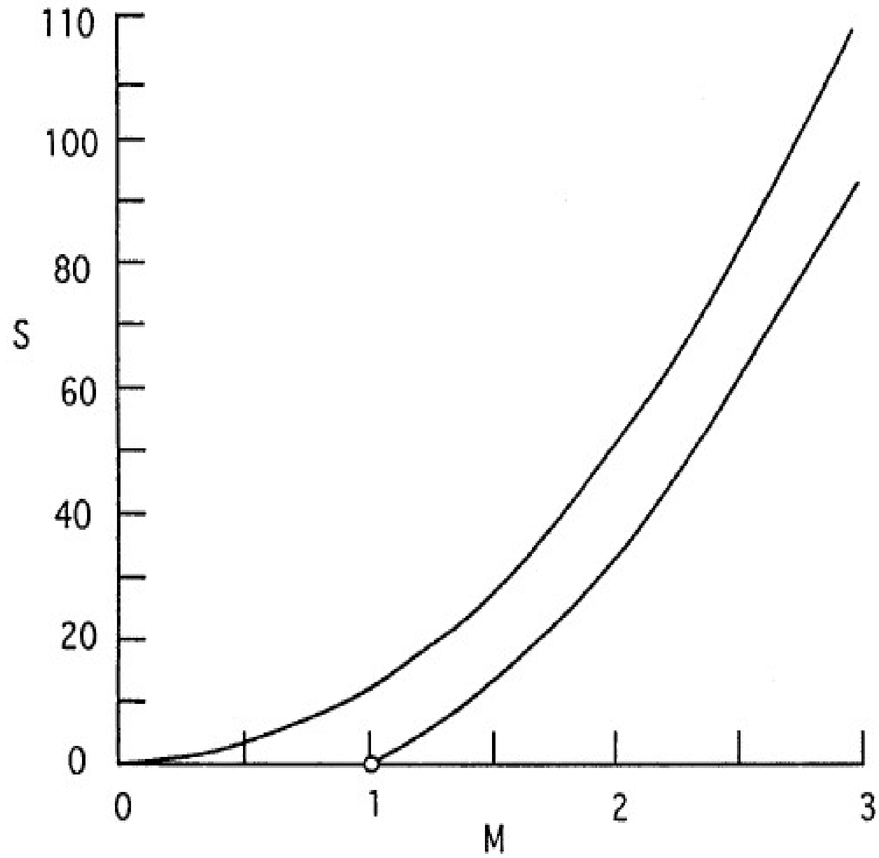

Figure 2.2: Black hole Entropy vs its Mass in Planck units with lower curve indicating GUP-corrected results. [*Source: Adler et.al. Gen.Rel.Grav. 33 (2001) 2101-2108*]

We would conclude here by examining closely the two situations described in GUP framework regarding the final stages of a black hole. The most important result derived here is the dominant influence that GUP seems to have on blackhole physics in describing its fate. It implies that a black-hole remnant, with entropy reaching zero and temperature increasing continuously as a minimum size approaches, is its favourable destination. It would be fair to argue that such blackholes might have existed in the early stages of our universe [56].

III) Quark-Gluon Plasma

As per QCD considerations, Quark-Gluon plasma(also called qurak soup) is a state of matter present in the scenarios of very high temperature and density [57]. With an asymptotic freedom , quarks and gluons move in a complex manner with certain equations of state governing their behaviour. Such scenarios are believed to have existed in the early stages of universe before the quark

confinenent of present universe initiated. It is believed that QGP can prove to be a platform to accomplish a consistant finite temperature field theory which studies the behaviour of elementary particles at very high temperatures. Such analysis is very essential to understand the evolution of early universe. Also, an extension of AdS/CFT duality to fluids, called fluid/gravity correspondence will possibly give a thorough insight into QGP. However, the main tool to understand the physics of QGP is the lattice gauge theory [58]. Since QGP physics is a vast area of high energy physics, we will constrain here ourselves to thermodynamics of QGP. In this section, we study the behaviour of QGP when minimal length is taken into consideration. We will use the quadratic generalized uncertainty principle to analyze the QGP thermodynamics. We consider a QGP equation of state for massless qurak flavours with GUP effect. We compute corrections to pressure and energy density. The case of bosons and fermions will be taken individually.
For bosons, the genreral form of grand canonical partition function reads as

$$Z_B = \prod_k \left[\sum_{l=0}^{\infty} \exp\left(-l \frac{E(k)-\mu}{T} \right) \right]^g \tag{2.1.42}$$

$$= \prod_k \left[1 - \exp\left(\frac{E(k)-\mu}{T} \right) \right]^{-g} \tag{2.1.43}$$

where E_k is the energy of the each quantum state with occupation number l, g the internal degrees of freedom, μ is the chemical potential and T is the temperature. The energy dispersion relation is $E_k = \sqrt{k^2+m^2}$ with k as wave vector [59]. For a vanishing chemical potential and mass, the dominant excitations are massless quarks and gluons. For particle with energy close to Planck energy, the dispersion relation with regard to momentum deformation is

$$E^2(k) = k^2c^2\left(1+\beta k^2\right)^2 + M^2c^4 \tag{2.1.44}$$

where M is the mass of the particle. As we are dealing with massless particle, we have (in natural units $\hbar = c = 1$)

$$E(k) = k\left(1+\beta k^2\right) \tag{2.1.45}$$

The summation$\sum_{l=0}^{\infty}$ for a large volume can be written in the form [60]

$$\sum_k \longrightarrow \frac{V}{2\pi^2} \int_0^{\infty} \frac{k^2 dk}{(1+\beta k^2)^4} \tag{2.1.46}$$

Thus from(2.1.43), we can write

$$\ln Z_B = \frac{-Vg}{2\pi^2} \int_0^{\infty} \frac{1}{6\beta(1+\beta k^2)} \left[\ln\left\{ 1 - \exp\left(\frac{-k(1+\beta k^2)}{T} \right) \right\} + \frac{k(1+3\beta k^2)}{T} \frac{1}{\exp\left(\frac{k(1+\beta k^2)}{T} \right) - 1} \right] dk \tag{2.1.47}$$

Upon some variable adjustment, we get to a first order in β the momentum k as

$$k \simeq xT - \beta x^3 T^3 \tag{2.1.48}$$

where $x = \left(\frac{k}{T}\right)(1+\beta k^2)$. We finally get the deformed relation for the boson partition function as

$$\ln Z_B = \frac{\pi^2}{90} V g T^3 - \frac{16\pi^4}{315} g \beta T^5 \tag{2.1.49}$$

This will reduce to the normal relation in case of $\beta \to 0$ [61].

Next we treat an ensemble of fermions with zero chemical potential with a chiral limit i.e. massless case. Keeping particle occupation for fermionic case in view, our final relation for fermion grand canonical partition function within first order in β reads as

$$\ln Z_f = \frac{7}{8}\frac{\pi^2}{90} V g T^3 - \frac{31\pi^4}{630} V g T^5 \tag{2.1.50}$$

In order to get the total grand canonical partition function for quarks and gluons is achieved by adding the bosonic(gluons) and fermionic(quarks) grand canonical partition from (2.1.49) and (2.1.50). Hence we get

$$\ln Z_{QGP} = \ln Z_B + \ln Z_f + \ln Z_v \tag{2.1.51}$$

where Z_v is the partition function of vacuum. We take Bag model into account, where instead of vacuum pressure, we take bag constant B [62]. Defining grand canonical potential as $\Omega = VB$, the GUP deformed equation of state and energy density of QGP are respectively given by [61]

$$\begin{aligned} P_{QGP} &= -\frac{\Omega}{V}\Big|_{QGP} \\ &= \frac{\sigma_{SB}}{3} T^4 - \left(g_g \frac{16\pi^4}{315} + g_q \frac{31\pi^4}{630} \right) \beta T^6 - B \end{aligned} \tag{2.1.52}$$

and

$$\varepsilon_{QGP} = \sigma_{SB} T^4 - 3\left(g_g \frac{16\pi^4}{315} + g_q \frac{31\pi^4}{630} \right) \beta T^6 + B \tag{2.1.53}$$

with $\sigma_{SB} = \left\{ g_g + \left(\frac{7}{8}\right) g_q \right\} \frac{\pi^2}{30}$ as SB constant.

These results show that even after including minimal length effects through GUP, pressure $P(T)$ is rapidly increasing function of temperature at $T \simeq T_c$ which is in full agreement with Monte-Carlo lattice calculations. Thus significant GUP effects can be studied on QGP. The important result which can be obtained from the above relations is the large bag pressure corresponding to fermion degrees of freedom $n_f = 2+1$ flavour. Also GUP deformed bag model can reproduce the results QCD lattice calculations and these things can pave way for studying other thermodynamic chracteristics of QGP physics [61].

2.2 Generalized Time-Energy Uncertainty

In this section, we study the extension of usual time-energy uncertainty principle to a generalized form like the one for position-momemtum relation. This leads us to the study the quantization of time where time is considered as quantum mechanical variable. It will also encourage us to analyze time crystals and minimum time uncertainty. This extension of energy-time unceratinty relation is very helpful to study the thermodynamic aspects of blackholes and various other decay processes.
From relation (1.0.9), if we divide it by speed of light c, we get the modified uncertainty relation for time-energy relation as

$$\Delta t \geq \frac{\hbar}{\Delta E} + t_p^2 \frac{\Delta E}{\hbar} \tag{2.2.1}$$

where t_p is Planck time approximately equal to $5.39 \times 10^{-44} s$. Using (2.2.1), one can get the equation in ΔE variable as

$$t_p^2 \Delta E^2 - \hbar \Delta t \Delta E + \hbar^2 \approx 0 \tag{2.2.2}$$

This leads us to the minimum energy uncertainty

$$\Delta E_{min} \approx \frac{\hbar \Delta t}{2t_p^2} \left[1 - \sqrt{1 - \frac{4t_p^2}{(\Delta t)^2}} \right] \tag{2.2.3}$$

We can approximate ΔE by (2.2.3) around t_p^2, thus we get [63]

$$\Delta E_{min} \approx \frac{\hbar}{\Delta t} + \frac{\hbar t_p^2}{(\Delta t)^3} \tag{2.2.4}$$

The discretization of time can be motivated from the deformed Wheeler-DeWitt equation [64]. We demonstrate a time discretization here by studying the deformed commutation relation between Hamiltonian and time which is also rooted in Planck scale physics.

2.2.1 Quantization of Time

We first consider the position-momentum deformed relation due GUP. As previously discussed, the deformation rendered to momentum operator in position space to first order in β can be written as

$$p_i = -i\hbar \left[1 - \hbar\beta \sqrt{-\partial^j \partial_j} - 2\hbar^2 \beta^2 \partial^j \partial_j \right] \partial_i \tag{2.2.5}$$

with the deformed position momentum relation

$$[x^i, p_j] = i\hbar \left[\delta^i_j - \beta |p^k p_k|^{\frac{1}{2}} \delta^i_j + \beta |p^k p_k|^{\frac{-1}{2}} p^i p_j + \beta^2 p^k p_k \delta^i_j + 3\beta^2 p^i p_j \right] \tag{2.2.6}$$

where β is the usual deformation parameter. Note that the nature of deformed momentum operator is nonlocal. By extending above algebra to four dimensions, one can then study the temporal part of deformed algebra, since the case under consideration is the time, while the deformed spatial part has been studied in [66]. We will investigate the effects of temporal deformation on physical systems. With time as an observable, the commutation relation between time and Hamiltonian reads as

$$[t,H] = -i\hbar \tag{2.2.7}$$

which is then deformed to the form

$$[t,H] = -i\hbar[1+f(H)] \tag{2.2.8}$$

where $f(H)$ is a function that depends upon system Hamiltonian.

Using principle of covariance, we can consider the deformation of momentum operator p to

$$p_\mu = -i\hbar\left[1-\hbar\beta\sqrt{-\partial^\nu\partial_\nu}-2\hbar^2\beta^2\partial^\nu\partial_\nu\right]\partial_\nu \tag{2.2.9}$$

with corresponding deformation spacetime commutation relation

$$[x^\mu,p_\nu] = i\hbar\left[\delta^\mu_\nu-\beta|p^\rho p_\rho|^{\frac{1}{2}}\delta^\mu_\nu+\beta|p^\rho p_\rho|^{\frac{-1}{2}}p^\mu p_\nu+\beta^2p^\rho p_\rho\delta^\mu_\nu+3\beta^2p^\mu p_\nu\right] \tag{2.2.10}$$

The Schrödinger equation corresponding to the temporal part of above relations is

$$H\psi = i\hbar\frac{\partial\psi}{\partial t}+\hbar^2\beta\frac{\partial^2\psi}{\partial t^2} \tag{2.2.11}$$

This will be highly effective to all quantum systems and produces the following uncertainty relation [67]

$$\begin{aligned}\Delta t\Delta E &\geq \frac{\hbar}{2}\left[1-2\beta\langle E\rangle+4\beta^2\langle E^2\rangle\right]\\ &\geq \frac{\hbar}{2}\left[1+\left(\frac{\beta}{\sqrt{\langle E^2\rangle}}+4\beta^2\right)\Delta E^2+4\beta^2\langle E\rangle^2-2\beta\sqrt{\langle E^2\rangle}\right]\end{aligned} \tag{2.2.12}$$

Now we will analyze the temporal deformation effect on a particle in a box. For the spatial case, it has been found that length of the box was quantized in terms of some fundamental length which is a necessary condition for the particle to exist in te box. This leads to quantization of space. We will take similar boundary conditions for the particle. Here the boundary conditions read as $\psi(0)=0$ and $\psi(T)=0$ where T is some particular fixed interval of time. Here, it will turn out that time is also quantized by treating this particle as a probe to measure time.

From (2.2.11), we can write

$$i\hbar\frac{\partial\psi}{\partial t}+\hbar^2\beta\frac{\partial^2\psi}{\partial t^2} = E\psi \tag{2.2.13}$$

By solving the above equation for ψ, we get the general solution as

$$\psi(t) = A\exp\left[\frac{-it\left(1+\sqrt{1-4\beta E}\right)}{2\beta\hbar}\right] + B\exp\left[\frac{-it\left(1-\sqrt{1-4\beta E}\right)}{2\beta\hbar}\right] \tag{2.2.14}$$

Employing boundary conditions $\psi(0) = 0$ gives $B = -A$ and second condition $\psi(T) = 0$ yields

$$A\left\{1-\exp\left(\frac{iT\sqrt{1-4\beta E}}{\beta\hbar}\right)\right\}\exp\left[\frac{-iT\left(1+\sqrt{1-4\beta E}\right)}{2\beta\hbar}\right] = 0 \tag{2.2.15}$$

Since A can't be zero(otherwise it becomes meaningless), thus the real and imaginary parts must be zero, either both or one of them. Setting imaginary and real parts to zero respectively gives

$$-2\cos\left(\frac{T}{2\beta\hbar}\right)\sin\left(\frac{T\sqrt{1-4\beta E}}{2\beta\hbar}\right) = 0 \tag{2.2.16}$$

and

$$-2\sin\left(\frac{T}{2\beta\hbar}\right)\sin\left(\frac{T\sqrt{1-4\beta E}}{2\beta\hbar}\right) = 0 \tag{2.2.17}$$

Setting both real and imaginary parts to zero, we get

$$\sin\left(\frac{T\sqrt{1-4\beta E}}{2\beta\hbar}\right) = 0 \tag{2.2.18}$$

which is only possible when

$$\frac{T\sqrt{1-4\beta E}}{2\beta\hbar} = n\pi \tag{2.2.19}$$

or

$$T = n\pi\frac{2\beta\hbar}{\sqrt{1-4\beta E}} \tag{2.2.20}$$

where n is an integer. By expanding the above relation for T, we can retain terms upto fourth order in β. Thus, we can write

$$T = 2n\pi\hbar\Big(\beta + 2E\beta^2 + 6E^2\beta^3 + \mathscr{O}(\beta^4)\Big) \tag{2.2.21}$$

This result indicates that the probing the time intervals in spacetime is a phenomenon that depends upon the energy of the probing particle and also on the bound on parameter β. Also, time is only possible to be measured in discrete intervals. This is quantization of time and is much more pronounced near Planckian energy scale. Taking a close look at above equation, it is clear that this time interval bears same order to the minimum time interval measured from the extended time-energy relation of (2.2.12). It might seem plausibe to conclude that this time interval diverges as the energy approaches

Planck scale, but then it is rendered physically meaningless because the deformation of Schrödinger equation in (2.2.13) is only upto first order in β. It is obvious from above that we can recover back the continuous time in the low energy limit $\beta \to 0$ [67]. The conclusion is that systems evolove with time in discrete jumps instead of conventional continuous manner. These results indicate a dramatic shift in the physical behaviour of systems at very high energy scales. This manifests the ubiquitous nature of gravity i.e. gravity pervades every physical system and decides its properties.

2.2.2 A Look at Spontaneous Emission

As a working example, we will study a two level quantum mechanical system subjected to quantized time and extended time-energy uncertainty realtion. It is a concrete system with a well defined characteristic features. The main conceptual entity of interest in such a system is the rate of spontaneous emission [68]. For the sake of notations, let's denote unperturbed Hamiltonian as H^0 with the eigenstates ψ_a and ψ_b with the following eigenvalue equations

$$H^0\psi_a = E_a\psi_a, \;\; H^0\psi_b = E_b\psi_b \tag{2.2.22}$$

In line with (2.2.14), one can write the solution to above system as

$$\psi(t) = C_a\psi_b \exp\left[\frac{-it\left(1+\sqrt{1-4\beta E}\right)}{2\beta\hbar}\right] + C_b\psi_b \exp\left[\frac{-it\left(1-\sqrt{1-4\beta E}\right)}{2\beta\hbar}\right] \tag{2.2.23}$$

This system is then subjected to a time-dependent perturbation $H^{'}(t)$. Hence, the changes rendered to the system are that the coefficients become time dependent $C_a(t)$ and $C_b(t)$, which are to be determined. The dynamical evolution is given by deformed Schrödinger equation as

$$\begin{aligned} H\psi &= H^0\psi + H^{'}(t)\psi \\ &= i\hbar\frac{\partial}{\partial t}\psi + \hbar^2\beta\frac{\partial^2}{\partial t^2}\psi \end{aligned} \tag{2.2.24}$$

Before solving the above equation, we first define the following terms:

$$\varepsilon_a = \frac{1}{2\beta}\left(1-\sqrt{1-4\beta E_a}\right), \text{ and} \varepsilon_b = \frac{1}{2\beta}\left(1-\sqrt{1-4\beta E_b}\right) \tag{2.2.25}$$

Since β is very small, we can neglect the terms of order $\hbar\beta$ and $\hbar^2\beta$ which are much smaller. Thus, from (2.2.23) and (2.2.24), we can write [67]

$$\begin{aligned} &C_aH^0\psi_a e^{\frac{-i\varepsilon_a t}{\hbar}} + C_bH^0\psi_b e^{\frac{-i\varepsilon_b t}{\hbar}} + C_aH^{'}\psi_a e^{\frac{-i\varepsilon_a t}{\hbar}} + C_bH^{'}\psi_b e^{\frac{-i\varepsilon_b t}{\hbar}} \\ &\qquad = i\hbar\left(\dot{C}_a\psi_a e^{\frac{-i\varepsilon_a t}{\hbar}} + \dot{C}_b\psi_b e^{\frac{-i\varepsilon_b t}{\hbar}}\right) + C_aE_a\psi_a e^{\frac{-i\varepsilon_a t}{\hbar}} + C_bE_b\psi_b e^{\frac{-i\varepsilon_b t}{\hbar}} \end{aligned} \tag{2.2.26}$$

Further mathematical analysis yields

$$\dot{C}_a = \frac{-i}{\hbar}\left(C_aH^{'}_{aa} + c_bH^{'}_{ab}e^{-i\omega_0 t}\right) \tag{2.2.27}$$

where

$$H'_{ab} = \langle \psi_a | H' | \psi_b \rangle \tag{2.2.28}$$

are matrix elements of perturbative Hamiltonian. Also

$$\begin{aligned} \omega_0 &= \frac{\varepsilon_b - \varepsilon_a}{\hbar} \\ &= \frac{\sqrt{1-4\beta E_b} - \sqrt{1-4\beta E_a}}{2\beta\hbar} \end{aligned} \tag{2.2.29}$$

Likewise

$$\dot{C}_b = \frac{-i}{\hbar}\left(C_b H'_{bb} + C_a H'_{ba} e^{i\omega_0 t}\right) \tag{2.2.30}$$

We can ignore the diagonal elements of H' in the above equations as they mostly vanish. Thus

$$\dot{C}_a = \frac{-i}{\hbar} C_b H'_{ba} e^{-i\omega_0 t} \tag{2.2.31}$$

and

$$\dot{C}_b = \frac{-i}{\hbar} C_a H'_{ba} e^{i\omega_0 t} \tag{2.2.32}$$

These ressults are similar to the systems of undeformed case except some modifications to ω_0. Let's subject this system to a time varying electric field $E = E_0 \cos wt$, then perturbation H' reads

$$H'(t) = -qE_0 r \cos(\omega t), \text{ and } H'_{ba} = -pE_0 \cos(\omega t) \tag{2.2.33}$$

Here $p = \langle \psi_b | r | \psi_a \rangle$ is the dipole radiation due to electric field [68]. The spontaneous emission rate of this deformed is found to be

$$A = \frac{\omega_0^3 |p|^2}{3\pi\varepsilon_0\hbar} \tag{2.2.34}$$

In deformed framework, this emission rate to a first order in β is given by

$$A = \frac{(E_b - E_a)^3 |p|^2}{3\pi\varepsilon_0\hbar^4} + \frac{(E_b - E_a)^3 (E_b - E_a) |p|^2}{3\pi\varepsilon_0\hbar^4}\beta \tag{2.2.35}$$

From this modified relation for emission rate, we can compute the bound on deformation parameter β. For that case, we shall take into consideration the ground and first excited state energies given by $E_1 = 13.6eV$ and $E_2 = 3.4eV$ with $|p| \sim qa_0$ where a_0 is the Bohr radius. This gives the value of A as

$$A \approx 6.2 \times 10^8 + 5.1 \times 10^{-9}\beta (\text{in } s^{-1}) \tag{2.2.36}$$

For hydrogen atom, the uncertainty in the measurement of emission constant is about $\pm 0.3\%$ [69], the bound on β_0 reads as

$$\beta_0 \leq 7.2 \times 10^{23} \tag{2.2.37}$$

which makes the time deformation effects measurable at this scale. From other particles, the bound set on this deformation parameter is not as stringent as the one obtained from hydrogen atom. Thus, it becomes clear that the minimal time effects can be first visible in hydrogen atom rather than the other particles [67].

2.3 Nonlocal GUP and Field Theories

In the physical description of certain systems, it is found that a particular configuration adopted by the system does not only depend upon the happenings in its immediate surroundings but also on the events or the configurations which are are far off from it both spatially as well as temporally. In this way, the system bears a close correspondence to its journey through time and space. Thus the resultant behaviour of this system shows a high degree of dependence upon its past and present circumstances. Such description of the system is history dependent. We call this feature of systems as a nonlocal behaviour. The notion of nonlocality has ever embarrassed conservative physicsists who stick to the principle of locality or local realism where system is only affected by its neighbourhood distribution. The locality assumption primarily rests on the causality notion of special relativity which dictates that no effect can preceed its cause and the assumption that particles must have a pre-existing value before any measurement is carried out. There is a possibilty that a nonlocal field theory might suffer from accausal solutions. But we are lucky to have so-called effective field theory description where the unphysical scenarios are removed efficiently and thus some interesting physics emerges out. Quantum nonlocality is thus in full agreement with special relativity and does not allow faster-than-light-communication [70]. Nonlocality can be incorporated into various physical theories to investigate for certain conceptual tests and can probably provide a better description of the system than a local one. The sources of nonlocality varies from case to case. Its consequences for quantum mechanics has given birth to nonlocal quantum mechanics where the nonlocality is essentially introduced into Schrödinger equation. The typical cases include Perey–Buck nonlocality [71], Feshbach nonlocality [72] etc. Mathematically, nonlocality is incorporated by a nonlocal operator on the right side of Schrödinger equation. In this section, we aim to study the the form of GUP that is motivated from nonlocal quantum mechanics and extended structure considerations in string theories. Hence the generic name 'nonlocal GUP'. We have proposed this form of GUP and later analysed its effects on the various systems. In addition, we will study the nonlocal formulations of various field theories by incorporating quadratic GUP in them. Also, a supersymmetric extension of Yang-Mills theory and SUSY breaking will be dealt in detail.

2.3.1 String-inspired Nonlocality

Before formulating nonlocal extension of a physical theory, it is necessary to have an idea of the source of nonlocality. As discussed in the introductory discussion of this section, nonlocality may arise in various contexts and eventually have significant and versatile impact on the physical description of the theory. One such kind of nonlocality is motivated from the string theories. String theories are fundamentally build upon the notion of extended structural existence in the form of string length at smallest scale in nature. The typical string length is about Planck length($L_p \sim 10^{-35}m$). In the wake of black hole information paradox, we see that a resolution by t'Hooft favours abandoning the locality assumption of quantum mechanics. Susskind's work later formulated the so-called *holographic principle* and *principle of black hole complementarity* which assemble some key features of black hole physics. The entropy relation for black hole indicates that it is area dependent rather than volume dependent as a local field theory would imply [73]. Nonlocality is believed to be able to play a role in resolving the paradox. By an argument from string theory perspective, if one considers commutators of two operators corresponding to two overlapped states with combined total energy E, then the string can stretch over a distance of

$$L \sim \frac{E}{M_{st}^2} \tag{2.3.1}$$

where $M_{st} = \frac{1}{l_{st}}$ represents string mass scale. This indicates that this commutator will have a nonlocal contribution as well. The reason for this emergence of nonlocality is the fact that total combined energy of these local operators acting on a vacuum deform the spacetime and produce a very large curvature where local field concepts cease to exist, at least in theory. This occurs around very high energy regimes close to Planck scale energy. Thus, locality is a strictly low energy assumption and breaks near Planck scale. This breakdown occurs if the extent of separation between these two operators is less than the Schwarzschild radius $R_s(E)$ of their center-of-mass energy,

$$L \lesssim R_s(E) \tag{2.3.2}$$

This is the criterion to witness the breakdown of local behaviour of a theory [74, 75]. Scattering processes in highy energy string physics are believed to display nonlocal effects. There is strong effect of distance and gravity in deciding the bound for locality in field theories. In a field theory, the commutativity condition

$$[O(x), O(y)] = 0 \; for \; (x-y)^2 > 0 \tag{2.3.3}$$

outside the lightcone, where $O(x)$ and $O(y)$ are gauge-invariant operators, signifies a local feature in it. However, this condition has no role to play when gravity is taken into consideration and forces one to use wavepackets instead of mere operators. Following this line, the gravitational considerations yields a bound on locality given by

$$|x-y| \gtrsim R_s(|p+q|) \tag{2.3.4}$$

where p and q are the momenta conjugate to x and y respectively. This marks the onset of nonlocality in a field theory in the event of quantum black hole formation. Thus, it is argued that a local field theory can be recovered as a low energy case to a more fundamental theory like quantum gravity. Alternatively, with the increase in energy, gravitational scattering amplitudes grow quickly, rendering the theory nonlocal, although it is fundamentally quantum mechanical. Thus, a local field theory is unable to predict exact number of degrees of freedom for a black hole [73, 76]. It is this type of string-inspired nonlocality that is our subject of interest. The consequences are far reaching. The type of Generalized Uncertainty Principle that arises in this context would be nonlocal. The nonlocality would be contained in the inverse powers of momentum operator(or inverse of gradiant). Since, the integration implies summing up 'point' elements, this gives rise to extended objects and hence nonlocality. The implications of this nonlocality have been studied in the context of Schrödinger equation. With a nonlocal term on RHS of Schrödinger equation, we have [71],

$$i\hbar\frac{\partial}{\partial t}\psi(x)+\frac{1}{2m}\hbar^2\partial^i\partial_i\psi(x)-V(x)\psi(x)=\int d^3x^{'}K(x,x^{'})\psi(x^{'}) \tag{2.3.5}$$

where $K(x,x^{'})$ is a nonlocal operator being a function of $(p^ip_i)^{-1}$. This helps us to achieve a nonlocal extension of field theories. For example, the nonlocal extension of gravity and other field theories have been shown to entail interesting physical consequences [77]- [80]. The point of interest here regarding nonlocality is that we are setting platform for the type of GUP that is essentially rooted in nonlocality through the modification of momentum operator representaion. For that matter, a massless scalar field theory deformed by nonlocality reads as

$$\hbar^2\partial^\mu\partial_\mu\psi(x)=\beta\int d^4xG(x-y)\psi(y) \tag{2.3.6}$$

where λ measures coupling of the nonlocal part of this framework and G is Green's operator given by

$$G(x-y)=\int\frac{d^4p}{(2\pi)^4}\frac{1}{p^2}e^{ip.(x-y)} \tag{2.3.7}$$

The term p^2 is given by $p^2=p^\mu p_\mu$ with spatial part p^ip_i. We can write the above field equation as

$$\begin{aligned}-\int d^4x\Big[\delta(x-y)\hbar^2\partial^\mu\partial_\mu\psi(y)-\beta G(x-y)\psi(y)\Big]&=\int\frac{d^4p\,d^4x}{(2\pi)^4}\left[\left(p^2+\frac{\beta}{p^2}e^{ip.(x-y)}\right)\right]\psi(y)\\&=0\end{aligned} \tag{2.3.8}$$

Hence by transforming $p^2\rightarrow p^2+\frac{\beta}{p^2}$, we can deform a scalar field theory to the nonlocal framework. Thus the term p^{-2} is the source of nonlocality. With this deformation of momentum operator containing inverse powers, our Generalized Unceratinty Principle becomes nonlocal. Thus the general deformation of momentum operator with both positive and inverse powers reads like

$$p_i \to p_i\Big(1+\sum\beta_{1r}(p^jp_j)^{r/2}+\sum\beta_{2r}(p^jp_j)^{-r/2}\Big) \tag{2.3.9}$$

It can be further generalized by choosing a particular sign for the coefficients β_{1r} and β_{2r}. Hence we can write the most general deformation of spatial part of momentum representation as [81]

$$p_i \to p_i\Big(1\pm\sum\beta_{1r}(p^jp_j)^{r/2}\pm\sum\beta_{2r}(p^jp_j)^{-r/2}\Big) \tag{2.3.10}$$

However, it is not possible to analyze the effect of such general deformation on quantum mechanics of physical systems as the calculations would highly complicated. Thus, we will study a specific limit of it in the inverse domain.

2.3.2 Physical Systems in a Nonlocal GUP

How does a nonlocal GUP affect the physics of quantum mechanical systems? This question is very interesting for the notion of nonlocality is the aim of this part of the section. It is a means to investigate the nonlocal extensions of quantum description of systems. We reiterate here that the fundamental quantum gravity extension of Heisenberg principle gives rise to a deformed quantum algebra. A general form of such position-momentum commuatator is

$$[x^i,p_j]=i\hbar\Big(1+f[p]^i_j\Big) \tag{2.3.11}$$

where $f[p]^i_j$ is specific tensorial function choosen in line with modified principle. It is very interesting to note that quadratic deformation coupled with Doubly Special Relativity(DSR) gives rise to a simultaneous linear and quadratic deformation of the momentum operator as discussed earlier with Hamiltonian

$$H=\frac{p^2}{2m}+V(x)+\beta_1H_1+\beta_2H_2 \tag{2.3.12}$$

where the new corrections H_1 and H_2 scale as p^3 and p^4 respectively.

It is highly interesting to note that in higher dimensions, linear and other powers of momentum will generate fractional derivative terms. As obvious from the relation for deformed momentum, $p_i\big[1+\beta_r(p^jp_j)\big]^{r/2}=-i\hbar\partial_i\big[1+\beta(\hbar^2\partial^i\partial_i)\big]^{r/2}$, odd values for r i.e $r=2n+1$ where n is an integer are bound to generate fractional derivatives. For even values, they are eliminated. These fractional derivatives are dealt with harmonic extension of functions [82]. They are known to occur in space fractional quantum mechanics where we replace Brownian trajectories in Feynman path integrals are replaced by Levey flights which can be used later in condensed matter physics [83]- [85]. Fractional quantum mechanics finds its role in optics to analyze dual Airy beams generated by off-axis longitudinal pumping [86]. These things motivate us to include fractional derivative terms in generalized uncertainty terms. With these fractional derivatives in Schrödinger equation, we can use harmonic extension of functions from $\mathbb{R}^3$ to $\mathbb{R}^3 \times (0,\infty)$ [87]. Thus, if $u : \mathbb{R}^3 \times (0,\infty) \to \mathbb{R}$ is harmonic function which is harmonic extension of wavefunction $\psi : \mathbb{R}^3 \to \mathbb{R}$, we can use Dirichlet condition to solve this problem

$$u(x,0) = \psi(x), \quad \nabla_4^2 u(x,y) = 0 \tag{2.3.13}$$

where ∇_4 is Laplacian operator in $\mathbb{R}^4$ with $x \in \mathbb{R}^3$ and $y \in \mathbb{R}$. We can use some special mathematical techniques to argue that these fractional derivative terms commute with ordinary derivatives [81]

$$\Big(\nabla_3^2\Big)^{1/2} \partial_i = \partial_i \big(-\nabla_3^3\big)^{1/2} \tag{2.3.14}$$

It can demonstrated that this fractional derivative operator $i(\partial^i \partial_i)^{1/2}$ is a self-adjoint operator. Consider $u_1(x,y)$ and $u_2(x,y)$ as harmonic extensions of wavefunctions $\bar{\psi}_1(x)$ and $\psi_2(x)$ such that for $|x|,|y| \to \infty$, these functions vanish. Thus we can write [88]

$$\begin{aligned}\int_{\mathbb{C}} \nabla_4 u_1(x,y).\partial^i \partial_{i4} u_2(x,y) dxdy &= \int_{\mathbb{C}} \nabla_4 \big[u_1(x,y).\partial^i \partial_{i4} u_2(x,y)\big] dxdy \\ &= \int_{\partial\mathbb{C}} u_1(x,y) \nabla_4 u_2(x,y) dxdy \\ &= -\int_{\mathbb{R}^3} u_1(x,y) \frac{\partial}{\partial y} u_2(x,y)\Big|_{y=0} dx\end{aligned} \tag{2.3.15}$$

where $\partial\mathbb{C}$ is the boundary contour of $\mathbb{C}$. Hence, for two harmonic functions $u_1(x,y)$ and $u_2(x,y)$, we can write an important property

$$\int_{\mathbb{C}} u_1(x,y) \nabla_4^2 u_2(x,y) dxdy - \int_{\mathbb{C}} u_2(x,y) \nabla_4^2 u_1(x,y) dxdy = 0 \tag{2.3.16}$$

In terms of wavefunctions $\bar{\psi}_1(x)$ and $\psi_2(x)$, the above relation reads [89, 90]

$$\int_{\mathbb{R}^3} \bar{\psi_1}(x) i(\partial^i \partial_i)^{1/2} \psi_2(x) dx = \int_{\mathbb{R}^3} \psi_2(x) i(\partial^i \partial_i)^{1/2} \bar{\psi}_1(x) dx \tag{2.3.17}$$

The source of fractional derivatives in the calculus like above one varies. Here, they emerge due to a gravitational correction to momentum operator representation. The peculiar feature of self adjointness or Dirac hermiticity is still preserved and has been investigated in many systems [91].

Taking note of such type of nonlocality in view, the most general form of Heisenberg algebra proposed in (2.3.10), it becomes very interesting to study the physics of resulting Hamiltonian. Very interestingly, though the momentum correction term is nonlocal due to inverse odd powers, the Hamiltonian is still local which is true for p^{-1} term as well. However, we might have to compromise certain secondary less stringent conditions like parity, a situation which resembles to those of decay processes. Taking a specific limit for the most general deformation in (2.3.10), we get

$$p \to p\Big(1 + \frac{\beta}{p}\Big) \tag{2.3.18}$$

which gives the resulting energy correction term to the Hamiltonian as

$$\beta H_h = \frac{\beta p}{m} \tag{2.3.19}$$

So the scaling for this part of Hamiltonian is $\sim p$. It is self-adjoint and hence this deformation is well defined. However, this Hamiltonian term violates the parity due to odd momentum powers. Here, existence of GUP effects near minimal length can be brought down to a lower energy regime than Planck scale. We fix bounds on β parameter using some experimental data. It is shown to exist at a scale between electroweak and Planck scale. The physical systems to be analyzed here include harmonic oscillator, Lamb shift, potential barrier, particle in a box and Landau levels. We find some interesting points about the minimal length physics.

I) Particle in a box: Length Quantization

The most fascinating thing about the linear deformation of momentum operator is that it predicts the quantization of space. In fact, a deformation of momentum in single power due to GUP leads to discretization of space [37]. This is argued from the result that particle existence is subjected to the length quantization of box of any length i.e. particle exists only for some integer multiple of the box length. Thus space is discrete. This result has been applied to relativistic Dirac equation which again shows the discretization of space [38]. For quadratic generalized uncertainty principle, such discretization does not occur. For this linear inverse GUP, modified Schrödinger equation reads

$$\frac{d^2\psi}{dx^2} + \left(\frac{2i\beta}{\hbar}\right)\frac{d\psi}{dx} + \frac{2mE}{\hbar^2} = 0 \tag{2.3.20}$$

which yields a solution for ψ as

$$\psi = Ae^{i\frac{k_1 x}{\hbar}} + Be^{-i\frac{k_2 x}{\hbar}} \tag{2.3.21}$$

where $k_1 = \sqrt{\beta^2 + 2mE} - \beta$ and $k_2 = \sqrt{\beta^2 + 2mE} + \beta$. Applying boundary conditions, for $x = 0$, $\psi = 0$ which gives $A = -B$. Thus,

$$\psi = A\left(e^{\frac{ik_1 x}{\hbar}} - e^{-i\frac{k_2 x}{\hbar}}\right) \tag{2.3.22}$$

Second boundary condition, $x = L$, $\psi = 0$ yields

$$A\left(e^{\frac{ik_1 L}{\hbar}} - e^{-i\frac{k_2 L}{\hbar}}\right) = 0 \tag{2.3.23}$$

Here since $A \neq 0$, therefore

$$\left(e^{\frac{ik_1 L}{\hbar}} - e^{-i\frac{k_2 L}{\hbar}}\right) = 0 \tag{2.3.24}$$

or

$$e^{\frac{i(k_1+k_2)L}{\hbar}} = 1 \tag{2.3.25}$$

Hence we get

$$\frac{(k_1+k_2)L}{\hbar} = 2n\pi \tag{2.3.26}$$

Therefore, we write the length of the box as

$$L = \frac{2n\pi\hbar}{k_1+k_2} \tag{2.3.27}$$

Upon using the values for k_1 and k_2, we finally write for the box length [81]

$$L = \frac{n\pi\hbar}{\sqrt{\beta^2+2mE}} \tag{2.3.28}$$

The above result depicts the quantization of box length in terms of a fundamental unit. Thus, there exists no particle in a box, if its length has not a discrete nature as shown in the above relation. It indicates that under such momentum deformation, space must be quantized in terms of a fundamental length, otherwise there is no meaning to existence of a particle. This inverse linear momentum deformation has the same effect as that of linear one [37]. If we look at the above relation, it speaks a volume about length measurement which is linked to probing of spacetime. It(probing) does depend upon the energy of the particle which means we have a space that assumes a structure as per energy of the probe. This effect is same as the gravity's rainbow effect where spacetime geometry depends upon the energy with which that region of space is probed [92]- [96]. In renormalization group flow, constants also flow, so the scale at which a field theory is probed depends upon the probe energy. This motivates us to look for gravity's rainbow from string theory [97]. Through a correspondence, we know that string theory can be regarded as two dimensional conformal field theory and target space can be taken as matrix of coupling constants of this 2-d CFT. This hints towards such possible effects in GUP which is also motivated from string theory [16].

As in this particular type of GUP, we find the microscopic structure of spacetime dependent upon probe energy, it can also affect the macroscopic structure. Before attempting to quantify those changes, we must first be able to construct a curved spacetime theory in this GUP. As mentioned earlier, spacetime possesses a discrete structure due to GUP, the possible effect due to this p^{-1} term would very helpful in constructing a setup for getting corrections to macroscopic geometry. It can be achieved by absorbing this energy dependence in the metric [81]. Some results in GUP have demostrated a modification of equivalence principle [98] which is the main motivation for DSR(and thus gravity's rainbow). This way one can expect similar effects due to nonlocal GUP as well [81].

II) Quantum Harmonic Oscillator

Harmonic oscillator is one of the very important toy model systems in physics. The generalized uncertainty principle has some interesting consequences for the harmonic oscillator. The deformed Hamiltonian for a quantum harmonic oscillator due to this inverse nonlocal GUP reads as

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2 + \frac{\beta p}{m} \tag{2.3.29}$$

Correction rendered to its ground state energy to a first order in β with unpurturbed wavefunction ψ_0 is given by

$$\begin{aligned} \Delta E_0 &= \int_{-\infty}^{+\infty} \psi_0^* \Big(\frac{\beta p}{m}\Big) \psi_0 dx \\ &= \frac{-i\hbar\beta}{m} \int_{-\infty}^{+\infty} \psi_0 \Big(\frac{d\psi_0}{dx}\Big) dx \end{aligned} \tag{2.3.30}$$

Now the ground state wavefunction is given by

$$\psi_0 = \Big(\frac{m\omega}{\pi\hbar}\Big)^{1/4} e^{-\alpha x^2} \tag{2.3.31}$$

where $\alpha = (\frac{m\omega}{2\hbar})$. Hence the energy correction is

$$\begin{aligned} \Delta E_0 &= \sqrt{\frac{m\omega}{\pi\hbar}} \Big(\frac{-i\hbar\beta}{m}\Big) \int_{-\infty}^{+\infty} e^{-2\alpha x^2} (-2\alpha x) dx \\ &= 0 \end{aligned} \tag{2.3.32}$$

This result indicates that there is no role of this first order perturbation in the ground state configuration of QHO. However, a second order perturbation does affect the ground state energy of the oscillator. For the second order perturbation, the correction terms read as

$$\Delta E_n^{(2)} = \sum_{m \neq n} \frac{\Big|\langle \psi_m \big| \frac{\beta p}{m} \big| \psi_n \rangle\Big|^2}{E_n^{(0)} - E_m^{(0)}} \tag{2.3.33}$$

For the ground state $n = 0$, we can write the above equation as

$$\begin{aligned} \Delta E_0^{(2)} &= \sum_{m \neq 0} \frac{\Big|\langle \psi_m \big| \frac{\beta p}{m} \big| \psi_0 \rangle\Big|^2}{E_0^{(0)} - E_m^{(0)}} \\ &= \sum_{m \neq 0} \frac{\Big|\frac{-i\hbar\beta}{m} \langle \psi_m \big| \frac{d}{dx} \big| \psi_0 \rangle\Big|^2}{E_n^{(0)} - E_m^{(0)}} \\ &= \frac{-2i\beta\hbar\alpha}{m} \sum_{m \neq 0} \frac{\Big|\langle \psi_m \big| x \big| \psi_0 \rangle\Big|^2}{E_0^{(0)} - E_m^{(0)}} \end{aligned} \tag{2.3.34}$$

Next we have some mathematical conditions

$$\langle\psi_m|x|\psi_n\rangle = 0, \ \text{ for } m \neq n \pm 1 \tag{2.3.35}$$

$$\langle\psi_m|x|\psi_n\rangle = \sqrt{\frac{n+1}{2\gamma}}, \ \text{ for } m = n+1 \tag{2.3.36}$$

$$\text{and } \ \langle\psi_m|x|\psi_n\rangle = \sqrt{\frac{n}{2\gamma}}, \ \text{ for } m = n-1 \tag{2.3.37}$$

where $\gamma = \frac{m\omega}{\hbar}$. Since from third condition, for $n = 0$, $m = -1$, which gives unphysical results. Thus this condition in neglected. Only first two are useful here. For $n = 0$, first two conditions give $m \neq 1$ and $m = 1$. For an unperturbed oscillator, the ground state and first excited state energies are $E_0 = \frac{\hbar\omega}{2}$ and $E_1 = \frac{3\hbar\omega}{2}$. We therefore write the second order correction to energy for $m = 1$ as

$$\Delta E_0^{(2)} = \frac{-\beta^2}{2m} \tag{2.3.38}$$

and for $m \neq 1$, we get

$$\Delta E_0^{(2)} = 0 \tag{2.3.39}$$

Thus there second order correction does not contribute anything for $m \neq 1$ case. However, a small correction is obtained for $m = 1$ case. Next, we calculate the bounds on β using some known experimental data. As physical systems like a charmonium(which is heavy meson system)can be modelled by harmonic oscillator [99], we can use its data to fix bound on β. For charmonium, mass $m_c = 1.3\frac{GeV}{c^2}$ with binding energy being equal to the energy gap between various adjacent levels with a value $\hbar\omega = 0.3GeV$. The current level of experimental precision is about 10^{-5} [100], we get the bound on β as $\beta \leq 10^{-21}$. So this value of β chracterising this gravitational deformation cannot exceed this value, otherwise it will violate the known experimental findings [81].

III) Landau Levels

We now consider the case of Landau levels to compute the gravitational deformation effects due $p \to p(1+\beta p^{-1})$ type. Landau levels are energy levels available for a charged particle in a magnetic field. This fixation of orbits for this charged particle occurs purely due to quantum mechanical effects. Electron occupancy in each level is dictated by the strength of external magnetic field. The modified Hamiltonian for this system with a vector potential A is

$$\begin{aligned} H &= \frac{(p-eA)^2}{2m} + \frac{\beta(p-eA)}{m} \\ &= H_0 + \beta H_h \end{aligned} \tag{2.3.40}$$

where $H_0 = \frac{(p-eA)^2}{2m}$ is original unperturbed Hamiltonian and βH_h is the perturbed energy term. We can rewrite the perturbed term in terms of original Hamiltonian H_0 as

$$H_h = \beta\sqrt{\frac{2H}{m}} \tag{2.3.41}$$

Applying perturbation theory, we get the first order corrections to the nth energy level as

$$\begin{aligned} \Delta E_n &= \langle\psi_n\Big|\beta\sqrt{\frac{2H}{m}}\Big|\psi_n\rangle \\ &= \beta\sqrt{\frac{2\hbar\omega}{m}}\Big(n+1/2\Big)^{1/2} \end{aligned} \tag{2.3.42}$$

Now the ratio of correction in the energy to original unperturbed energy is

$$\frac{\Delta E_n}{E_n^{(0)}} = \frac{\beta}{\hbar\omega}\sqrt{\frac{2\hbar\omega}{m}}\frac{\Big(n+1/2\Big)^{1/2}}{\Big(n+1/2\Big)} \tag{2.3.43}$$

For the lowest level, $n=1$, we get the correction as

$$\frac{\Delta E_1}{E_1^{(0)}} = \frac{2\beta}{\sqrt{3\hbar m\omega}} \tag{2.3.44}$$

Thus for this nonlocal GUP deformation of momentum, Landau levels receive corrections in first order. From the experimental point of view, some measurements for Landau levels in scanning tunneling microscope(STM) with an electron kept in a magnetic field of $10T$, the angular frequency is $\omega = 10^3 GHz$. This fixes a bound on β as $\beta \leq 10^{-22}$. Hence, the value of β can not be greater than this value, otherwise it would violate experimentally known results for Landau levels [81].

IV) <u>Lamb Shift</u>

This effect occurs in Hydrogen atom when the orbital electron interacts with the vacuum fluctuation energy. This is a purely quantum mechanical effect and hence a quantum theory of hydrogen atom will be empoloyed to calculate the changes in the wavefunction. So the point of our interest will be the wavefunction analysis within this deformation scheme. Te potential energy of hydrogen atom which is spherically symmetric system is $V(r) = \frac{-k}{r}$. The deformed Hamiltonian in this case is [81]

$$H = \frac{p^2}{2m} - \frac{k}{r} + \frac{\beta p}{m} \tag{2.3.45}$$

Using perturbatice calculations, we get the first order corrected wavefunction for hydrogen atom as

$$|\psi_{nlm}\rangle_1 = |\psi_{nlm}\rangle + \sum_{n'l'm' \neq nlm} \frac{\varepsilon_{n'l'm'|nlm}}{E_n^{(0)} - E_{n'}^{(0)}} \tag{2.3.46}$$

where $\varepsilon_{n'l'm'|nlm} = \langle \psi_{n'l'm'}|\frac{\beta p}{m}|\psi_{nlm}\rangle$ are the matrix elements of perturbation hamiltonian. Our calculations for ground and first excited state ask for the respective wavefunctions. For ground state, $n = 1, l = 0, m = 0$, wavefunction is given by

$$\psi_{100} = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0} \tag{2.3.47}$$

where a_0 is the Bohr radius. For $n = 2, l = 0, m = 0$ state, it is given by

$$\psi_{200} = \frac{1}{\sqrt{8\pi a_0^3}} \left(1 - \frac{r}{2a_0}\right) e^{-r/2a_0} \tag{2.3.48}$$

Since the system is spherically symmetric, we need only radial momentum for the calculations. It is given by

$$p = -\frac{i\hbar}{r}\frac{\partial}{\partial r}(r) \tag{2.3.49}$$

The eigenvalues thus read as

$$\begin{aligned}
\varepsilon_{200|100} &= \int \psi^*_{200}\left(\frac{\beta p}{m}\right)\psi_{100} d\tau \\
&= \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{1}{\sqrt{8\pi a_0^3}}\left(1 - \frac{r}{2a_0}\right) e^{-r/2a_0}\left(\frac{-i\hbar\beta}{m}\right)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}\right) r^2 \sin\theta dr d\theta d\phi \\
&= \frac{-i4\hbar\beta}{\sqrt{8} m a_0^3}\int_0^\infty \left(1 - \frac{r}{2a_0}\right) e^{-r/2a_0}(r)\frac{\partial}{\partial r}\left(re^{-r/2a_0}\right) dr
\end{aligned}$$

The derivative inside the integral comes out to be

$$\frac{\partial}{\partial r}(re^{-r/a_0}) = \left(1 - \frac{r}{a_0}\right) e^{-r/a_0}$$

Thus we can write

$$\begin{aligned}
\varepsilon_{200|100} &= \frac{-i4\hbar\beta}{\sqrt{8} m a_0^3}\int_0^\infty r\left(1 - \frac{r}{2a_0}\right)\left(1 - \frac{r}{a_0}\right) e^{-\frac{3r}{2a_0}} dr \\
&= \frac{-i4\hbar\beta}{\sqrt{8} m a_0^3}\left[\int_0^\infty re^{-\frac{3r}{2a_0}} - \frac{3}{2a_0}\int_0^\infty r^2 e^{-\frac{3r}{2a_0}} + \frac{1}{2a_0^2}\int_0^\infty r^3 e^{-\frac{3r}{2a_0}}\right]
\end{aligned} \tag{2.3.50}$$

Using Gamma functions

$$\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$$

Eq.(2.3.50) yields

$$\varepsilon_{200|100} = \frac{-i\hbar\beta\sqrt{32}}{27 m a_0}$$

First order correction to the ground state wavefunction is given by

$$\begin{aligned}\Delta\psi_{100}(r) &= \psi_{100}^{(1)} - \psi_{100}^{(0)} \\ &= \frac{\varepsilon_{200|100}}{E_1^{(0)} - E_2^{(0)}}\end{aligned}$$

Now for hydrogen atom,$E_n = \frac{-E_0}{n^2}$ with $E_0 = -13.6eV$. Therefore,

$$\begin{aligned}\Delta\psi_{100}(r) &= \frac{-i\hbar\beta\sqrt{32}}{27ma_0}\frac{1}{(-E_0+E_0/4)}\psi_{200}(r) \\ &= \frac{i\hbar\beta 8\sqrt{8}}{81ma_0E_0}\psi_{200}(r)\end{aligned} \tag{2.3.51}$$

So the wavefunction receives the correction as above. This also changes the Lamb shift as that is related to wavefunction shift. The change in the Lamb shift due this type of momentum deformation is given by

$$\Delta E_n^{(1)} = \frac{4\alpha^2}{3m^2}\left(\ln\frac{1}{\alpha}\right)\left|\psi_{nlm}^{(0)}\right|^2 \tag{2.3.52}$$

Now varying $\psi_{nlm}^{(0)}$ helps us to compute the other additional changes due this momentum deformation with regard to its original value as [15]

$$\frac{\Delta E_{n(c)}^{(1)}}{\Delta E_n^{(1)}} = 2\frac{\Delta|\psi_{nlm}(0)|}{\psi_{nlm}(0)} \tag{2.3.53}$$

where $\Delta E_{n(c)}^{(1)}$ is the energy corrected due to momentum deformation. This deformation effect for ground state can be written as

$$\frac{\Delta E_{1(c)}^{(1)}}{\Delta E_1^{(1)}} = 2.7\beta\times 10^{23} \tag{2.3.54}$$

From the experimental data, the precision for Lamb shift is about 10^{-12} which yields a bound on β as $\beta \leq 10^{-35}$. So our value for β parameter must be smaller than this to be in agreement with the experimental findings. It is the lowest bound so far obtained with p^{-1} deformation [81].

V) <u>Potential Barrier</u>

This part is devoted to the analysis of potential barrier and tunneling current in the minimal length physics of nonlocal GUP with p^{-1} extension. Quantum tunneling phenomenon occurs in scanning tunneling microscope(STM). Now the modified Schrödinger equation for potential barrier problem reads

$$\frac{d^2\psi}{dx^2} + \left(\frac{2i\beta}{\hbar}\right)\frac{d\psi}{dx} - \frac{2m(V-E)}{\hbar^2}\psi = 0 \tag{2.3.55}$$

We will do a regional analysis as usual to solve this equation.

In the first region, particle is free i.e. potential energy $V = 0$. The above equation is

$$\frac{d^2\psi_1}{dx^2} + \left(\frac{2i\beta}{\hbar}\right)\frac{d\psi_1}{dx} + \frac{2m(E)}{\hbar^2}\psi_1 = 0 \tag{2.3.56}$$

which yields te solution as

$$\psi_1 = Ae^{\frac{i}{\hbar}(\sqrt{\beta^2+2mE}-\beta)x} + Be^{\frac{-i}{\hbar}(\sqrt{\beta^2+2mE}+\beta)x} \tag{2.3.57}$$

The first part is incident wave and second part reflected wave from the barrier. But the condition for first part to be incident wave is that

$$|\sqrt{\beta^2 + 2mE}| > |\beta| \tag{2.3.58}$$

We can write the solution as

$$\psi_1 = Ae^{ik_1x} + Be^{-ik_2x} \tag{2.3.59}$$

with $k_1 = \frac{(\sqrt{\beta^2+2mE}-\beta)}{\hbar}$ and $k_2 = \frac{(\sqrt{\beta^2+2mE}+\beta)}{\hbar}$. Potential in the second region is $V = V_0$. The Schrödinger equation with modified version is given by

$$\frac{d^2\psi_2}{dx^2} + \left(\frac{2i\beta}{\hbar}\right)\frac{d\psi_2}{dx} - \frac{2m(V_0-E)}{\hbar^2}\psi_2 = 0 \tag{2.3.60}$$

The solution can be written as

$$\psi_2 = Ce^{ik_3x} + De^{-ik_4x} \tag{2.3.61}$$

where $k_3 = \frac{(\sqrt{\beta^2-2m(V_0-E)}-\beta)}{\hbar}$ and $k_4 = \frac{(\sqrt{\beta^2-2m(V_0-E)}+\beta)}{\hbar}$. There is no reflected wave in the third region. Hence we can write the solution in that region as

$$\psi_3 = Ee^{ik_1x} \tag{2.3.62}$$

Here $k_1 = \frac{(\sqrt{\beta^2+2mE}-\beta)}{\hbar}$.

The concept of transmission coefficeint T for this analysis is very important. We need to show how this T transforms in presence this momentum deformation. For that matter, we need incident current density J_I and transmitted current density J_T given by

$$J_I = \frac{\hbar k_1}{m}|A|^2$$

and

$$J_T = \frac{\hbar k_1}{m}|E|^2$$

Now we can expand the term within the square root in k_1 and retain terms upto first order in β. This gives the values of constants k_1, k_2, k_3 and k_4 as

$$k_1 = \frac{1}{\hbar}(\sqrt{2mE} - \beta),$$

$$k_2 = \frac{1}{\hbar}(\sqrt{2mE} + \beta),$$

$$k_3 = \frac{1}{\hbar}(\sqrt{2m(V_0 - E)} - \beta),$$

and

$$k_4 = \frac{1}{\hbar}(\sqrt{2m(V_0 - E)} + \beta) \tag{2.3.63}$$

The coefficient $\left(\frac{E}{A}\right)$ needs to be calculated for computing transmission coefficient using standard analysis. It turns out to be

$$\frac{E}{A} = \left[e^{-i(k_1+k_4)a}(k_3k_1 + k_1k_4)\right]\Bigg[(k_3 - k_1)\left\{k_2(k_1 + k_4)\left(1 - e^{-i(k_3+k_4)a}\right) - k_4(k_1 + k_2)\right\} + k_3(k_1 + k_4)e^{-i(k_3+k_4)a}\Bigg]^{-1} \tag{2.3.64}$$

Now, for the sake of comparison, lets consider that the original transmision coefficient T_0 and the new one after deforming momentum operator is T, then [81]

$$T = \frac{J_T}{J_I} = \left|\frac{E}{A}\right|^2 \tag{2.3.65}$$

Also considering I_0 is the original current and I to be the deformed one, one can write [15]

$$\frac{I}{I_0} = \frac{T}{T_0} = \frac{1}{T_0}\left|\frac{E}{A}\right|^2 \tag{2.3.66}$$

Therefore the excess tunneling current generated from gravitational deformation of momentum is

$$\frac{I - I_0}{I_0} = \left[\frac{1}{T_0}\left|\frac{E}{A}\right|^2 - 1\right] \tag{2.3.67}$$

With this relation in consideration, some very precise experimental instruments can possibly detect these changes if minimal length effects exist at such low energy scale. This way we can testify the Planck scale physics in the lab [81].

To this end, we have described how we can use the deformed Heisenberg algebra from Generalized Uncertainty Principle to get a glimpse of various quantum gravity approaches i.e. whether these approaches are going in the right direction or do we just need to abandone the process . We have been successful here to perform analysis from a new proposed nonlocal GUP by using p^{-1} type deformation which has never been done before. We can also compare the results from other positive deformations to this one in order to achieve more progress in this direction. This also completes our studying GUP as a quantum gravity phenomenology.

2.3.3 Supersymmetry with a Minimal Length

The effect of Generalized Uncertainty Principle(GUP) on the low energy quantum mechanical systems has been the subject matter in the previous analysis. Now we turn our attention to the other aspect of minimal length physics. The modification in the momentum operator representation due to GUP can be easily incorporated into various field theories. It is expected to modify the corresponding dynamics these theories describe. It is an extended formulation of quantum field and gauge theories. A realistic formulation of a QFT with interaction between particles and fields with a minimal length incorporated would be very interesting feature when taking GUP into consideration. Examples include the modification of standard model dynamics. Since, interaction theories are based on gauge principles, a minimal length consideration here would mean a possible relation between gauge principle and Generalized Unceratinty Principle other side [101]. Gauge principles are key ingredients related to the Lagrangian of a field theory, a possible incorporation of GUP into a field theory can lead to modified Lagrangian. In this section, we aim to investigate a field theory with supersymmetric extension subjected to quadratic GUP, $p \to p(1+\beta p^2)$ deformation. A Lifshitz extension of this theory would also be taken into account. In Lifshitz scaling, space and time scale differently which is written as

$$\boxed{x \to bx}$$

and

$$\boxed{t \to b^z t} \tag{2.3.68}$$

where z is a parameter that measures the extent of anisotropy and b is the scaling factor. Lifshitz scaling has proved to be highly encouraging in studying the physical systems where breaking of Kondo effect is observed as in Kondo lattices [102]. The Kondo effect is a prime example of sources witnessing supersymmetry in lower dimensional systems including graphene [103, 104]. It has also been used in the study of van Casimir and der Waals interaction, graphene and a molecule, plate and the single wall and CNT, and graphene and a material plate [105]. Yet another point regarding Lifshitz scaling is the Nambu-Jona-Lasinio type four-fermion coupling with $z = 3$ asymptotically free Lifshitz point which gives rise to mass scale dynamically [106]. Also a fermionic theory with $z = 2$ requires a nonlocal differential operator. It must be noted that one requires harmonic extension of functions to deal with nonlocal differential operator as illustrated earlier [87]. Lifshitz theories with GUP have also been analyzed in detail [107]. Now the deformation of algebra or GUP is motivated from the Planck scale physics which dictates that spacetime is not a differential manifold beyond Planck scale due to flutuating geometry. Every quantum gravity approach predicts this minimal length. Given a Lifshitz scaled field theory, we aim to incorporate GUP into it. This can be done by writing its Lagrangian for this theory. GUP algebra has been discussed before. So we only write a momentum deformation here as

$$p_i = -i\hbar\partial_i(1+\beta\partial^i\partial_i)$$

A Lifshitz bosonic action in three dimensions taking GUP into account is [107]

$$S_b = \frac{1}{2}\int d^3x(\phi\partial^0\partial_0\phi - \kappa^2\partial^i\phi\mathbb{T}_\partial^2\partial_i\phi) \tag{2.3.69}$$

where $\mathbb{T}_\partial$ is the nonlocal fractional derivative operator and is given by

$$\begin{aligned}\mathbb{T}_\partial &= T_\partial(1-\beta\partial^j\partial_j)\\ &= \sqrt{-\partial^i\partial_i}(1-\beta\partial^j\partial_j)\end{aligned} \tag{2.3.70}$$

Since β does not scale suitably with time and space, hence this deformation breaks down Lifshitz scaling. If we instead promote this β parameter to a background field, we will be able to preserve the Lifshitz scaling. This will then scale as [107]

$$\beta \to b^2\beta$$

The nonlocal differential operator is seen as effectively local by using harmonic extension of functions. Using some sophisticated mathematical techniques we can finally write this bosonic action as

$$S_b = \frac{1}{2}\int d^3x\, i\partial^\mu\phi G_{\mu\nu}\partial^\nu\phi \tag{2.3.71}$$

where $G_{\mu\nu}$ is some specific matrix [108] related to local gamma matrices as

$$[\Gamma_\mu, \Gamma_\nu] = 2G_{\mu\nu}$$

Now a GUP based fermionic operator can be written as

$$\Gamma^\mu\partial_\mu = \gamma^0\partial_0 + \gamma^i\kappa\mathbb{T}_\partial\partial_i \tag{2.3.72}$$

Moreover, we have other other related product as

$$\Gamma^\mu\partial_\mu\Gamma^\nu\partial_\nu = \partial^0\partial_0 - \kappa^2\left[\partial^i\partial_i(1-\beta\partial^k\partial_k)\right]^2 \tag{2.3.73}$$

In the same manner, we now write the fermionic action as follows

$$\begin{aligned} S_f &= \frac{1}{2}\int d^3x \psi^a (\Gamma^\mu \partial_\mu)_a^b \psi_b \\ &= \frac{1}{2}\int d^3x \psi^a (\gamma^0 \partial_0 + \gamma^i \kappa \mathbb{T}_\partial \partial_i)_a^b \psi_b \end{aligned} \tag{2.3.74}$$

where ψ^a and ψ_a are 3-d spinor fields given by

$$\psi_a = \psi^b C_{ba} \text{ and } \psi^a = C^{ab}\psi_b$$

with the condition

$$C_{ab}C^{cd} = \delta_a^c \delta_b^d - \delta_b^c \delta_a^d \text{ and } \psi^2 = \frac{\psi^a \psi_a}{2}$$

With these two GUP deformed actions, it would be possible to construct a supersymmetric theory with $\mathcal{N} = 1$ supersymmetry [108]. The generator of this $\mathcal{N} = 1$ supersymmetry Lifshitz theory with GUP deformation is given by

$$Q_a = \partial_a - (\gamma^0 \partial_0 \theta + \gamma^i \kappa \mathbb{T}_\partial \partial_i \theta)_a \tag{2.3.75}$$

Now consider a harmonic extension of a function $f(x)$ represented by $u(x,y)$ and $\partial_i u(x,y)$ will be the corresponding harmonic extension of $\partial_i f(x)$. Thus, we can write

$$\begin{aligned} T_\partial \partial_i f(x) &= -\partial_y \partial_i u(x,y)|_{y=0} \\ &= -\partial_i u(x,y)|_{y=0} \end{aligned} \tag{2.3.76}$$

It can be shown that te operator T_∂ commutes with ∂_i, so we can write

$$T_\partial \partial_i f(x) = \partial_i T_\partial f(x) \tag{2.3.77}$$

It is also possible to construct a superderivative D_a that will commute with the generator of $\mathcal{N} = 1$ supersymmetry theory as

$$D_a = \partial_a - (\gamma^0 \partial_0 \theta - \gamma^i \kappa \mathbb{T}_\partial \partial_i \theta)_a \tag{2.3.78}$$

The nonlocal supersymmetric algebra reads as

$$\{Q_a, Q_b\} = 2(\gamma^0 \partial_0 + \gamma^i \kappa \mathbb{T}_\partial \partial_i) \tag{2.3.79}$$

$$\{D_a, D_b\} = -2(\gamma^0 \partial_0 + \gamma^i \kappa \mathbb{T}_\partial \partial_i) \tag{2.3.80}$$

$$\{Q_a, D_b\} = 0 \tag{2.3.81}$$

Symmetric states with respect to certain operation here are annihilated here by generators of thet symmmetry. This implies that taking trace of $\langle E|\{Q_a,Q_b\}|E\rangle$ will yield a vanising ground state for this deformed SUSY theory. The commuting property of GUP deformed Lifshitz momentum and generators of SUSY indicates a mass degeneracy of these two states through SUSY generators. It is known that these variations violate Leibniz rule due to nonlocal differential operator contained in Q_a. Differentiating a product of superfields is not equal to differentiation of individual superfields. But, for free theories, it doesn't exist because in the Lagrangian, we can shift one differential operator from one field to other. It means that we can still use superspace formalism to construct a SUSY theory with a nonlocal term for GUP deformed Lifshitz theories. However, interactions break the supersymmetry. We discuss here few features of superspace used for constructing SUSY nonlocal field theories [108].

The product of two supersymmetric derivatives is defined by

$$D_aD_b = -C_{ab}D^2(\gamma^0\partial_0 - \gamma^i\kappa\mathbb{T}_\partial\partial_i)_{ab} \tag{2.3.82}$$

Also

$$2D_aD_b = D_a\{D_b,D_c\}+D_b\{D_a,D_c\}+D_c\{D_a,D_b\} \tag{2.3.83}$$

Other properties include

$$D^aD_bD_c = 0,\ \ D^2D_a = -D_aD^2 \tag{2.3.84}$$

where $D^2D_a = (\gamma^0\partial_0 D+\gamma^i\kappa\mathbb{T}_\partial\partial_i D)_a$. It is obvious that the antisymmetrization of three 2-d indices is lost.

Before writing an action for a free SUSY theory of $\mathcal{N}=1$ generator, we must emphasize the point that one needs to expand superfield Φ as

$$\Phi = \phi + \psi^a\theta_a - \theta^2 F \tag{2.3.85}$$

where $F=[D^2\Phi]$, $\phi=[\Phi]_|$ and $\psi_a=[D_a\Phi]_|$ with the sign $'|'$ means to set $\theta_a=0$ at the end of the calculations. Lets first write the nonlocal SUSY algebra generated by ε^aQ_a with sperspace $\mathcal{N}=1$ as

$$\varepsilon^aQ_a\phi = \varepsilon^a\psi_a\ ,$$

$$\varepsilon^aQ_a\psi_a = -\varepsilon^b[C_{ab}F+(\gamma^0\partial_0+\gamma^i\kappa\mathbb{T}_\partial\partial_i)_{ab}\phi]$$

and

$$\varepsilon^aQ_a\phi = -\varepsilon^b(\gamma^0\partial_0+\gamma^i\kappa\mathbb{T}_\partial\partial_i)_a^b\psi_b\mathcal{N} \tag{2.3.86}$$

With these considerations, the free SUSY action in minimal length scenario reads

$$\begin{aligned}
S_{free}[\Phi] &= \frac{1}{2}\int d^3x D^2[\Phi D^2\Phi]_| \\
&= \frac{1}{2}\int d^3x[D^2\Phi D^2\Phi + D^a\Phi D_a D^2\Phi + \Phi(D^2)^2\Phi] \\
&= \frac{1}{2}\int d^3x\Big[F^2 + \phi\Big\{\partial^0\partial_0 - \kappa^2[\partial^i\partial_i(1-\beta\partial^j\partial_j)]^2\Big\}\phi + \psi^a(\gamma^0\partial_0 + \gamma^i\mathbb{T}_\partial\partial_i)_a^b\psi_b\Big] \\
&= S_a + S_b + S_f
\end{aligned} \tag{2.3.87}$$

where S_a is the auxiliary field, S_b the GUP deformed bosonic action and S_f the deformed fermionic action. Like usual SUSY theories, temporal part of SUSY variations cancel out here as well. Also a part of the this nonlocal fermionic action in SUSY variation cancels out a term of nonlocal bosonic action. However a part of fermionic action with nonlocality does not cancel a part of bosonic action. For this matter we treat nonlocality with harmonic extension of functions. If u_1 and u_2 are harmonic functions of $f_1(x)$ and $f_2(x)$, then for $|x| \to \infty$ and $|y| \to \infty$, we can write [89]

$$\int_C u_1(x,y)\partial^2 u_2(x,y)dxdy - \int_C u_2(x,y)\partial^2 u_1(x,y)dxdy = 0 \tag{2.3.88}$$

Therefore we have

$$\int_{R^2}\big[u_1(x,y)\partial_y u_2(x,y) - u_2(x,y)\partial_y u_1(x,y)\big]_{y=0}dx = 0 \tag{2.3.89}$$

Translating into f_1 and f_2, we can write

$$\int_{R^2}\big[f_1(x)\partial_y f_2(x) - f_2(x)\partial_y f_1(x)\big]dx = 0 \tag{2.3.90}$$

When the operator $\mathbb{T}$ is moved from $f_2(x)$ to $f_1(x)$, we have

$$\int_{R^2} f_1(x)\mathbb{T}_\partial f_2(x) = \int_{R^2} f_2(x)\mathbb{T}_\partial f_1(x) \tag{2.3.91}$$

Now the cancellation of nonlocal SUSY variation of part of fermionic part can be now cancelled by its bosonic counterpart. ALso the terms generated due to nonlocality in fermionic action can be cancelled by the nonlocal SUSY variation of auxiliary fields. This way the generalized uncertainty principle is incorporated into the Lifshitz scaled $\mathcal{N}=1$ supersymmetric theory. It is noteworthy that even after adding the SUSY invariant mass term $\frac{mD^2[\Phi^2]_|}{2} = m\psi^2 + mAF$ to the original SUSY theory Lagrangian, it is still a GUP deformed Lifshitz scaled theory.

As interaction terms in Lagrangian break the supersymmetry of this Lifshitz scaled theory, we can use functional integral method to quantize a free field theory thereby obtaining Feynman graphs.

The functional integral for free theory reads

$$Z_0[J] = \frac{D\Phi\exp\Big\{iS_{free}[\Phi] + J\Phi\Big\}}{D\Phi\exp\Big\{iS_{free}[\Phi]\Big\}} \tag{2.3.92}$$

Here $J\Phi$ is given by

$$J[\Phi] = \int d^3xD^3[J\Phi]_| \tag{2.3.93}$$

Hence we can write

$$Z[J] = \exp\Big\{-i\int d^3xD^3[J(D^2+m)^{-1}J]_|\Big\} \tag{2.3.94}$$

We now write superfield propagator as follows

$$\langle\Phi(p,\theta_1)\Phi_2(-p,\theta_2)\rangle = \frac{D^2-m}{p^0p_0-\kappa^2[p^ip_i(1-\beta p^kp_k)]^2-m^2}\delta(\theta_1-\theta_2) \tag{2.3.95}$$

As interaction terms in the Lagrangian cause a supersymmetry breaking , it is not possible to write the nonlocal SUSY variation of product of two or or more subfields as individual SUSY variation of the subfields. For the sake of argument, we consider here a simple interaction of the form

$$S[\Phi] = S_{free}[\Phi] + S_{int}[\Phi] \tag{2.3.96}$$

The interaction action here reads as

$$\begin{aligned} S_{int}[\Phi] &= \frac{\lambda}{6}\int d^3D^2[\Phi^3]_| \\ &= \frac{\lambda}{2}\int d^3(\phi\psi^a\psi_a+\phi^2F) \end{aligned} \tag{2.3.97}$$

Under a nonlocal SUSY variation generated by ε^aQ_a, this type of action is not gauranteed to be invariant. In normal SUSY theories, we usually show that $\varepsilon^a\psi^b(\gamma^\mu\partial_\mu)_{ab}\phi^2 = 2\varepsilon^a\psi^b\phi(\gamma^\mu\partial_\mu)_{ab}\phi$. However, it turns out here that interactions terms in nonlocal SUSY variation don't cancel each other [108]. Hence, we have

$$\varepsilon^a\psi^a(\gamma^i\kappa\mathbb{T}_\partial\partial_i)_{ab}\phi^2 \neq 2\varepsilon^a\psi^b\phi(\gamma^i\kappa\mathbb{T}_\partial\partial_i)_{ab}\phi \tag{2.3.98}$$

We summarize here by noting that a supersymmetric theory in a nonlocal generalization will only be symmetric if the Lagrangian does not contain any interaction term, which otherwise breaks the supersymmetry. Also the fractional derivatives introduced by nonolocalities can be effectively dealt with harmonic extension of functions from R^2 to $R^2\times(0,\infty)$. This concludes our discusion of nonlocality in field theories due to string considerations. In addition to GUP, we have done a thorough analysis of boundary effects on a supersymmetric Yang-Mills theory field theory [124].

Apart from the formalism of GUP and minimal length as potential quantum gravity phenomenogical model, there are some other ones going on with a sinificant progress. These are build essentially due the considerations of Lorentz symmetry breaking which is expected around Planck scale. The group structure of Lorentz symmetry suffers some modifications. One such class includes a spacetime with a Lorentz violating background where only a subgroup of the total Lorentz symmetry group and some spacetime translation acts as fundamental symmetric group of nature. This idea was put forward by Cohen and Glashow and is known as Very Special Relativity(VSR) [125]. This leads to some interesting results in quantum field and gauge theory. For example, a study of spontaneous symmetry breaking (Higgs mechanism) in this background spacetime indicates that photons attain mass which is very unusual. However, the value of this mass is well below the upper experimental bound [126]. Some essential implications of the origin of neutrino mass in VSR framework has been discussed in [127]. However, till date, there is no substantial experimental evidence for any of these predictions by any QG phenomenogical model.

Chapter 3

Quantum Entanglement and Minimal Length

"We live on an island surrounded by a sea of ignorance. As our island of knowledge grows, so does the shore of our ignorance."———John Archibald Wheeler

One of the enigmatic attributes of quantum mechanics that startled physicists from the very beginning is the idea of quantum entanglement. It is a strange phenomenon where by a group of particles are generated or interact in a way such that effect of one system is felt by another at a large distance distance simultaneously. So their existence is tied up in such a way that one can never separate out their states and study them individually. So this simultaneous effect seems to have no respect for the distance, even if it is million light years. Wouldn't this violate the causality notion of special relativity? This was the very situation that embarrased Einstein and he started descrediting quantum mechanics. He believed that quantum mechanics was an incomplete theory and it needed certain 'hidden' parameters to describe the real physical universe.

While dealing with a composite system of two or more quantum systems, we come to know that there exist global states which can not be written as product of individual quantum states. We say such states are entangled [26]. So it is a highly non-classical correlation between the systems manifested by quantum formalism. Einstein criticised the idea and called it a "a spooky action at a distance". in a well known paper with his students Podolsky and Rosen [27]. It came to be later known as EPR paradox. It was John bell who later turned this puzzle into a thought experiemnt [28]. He accepted the EPR conclusion as his working hyphothesis and later showed that if there would a local hidden variable theory as predicted by Einstein, then some measurement correlations in the form of inequality(Bell inequality would have some upper bounds. These thought experiments were later realised in some lab experiments [29]- [30]. Everytime, Bell inequality was violated. This confirmed beyond doubt that the predictions made by quantum description were right.

Nowadays, quantum entanglement forms a basis for all modern quantum information processing systems. With the advent of quantum information and computation theory, we find its uses in quantum teleportation, quantum dense coding, quantum error correction codes.

The aim of this chapter is to study quantum entanglement in the framework of minimal length physics of Generalized Uncerrainty Principle. An explicit investigation of quantum entangled states with GUP and minimal length has been carried out. It is very interesting to include noncommutative geometry within this study because spacetime with noncommutative features has been found to be highly useful in constructing entangled states with an improved entanglement. Hence, we will incorporate minimal length and noncommutative spacetime simultaneously to study the behaviour of the entangled states.

3.1 Quantifying Entanglement: The Entanglement Entropy

Known as von Neumann entropy for bipartite systems, entanglement entropy helps us to compute the information content in entangled systems. It also tells us how the information is stored through the process of quantum entanglement. Though entropy of entanglement is a subject of study in traditional quantum mechanics, however, it has also been very thoroughly extended to quantum field theory and quantum gravity scenarios within holography and AdS/CFT correspondence. Through the notion of pure and mixed states in quantum mechanics, we are able to define density operator which helps to write a mathematical relation for entropy. By a pure state, we mean, it is the one which gaurantees us to furnish us all possible information about the state of a system. We don't have any uncertainty about the state of a system. Thus a pure state can be represented by vector in Hilbert space with unit norm. Thus, for a pure state $|\psi\rangle$ in some arbitrary basis $\big(|u_1\rangle, |u_2\rangle, ..., |u_n\rangle\big)$, we can always have a representation

$$|\psi\rangle = \alpha_1|u_1\rangle + \alpha_2|u_2\rangle + ...\alpha_n|u_n\rangle \tag{3.1.1}$$

where $\sum_{i=1}^{n}\left|\alpha_n\right|^2 = 1$ guarantees the normalization condition. We now consider two systems characterised by two state vectors $|\psi\rangle$ and $|\phi\rangle$ as two pure states with H_n and H_m as their Hilbert spaces which are entangled to each other. Suppose $\big(|u_i\rangle, i = 1,2,....,n\big)$ and $\big(|v_j\rangle, j = 1,2,....,m\big)$ be the orhtonormal basis for these two systems respectively, in their repective Hilbert spaces,then combined Hilbert space of these two systems is tensor product of these two Hilbert spaces $H_{nm} = H_n \otimes H_m$ with orthonormal basis$\big(|u_i\rangle \otimes |v_j\rangle, i = 1,2,....,n, j = 1,2,....,m\big)$ where $|u_i\rangle \otimes |v_j\rangle = |u_i\rangle|v_j\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m}\gamma_{ij}|u_i\rangle|v_j\rangle$ and $\sum_{i=1}^{n}\sum_{j=1}^{m}\left|\gamma_{ij}\right|^2 = 1$ [109].

A pure state $|\chi\rangle \in H_{nm}$ is a separable state if and only if we can write it as Kronecker product of two pure states $|\psi\rangle \in H_n$ and $|\phi\rangle \in H_m$ given by

$$|\chi\rangle = |\psi\rangle \otimes |\phi\rangle \tag{3.1.2}$$

where the individual states $|\psi\rangle$ and $|\phi\rangle$ have the structure $|\psi\rangle = \sum_{i=1}^{n}\alpha_i|u_i\rangle$ and $|\phi\rangle = \sum_{j=1}^{m}\beta_i|v_j\rangle$

Now for the idea of a mixed state, we introduce the concept of a density matrix is. It is among the very tenets of quantum mechanics that Heisenberg uncertainty principle restricts us to acquire complete knowledge about a system, given its state vector. It is due to the inherent nature of quantum world that we always have a loss of information about a system when we try to make a measurement on it since our act of measurement forces the system to take a definite stand(wavefunction collapse). Thus, this uncertainty of knowledge about such a system has nothing to do with our measurement apparatus or procedure. On contrary, if a system is produced in such a way that it has a probablistic distribution of different states, we will always have incomplete knowledge about it owing to our inefficient preparation procedure of its state. For example, a machine generating atoms in ground state 90% of times and 10% in excited state will make us difficult to know which state an atom is in when choosen randomly from the sample. So we can only give a statistical distribution for such state. So, comes the density matrix. For a pure state, density matrix is $\rho = |\psi\rangle\langle\psi|$. Mixed states are statitical mixtures of pure state density matrices. Density operator for pure states is a projection operator in one dimensional space and satisfies the property $\rho^2 = \rho$. Thus, for a mixture of pure states, density matrix is

$$\rho = \sum_{i=1}^{n} p_i \rho_i \tag{3.1.3}$$

where $\rho_i = |\psi_i\rangle\langle\psi_i|$ and p_i are the probabilities for pure density matrices ρ_i.

3.1.1 Entropy in Quantum Mechanics

The notion of quantum entanglement emerged during the early development of quantum mechanics. So the concept of entropy associated with entanglement follows from the very foundational aspects of quantum mechanics. We discuss the first conception of entropy of entanglement from the quantum mechanical point of view.

Consider the system divided into two subsystems A and B with Hilbert spaces H_A and H_B respectively, then total Hilbert space of this combined system reads as

$$H = H_A \otimes H_B$$

Now the reduced density matrix of ρ_A of system A is

$$\rho_A = tr_B \rho_{tot} \tag{3.1.4}$$

where we traced out the degrees of freedom in system B. Now the entanglement entropy of this subsystem A with reduced density matrix ρ_A is given by

$$S_A = -tr_A \rho_A \log \rho_A \tag{3.1.5}$$

This is von Neumann entropy for this entangled system. It can be clearly seen that it vanishes for a pure ground state [110]. Now consider $|i\rangle_A$ and $\rangle j_B$ as orthonormal states of systems, then the pure ground state is

$$|\Psi\rangle = \sum_{ij} C_{ij} |i\rangle_A \otimes |j\rangle_B \tag{3.1.6}$$

Here if $C_{ij} \neq C_i^A C_j^B$, this state is not separable and is hence called entangled states. Hence a reduced density matrix $(\rho_A)_{ij} \equiv [\langle i|\rho_A|j\rangle]_A = \sum_l C_{il} C_{kl}^*$ corresponds to case of vanishing entanglement entropy. So this is measure of how much system A is entangled with system B.

For the sake of an example, consider two spin half particles with spin up and spin down configuration in an entangled state as

$$|\Psi\rangle = \frac{1}{2}\Big[|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B\Big] \tag{3.1.7}$$

Now we write the reduced density for system A as

$$\rho_A = \frac{1}{2}\Big[|\uparrow\rangle_A \langle\uparrow|_A + |\downarrow\rangle_A \langle\downarrow|_A\Big] \tag{3.1.8}$$

It is very interestring to note here that ρ_A is a pure state rather ρ_{tot} is indeed. Therefore the entanglement entropy now reads

$$S_A = \log 2 \tag{3.1.9}$$

So it does not vanish for ρ_A. CLosely associated with von Neumann entropy is the concept of Rényi entropy which is actually a generaization of entanglement entropy. It reads

$$S_n(A) = \frac{1}{1-n} \log\, tr_A \rho_A^n \tag{3.1.10}$$

where n is just a label. In the limit of $n \longrightarrow 1$, we recover the usual von Neumann entropy [110].

$$S_A = \lim_{n\to 1} S_n(A) \tag{3.1.11}$$

Unlike von Neumann entropy which tells us the extent of entanglement, Rényi entropy gives us information about the reduced density matrix eigenvalues. One of the important proerties of entanglement entropy is that for a subsystem with a pure ground state, it equals to that of its complement $B = \bar{A}$.

$$S_A = S_B$$

For three subsystems labelled $A, B$and C with no intersection, the following inequalities, called *strong subadditivity* are observed [111]

$$S_{A+B+C} + S_B \leq S_{A+B} + S_{B+C},$$

and

$$S_A + S_C \leq S_{A+B} + S_{B+C}$$

Summarizing the von Neumann entropy properties, it becomes evident that it becomes directly dependent upon dimensions of space. For example, for d-dimensional Hilbert space, its maximum value would be log d [112]. Several other forms of entropty are also discused in the framework of quantum information and computation, but those don't fall under the domain of our interest.

3.1.2 Holographic Entropy

This is an extension of entanglement entropy to the quantum gravity and string theory regimes. Holography is an remarkable conceptual landmark in string theories and other theories of gravity. Holographic field theories are connected to gravity physics through the beautiful Maldacena or gauge/gravity duality. The essential proposition of this duality is that a field theory in d-dimensional spacetime can be described in terms of a string theory in $d+1$ dimensions but with some gravitational ingredients. This way the physical effects manifested at a low energy scale are finally attributed to gravitaional interactions [113]. Now in the description of black hole thermodynamics, the famous Bekenstein-Hawking entropy has been found to have a close resemblence to entanglement entropy. For an observer outside black hole, there is no possible probe to get information about the infalling body. Thus by obtaining a reduced density matrix of the system outside the horizon which is in entanglement with the system inside of the horizon gives us the information about the infalling system [114,115]. Here we present a view of the entanglement entropy primarily built upon the AdS/CFT correspondence and can be regarded as the so-called gravitational interpretation of entropy of entanglement.

In a conformal field theory(CFT) with $\mathbb{R}^{1,d}(or\ \mathbb{R}\times S^d)$ of a subsystem A corresponding to $d-1$ dimensional boundary, the area law or entropy relation reads

$$S_A = \frac{\text{Area of } \gamma_A}{4G_N^{(d+2)}} \tag{3.1.12}$$

where $G_N^{(d+2)}$ is Newtons gravitational constants in $d+2$ dimensions with d-dimensional γ_A as a static minimal surface in AdS_{d+2} having the boundary surface ∂A. This means that the role of factor γ_A is to act as a holographic screen for an observer who has an access to the outside system A. Dimensionality d of space plays a very important role here for computing entropy. For example, in a $d=2$ dimensional space i.e AdS_3/CFT_2, it becomes [116]

$$S_A = \frac{L_{\gamma A}}{4G_N^{(3)}} = \frac{R}{2G_N^{(3)}}\int_{\varepsilon}^{\pi/2}\frac{dS}{\sin s} = \frac{c}{3}\log\left(\frac{l}{a}\right) \tag{3.1.13}$$

where R is the radius of the AdS_3 space, ds corresponds to the metric of AdS_3 space which in global coordinates (t,ρ,θ) is

$$ds^2 = R^2(-\cosh\rho^2 dt^2 + d\rho^2 + \sinh\rho^2 d\theta^2)$$

and is divergent at $\rho \to \infty$ of the AdS_3 space Also c is the central charge given by

$$c = \frac{3R}{2G_N^{(3)}}$$

For higher dimensional cases with $d \geq 2$, the geodesic lines are replaced with miniml surfaces which makes the computations analagous to Wilson loops [117]- [119]. Working out in the Poincaré metric for the space AdS_{d+2}

$$ds^2 = R^2 z^{-2}(dz^2 - dx_0^2 + \sum_{i=1}^{d} dx_i^2)$$

Here the entropy will correspod to two cases depending upon the shape of A. For a straight belt with $A_S = \{x_i | x_1 \in [-l/2, l/2], x_{2,3,\ldots,d} \in [-\infty,\infty]\}$, it reads

$$S_{A_S} = \frac{N^2 L^2}{2\pi a^2} - 2\sqrt{\pi}\left(\frac{\Gamma 2/3}{1/6}\right)^3 \frac{N^2 L^2}{l^2}$$

and for a disk like surface with

$$S_{A_D} = N^2\left[\frac{l^2}{a^2} - \log\left(\frac{l}{a}\right) + \mathscr{O}(1)\right]$$

In this analysis, we took Type IIB string on $AdS_5 \times S^5$ with a dual CFT as a $4D$ $\mathscr{N} = 4$ $SU(N)$ super-Yang-Mills theory [116]. Guided by this computation, we can easily get the near horizon limit case of $AdS_4 \times S^7 (AdS_7 \times S^4)$ of $M5$ branes

$$S_{A_S}^{M2} = \frac{\sqrt{2}}{3} N^{3/2}\left[\frac{l}{a} - \frac{4\pi^3}{\Gamma(1/4)^4}\frac{L}{l}\right],$$

$$S_{A_D}^{M2} = \frac{\sqrt{2}\pi}{3} N^{3/2}\left[\frac{l}{a} - 1\right],$$

$$S_{A_S}^{M5} = \frac{2}{3\pi^2} N^3\left[\frac{L^4}{a^4} - 16\pi^{5/2}\frac{\Gamma(3/5)^5}{\Gamma(1/10)^5}\frac{L^4}{l^4}\right]$$

and

$$S_{A_D}^{M5} = \frac{32}{9} N^3\left[\frac{1}{4}\frac{l^4}{a^4} - \frac{3}{4}\frac{l^2}{a^2} + \frac{3}{8}\log\left(\frac{l}{a}\right)\right] \tag{3.1.14}$$

Now consider a super-Yang-Mills theory with $\mathcal{N}=4$ superspace at finite temperature T defined on $\mathbb{R}^3$ with the metric [120]

$$ds^2 = l^2\left[\frac{du^2}{hu^2} + u^2(-hdt^2 + dx_1^2 + dx_2^2 + dx_3^2) + d\Omega_5^2\right]$$

For such a case, entropy is given by

$$S_{finite} \simeq \frac{\pi^2}{2}N^2T^3L^2l = \frac{\pi^2 N^2 T^3}{2} \times (\text{Area of } A_S) \tag{3.1.15}$$

which is like Bekenstein-Hawking entropy of black hole. It is noteworthy here that in higher dimensions $d \geq 2$, quantitaive comparison is highly difficult to be drawn. The reason for this is primarliy due to the fact that a strongly coupled gauge theory with unknown entanglement entropy is related to a gravity theory [116]. The bottom line of this discussion is that in case of a high gravity regime like a black hole, the entanglement entropy is calculated for a spatial region of the boundary for it.

3.2 Entanglement in Non-commutative Spaces

There is an interesting feature associated with the squeezed coherent states of harmonic oscillator. By squeezed states in quantum mechaics, we mean the eigenstates that correspond to the minimum value of Heisenberg uncertainty principle, i.e.

$$\Delta x \Delta p = \frac{\hbar}{2}$$

In case of harmonic oscillator, the ground state $|0\rangle$ is the simplest example of it. In general, the states which satisfy the above relation constitute squeezed states. An general expression for a squeezed state (in $\hbar$ units) can be written as

$$\psi(x) = C\exp\left\{-\frac{(x-x_0)^2}{2\omega_0^2} + ip_0 x\right\}$$

with C as the normalization constant. Such a state is an eigenstate of operator $\hat{x} + i\hat{p}\omega_0^2$. What is more interesting is the the entanglement of squeezed states in high energy scenario. In this connection, we shall not restrict ourselves to minimal length only. In addition to the Planck length considerations, we shall also take into account the noncommutative nature of spacetime which essentially emerges around this scale. As previously discussed, as soon we reach this minimal length scale, the geometrical structure of spacetime suffers radical changes such that the usual commuting variables do no longer commute. Due to this, we replace the background spacetime by a noncommutative one. We choose here these squeezed states of harmonic oscillator and calculate the linear entropy of entanglement for it.

For a brief overview of the noncommutative algebra, let us first write down the Hamiltonian for a system in noncommutative fabric of spacetime as [121]- [123]

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2 - \hbar\omega\Big(\frac{1}{2} + \frac{\tau}{4}\Big) \tag{3.2.1}$$

which accomodates the following spacetime algebra in momentum representaion

$$[X,P] = i\hbar(1+\check{\tau}P^2),\ \ X = (1+\check{\tau}P^2)x \ \text{ and } \ P = p \tag{3.2.2}$$

In the above relations, the parameter τ chracterizes the noncommutative regime and is given by

$$\check{\tau} = \Big(\frac{\tau}{\hbar m\omega}\Big)$$

Such algebra closely resembles the GUP algebra for its motivation is the same minimal length as that of GUP. The peculiar property of this Hamiltonian in (3.2.1) is that it is non-hermitian if seen from an inner product perspective, however the eigenvalues are gauranteed to be real. Through a similarity transform, one can relate non-hermitian Hamiltonian to that of hermitian one. i.e. given a Dyson map η, we can write $h = \eta H \eta^{-1}$ where h is hermitian. For the noncommutative algebra described above, the corresponding Dyson map can be written as $\eta = (1+\check{\tau}p^2)^{-1/2}$. Acting as a metric, η helps us to write the Hamiltonian h as [128]

$$h = \eta H \eta^{-1} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \frac{\omega\tau}{4\hbar}(p^2x^2 + x^2p^2 + 2xp^2x) - \hbar\omega\Big(\frac{1}{2} + \frac{\tau}{4}\Big) + \mathscr{O}(\tau^2) \tag{3.2.3}$$

We construct the coherent states for harmonic oscillator in these noncommutative spaces. Hence we call it noncommutative oscillator. Invoking techniques from Rayleigh-Schrödinger perturbation theory, we can write the corrected wavefunctions of this noncommutative oscillator as

$$|\alpha\rangle = \frac{1}{N(\alpha)}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{\rho(n)}}|\phi_n\rangle \tag{3.2.4}$$

with $\frac{1}{N(\alpha)}$ as normalization constant. Here the states $|\phi_n\rangle$ have the form

$$|\phi_n\rangle = |n\rangle - \frac{\tau}{16}\sqrt{(n-3)^{(4)}}|n-4\rangle + \frac{\tau}{16}\sqrt{(n+1)^{(4)}}|n+4\rangle + \mathscr{O}(\tau^2) \tag{3.2.5}$$

So in the expanded form, the state in (3.2.4) reads as

$$\begin{aligned}
|\alpha\rangle &= \frac{1}{N(\alpha)}\left[\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{\rho(n)}}\left(|n\rangle + \frac{\tau}{16}\sqrt{\frac{(n+4)!}{n!}}|n+4\rangle\right) - \frac{\tau}{16}\sum_{n=4}^{\infty}\frac{\alpha^n}{\sqrt{\rho(n)}}\sqrt{\frac{n!}{(n-4)!}}|n-4\rangle\right]\\
&= \frac{1}{N(\alpha)}\left[\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{\rho(n)}}\left(1 - \frac{\tau}{16}\alpha^4\frac{f(n)!}{f(n+4)!}|n\rangle\right) + \frac{\tau}{16}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{\rho(n)}}\sqrt{\frac{(n+4)!}{n!}}|n+4\rangle\right]\\
&= \frac{1}{N(\alpha)}\left[\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{\rho(n)}}\left(1 - \frac{\tau}{16}\alpha^4\frac{f(n)!}{f(n+4)!}|n\rangle\right) + \frac{\tau}{16}\sum_{n=0}^{\infty}\frac{\alpha^{(n-4)}}{\sqrt{\rho(n)}}\frac{n!}{(n-4)!}\frac{f(n)!}{f(n-4)!}|n\rangle\right]\\
&= \frac{1}{N(\alpha)}\sum_{n=0}^{\infty}\frac{\mathscr{C}(\alpha,n)}{\sqrt{\rho(n)}}|n\rangle
\end{aligned}$$

where

$$\mathscr{C}(\alpha,n)=\begin{cases}\alpha^n-\frac{\tau}{16}\alpha^{n+4}\frac{f(n)!}{f(n+4)!}, & \text{if } 0\leq n\leq 3\\ \alpha^n-\frac{\tau}{16}\alpha^{n+4}\frac{f(n)!}{f(n+4)!}+\frac{\tau}{16}\alpha^{n-4}\frac{n!}{(n-4)!}\frac{f(n)!}{f(n+4)!}, & \text{if } n\geq 4\end{cases}$$

Since our interest is in constructing the squeezing states for QHO, the squeezed coherent state for it reads [129]

$$|\alpha,\zeta\rangle=\frac{1}{N(\alpha,\zeta)}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\left(\frac{\zeta}{2}\right)^{n/2}H_n\left(\frac{\alpha}{\sqrt{2\zeta}}\right)|n\rangle \tag{3.2.6}$$

where $H_n(\alpha)$ are Hermite polynomials.

Now we turn to the study entanglement in squeezed states given by (3.2.6) in noncommutative space. The measurement of entanglement can be done via many ways like entanglement distillation, concurrence, relative entropy etc. However, we compute von Neumann entropy here for these squeezed states because a given density matrix ρ easily helps to compute it. It is given by

$$S(\rho)=-Tr(\rho\log_2\rho)=-\sum_x\lambda_x\log_2\lambda_x \tag{3.2.7}$$

while the linear entropy would be

$$S=1-Tr(\rho_A^2) \tag{3.2.8}$$

Here ρ_A is the reduced density matrix of system A with respect to system D and density matrix ρ_{AD}. Hence for this case, the reduced density matrix ρ_A reads

$$\rho_A=\frac{1}{N^2(\alpha,\zeta)}\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}\sum_{m=0}^{\infty-\max(q,s)}\frac{\mathscr{S}(\alpha,\zeta,m+q)\mathscr{S}^*(\alpha,\zeta,m+s)}{m!\sqrt{q!s!}f(m+q)!f(m+s)!}t^q\bar{t}^s r^{2m}|q\rangle\langle s| \tag{3.2.9}$$

This yields the linear entropy as [129]

$$S=1-\frac{1}{N^4(\alpha,\zeta)}\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}\sum_{m=0}^{\infty-\max(q,s)}\sum_{n=0}^{\infty-\max(q,s)}|t|^{2(q+s)}|r|^{2(m+n)}\times\frac{\mathscr{S}(\alpha,\zeta,m+q)\mathscr{S}^*(\alpha,\zeta,m+s)\mathscr{S}(\alpha,\zeta,n+s)\mathscr{S}^*(\alpha,\zeta,n+q)}{q!s!m!n!f(m+q)!f(m+s)!f(n+s)!f(n+q)!} \tag{3.2.10}$$

It is very difficult to carry out the analytical study of the above relation for entropy. However, when one of the input states in a quantum beam splitter is a squeezed state in noncommutative space, the linear entropy shows a interesting behaviour as shown in the Figure 3.1.

For the input with squeezed state, the relationship for entropy with dependence on α and τ is graphically depicted below in Figure 3.2.

The relationships shown graphically above speak a volume about the behaviour of entangled states in noncommutative spaces when the input to the beam splitter are squeezed states. It is very clear from

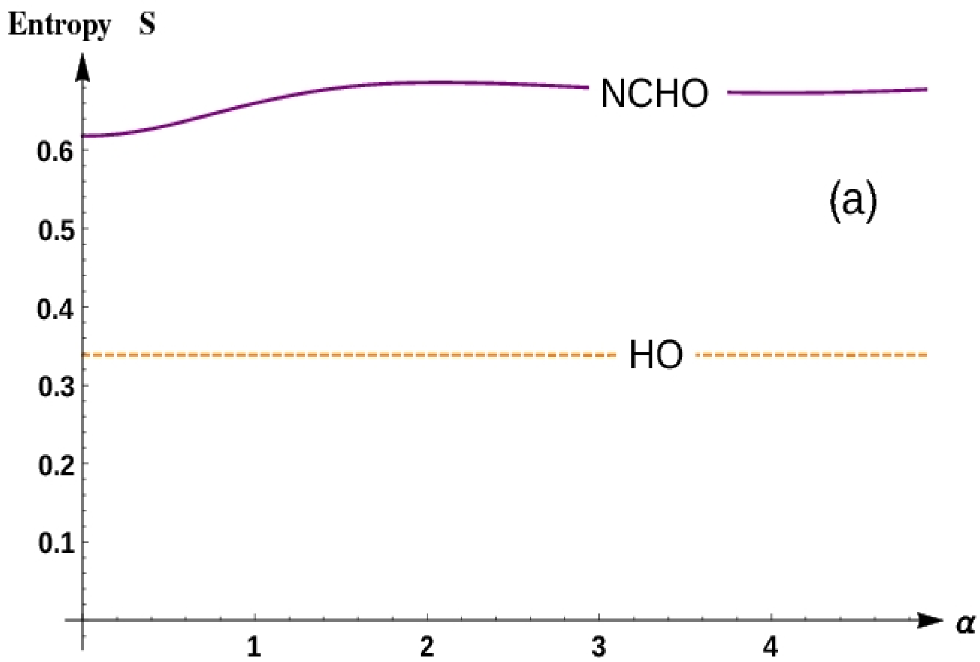

Figure 3.1: Linear entropy of a squeezed state as input in a noncommutative space plotted against α for different values of $\tau = 0.5$ and $\zeta = 0.75$ $\big[$*Source: Sanjib Dey et.al. Phys. Rev. D 91, 124017 (2015)*$\big]$

the above discussion that the noncommutative parameter τ shows a deep effect on the entangled states. The entropy increases continuously for increasing noncommutativity till it saturates irrespective of α. The discussion is also true for coherent input states [129]. This means that within the noncommutative spaces, the quantum systems show a higher degree of entanglement as compared to the ordinary spaces which is a very fascinating result.

3.3 Entangled States with GUP Algebra

It is very interesting to see that if one works out for entanglement in GUP framework instead of noncommutative spaces, the results obtained completely agree with that of noncommutative spaces. For that purpose, we first consider the uncertainty relations for entangled states and then obtain the modified ones with straightforward application of GUP.

Lets assume that x and p are the positions and momenta of n identical particles entangled to each other such that

$$x = \sum_{i=1}^{n} x_i \text{ and } p = \sum_{i=1}^{n} p_i \tag{3.3.1}$$

where x_i is the position of ith particle and p_i its momentum. The commutation relation as usual reads

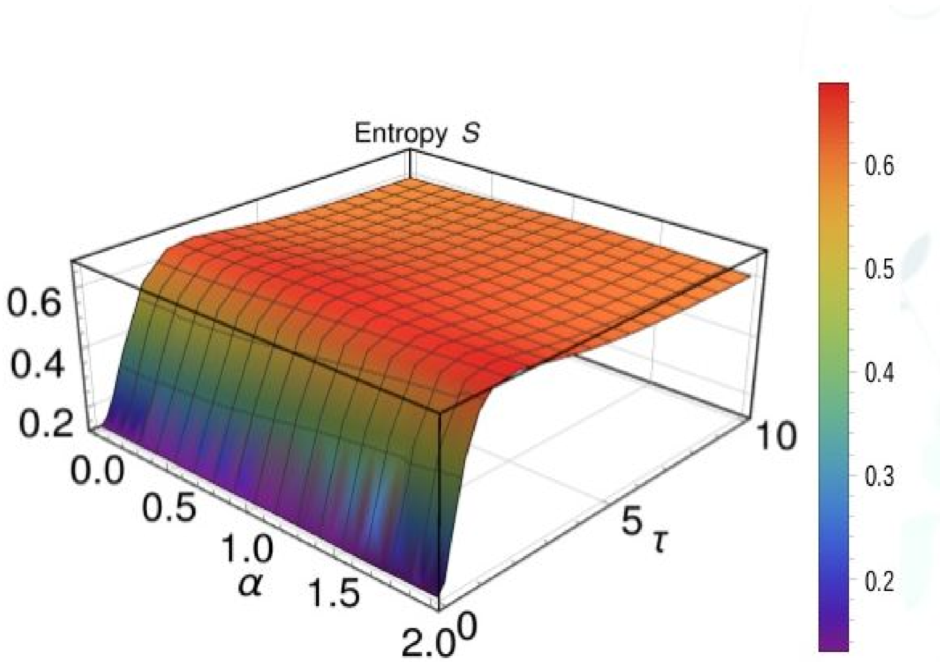

Figure 3.2: Linear entropy plotted against α and τ, for $\zeta = 0.5$ with a squeezed state input for a noncommutative harmonic oscillator with 10 energy levels. $\big[$*Source: Sanjib Dey et.al. Phys. Rev. D 91, 124017 (2015)*$\big]$

as

$$[x_i, p_j] = i\hbar\delta_{ij}, \tag{3.3.2}$$

Now we can write here for n particle system

$$[x, p] = [x_1, p_1] + [x_2, p_2] + ... + [x_n, p_n] = n(i\hbar) \tag{3.3.3}$$

It can shown that for such n entangled systems, the Heisenberg uncertainty relation reads

$$(\Delta x)^2(\Delta p)^2 \geq \frac{n^2\hbar^2}{4} \tag{3.3.4}$$

We can restrict here ourselves by assuming only two particles (bipartite case) for which $n = 2$. The following relations hold for it

$$(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2 \tag{3.3.5}$$

As $x = x_1 + x_2$, we have from above equation

$$(\Delta x)^2 = (\Delta x_1)^2 + (\Delta x_2)^2 + 2\langle x_1 x_2 \rangle - 2\langle x_1 \rangle \langle x_2 \rangle \tag{3.3.6}$$

Similarily, for momentum for this bipartite system, we write

$$(\Delta p)^2 = (\Delta p_1)^2 + (\Delta p_2)^2 + 2\langle p_1 p_2 \rangle - 2\langle p_1 \rangle \langle p_2 \rangle \tag{3.3.7}$$

Using (3.3.6) and (3.3.7) in (3.3.4), following a rigorous mathematical analysis using some suitable quantum states for variables x and p and Schwarz inequality, we finally get the uncertainty for this bipartite system comprising two identical particles [133]

$$\boxed{\Delta x_i \Delta p_i \geq \frac{\hbar}{4}} \text{ with } i = 1,2 \tag{3.3.8}$$

This is very important result for entangled particles which demonstrates that entangled particles decrease the uncertainty in position and momentum and hence a decrease in the lower bound for HUP which is expected. We can see similar conclusions regarding entangled systems in the work by Mario Berta et.al [130] The result should not be mistaken. It is for a combined state for two entangled particles not for a single particle state.

There is a mathematical condition that suffices to draw a line between separable and inseparable (entangled) systems and helps to quantify the extent of entanglement between participating systems. These results have been explicitly discussed in [131].

Considering a density operator ρ for a separable composite system of two individual systems with modes 1 and 2 such that

$$\rho = \sum_i \eta_i \rho_{i1} \otimes \rho_{i2} \tag{3.3.9}$$

where $\sum_i \eta_i = 1$ and $\eta_i \geq 0$. Next we consider a EPR-like operators $\hat{u}$ and $\hat{v}$ given by

$$\hat{u} = |a|\hat{x}_1 + \frac{1}{a}\hat{x}_2 \text{ and } \hat{v} = |a|\hat{p}_1 - \frac{1}{a}\hat{p}_2 \tag{3.3.10}$$

where a is a non-zero real number and x_i and p_i are usual position and momentum variables. Following usual uncertainty relation and other rigourous mathematical results, we calculate the total variance of these two EPR-like operators. We find that for two systems to be inseparable(entangled)

$$\boxed{\langle(\Delta\hat{u})^2\rangle_\rho + \langle(\Delta\hat{v})^2\rangle_\rho < \left(a^2 + \frac{1}{a^2}\right)} \tag{3.3.11}$$

Any violation of the above inequality means an possible separation of systems. This means that for any bipartite entangled system, this relation must hold for the states to remain inseparable. For separable systems, the condition is [131]

$$\boxed{\langle(\Delta\hat{u})^2\rangle_\rho + \langle(\Delta\hat{v})^2\rangle_\rho \geq \left(a^2 + \frac{1}{a^2}\right)} \tag{3.3.12}$$

We now turn to the situation where the above algebra is subjected to GUP analysis. Interestingly, one finds an increase in the upper bound given by the relation (3.3.11). Here the modified commutator algebra reads

$$[x_i, p_j] = i\hbar\delta_{ij}[1 + \beta p_i^2], \tag{3.3.13}$$

and

$$[x_i, x_j] = [p_i, p_j] = 0$$

In this connection, the GUP-modified uncertainty relation for bipartite entangled system to a first order in β becomes

$$\boxed{\Delta x_i \Delta p_i \geq \frac{\hbar}{4} + \frac{\hbar}{4}\beta(\Delta p_i)^2}, \; i = 1,2 \tag{3.3.14}$$

where we have assumed that the uncertainties in position and momentum of one system are equal to their counterparts for other system. This is for the sake of simplification of the relations.

Now to proceed further with with this modified algebra, we consider the experimental findings by Kim and Shih [132]. The results in their work indicate a possible violation of uncertainty principle by obtaining a seemingly paradoxical value as $\Delta y \Delta p_y < \hbar$ for two entangled photons through the consideration of Popper's thought experiment. However, the error lies in the invoking of two-particle physics to investigate the physics of single quantum particle as both EPR and Popper did. Thus, it is noteworthy that their experimental observations clearly correspond to standard quantum predictions and hence there is no violation of uncertainty principle. Thus we can say that quantum mechanics does not stop one to obtain $\Delta y \Delta p_y < \hbar$ for the coincidence measurements in identical entangled particles. However, it is never possible for a single quantum particle. We can see an alternative theoretical explaination to this experiment in the works by Rigolin [133, 134].

Now the presence of extra term in the modified relation (3.3.14) makes it conditional to decide whether the two entangled particle state does or does not violate uncertainty principle. We can consider two cases here. If in the relation (3.3.14),

$$\left[\frac{\hbar}{4}\beta(\Delta p_i)^2\right] > \frac{\hbar}{4}$$

or

$$\beta(\Delta p_i)^2 > 1$$

then there is no violation of standard uncertainty relation $\Delta x \Delta p > \frac{\hbar}{2}$. On contrary, if

$$\left[\frac{\hbar}{4}\beta(\Delta p_i)^2\right] < \frac{\hbar}{4}$$

or

$$\beta(\Delta p_i)^2 < 1$$

then the entangled states produce a disagreement with the uncertainty principle. In case of Kim and Shih's experiment, if we consider $\Delta p_i \sim \frac{\hbar}{\Delta y}$ with $\Delta y = 0.16\text{mm}$, then $\beta(\Delta p_i)^2 < 1$, then we have a disagreement with HUP. However, the presence of GUP term lessens the disagreement with HUP since it is positive. Also it has been demostrated in [134] that as the number of identical entangled particles n increases, the lower bound of uncertainty relation tends to decrease towards zero indicating the systems tendency towards classicality. We observe here that the presence of GUP term prohibits this decrease for a large number of systems tending to make the system more quantum mechanical.

Now the result of GUP-corrected separability condition is very interesting. Following some rigorous mathematical arguments, we get the GUP-corrected relation for separability condition to the relation $(3.3.11)$ as

$$\boxed{\langle(\Delta\hat{u})^2\rangle_\rho + \langle(\Delta\hat{v})^2\rangle_\rho < \left(a^2 + \frac{1}{a^2}\right) + \frac{\beta}{4}(\Delta P_i)^2} \tag{3.3.15}$$

The comparison between the relations $(3.3.11)$ and $(3.3.15)$ shows that the range of inseparability of particle states increases in presence of GUP. This in other words means that the entanglement between the particles increases. Note that with the noncommutative spaces, we witnessed an increase in entanglement as soon as we approach towards minimal length scale. Now the onset of GUP scale characterized by the deformation parameter β also indicates an increase in entanglement. Thus there is a strong agreement between results obtained from the GUP-framework and noncommutative spaces. There is an underlying hint in this agreement that speaks about the credibility of minimal length scale in high energy physics. These results could be of great significance in the construction of phenomenogical models of quantum gravity. We hope to have drawn interesting and bold conclusions from these results that would possibly enhance the advancement in this area of physics to a great extent.

Chapter 4

Results and Discussions

"Knowledge is the conformity of the object and the intellect."———Ibn Rushd

This section will encompass the summary and discussion of all of the important physical implications derived from the the preceeding sections. Although the attempts have been made to present the results in the respective sections to a greater comprehensible level. We present the results in a hierarchial manner starting from the first chapter onwards.

4.1 The Quest for Signatures

The essential aim of this thesis has been to gather all possible works for getting a better understanding of phenomenogical models of quantum gravity. Since all efforts till date regarding quantum gravity have not been completely satisfactory raising doubts regarding our understanding of physical world. Is general relativity a fully consistent framework when it comes to experimentation and observation or does the very tenets of quantum mechanics lack a plausible structure? Nobody really knows with certainty. Drawing a comparison between the quantum mechanics and general relativity shows that QM has till date met with remarkable success while GR is under intense investigation though recent progress in it has been very significant especially with the discovery of gravitational waves. This forces us to take a pause and look back. The standard techniques of quantization employ the principles of quantum mechanics to the fundamental interactions of nature including EM, Strong and weak interaction. But, gravity has so far refused to follow this line. While the process of formulating a good candidate theory for QG is under way, nothing stops in principle to extract some fundamental principles from these theoretical structures that run parallel to each other so that we can construct a simpler model which would possibly entail some testable predictions. So we look for phenomenogical models and hence experimental signatures. The most promising candidate recipe would be the one that matches the experimental findings to a greater extent.

The model of minimal length and Generalized Unceratinty Principle(GUP) has been rigourously pursued for the last couple of decades with different variants. Some experimental results regarding the deformed quantum commutators motivated from minimal length and GUP have also been reported [135]. In addition, a new class of spaces known as noncommutative spaces can also be built from this minimal length notion, though it is subfield of a very wide class of mathematics called noncommutative geomtery which is an extension of commutation algebra of spaces to noncommutative setting.

The framework of GUP and minimal length rests fundamentally on the notion of minimal length which is a common feature of almost all of the proposals that seek merger of quantum mechanics and gravity. Minimal length is often contended to be of the same extent as Planck length scale. The incompatibility between the minimal length and standard principle of uncertainty leads us the generalized form of HUP

$$\Delta x \Delta p \geq \frac{\hbar}{2}(1+\beta p^2)$$

Not only a quadratic deformation, we have some other formulations that have different powers in p (see for instance [81]).

In chapter 2, we presented various kinds of GUP models and their physical implications. Attempts have been made to discuss the bounds on the deformation parameter β to express the possible experimental implementation. The first kind of GUP models that is quadratic in p is the one proposed by Kempf *et.al.* [16]. Our first attempt was to apply the results to a Hydrogen atom in section 2.1.2. The relation for energy from equation (2.1.31) shows that it depends explicitly on l and is maximum for ground state ($n = 1$). A study of fine and hyperfine structure with this GUP would require us to solve Dirac equation for H-atom including higher angular momentum states. This can be later applied to harmonic oscillators which shows that results match with H-atom case in the sense that it again shows a dependence on l. This shows that the energy levels of an oscillator are splitted with a certain rule as per the value of l. The independence of corrected energy on the value of azimuthal quantum number m shows each level still remains $(2l+1)$ degenerate [42]. Thus a non-pointlike structure consideration for harmonic oscillator and H-atom indicates the extended structure for electrons with an upper finite accuracy. We also presented the black hole dynamics and studied the modified relations for temperature and entropy

$$T_{GUP} = \frac{Mc^2}{4\pi}\left[1 \mp \sqrt{1-\left(\frac{M_p^2}{M^2}\right)}\right]$$

and

$$S_{GUP} = 2\pi\left[\frac{M^2}{M_p^2}\left(1-\frac{M_p^2}{M^2}+\sqrt{1-\frac{M_p^2}{M^2}}\right) - \log\left(\frac{M+\sqrt{M^2-M_p^2}}{M_p}\right)\right]$$

that predict a halt to the evaporation process of black holes beyond a certain scale typically around Planck scale. It is also observed that the mass and energy outputs become finite unlike the ordinary case where it is infinite at the endpoint of mass-time graph depicting the GUP-modified results to be more physically reasonable than the standard case. Temperature in this modified case comes out to become meaningless beyond Planck scale and Schwarzschild radius of $2L_p$ [56]. We observed that the minimal length considerations for quark gluon plasma(QGP) indicate the same behaviour for presssure P(T) as being an increasing function of temperature(T) at around critical temperature T_c sowing a close agreement with the results obtained from Monte Carlo simulations. Thus incorporation of minimal length can lead to very interesting thermal properties of QGP. It is noteworthy that the GUP-corrected relation for entropy of QGP reads [136]

$$\frac{S_{QGP}}{T^3} = 4g_{QGP}\frac{\pi^2}{90} + 5g_{QGP}\alpha_1 T \tag{4.1.1}$$

which shows that entropy is independent of bag pressure.

Section 2 presented the generalization of time-energy uncertainty relation. In this setting, the interesting result is the emergence of time quantization and time crystals. The traditional box problem in this scenario shows that time can be measured only in discrete steps given by the relation

$$T = 2n\pi\hbar\Big(\beta + 2E\beta^2 + 6E^2\beta^3 + \mathscr{O}(\beta^4)\Big)$$

unlike the ordinary notion of continuum. The bound on β_0 obtained from consideration of spontaneous emmision in hydrogen atom is found to be

$$\beta_0 < 7.2 \times 10^{23}$$

which the onset of temporal deformation effects on hydrogen atom. An interesting consequence of this result has been studied for evolution of universe which shows that universe takes discrete jumps rather than a continuous evolution. Further, it shows that the fundamental structure of space-time has to be discrete in case of quantum gravity. Also here M-2 branes show an extended structure and act like probes for this spacetime geometry which lessens the accuracy for resolution with which this geometry can be probed [64].

4.2 Nonlocality, Minimal Length and Fields

Nonlocality essentially arises in physical systems where a particular event shows dependence on the the events that are far off from it spatially as well as temporally i.e. equations that govern the physical dynamics and evolution of the system show dependence on certain values of field that arise not only at a given location and time but also on the entire past history and on very distant regions of spacetime. Such description of cosmos would give rise to a nonlocal universe. In our work, we have studied the nonlocality that arises in the context of string theory. In particular, the appearance of minimal length which is also the typical string size indicates that the universe at the fundamental level possesses an extended structure, hence gives rise to nonlocality. This string-inspired nonlocality has been incorporated into verious gauge and field theories. Schrödinger equation with nonlocality term on the R.H.S essentially helps us to study the nonlocal extension of field theories with the help of Green's operator. We studied the nonlocal form of GUP proposed by S.Masood *et.al* [81] and its implications on various quantum systems. This is achieved by writing an inverse form of momentum by the transformation

$$p \to p(1+\beta p^{-1})$$

from the most general form of momentum deformation

$$p_i \to p_i\Big(1 \pm \sum \beta_{1r}(p^j p_j)^{r/2} \pm \sum \beta_{2r}(p^j p_j)^{-r/2}\Big)$$

Within this nonlocal GUP, we analyzed particle in a box problem which shows that space is quantized unlike a continuum sheet with box length given by

$$L = \frac{n\pi\hbar}{\sqrt{\beta^2+2mE}}$$

It also turns out that structure of spacetime depends on the energy with which it is probed. This resembles the case of gravity's rainbow. From the harmonic oscillator algebra, we calculated the bound to be

$$\beta < 10^{-21}$$

and from the Landau levels, it comes out to be $\beta < 10^{-22}$. The more acccurate bound on β comes from the Lamb shift which is about $\beta < 10^{-35}$. We also got the excess tunneling current as

$$\frac{I-I_0}{I_0} = \left[\frac{1}{T_0}\left|\frac{E}{A}\right|^2 - 1\right]$$

which can be used to determine the existence of deformation in the physical systems, otherwise the known experimental results would be violated. The above form of momentum deformation has been applied to different one dimensional potentials including delta potential well and barrier, Coulomb potential. Some interesting expressions have been computed for wavefunctions which show that some extra phase factor emerges for each of them. The excess tunneling current for barrier potential is found to be prportional to square of deformation parameter ($\propto \beta^2$). The energy corrections have the same form $\big(-\beta^2/2m\big)$ [137]. This string-motivated nonlocal GUP containing inverse powers of momentum has also been extended to teleparallel cosmology [138], higher order theories of gravity [139], black hole physics [140] which have provided some valuable physical insights.

In section (2.2.3), we discussed the supersymmetric extension of a field theory while taking into account the Lifshitz scaling (space and time scale differently) and GUP. Lifshitz scaling helps in describing the physical systems like Kondo lattices with broken Kondo effect. We discussed both bosonic and fermionic versions of the theory While taking GUP into the Lagrangian of this modified theory, we get fractional derivative given by

$$
\begin{aligned}
\mathbb{T}_\partial &= T_\partial(1-\beta\partial^j\partial_j) \\
&= \sqrt{-\partial^i\partial_i}(1-\beta\partial^j\partial_j)
\end{aligned}
$$

with the bosonic action

$$S_b = \frac{1}{2}\int d^3x(\phi\partial^0\partial_0\phi - \kappa^2\partial^i\phi\mathbb{T}_\partial^2\partial_i\phi)$$

while the fermionic action reads

$$
\begin{aligned}
S_f &= \frac{1}{2}\int d^3x\psi^a(\Gamma^\mu\partial_\mu)_a^b\psi_b \\
&= \frac{1}{2}\int d^3x\psi^a(\gamma^0\partial_0 + \gamma^i\kappa\mathbb{T}_\partial\partial_i)_a^b\psi_b
\end{aligned}
$$

It is thus observed that the SUSY Lagrangian only stands symmetric if it does not contain any interaction term [108]. Further, it has found that breaking of supersymmetry at very high energies by a noncommutative deformation leads to generalized uncertainty principle and a field theory constructed from GUP and noncommutative deformation resembles Lee-Wick field theory [141].

4.3 Entanglement with Gravity

Chapter 3 dealt with the short distance or minimal length modification to the behaviour of quantum entangled systems. We pursued the minimal length based approach of both noncommutative space and GUP approach to study these systems. Prior to that we introduced the notion of entanglement and the methods to quantify it: entanglement entropy. The relation for entanglement entropy is the von Neumann entropy given by

$$S_A = -tr_A\rho_A\log_{\rho_A}$$

There are other types of entropy like Renyi entropy that also serve as alternatives to von Neumann entropy. It turns out that not only in ordinary quantum mechanics but the concept of entanglement can be extended to field theory and later to holographic context. We obtained an entropy relation for a subsystem A in Conformal Field Theory with $\mathbb{R}^{(1,d)}$ corresponding to $d-1$ dimensions as

$$S_A = \frac{\text{Area of }\gamma_A}{4G_N^{(d+2)}}$$

This relation comes from the consideration of Maldacena duality or AdS/CFT correspondence. We got to know that in a very strong gravity area in space, entanglement entropy is calculated by taking into account the spatial region of certain area rather than volume as also elucidated by Bekenstein-Hawking entropy relation. Section (3.2) was devoted to the extention of entanglement to noncommutative spaces. Considering a non-commutative harmonic oscillator and its coherent states, we have entropy given in (3.2.10)

$$S = 1 - \frac{1}{N^4(\alpha,\zeta)} \sum_{q=0}^{\infty}\sum_{s=0}^{\infty} \sum_{m=0}^{\infty-\max(q,s)} \sum_{n=0}^{\infty-\max(q,s)} |t|^{2(q+s)}|r|^{2(m+n)} \times \frac{\mathscr{S}(\alpha,\zeta,m+q)\mathscr{S}^*(\alpha,\zeta,m+s)\mathscr{S}(\alpha,\zeta,n+s)\mathscr{S}^*(\alpha,\zeta,n+q)}{q!s!m!n!f(m+q)!f(m+s)!f(n+s)!f(n+q)!}$$

which is a very complicated relation and hard to physically interpret. However, the relationship for linear entropy plotted in Figure (3.1) shows that the entropy does not change with parameter α for an ordinary oscillator. However, for a noncommutative one, it increases with α. This is also supported by the behaviour of entangled states in noncommutative spaces from Figure 3.2, where it is plotted against both α and τ. We get a stunning result that when the background spacetime has a noncommutative behaviour, the entanglement between systems increases. Not only entangled states, further analysis of PACS *(Photon Added Coherent States)* for a noncommutative harmonic oscillator with q-deformed algebra also provides additional hint in this direction. We observe that squeezing property of states increases as noncommutativity increases. This means that noncommutative spaces are very helpful in constructing quantum states with enhanced properties unlike ordinary spaces and thus can have significant effect if employed in quantum optics [142]. But this requires a very sophisticated methodology, both theoretical as well as experimental to achieve this goal. For other interesting implications of noncommutative spaces, see for instance [143]- [145].

Now the same analysis has been carried out in section (3.3). We considered GUP-based quantum algebra for two entangled particles. From the relationsip (3.3.15), it is very easy to see that the extra term due to minimal length on the R.H.S shows beyond doubt that inseparability increases with GUP, which in other words indicates the increase of entanglement between quantum systems, fully conforming to results of noncommutative case. Apart from this quantum gravity and minimal length consideration, however, there is an interesting experiment proposed in [146] which seeks the effects of motion and gravitational field strength on quantum entanglement. Two Bose-Einstein condensates entangled to one another kept in two different satellites suddenly experience a change in distance between them resulting in a change in gravitaional field strength they experience. This is done by changing orbit of one of the satellites. The effects are observable in a space-based setup. It is found that entanglement oscillates periodically with difference between the gravitational potential or orbits. We can conserve entanglement in this setup at a particular location of satellites. This result can dictate future quantum communication technologies in space by considering relativistic effects on quantum systems. It will possibly give another insight into the possible overlap between quantum mechanics and relativity.

Chapter 5

Conclusion and Outlook

"Out beyond the ideas of wrongdoing and rightdoing there is a field. I'll meet you there. When the soul lies down in that grass, the world is too full to talk about." ———*Rumi*

A possible overlap between general relativity and quantum mechanics has been a subject matter of great scientific debate for decades now. Falling short of any experimental clues, we are eventually compelled to look for phenomenogical models. Though not possible to do justice with the subject in such a short piece of work, our aim was to bring up a panoramic view of minimal length physics which serves as one of the leading proposals for quantum gravity phenomenology. The methods of Generalized Uncertainty Principle(GUP) and noncommutative spacetime that follow from this minimal length scale lead us possibly to a position where things can be boiled down to lab-based predictions, to whatever degree. We investigated various quantum mechanical systems within GUP framework leading to new upper bounds for deformation parameter signifying the extent of gravity effects on them. In addition, we dealt with entangled systems in a spacetime background with noncommutative properties and also with modified Heisenberg algebra of quantum mechanics from GUP. The chapterization was done in a most possible hierarchial manner.

In chapter (1), we introduced and reviewed the literature of the problem of quantum gravity, minimal length scale (Planck length) and QG models of GUP and noncommutative spaces. It turns out that certain Gedanken experiments in physics indicate that notion of spacetime breaks down beyond a certain fundamental length scale or typical energy scale. So understanding general relativity at very small scales where quantum effects start dominating is really a difficult task. Further, the measurement in spacetime is governed by Heisenberg uncertainty principle, which does not put any constraint on the resolution with which we can probe space. Hence the incompatibility of HUP and minimal length scale follows in a most natural way. A mathematical relation for well studied form of GUP as been presented in $(1.0.11)$ alongwith a graphical representation in Figure(1.1) conforming to each other.

Chapter (2) contains a full-scale analytical study of various GUP models with the corresponding motivation and algebra. We discussed linear and quadratic forms of GUP and their consequences for physical systems comprising Hydrogen atom, black holes thermodynamics and QG plasma. The interesting result for black hole is that final stages appear to come as a halt to the evaporation process forbidding further elimination, depicting again the power of minimal length physics. Section (2.2) presented the extension of modification to time-energy principle. Time crystals which is a consequence of time quantization shows that its measurement is fundamentally possible in discrete units rather a continuum. As an example case, we studied the spontaneous emission and got the bounds on β parameter. Nonlocality arising out of string theory and minimal length is discussed in section (2.3). The physical systems under a nonlocal GUP of linear inverse form of momentum p hinted at new extent and understanding of quantum gravity effects on low enery systems. we conclude here that extended structure consideration provides extra means for system to aquire different configurations perturbatively. We see a resemblance between probing of spacetime with linear inverse GUP and rainbow gravity case in the sense that the fundamental strucure of spacetime picks up a geometry that depends upon the probing energy. Minimal length supersymmetry with Lifshitz scaling sheds light on the incorporation of nonlocal features of spacetime in gauge field theories.

Chapter (3) was fully dedicated to minimal length motivated noncommutative spaces and GUP based study of quantum entangled systems. The quantifying procedure for entanglement, the entanglement entropy, plotted against noncommutativity in Figure (3.1) and Figure (3.2) demostrates the edge of using noncommutative spaces as states seem more entangled on such spaces as compared to ordinary ones. These results also agree with the GUP-deformed relation for inseparability involving two separate variables (representing two physical quantities like position and momentum). We conclude here by stating that in a spacetime region of very high gravity, the entanlement between systems increases. Coming the other way round, the role of gravity at very small scales in quantum systems is to enhance their quantum mechanical nature like the property of quantum enatnglement. Hence we see a complete agreement between the results derived from two different recipes which is expected because both share the same bedrock, minimal length. We assembled the results of the preceeding sections in chapter (4) along with an extra analysis and discussion of the results while invoking further scientific literature with an advanced outlook over the subject.

Regarding future investigations, we would like to point out that the physical implications of Generalized Uncertainty Principle have been well studied in many systems over the past few decades. In this connection, we believe that the main points of study in the work related to nonlocal GUP proposed by S.Masood *et.al* [81] would be highly encouraging. For instance, whole analysis of the entangled particle system done in section (3.3) of chapter (3) can be repeated with this linear nonlocal GUP. Further, these assessments can be carried out for black hole thermodynamics, various different potentials in quantum systems which carry a great importance in experimental studies and a lot more. We can even compare the results drawn in section (3.3) regarding enatnglement with minimal length and the behaviour of enatanglement in space based experiments as prposed in [146].

With this piece of hope and optimism, we believe that the studies presented in this thesis would serve as a suitable motivation and source for the people to make the efforts in a well-defined direction and to take the advancement in this area of physics to a logical conclusion.

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Publications

1) **Syed Masood**, Mir Faizal, Zaid Zaz, Ahmed Farag Ali, Jamil Raza, Mushtaq B. Shah, *The Most General Form of Deformation of the Heisenberg Algebra from the Generalized Uncertainty Principle, Phys.Lett. B763 (2016) 218-227.*

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3) **Syed Masood**, Mushtaq B. Shah, and Prince A. Ganai, *Spontaneous Symmetry Breaking in Lorentz Violating Background, Int. J. Geom. Methods Mod. Phys. 15, 1850021 (2018).*

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