# The Application of Q-deformed Quantum Mechanics to Quantum Entangled States

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This thesis is dedicated to my mother for their love, support and pray.

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I am grateful to my mother for her love and support. I feel sorry that I can,t be able to share this great moment with my late father. I keep praying for both of them that “ O Allah! Forgive them for their mistakes after all they are human beings ”. I am also thankful to my brothers and my sisters for their love specially to Raja Muhammad Imran who supported me a lot during my research period.

# Abstract

Quantum observables provide algebraic structures only for their eigen basis and copy of them implies structures that are of complementary nature. Qubit state space which inherit two basis vectors and complementary algebraic structures there off takes us to the graphical calculi known as ZX-calculus based on the interaction of their complementary algebraic structures. These structures yields an unexpected connection to the area of Hopf algebras and also to the quantum group. By using this calculus derivations took place in few steps which may proved to be very involved and tedious otherwise. It also takes to the representation and analysis of the states specially entangled one in an intuitive way and also the explanation of physical phenomenons of quantum teleportation and non-locality. Dagger Symmetric Monoidal Categories(dsmcs) provides an exact arena of mathematical framework for further enhancement and elaboration. Strong form of complementarity i.e. scaled variant of a bialgebra yields commutation for the non-commuting observables.

# Contents

# List of Figures

# Chapter 1

# Introduction

Quantum information and computation (QIC) is a branch with main feature of entanglement proves to be step ahead of quantum mechanics in sense that it removes ambiguities and gave answer to the questions raised regarding quantum foundation within the framework of quantum mechanics. It also play their part in other areas of physics e.g. condensed matter physics and statistical physics [1], producing new results for them via computation and/or information theoretical perspective [2]. Importantly, all this glimpse that still much to remain concerning quantum world to be discovered and how we reasoning about it.

First the language introduced to understand quantum theory and their concepts such as observables and complementarity there off is that of Hilbert spaces by von Neuman's in 1932 but just three years after publishing his work saying that he was not so convinced about the idea of Hilbert spaces creates a big doubt and question mark regarding his work and its correctness. Hilbert space formalisms seems to be somewhat ad hoc from conceptual perspective also but the framework of QIC put all these things aside because it present quantum theory in a new way therefore proposed new concepts and paradigms. Another major issue with Hilbert space formalisms is that it can,t address compoundness for quantum systems which proves to be an essential part even while they are given individually. Within the framework of QIC *compoundness* seems to come first as it conceptually analysis the protocol of 'quantum teleportation', find the hidden structures fall behind the no-cloning theorem and diferent computational schemes such as quantum computing based on measurements [3]. Historically speaking, Schrodinger was the first who stressed up on compoundness as early as 1935 [4].

All this paid our attention that Hilbert space formalisms lacks in something which is being fulfilled by the framework of QIC. In that context, we focus our attention to the framework of QIC which associate graphical languages with it and seems to be a right choice for the replacement of conventional Hilbert space formalism as manifested in the following explanation. No-cloning theorem argues that quantum states can not be cloned which on positive side reveals that they may be copied only if they lie in an eigen basis of the observables and also their structures corresponding to copy of eigen vectors are complementary or non-commuting observables structures. Qubit state space which inherit two basis vectors and complementary algebraic structures there off takes us to the graphical calculi based on the interaction of these complementary observable structures known as ZX-calculus comes up with Pauli Z and X spin observables and a set of graphical equational rules. These complementary observable structures yields an unexpected connection to the area of Hopf algebras and even their deformed version i.e. Quantum groups [5][6][7]. Using correspondence between the mathematical framework of Dagger Symmetric Monoidal Categories (dsmcs) and graphical calculi based on this framework provides required tool for further elaboration and enhancement. This also provides an intuitive and visualize way for deriving equalities, proofs and computations.

In this thesis we introduce a simple, intuitive and high-level graphical language based on the interaction of complementary observables structures and a set of graphical equational rules with pure qubit as post selected measurements takes us to the representation of quantum states specially entangled one and do their analysis graphically. Analogies between diagrammatic languages and the framework of Dagger Symmetric Monoidal Categories (dsmcs) [8], comes from the work of Penrose's done on the tensor networks [9]. Formulating quantum theory within the mathematical framework of dsmc which associate graphical languages now has become an active area of research [10][3].

Going deeper into that, in most axiomatic approaches complementarity (incompatibility) is taken to be as a negative property e.g. for observables operators that they do not commute and lattices that they do not enjoy distributivity and so can,t be analyzed in full detail. However here we study the positive aspects of it and show how it ease out things for us to understand specially within the area of QIC. We know from quantum mechanics that a maximally incompatible observable pair called complementary or unbiased pair can,t admit a sharp value at same instant in contrast to compatible observables as in classical physics which shows viceversa. Many algorithms, protocols and computations done before with elementary quantum logic gates now do with ease through this approach of interacting observables.

Specific physical concepts know from the framework quantum mechanics give rise to specific kinds of equations over diagrams e.g. sequential composition of two controlled

X-gates is equivalent to the identity map which give rise to the following equations over diagram:

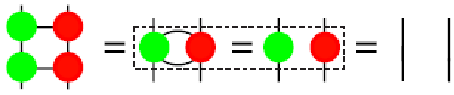


Figure 1.1: Sequential Composition of controlled NOT Gate

*Complementary observables present change in their topology* shown by disconnection between components of green and red dots while compatible observables shows connectedness or contraction.
Now for both the cases the situation arises like this:

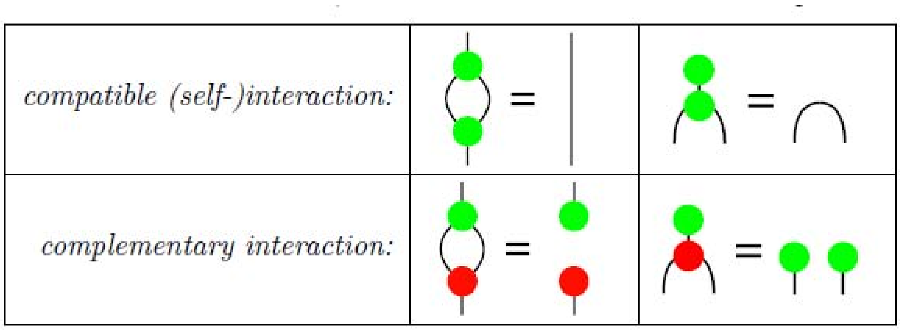



Figure 1.2: Compatible and Complementary Interaction

Clearly indicates that complementarity yields disconnectedness while compatibility shows connectedness between components of observables structures. This topological difference has very interesting consequence in the area of quantum information showing the absence of information flow from one part to the other in a pair of complementary observable structures. In other words, complementary observables has no knowledge of each other in a pair if any of them has knowledge of itself individually implies dynamic counterpart of information flow.
Observable structure phase shift concept takes us to the analysis of their *phase group*. By combining together both the complementarity and phase shifts we can do analysis of multiple two-level quantum systems, qubits.

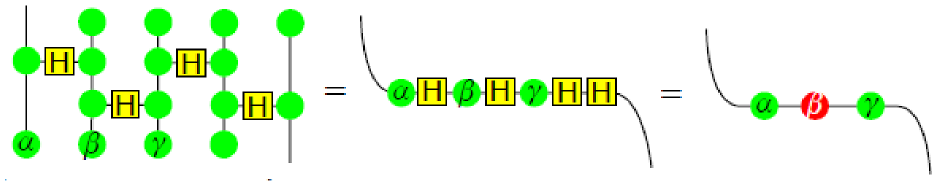
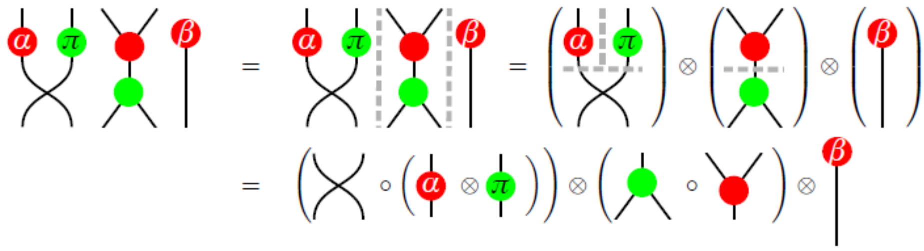



Figure 1.3: Arbitrary One Qubit Gate

We now explain how conceptual analysis of the no-cloning theorem leads to the concept of algebras for observables: Frobenius algebras, bi-algebras etc via taking its *contrapositive* argument. No-cloning theorem reveals fundamental restriction of QIC compared to its classical counterpart that arbitrary quantum states can not be cloned and if it is than it must lie in a given basis or known to be the eigenstates of some non-degenerate observable. So we can treat eigenstates of observables as classical data and copied them freely.
More precisely, consider an observable A with finite number of dimensions termed as the basis of A. The copy operation provides assurance for the copied states that they are only the eigenbasis of A from contrapositive fact of the no-cloning theorem.

$$\delta : |a_i\rangle \longrightarrow |a_i\rangle \otimes |a_i\rangle$$

Now, assume $\epsilon$ be the linear function on H characterized by $\{|a_i\rangle\}_i \longmapsto 1$. In more thoughtful way, $\epsilon$ is the function which uniformly wipe out elements of the basis of A. In algebraic terms, $\epsilon$ defined to be the co-unit for the co-multiplication operation $\delta$.
From a pragmatic point of view, algebras of interacting observables i.e. ZX-calculus with the mathematical framework of Dagger Symmetric Monoidal Categories (dsmcs) seems to provide an exact pattern for manipulating the operations of linearity which are the basic elements of quantum mechanics. Replacing more conventional types of notations such as quantum circuits [11] or the measurement calculus with the meaningful notations of ZX-calculus based on measurements unify all of them in one setting [12].



giving the linear map

$$D = \left( \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \left( e^{-i\frac{\alpha}{2}} \begin{pmatrix} \cos\frac{\alpha}{2} & i\sin\frac{\alpha}{2} \\ i\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \right)$$
$$\otimes \frac{1}{\sqrt{2}} \left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \right) \otimes e^{-i\frac{\beta}{2}} \begin{pmatrix} i\sin\frac{\beta}{2} \\ \cos\frac{\beta}{2} \end{pmatrix}.$$

Figure 1.4: Measurement Based Quantum Computing

Implies computation of a matrix of the order of $16 \times 32$ according to formalism of Hilbert spaces. However by adopting this axiomatic approach of quantum computation the models can be straight forwardly translated into a simple circuit as shown above. It provides a visualized interpretation for the interacting quantum systems and computations, and their associated diagrammatic powerful graphical equational rules done this computation only in few steps which found to be very involved and tedious according to the framework of matrix mechanics. Explain phenomena of quantum teleportation, provides structural origin to the phenomena of *quantum non-locality* and exposes limitations of the no-cloning theorem. Logically, it supports the idea of automation as it present graphical interpretation to many of the computational schemes related to quantum theory. So the hurdle comes in the way to analyze quantum phenomena in full detail is to find other necessary additional structures including these that nature provides us which can turn mathematical entities subject to the law of quantum theory into their structural form. Finding these additional structures is over subject of thought and also current area of research.

In the next chapter we present a review on different types of high level graphical languages including ZX-calculus.

Chapter 3 mainly focuses on the necessary theoretical background of category theory in particular Dagger Symmetric Monoidal Categories required for the representation of these high level graphical languages. String diagrams expose its power by finding hidden structures fall behind the algebraic equalities and axioms. Graphical languages also gave the idea of equal up to graph isomorphism.

In chapter 4 we make a detailed analysis of ZX-calculus and with their pure qubit as post selected measurements we comes up with a set of powerful graphical equational rules. We show the completeness for ZX-calculus as it straightforwardly transformed to arbitrary one qubit unitary and controlled NOT Gate and soundness of graphical equational rules. We also show how the interaction of a pair of complementary observable structure leads to the concept of Hopf algebra and even more their closed form takes to the powerful commutation property for complementary observable structures i.e scaled variant of bialgebra.

Last chapter shows the applications of ZX-calculus to various fields including the important one, entangled states. Further we do their physical analysis by using this graphical language.

# Chapter 2

# Graphical Languages

*Quantum foundation* as a branch of quantum physics is mainly concerned with finding similarities and differences between quantum and classical physics. Quantum computation in a way also like that concerned with the efficiency of solving mathematical problems by considering quantum system rather than its classical counterpart. Within quantum computation we have to consider only an idealized system rather than focus on real systems whereas quantum foundation is more generalized assume rather general feature as well. Furthermore within the framework of quantum computation we consider discrete controlled time steps i.e. the gates as shown in quantum circuit model for measurements calculus or measurement-based quantum computing. Quantum state of some finite-dimensional system can be taken as the joint state of some finite number of qubits. Therefore qubit-based quantum computing yields insights about more general systems as well.

## 2.1 Reasons for Graphical Languages

One of the dominant formalism for quantum theory since its inception being the matrix mechanics, and it is the formalism for quantum computation as well. However there are some issues when we consider matrix mechanics for quantum computing which need to be addressed. First of all, it is not all obvious when we consider matrix mechanics to determine conceptual properties for quantum operations e.g. whether a matrix acting on multiple systems embodies a local transformation or not. Important physical phenomena e.g. quantum teleportation take more than 60 years to understand after the introduction of matrix mechanics [13].
For some fragments of quantum theory e.g. the stabilizer quantum mechanics there exists more efficient descriptions but it is only for the section of quantum theory. Thus, a general high-level languages are required for the analysis of quantum computation

and quantum foundations. Specifically the languages that uses two dimensions: one for parallel composition which allow application of different processes to different systems at the same instant while sequential composition allow application of different processes to the same system at different instances. Two-dimensional languages allow designating of one dimension to space while the other to the time respectively. This makes suitable for the study of network and multiply-connected processes. In contrast is the *algebraic terms* written on line and thus one-dimensional [14].
Here, we consider *some high-level graphical languages*

## 2.2 High-level Graphical Languages

### 2.2.1 Quantum Circuit Notation

Quantum circuit notation is the most basic and commonly used language for quantum computation derived from classical logic gates. Arbitrary unitary gates for single-qubit and two-qubit controlled-NOT gate are the most frequently used gate set for quantum circuits known as universal quantum gate set. Any unitary operation on a finite number of qubits can be expressed as a quantum circuit using universal quantum gate set. The number of gates in a model is the measure of its complexity. In quantum circuit notation, gates are (commonly) represented by labelled boxes with same number of inputs and outputs wires say n respectively on the left and right side, where n is some positive integer [15]. Number of inputs (or outputs) wires are the representation for single, double or multiple qubits gates. A piece of wire alone without any gates indicate the identity transformation, thus the length of wires has no mean to the interpretation of a quantum circuit diagram. Gates can be joined by stacking them horizontally, representing the tensor product of the corresponding matrices i.e. application of different gates to different systems at the same time. Similarly, the output of one gate can be merged into the intput of the next, which shows matrix multiplication: application of different gates to the same system at different times. An example of circuit with one-, two-, and three-qubit gates is

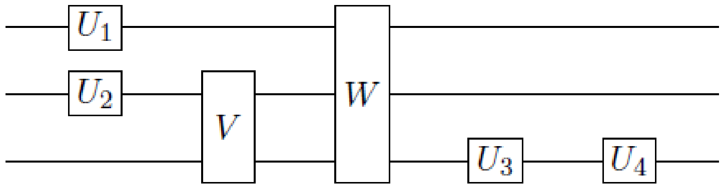



Figure 2.1: Quantum Circuit Diagram

**Issues** with quantum circuit notation:

- Neither defined in exact manner nor widely accepted rules for them to show whether the two circuits are equal: to test equality, we have to revert them back into matrices form.
- Evolution of the quantum systems under quantum circuit model happens to be in discrete steps, as indicated by the discrete gates. Furthermore, it also assumes that systems persist in the same state till acted upon by another gate.
- Quantum circuits do not allow curved wires or cycles, strictly distinguish between the inputs and outputs wire of a gate. For *quantum processes*, this distinction is not a very natural one due to *map-state duality* whereas for quantum circuits it is the representation of time reversal e.g. atemporal diagrams. Furthermore, cycles represents the process of tracing out subsystems from a quantum processes.

### 2.2.2 Stabilizer Graphs

Pure qubit stabilizer states can be represented as decorated graphs known as stabilizer graphs [16]. *Stabilizer states* are the quantum states that are simultaneous eigenstates of a group of Pauli products: tensor products of the identity matrix with the Pauli spin matrices for qubits (see the appendix 1).
Specifically, *Graph states* can be considered as a particular class of stabilizer states whose structure of entanglement is related to that of a finite simple graph ( having no self-loops and also there is at most one edge used for the connection between any two vertices ): the *qubits* represent the vertices and *entanglement* represents by the edges (see the appendix 1).
Any stabilizer state via a local Clifford operation is related to some graph state i.e.

an operation that decomposes into a tensor product of single-qubit Clifford unitaries [17] (see the appendix 1).

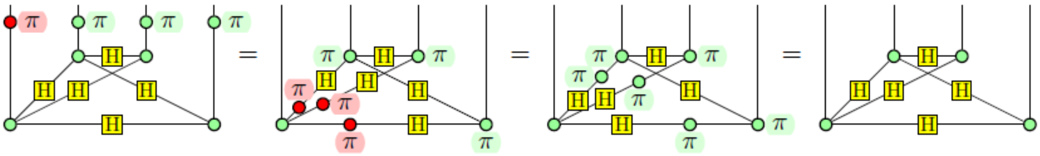


Figure 2.2: Arbitrary Stabilizer State

*Stabilizer graph notation* extends the graph state notation to generalize stabilizer states by using decorations on the graph vertices to denote the unitary applied to the corresponding qubit. Thus, vertices in stabilizer graphs can be empty or filled, have a minus sign or not, and they can have a self-loops or not.

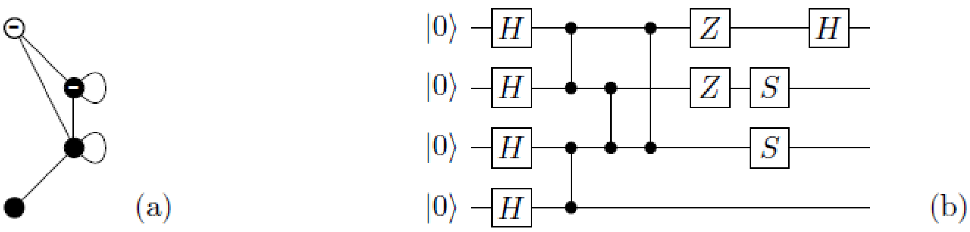



Figure 2.3: Stabilizer Graph

Stabilizer graphs are more efficient than the standard notation because it is represented interms of computational basis states for stabilizer states. Some symmetries are easier to see in stabilizer graphs than in other formalisms. Yet, unlike the other graphical languages introduced here, stabilizer graph notation represents quantum states rather than more general transformations.

### 2.2.3 Atemporal Diagrams

Atemporal diagrams are the generalization of quantum circuit diagrams by dropping any notion of time ordering which shows time reversal. Its not just restricted to qubits but allow arbitrary state spaces.
Large labelled circles called *centres* is the representation of transformations, or measurements, whereas small labelled circles called nodes represents quantum states. The

latter, which are connected by directed edges to the centres, represent the Hilbert spaces occupied by a system in a process. The direction of the edges, together with the decoration upon the nodes open (i.e. empty) / closed (i.e. filled) indicates whether a process involves a Hilbert space or its dual space. The *adjoint* of an atemporal diagrams can be created by changing the orientation of nodes between filled and empty ones also flipping the direction of arrows, and adding † symbols to the labels of boxes with the rule that two daggers on the same label cancel. Tensor product of the corresponding transformations shown by putting their diagrammatic representation next to each other. An open and a closed node having the same label, is the representation of a Hilbert space and its dual, can be merged together; corresponds to an inner product or a (possibly partial) trace. Atemporal diagrams proves to be helpful for showing correspondence between maps and states.

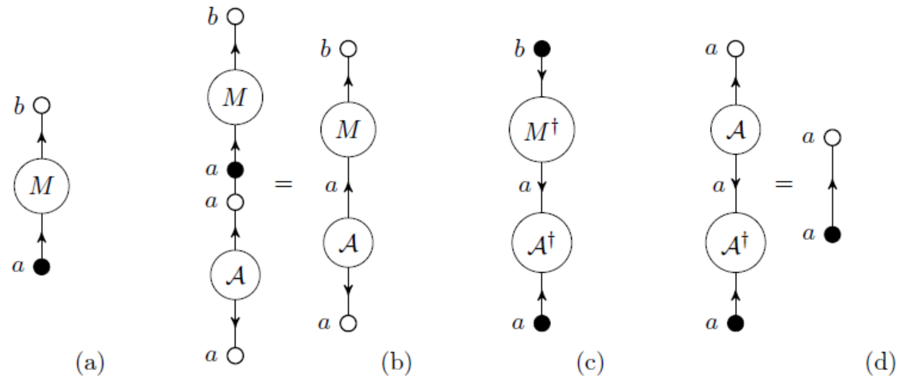



Figure 2.4: Atemporal Diagrams

### 2.2.4 The ZX-calculus

Several graphical languages including ZX-calculus based on *category theory* [18]. Qubit state space inherit two basis vectors and complementary algebraic structures there of on the bases of no-cloning theorem takes us to the calculi of complementary observables from the virtue of their structures interaction known as ZX-calculus and a set of rules interms of their graphical representation with pure qubit as post selected measurements. Phase shift analysis of these complementary algebraic structures by taking pure qubit phase shifted version as their post selected measurements takes us to the Pauli Z and X spin observables. Dagger Symmetric Monoidal Categories mathematical framework with associated graphical calculi provides strucutres for further expansion to the ZX-calculus.

Diagrams with green and red nodes having arbitrary many inputs or outputs and phase label attached to them called *spiders*. Yellow nodes with the restriction of just one input and output represent Hadamard nodes. The green and red nodes shows maps in the computational and Hadamard basis respectively. Nodes can be joined in any way while edges are allowed to cross or curve.
Ignoring normalisation, the ZX-calculus is more versatile than quantum circuit notation as it straightforwardly convert quantum circuits consisting of controlled-NOT, Hadamard, and generalised phase gates into the ZX-calculus but other way around most quantum circuits diagrams do not arise from ZX-calculus in this way.

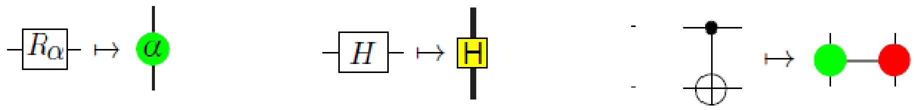



Figure 2.5: Controlled Phase Gate,Hadamard Gate,Controlled NOT Gate

States, post-selected measurements as well as unitary transformations represented by using ZX-calculus, hence measurement-based quantum computations (MBQC) can be straightforwardly transformed into ZX-calculus diagrams in a much more natural way than their translation into circuits. *Graph or cluster states* ( the underlying resource for MBQC) can be transformed into the ZX-calculus with green node as an output, edges are represented by yellow nodes connected along the way to two qubits. In this thesis we present and analyze states even entangled one by adopting this approach which provide an intuitive, rigorous way to understand.

# Chapter 3

# Making Graphical Languages Rigorous and Thorough

Graphical notations often introduced to develop an intuitive understanding of a problem but it become meaningless if we have to revert it back to the original formalism to confirm its correctness. This means executing the same task twice: once by the method of graphical language, than back to the original formalism. An alternative and useful approach would be to make the graphical languages themselves rigorous, so that the results derived graphically found to be true in alternative formalisms without any back transformation.
The graphical languages introduced so far except stabilizer graphs have many things in common: in all of these languages systems are shown by some kind of wire while processes are denoted by some kind of node or box. These languages can be made more thorough using framework of category theory, an approach pioneered by Joyal and Street who examined a series of graphical notations from various diagrams i.e. Feynman diagrams to *Petri Nets* and gave them rigorous underpinnings. Graphical languages can be made rigorous by using the framework of category theory and seems to be the natural formalism for their representation. *Monoidal Categories* be the more general framework used to analyze quantum networks because it incorporates both parallel and sequential composition operations of transformations. Introducing the concepts from Monoidal Categories provide an exact arena of mathematics for establishing corresponding high level graphical languages rigorously and thoroughly. The standard textbook for study is Categories for the working mathematician by Mac Lane [19]. Another textbook which put category theory, graphical languages, and quantum theory side-by-side is Picturing quantum processes by Coecke [20].
Category theory is a mathematical framework equipped with graphical language that provides an intuitive way to perform mathematical reasoning e.g. Interchange law,

hidden structures behind identity map give the idea of diagrammatic equalities upto graph isomorphisms and even calculations made easy using this graphical objects concept instead of considering mathematical entities of abstract nature. Category theory has only recently beings comes up to model the concept of quantum mechanics by using their mathematical framework. Making connection of already established areas of quantum circuit theory and tensor network states to the mathematical framework provide by category theory seems to provide an exact framework of mathematics to diagrammatically reason about complex quantum networks [21].

## 3.1 Category Theory

A category $C$ found to be consists of:

- Objects A, B, C, ... $\in$ Ob($C$),
- For each pair of objects A, B $\in$ Ob($C$), a set of arrows $C$(A, B),
- For each object A $\in$ Ob($C$), there exist an identity arrow $1_A \in C$(A, A), and
- For each triple A, B, C $\in$ Ob($C$), there exists *sequential composition* operation

$$(\text{-}, \text{-}) : \quad C(\mathrm{A, B}) \times C(\mathrm{B, C}) \rightarrow C(\mathrm{A, C});$$

  satisfying the following axioms:

- Sequential composition is associative for arrows, i.e. for any f $\in (A, B)$, g $\in$ (B, C), h $\in (A, C)$

$$f \circ (g \circ h) = (f \circ g) \circ h$$

- The identity arrows are units for sequential composition, i.e. for all f $\in C$(A, B)

$$1_A \circ f = f = f \circ 1_B$$

  **Examples**

- Any physical theory could be formalised as a category where the objects correspond to physical systems and transformations between them as physical process represented by arrows. In this case the sequential composition corresponds to simply applying one transformation after the other and the identity arrow leaves a system invariant.
- The category **Set** defined sets to be as their objects and functions as their arrows. Their composition is sequential application of functions i.e. for f : A → B and g : B → C

  $$g \circ f : A \longrightarrow C :: a \mapsto g\left(f\left(a\right)\right)$$

  The identity arrows are the identity functions, i.e. for A ∈ob($C$):

  $$1_A : A \to A :: a \mapsto a$$

- The category **Rel** again same as that of category Set has sets defined to be as their objects but here in contrast relations are arrows. A relation $A \xrightarrow{R} B$ can be thought of as a subset of the *Cartesian product* A × B. The sequential composition operation in **Rel** is that of relational composition, i.e.

  $$S \circ R = \{(a, c) \mid b \in B \text{ such that } (a, b) \in S \text{ and } (b, c) \in R\} \subseteq A \times C$$

- The category **Hilb** has complex Hilbert space as their objects representation and arrows are found to be the linear maps of boundness. The sequential composition operation is the composition of these bounded linear maps interms of functions. Identity arrows are the usual linear maps of identity. The categories **FRel** and **FHilb** are defined by restricting the objects of **Rel** to finite sets and of **Hilb** to finite-dimensional Hilbert spaces respectively because maps are not bounded when $H$ is infinite-dimensional. Classical deterministic physics is modelled in the category **Set** while **Hilb** and **FHilb** are there for quantum mechanics.

## 3.2 Functor

Relating categories via transformations:
Let $C$ and $D$ be categories. For a map F : C → D to be a functor following should be satisfied:

- F allocates an object FA $\in$ Ob($D$) for each object A $\in$ Ob($C$),
- F allocates an arrow Ff $\in$ $D$(FA, FB) to each arrow f $\in$ $C$(A, B),
- Composition of arrows preserves distributive:

$$\mathrm{F(f \circ g) = Ff \circ Fg}$$

- Identity arrow shows its preservation corresponding to F:

$$F1_A = 1_{FA}$$

**Examples**

- Category of physical systems evolve according to classical deterministic physics, a functor sending each physical system to its corresponding set of states describes its evolution to the category of **Set**, and each transformation between physical systems to the corresponding function between state sets.
- The map from category **Set** to **Rel** sends each of the function f : A $\rightarrow$ B of a set to a relation $R_f \subseteq$ A $\times$ B given as:

$$R_f = \{(a, f(a)) \mid a \in A\}$$

A *basic category* permits only sequential composition of transformations, i.e. applying transformations one after the other. There is no idea of the way to express two systems standing side by side and applying different transformation to the them "at the same time" .To handle this situation a new concept being considered which added *new structure* to category, called *parallel composition.*

## 3.3 Strict Monoidal Category

A ***strict monoidal category*** is a category $C$ including the operation of parallel composition for objects, denoted by A $\otimes$ B, a unit object I and a parallel composition operation for arrows:

$$(-,-) : C(A,B) \times C(C,D) \mapsto C(A \otimes B, C \otimes D)$$

such that for any objects A, B, C $\in$ Ob($C$) and any arrows f, g, h, i composable in the required ways, the following hold:

- The parallel composition for objects is associative , and I is a unit for them:

$$(A \otimes B) \otimes C = A \otimes (B \otimes C)$$

$$A \otimes I = A = I \otimes A :$$

- The parallel composition for arrows is also associative , and $1_I$ is a unit for them:

$$h \otimes (g \otimes f) = (h \otimes g) \otimes f$$

$$f \otimes 1_I = f = 1_I \otimes f :$$

  In the above definition the term "strict" refers to the fact that both associative and unit laws for *parallel composition* of objects and arrows involve same structures rather than being exact equalities .

- Parallel and sequential composition both at a time satisfies the interchange law:

$$(g \circ f) \otimes (i \circ h) = (g \otimes i) \circ (f \otimes h) :$$

  The coherence equations for monoidal categories imply this interchange law.
  **Examples**

- Category **Set** becomes a monoidal category by taking the *cartesian product* of sets as the objects of parallel composition. The unit object is the one-element set. Parallel composition of functions relates to element-wise application: Given functions f : A → B and g : C → D, the parallel composite of f and g is:

$$f \otimes g : (A \otimes C) \to (B \otimes D) :: (a, c) \to (f(a), g(c))$$

  **Rel** can be made in the same way.

- The category **Hilb** can become a monoidal category by taking the usual tensor product as the parallel composition operation. The unit object for the category is the one-dimensional Hilbert space. The same holds for **FHilb**.

- The Strict Monoidal Category (SMC) equivalent to **FHilb**, denoted by $Mat_C$, has objects represented by natural numbers (thought to be as the dimension of the Hilbert space). Complex matrices of size m by n as arrows in $Mat_C$(n,m), with multiplication of matrices as sequential composition and identity matrices as identity arrows. The parallel composition of objects given by product of natual numbers: $n \otimes m = nm$ the multiplication of numbers, with 1 as the unit object. Arrows compose in parallel by Kronecker product of matrices. An object in $Mat_C$ thought to be as the Hilbert space with a chosen basis, which admits linear maps to be uniquely expressed as matrices in terms of those chosen bases.
  Strict monoidal categories are found to be sufficient for explaining circuit diagrams with their *rigid structure*.

## 3.4 Strict Symmetric Monoidal Category

A ***Strict symmetric monoidal category*** is a form of strict monoidal category $C$ which includes the operation of swapping $\sigma_{A,B}$ for any pair of objects i.e. A, B ∈ Ob($C$) for which the following axioms hold.

- Swapping two systems and then swapping them again is equivalent to doing nothing:

$$\sigma_{B,A} \circ \sigma_{A,B} = 1_A \otimes 1_B$$

For swapping operation following of the axioms hold :

- Swapping two objects and after that applying two arrows in parallel is same as interchanging the arrows and then swapping, i.e. if f : A $\rightarrow \acute{A}$ and other g : B $\rightarrow \acute{B}$ than:

$$(f \otimes g)\circ\ \sigma_{A,B} = (g \otimes f) \circ \sigma_{\acute{B},\acute{A}}$$

- Swapping an object with the unit object I is the same as doing nothing:

$$\sigma_{A,I} = 1_A$$

- Swapping of composite object with a single object with is the same as component-wise swapping. Again, the strict symmetric monoidal category is actually found to be as a special case of symmetric monoidal category, in which most of the axioms involve isomorphisms rather than being exact equalities. The verification of the swap arrow structure is being confirmed by the equalities of compact closed categories.

  **Examples**

  - The category **Set** is symmetric with swap arrow given as:

  $$\sigma_{A,B} : (A \otimes B) \rightarrow (B \otimes A) :: (a, b) \rightarrow (b, a).$$

    The corresponding relation is a swap arrow for **Rel**, and similarly for **FRel**.
  - The category **Hilb** is symmetric with the swap arrow $\sigma_{H,\acute{H}}$ for two Hilbert spaces H, $\acute{H}$ being the unique linear map satisfying the following interchange:

  $$|\phi\rangle \otimes |\varphi\rangle = |\varphi\rangle \otimes |\phi\rangle$$

    Circuit diagrams, including quantum circuits, can be modelled in strict symmetric monoidal categories.

## 3.5 Compact Closed Category

A strict symmetric monoidal category $C$ is called a *compact closed category* if for every object A $\in$ Ob($C$) we found another object $\acute{A} \in Ob(C)$ called the *dual* of A with arrows $\eta_A$: $I \rightarrow A \otimes \acute{A}$ and $\epsilon_A : A \otimes \acute{A} \rightarrow I$ for which the following properties holds [23] [25]

:

$$(\epsilon_{\mathrm{A}} \otimes 1_A) \circ (1_A \otimes \eta_{\mathrm{A}}) = 1_A$$

also

$$(1_{\acute{A}} \otimes \epsilon_{\mathrm{A}}) \circ (\eta_{\mathrm{A}} \otimes 1_{\acute{A}}) = 1_{\acute{A}}$$

**Examples**

- The category **Set** is not compact closed because there is only a single function from any object A to the one-element set but converse there are arbitrary many. It is therefore impossible to find arrows satisfying the equalities in Definition.
- The category **Rel** on the other hand can be given a compact structure: take each set to be self-dual, i.e. $\acute{A}$ = A for all A $\in$ Ob(**Rel**) [22]. Denote the one-element set by $\{\bullet\}$.Consider the relations $\eta_A$ and $\epsilon_A$ as subsets of $\{\bullet\} \times \{A \times A\}$ and $\{A \times A\} \times \{\bullet\}$ respectively.

$$\eta_A = \{(\bullet, (a, a)) \,|a \in A\}$$

$$\epsilon_A = \{((a, a), \bullet) \,|a \in A\}$$

- The category **FHilb** ,for Hilbert spaces of finite-dimensions is compact closed.

$$\eta_H = \sum\nolimits^i |i\rangle \otimes |i\rangle$$

$$\epsilon_H = \sum\nolimits^i \langle i| \otimes \langle i|$$

The category **Hilb** cannot be given a compact structure because *maps are not bounded* when $H$ is infinite-dimensional.

## 3.6 Dagger Functor

A *dagger functor* on a category $C$ $(-)^{\dagger} : C \longrightarrow C$ defines to be an identity-on-objects while reversing the direction of arrows satisfies the following conditions on arrows.

- The dagger functor inverts the directions of arrows f : $A \longrightarrow B$ to an adjoint $f^{\dagger} : B \longrightarrow A$.

$$(f \circ g)^{\dagger} = g^{\dagger} \circ f^{\dagger}$$

  also referred to as contra variance of the functor.

- The dagger functor shows the property of involution:

$$\left(f^{\dagger}\right)^{\dagger} = f$$

  A *dagger compact closed category* is found to be compact closed category $C$ with a dagger functor $(-)^{\dagger}$ which satisfies the following properties

- The dagger of the parallel composition of two arrows split across each arrow individually:

$$(f \otimes g)^{=} f^{\dagger} \otimes g^{\dagger}$$

- The dagger of the swap arrow shows its inverse:

$$\sigma^{\dagger}_{A,B} = \sigma_{B},_{A}$$

- The maps associated with the compact structure are related to each other via dagger funtor:

$$\epsilon^{\dagger}_{A} = \eta^{*}_{A}$$

  This type of category was also named as "strongly compact closed category" [25]. **Examples**

  - The category *FHilb* is dagger compact closed with Hermitian adjoint of linear maps as the dagger

– The category *Rel* is dagger compact closed having relational converse as their dagger. For a relation R define as R$\subseteq$ A $\times$ B relational converse is

$$R^{\dagger} = \{(b, a) \,|(a, b) \in R\} \subseteq B \times A$$

Dagger compact closed categories are the setting for categorical quantum mechanics, i.e. doing the analysis of quantum theory and similar theories as categories.

*Dagger compact closed categories are also appropriate for formalizing graphical languages based on string diagrams.*

## 3.7 Diagrammatic intuition of Algebraic Equalities

Languages that represent processes as string diagrams ( processes shown diagrammatically by boxes denote maps on strings, and wire as it is denote systems or equivalently defined to be the identity maps on the systems). There are two ways for making a graphical language rigorous: one is to give an intuitive transformation between string diagrams and their corresponding algebraic terms. Second the important one need to have is that if two diagrams that look like to be intuitively equal revert back to the terms of algebras that are actually equal.

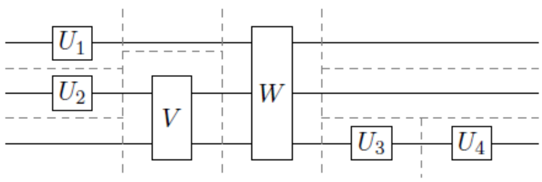



Figure 3.1: String Diagram

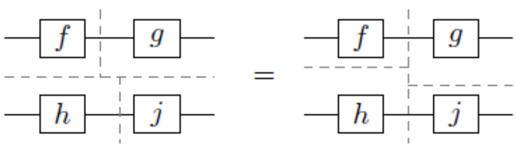



Figure 3.2: String Diagram

By cutting the diagrams up into segments *containing exactly one map* using only horizontal and vertical cuts, string diagrams can be converted into algebraic terms. *Each map is then represented by a symbol*, and symbols are combined using - *in the quantum case - tensor product and matrix multiplication.* The correspondence between cut direction and mode of composition depends on the direction of the diagrams; *for quantum circuits*, horizontal cuts correspond to tensor products and vertical cuts to matrix multiplication.
To make this rigorous, *one needs to show that the terms resulting from different decompositions of the same diagram are all equal.* This is ensured by the requirement that maps f, g, h, j in any monoidal category satisfy the condition of interchange law i.e.

$$(g \circ f) \otimes (j \circ h) = (g \otimes f) \circ (j \otimes h)$$

*Interchange law is tautological in diagrams ( processes )*, as demonstrated in Figure, whereas it is not at all obvious in the algebraic notation.
There are other category-theoretical structures which becomes intuitive when represented diagrammatically. Structures of compact closed category denoted by the symbols of $\eta$ and $\epsilon$, represented diagrammatically by the bending of wires: the cups and caps. By the definition of compact closed categories, the structures of compact closed categories obeys two equations when represented diagrammatically called snake equations. While the *snake equations are not tautological in string diagrams*, even though they do correspond to same equalities is covered by the *idea of equal up to graph isomorphisms* . By means of graph isomorphism, we mean that one can be transformed into the other by **topological transformations** such as crossing or uncrossing wires, moving wires around the nodes or moving nodes along wires while keeping their connections the same. Further extension of this since the compact structures allows bending of wires both way around without any effect its means that they can also be used to swap two objects also represented by two crossing lines. Even though not identical graphically but they do correspond to each other while up to graph isomorphism. Two string diagrams are equal up to graph isomorphism if there exists a bijection between the two. In particular, two diagrams are isomorphic if one can be converted into the other by the application of **topological transformations** while keeping the inputs and outputs fixed such as crossing or uncrossing wires, moving wires around the diagram or moving nodes along wires while keeping their connections the same. There are many equalities in category theory which becomes much more intuitive when expressed diagrammatically as shown above.

## 3.8 Graphical Language and Algebraic Reasoning

A graphical language is useful only if all the intuitive diagrammatic equalities (equalities that hold up to graph isomorphisms) correspond to true equations derived from the axioms of the category. In fact for categories of several different classes, it is possible to define a corresponding graphical language.

- Any equality follow directly from the axioms of that category holds up to the graph isomorphism of the diagrams in the graphical language.
- If two diagrams are equal up to graph isomorphism, after that the corresponding algebraic terms are equal by the axioms of the category

Thus the graphical language is completely equivalent to algebraic reasoning for those categories. This result also holds for the dagger compact closed categories. The graphical language for a dagger category generally has boxes that are asymmetrical, taking the dagger corresponds to flipping the diagram upside-down and mirroring the boxes representing the maps. While quantum theory can be modelled as the way of dagger compact closed category, not all equalities between linear maps follow from the axioms of the categories of dagger compact closed (means can,t be able to show diagrammatically). In fact, the equations implied from the axioms of the categories of dagger compact closed (shown diagrammatically) are only those that involve the interplay of swap maps, cups and caps, and general maps, or the relationship between a map and its dagger.

# Chapter 4

# ZX-Calculus

Introducing various high level graphical languages in chapter 2, we now focus on the ZX-calculus in detailed way presenting their notations, interpretations and graphical equational rules which arises with pure qubit state as post-selected measurements of algebraic structures defined in the following sections. Furthermore combination of ‘green’ with ‘red’ nodes and arbitrary phase apply to them allows us to write down any kind of unitary relevant to one-qubit in terms of its Euler-angle decompositions and also representation of controlled not gate confirms ZX-calculus *universality.* Rewrite rules argue that they are *sound*.

## 4.1 ZX-calculus Notations

The notations used for the ZX-calculus diagrams basically are *string diagrams* of which wires represent system or identity map on system and any kind of labelled nodes with phase and color upon them shows some kind of deformation or process. They are always read from bottom to top. The bottom wire represent input while the wires at the top of some kind of node represents outputs. The sequential composition operation is equivalent to joining wires one after the other vertically while parallel composition is to put them side-by-side.

## 4.2 Basic Elements of ZX-calculus

Basic elements for the ZX-calculus are as follows

- wires,
- green nodes with many number of inputs/outputs and a label $\alpha$ such that $\alpha \in (\pi, -\pi)$

- red nodes with many numbers of inputs/outputs and a label $\alpha$ such that $\alpha \in (\pi, -\pi)$
- yellow square nodes, having only one input and one output.

There are arbitrary many number of inputs/outputs for green and red nodes. The green and red nodes due to their many legs are called *spiders*. Yellow square nodes having only one input and one output whereas black diamonds nodes are just the scalar no input/output.

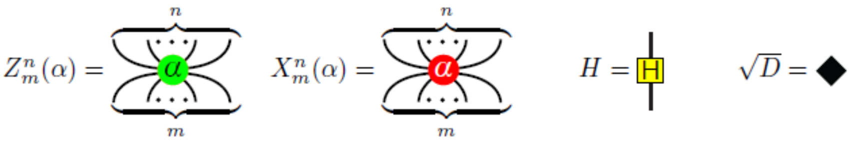


Figure 4.1: Basic Element of ZX-calculus

## 4.3 Universality of the ZX-calculus

ZX-calculus is universal as it straightforwardly convert a universal set of quantum gates into the ZX-calculus. combination of 'green' with 'red' nodes and arbitrary phase apply to them allows us to write down any kind of unitary relevant to one-qubit in terms of its Euler-angle decompositions [26]. The controlled-NOT gate shown diagramatically in ZX-calculus as follows:

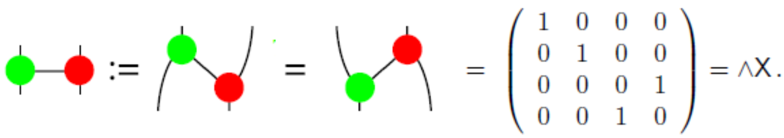


Figure 4.2: Controlled NOT Gate

On the same footing arbitrary one qubit unitary is shown as follows

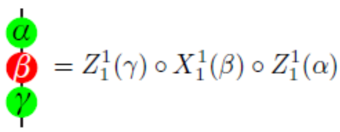


Figure 4.3: Arbitrary One Qubit Unitary

Quantum circuits are built from controlled-not gates and single-qubit gates are universal for unitary operators on qubits . Arbitrary n-qubit states can therefore be transformed as the projection of any n-qubit pure states under taking a well-defined unitary. Quantum computing based upon measurement can also be straightforwardly converted into ZX-calculus diagrams.

## 4.4 The ZX-equational rules

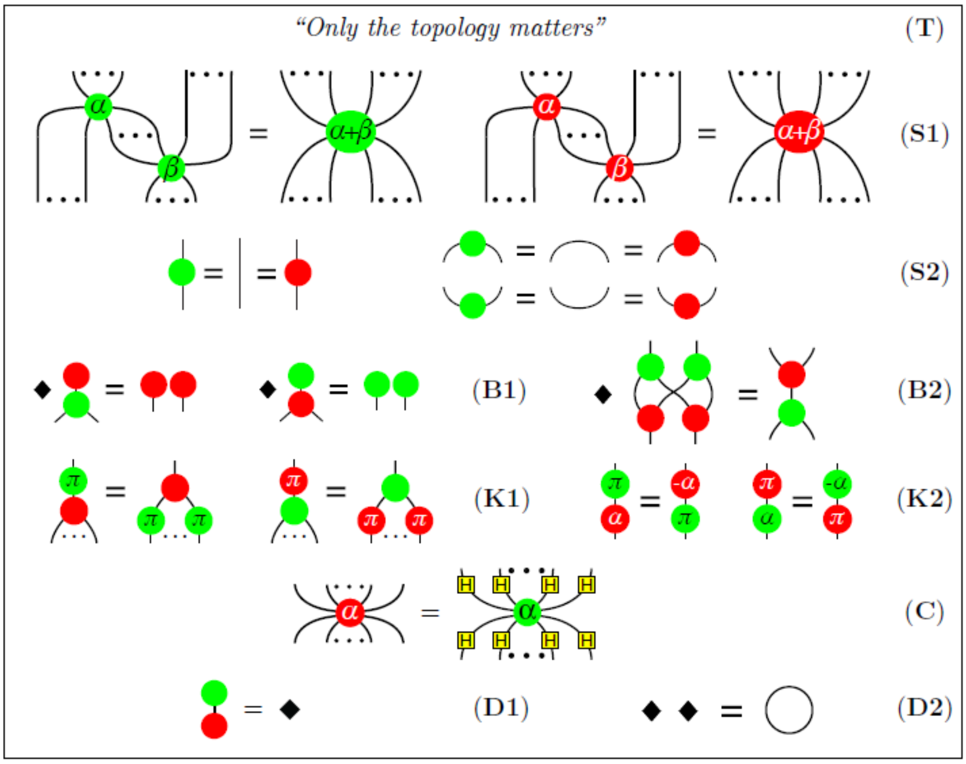



Figure 4.4: ZX Equational Rules

**Topology-rules**

An explicit statement of the fact that the two diagrams are same whenever they are equal up to graph isomorphism is expressed interms of "topology rule". The topology rules says that provided the connections are maintained the wires can be arbitrarily twisted, stretched, bent, tied in knots, etc, *without creating any change to the meaning of a diagram.* More specifically, after the identification (e.g. by enumerating) of the inputs and the outputs , the internal structure can go any kind of topological transformation and after that the structure yields a network that is equivalent to original one.

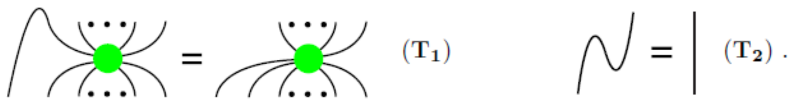



Figure 4.5: T Rules

**The Spider rules**

The “spider” rules states how dots having the same color with arbitrary phases interact with each other. First rule states that same color dots with some arbitrary phases when connected in sequence one after the other they results into a single dot with summing of their phases, conversely, a dot decomposition can be happened in one or more connecting wires. When spiders are found to be of trivial nature dots with degree 2 and phase $\alpha = 0$ can be introduced or other way round removed as mention in second part of spider rules. More over for spiders the following properties hold:

- It is being associative

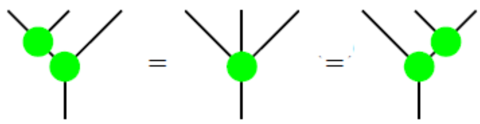


Figure 4.6: Associative Rule

- Commutative

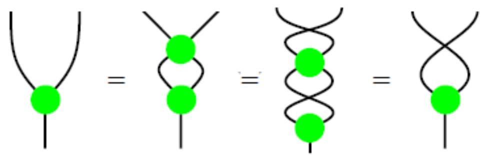


Figure 4.7: Commutation Operation

**The B-rules**

The B1-rule states that green do copies of red points and on the same footing red points can do copies of green in both cases upto a constant of proportionality of a diamond. Altering cycles of red and green dots changes themselves into a simpler graphs as shown in the statement of B2-rule is powerful commutation rule for complementary observables. An important derivation from the B-**rules** give rise to the Hopf Law:

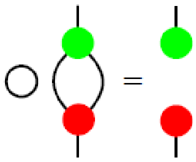
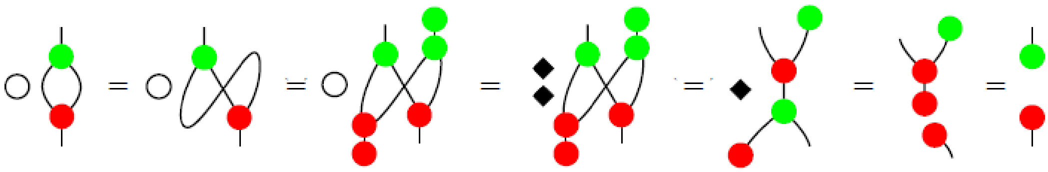


Figure 4.8: Hopf law

**Derivation:**



We can conclude from these powerful B-rules that different colored spiders can interact with each other in a way that a structure of scaled variant bialgebra is found called *scaled bialgebra*, which differs from the interaction of same colored spiders bialgebra only upto a normalising factor.

**The K-rules**

Special properties shown by the dots with phase $\alpha = \pi$ and the spiders. First rule argues that dots with phase $\Pi$ of one color do commutes with the other color spiders, i.e. $X_1^1(\Pi)$ is a homomorphism of the comultiplication $Z_1^2(\Pi)$ and viceversa.

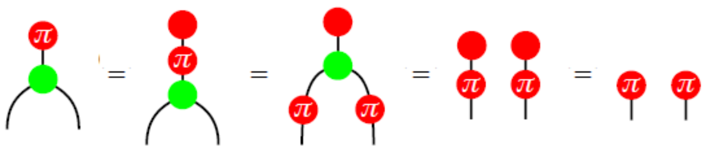



Figure 4.9: K Rule

Other rule shows the property of dot again with phase $\Pi$ of one color that it inverts the phase of the other color.

**The C-rule**

Conversion of one color structures into the other i.e. from green to red and viceversa via Hadamard gate and also by joining two Hadamard gates together results in identity operation which asserts that Hadamard gate is self inverse.

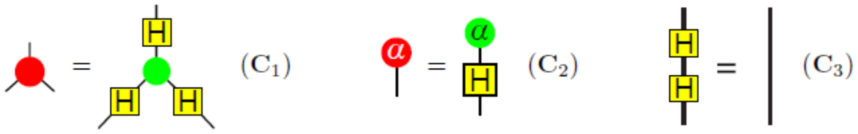



Figure 4.10: C Rules

**The D-rules**

D2 rule argues that a loop of wire results as a composition of a cup and cap is equivalent to two black diamond. Since loop is the representation of the dimension lies under the Hilbert space of an observable so its being justified that the *spacial juxtaposition of scalars is a form of multiplication* as shown below, justifying the label $\sqrt{D}$ used for the diamond notation. The (D1) rule is a follow up form of other rules:

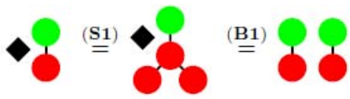
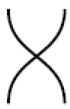



Figure 4.11: D Rules

## 4.5 Interpretation of the ZX-calculus

To interpret ZX-calculus diagrams as a matrix, functor from the category of ZX-calculus to $Mat_C$ is being defined which actually shows the image of that diagrams into $Mat_C$ ,where for reasons of legibility the matrices are represented in bra-ket notation: A wire crossing is a swap operation:



$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Figure 4.12: Swap Operation

and cups and caps relates to (non-normalised) Bell states and effects, respectively:

 $= |00\rangle + |11\rangle$  $= \langle 00| + \langle 11|$

Figure 4.13: Bell State and Effct

Let D and $\acute{D}$ denotes the general transformations:

Figure 4.14: General Operation

Than following diagrams represent parallel and sequential composition operation respectively:

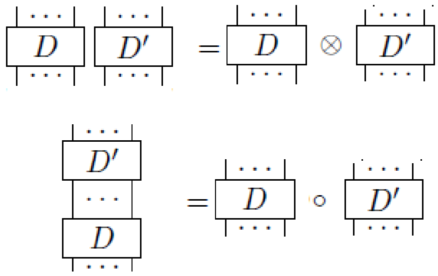
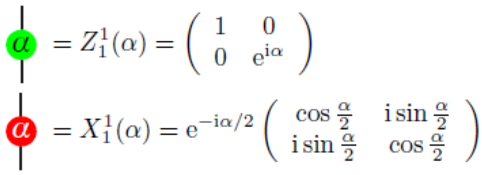


Figure 4.15: Parallel and Sequential Composition

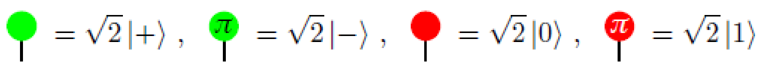
- A red or green spider which have only one input and one output and arbitrary phase to them represent their phase shifted version. The generators $Z_1^1(\pi)$ and $X_1^1(\pi)$ are special case of phase shifts leads to *pauli matrices.*



$$= Z_1^1(\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$

$$= X_1^1(\alpha) = e^{-i\alpha/2} \begin{pmatrix} \cos\frac{\alpha}{2} & i\sin\frac{\alpha}{2} \\ i\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}$$

- The Z and X bases vectors are reresented as follows:

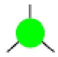
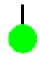


$$= \sqrt{2}\,|+\rangle\,, \quad = \sqrt{2}\,|-\rangle\,, \quad = \sqrt{2}\,|0\rangle\,, \quad = \sqrt{2}\,|1\rangle$$

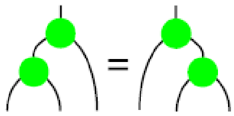
Figure 4.16: Z and X Bases Vectors

- A *state diagram* in a ZX-calculus with no inputs and a non-zero number of outputs which are in one-to-one correspondence with (not necessarily normalised) pure quantum states. Other way around, an *effect* is a diagram with no outputs and a non-zero number of inputs. A *scalar diagram* in ZX-calculus is define to be a diagram which have neither inputs nor any output.

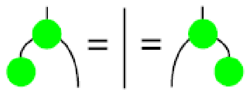
## 4.6 Observables Algebraic Structures

An observable algebraic Structure $\left(A, \delta =\right.$  $, \epsilon =$  $\left.\right)$ is a special commutative dagger-Frobenius algebra with associated phase takes us to their phase deformed version fulfill the following properties.

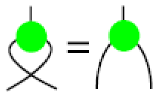
- Associative



- Sequential composition of comultiplication with its counit is equivalent to the identity

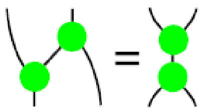


- Invariance under topological transformation i.e; cocommutative

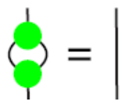


- Frobenius law is also satisfied by the observable structure i.e.

$$\left(\delta^{\dagger} \otimes 1_A\right) \circ \left(1^A \otimes \delta\right) = \delta \circ \delta^{\dagger}$$

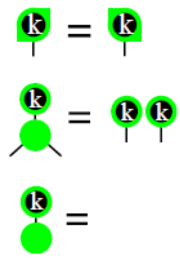


- Special in a sense that sequential composition of comultiplication with its dagger is also equivalent to identity i.e;



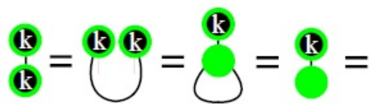
### 4.6.1 Classical Points

Consider an observable algebraic structure $(A, \delta, \epsilon)$ for which k : I⟶A morphism defined to be a point of classical nature if and only if their conjugate comes up with itself and also a homomorphism graphically shown as:




First figure shows invariance under conjugation while second is the representation of homomorphism.

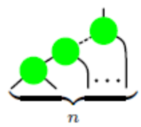
- These points found to be normalised: As seen before that the point of classical nature is self conjugate so its adjoint $k$ do coincide with its transpose $k^*$ i.e;



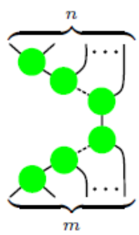

### 4.6.2 Spiders

Recursion for an algebraic structure $(A, \delta, \epsilon)$ defined to be $\delta_n$: $A \longrightarrow A^n$

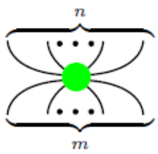
$$\delta_n = (\delta_{n-1} \otimes 1_A) \circ \delta$$




Composition with their adjoint appears to be the following

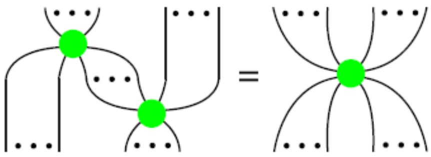


Hence sequential composition depends only upon the object we have and the inputs and outputs which it contain. It appears as if it is the representation of n+m legged spider.

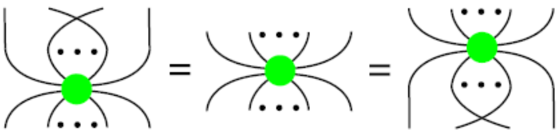



For spiders following rules are obeyed

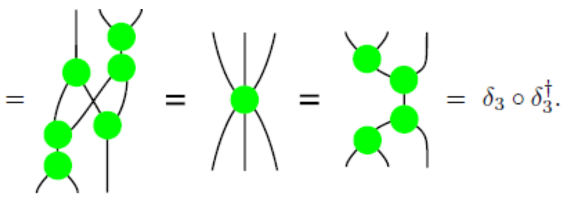
- Composition rule



- Invariance is shown by the spider under leg swapping



- Merger of all this shows the ease for us to solve complicated derivations using these rules



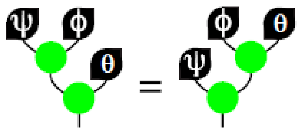

### 4.6.3 Points on Monoid Structure

For an observable algebraic structure define to be $(A, \delta, \epsilon)$ with underlying set of points $C(I, A)$, multiplication on these points $C(I, A)$ is as follows $\psi \odot \phi = \delta^\dagger \circ (\psi \otimes \phi)$ graphically

where $\psi, \phi \in C(I, A)$ be the points in $\dagger$ -smc under the observable structure $(A, \delta, \epsilon)$. Their shape indicate that they are variant under conjugation.

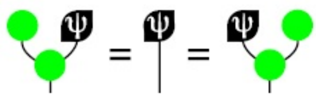
$\left(C(I,A), \odot, \epsilon^{\dagger}, (-)_{*}\right)$ under the observable structure $(A, \delta, \epsilon)$ is an involutive commutative monoid .
Multiplication on the underlying set of points for the observable algebraic structure $(A, \delta, \epsilon)$ satisfies the following properties

- Associative:




- Commutative:

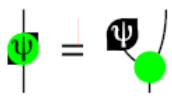

- $\epsilon^{\dagger}$ defines the monoid's unit, i.e.:




Since $\delta^{\dagger}$ is self-conjugate so conjugation is an involution for monoid.

For each of the point $\psi$ define a morphism $\lambda$ such that

$$\lambda(\psi) = \delta^{\dagger} \circ (\psi \otimes 1_A)$$

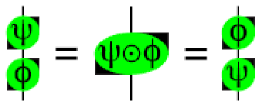



where $\psi \in C(I, A)$ be the point in $\dagger$-smc $C$ under the observable structure $(A, \delta, \epsilon)$.

Map $\lambda$ be an isomorphism of the monoids structure $\left(C\left(I,A\right), \odot, \epsilon^{\dagger}, \left(-\right)_{*}\right)$ characterized by their involutive commutative nature.

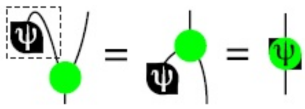
- Commutation is preserved for map $\lambda$.

$$\lambda\left(\psi\right) \circ \lambda\left(\phi\right) = \lambda\left(\psi \odot \phi\right) = \lambda\left(\phi\right) \circ \lambda\left(\psi\right)$$

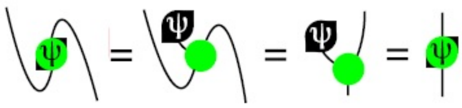


- Conjugation of point implies adjoint of endomorphisms.

$$\lambda\left(\psi_{*}\right) = \lambda\left(\psi\right)^{\dagger}$$

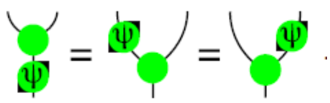


Their justification is given by the property of map transpose. Each map $\lambda$ is equal to its transpose.

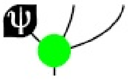


- Hence endomorphisms "$\lambda$" under 180 degree rotation are invariant.

- Another very interesting property about endomorphisms is the following $\lambda\left(\psi\right) \circ \delta^{\dagger} = \delta^{\dagger}\left(1_{A} \otimes \lambda\left(\psi\right)\right) = \delta^{\dagger}\left(\lambda\left(\psi\right) \otimes 1_{A}\right)$

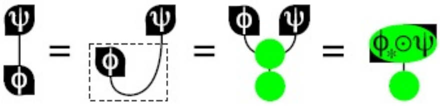


Proof. The proof simply relies upon the spider property that all of these merge to the following




so they are all equal.

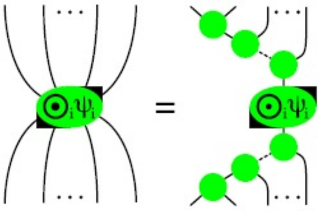
- $\langle \phi \,|\psi\rangle = \phi^{\dagger} \circ \psi = \epsilon \circ (\phi_{*} \odot \psi)$



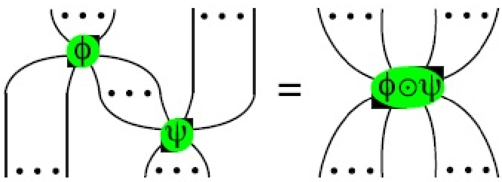

Which again follow from the property of the spider.

### 4.6.4 Generalized spiders

Sequential composition of recursions, their adjoints and points between them for an observable structure $(A, \delta, \epsilon)$ reduced to the following form



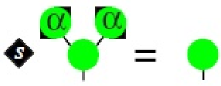

They obey the following compositional rule for an observable structure



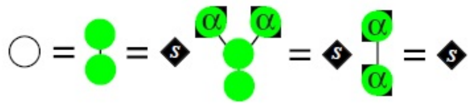

### 4.6.5 Unbiased Points

If $|\psi\rangle = \sum_i a_i \,|i\rangle$ than $\lambda(\psi)$ defines the diagonal matrix with $n \times n$ entries and $a_1, a_2, a_3, ..., a_n$ be their diagonal entries. Hence $\lambda(\psi)$ is unitary if $\bar{a_1} a_1 = \bar{a_2} a_2 = \bar{a_3} a_3 = ... = \bar{a_n} a_n = 1$ i.e. $|\psi\rangle$ is unbiased for $a_1, a_2, a_3, ..., a_n$.

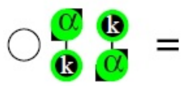
Point $\alpha$ is unbiased if there exsists a scalar s such that



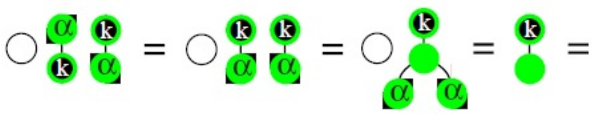
For an unbaised and normalized point $\alpha$ i.e. $\alpha^{\dagger} \circ \alpha = 1_I$ scalar s defines the dimension of observable equivalent to D. Other way round if $|\alpha|^2 = \alpha^{\dagger} \circ \alpha$ =dim A=D than scalar s will be equivalent to $1_I$.



If $\alpha$ be the unbaised and normalised points and $k$ being the classical point; then $D\, |\langle k|\, \alpha\rangle^2 = D\left(k^{\dagger} \circ \alpha\right)\left(\alpha \circ k^{\dagger}\right)$

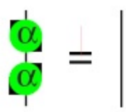


Proof.

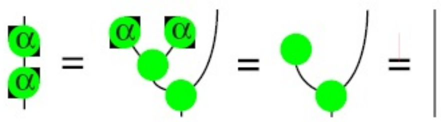


where we use the fact to reach this conclusion is that scalars are self transpose.

A point $\alpha$ is unbiased with $|\alpha|^2 = D$ if and only if $\lambda(\alpha)$ is found to be a unitary map.
Proof.
commutation property for $\lambda$ implies that we need to check only one property i.e. $\lambda(\alpha) \circ \lambda(\alpha)^{\dagger} = 1_A$



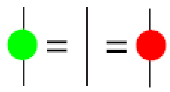
Start it by saying that $\alpha$ is unbaised



Reversibly , we revert back to the original concept for this group structure to unbiased points.

### 4.6.6 Compact Structure

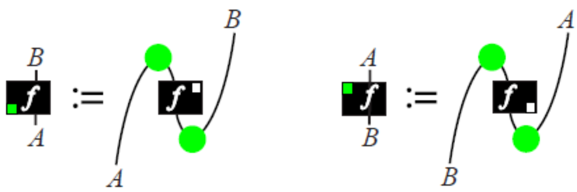
Since †-compact observable structures comes out of Z,X-observables structures that copy each other bases do coincide.
The following equations are found to be correct because of the property of counit defines for the observable algebraic structure and the reality that there exist only one map of identity .



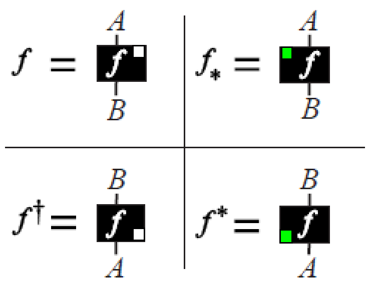

This is not the case for Z,Y or X,Y bases. The following is not derivable from the axioms of observable structures but consequence of the fact relevant to observable structures.


For a morphism $f : A \mapsto B$, transpose$f^* : B \mapsto A$ and conjugate $f_* : A \mapsto B$ defined to be:

$$f^* := \left(1_A \otimes \eta_B^\dagger\right) \circ (1_A \otimes f \otimes 1_B) \circ (\eta_A \otimes 1_B)$$
$$f_* := \left(1_B \otimes \eta_A^\dagger\right) \circ \left(1_B \otimes f^\dagger \otimes 1_A\right) \circ (\eta_B \otimes 1_A)$$




Above diagrams shows representation of transpose and conjugate with respect to a map and their adjoint. We may summerize these things in a way that conjugation is a vertical reflection of a map while transpose is 180 degree rotation of that map.



which is coherent with the reality such as

$$f^* = (f^\dagger)^* = (f^*)^\dagger$$
$$f^\dagger = (f_*)^* = (f^*)_*$$

### 4.6.7 Phase Group

The points which are unbiased for $(A, \delta_z, \epsilon_z)$ are of the form $|0\rangle + e^{i\alpha}|1\rangle$ for which range $0 \subseteq \alpha \subseteq 2\pi$ leads to the group of phase deformed version represented in matrix form as follows [27]:

$$\Lambda^Z(|\alpha_Z\rangle) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} =$$ 

This is the representation of pauli-Z matrix when $\alpha = \pi$. Its adjoint represented as follows

 $= \Big($  $\Big)^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} =$ 

This implies adjoint is equivalent to dropping the corner from diagrammatic notation and writing negative sign in front of angle.

The same also holds for X observable.

$$\Lambda^X(|\alpha_X\rangle) = \begin{pmatrix} \cos\frac{\alpha}{2} & \mathrm{i}\sin\frac{\alpha}{2} \\ \mathrm{i}\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix} =$$ 

This whole summerize to the following for qubits

| Observable | Classical Points | Unbiased Points | Phase Group |
| --- | --- | --- | --- |
| $Z = (\delta_Z, \epsilon_Z)$ | $\|0\rangle , \|1\rangle$ | $\|0\rangle + e^{i\alpha} \|1\rangle$ | $Z_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$ |
| $X = (\delta_X, \epsilon_X)$ | $\|+\rangle , \|-\rangle$ | $\cos\frac{\alpha}{2} \|0\rangle + i \sin\frac{\alpha}{2} \|1\rangle$ | $X_\alpha = \begin{pmatrix} \cos\frac{\alpha}{2} & i\sin\frac{\alpha}{2} \\ i\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}$ |

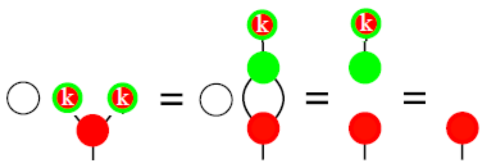
| Observable | Classical Subgroup | Automorphism Action |
| --- | --- | --- |
| $Z = (\delta_Z, \epsilon_Z)$ | $1, \mathsf{X}$ | $\mathsf{X} : Z_\alpha \mapsto Z_{-\alpha}$ |
| $X = (\delta_X, \epsilon_X)$ | $1, \mathsf{Z}$ | $\mathsf{Z} : X_\alpha \mapsto X_{-\alpha}$ |

## 4.7 Hopf and Scaled Bialgebra

Classical points k : I$\longrightarrow$A are the only which are copied by an observable algebraic structure of one color, say green, will be unbiased with respect to the other color i.e. red is shown graphically as follows

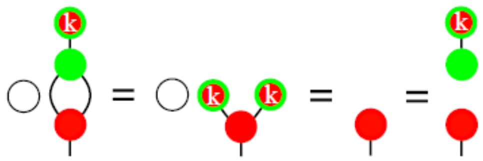
### 4.7.1 Hopf algebra

For observable structures $(A, \delta_Z, \epsilon_Z)$ and $(A, \delta_X, \epsilon_X)$ whose compact structures coincide to be complementary if their sequential composition obeys Hopf law [6] i.e.

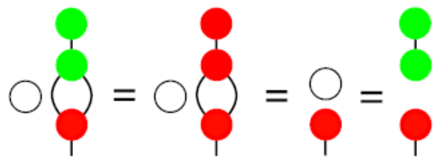


Going further deep into this if for observables $(A, \delta_Z, \epsilon_Z)$ and $(A, \delta_X, \epsilon_X)$ we know that they are complementary and one of them at least is either the representation of

their vector or on the other hand state basis than the Hopf law satisfied.
Proof:
We can straightforwardly proof this by just taking the sequential composition of left and right-hand side of the Hopf law to a vector basis element and saw that the equality is preserved

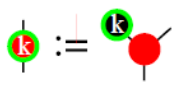



Same is the case for the sate basis



### 4.7.2 Scaled Bialgebra

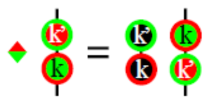
When we consider a special class in which classical points k of complementary observable algebraic structures rather than assume to be normalised have length square root of D (where D represents dimension) the forms we obtain from specific pairs of observable algebraic structures which are of complementary nature form a scaled variant of the *bialgebra*. The important equations for them being the commutation property define for their multiplication and (co)multiplication algebras structures of opposite nature, as well as the (co)multiplication of one algebra with the (co)unit of the other.



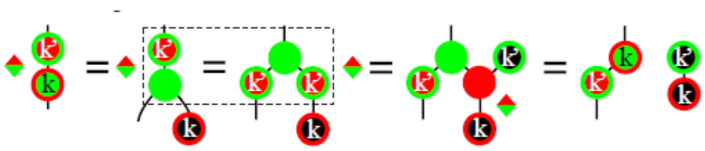

- The kind of observables assumed here are complementary, and in the usual sense they are non-commuting. What we want to show here is that certain alternative notions of commutation do happens as shown below corresponding specific structures of complementary observables. Commutation for these observable stuctures are as follows of which one of them is the important one that is B-2 rule. When k is found to be unbiased for structure $(A, \delta_X, \epsilon_X)$ having length square root of D rather than being normalised. The morphism corresponding to it is unitary iff $k \circ k^{\dagger} = \mathrm{D}$. The laws administrating classical points of comonoid homomorphism are as follows:

also

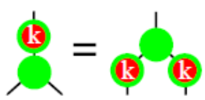
- Observable structures $(A, \delta_Z, \epsilon_Z)$ and $(A, \delta_X, \epsilon_X)$ with points $K_Z$ and $K_X$ of classical nature respectively obey following commutation relation iff k$\in K_Z$ and $\acute{k} \in K_X$



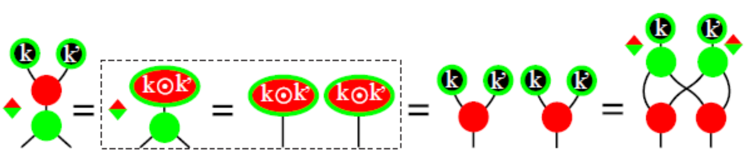
Proof:
Commuting morphisms



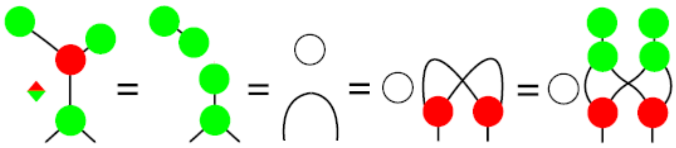
- Obey comultiplicative commutation with classical points belongs to $K_Z$



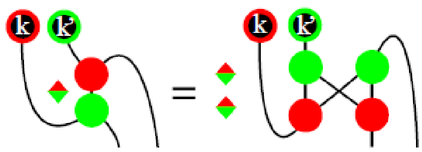
- Scaled Bialgebra defines commutation property for their multiplication and (co)multiplication algebraic structures of opposite nature.
Bialgebra Proof:



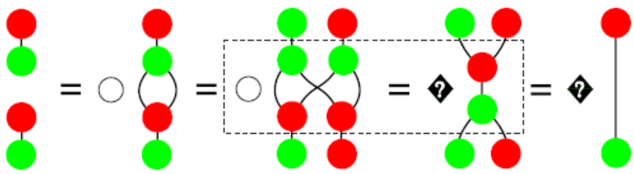
Classical points defines vector basis for an observable structure green or red and monoidal tensor for these basis vectors lifts to a basis vector takes us to the bialgebraic commutation relation.



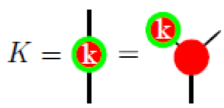
Now we find bialgebraic commutation relation for complementary observable structures from complementary operator strucutres. Again we use the same properties of classical points which constitute a vector basis for structure of green and red observables and the fact relevant to monoidal tensor is their lifting to a vector basis comes up with bialgebraic equation:



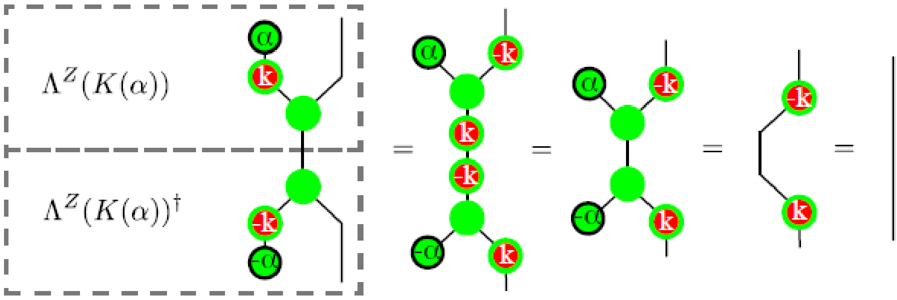
The above result also follows from merger of induced dagger compactness. Here we find that the scalars choice is not arbitrary one,



- Consider k $\in k_Z$ and the map K=$\lambda^X(k)$ then K implies a group automorphism for $U_Z$.

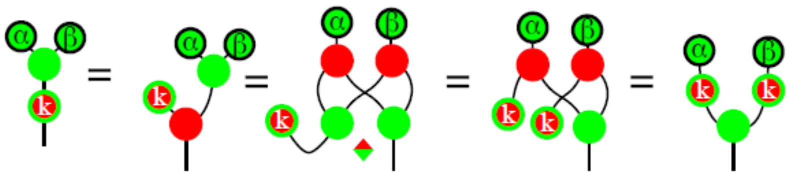



Proof:
Since K is known to be unitary, and so it will be invertible as well. It remains to show when $\alpha$ $\in U_Z$ so does the $K \circ \alpha \in U_Z$. This is justified if $\lambda^Z(k \circ \alpha)$ is unitary.



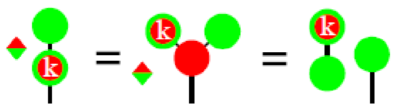

Now we have to prove that K defines a homomorphism for the following group structure.

1. $K \circ (\alpha \odot_Z \beta) = K \circ (\alpha) \odot_Z K \circ (\beta)$



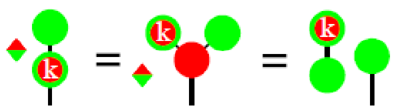
   The bialgebra law proves to be the main source to reach this conclusion.

2. $K \circ \epsilon^\dagger = \epsilon^\dagger$



   The result appears by dividing them by the scalar factor on both side of equations.

3. $(K \circ \alpha)^{-1} = K \circ (\alpha)^{-1}$



   From this property we can conclude that K belongs to unbiased for Z .Hence $(K \circ \alpha)^{-1}$ is also unbiased for Z

Hence K is an automorphism of unbiased Z.

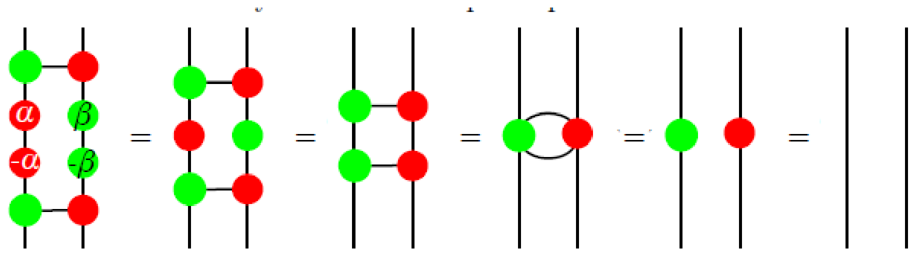
# Chapter 5

# Applications of Hopf and Scaled Bialgebra

## 5.1 Adjoints and their Composition

Arbitrary unitary map with their adjoint diagrammatically shown as:


Sequential composition of map with its adjoint equivalent to identity map:




For first equality we use the property of the spider that the dots of same color merge into each other summing up their phases while the second equality based on identity rule. Lastly we use Hopf law which shows the map is unitary.

## 5.2 Quantum Circuits

**The $\wedge$X Gate**

Diagrammatically shown as:

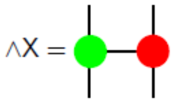



Figure 5.1: Controlled NOT Gate

It is *self-adjoint* as manifested by the diagram [28].Unitary is being confirmed by the sequential composition of it with their self adjoint which is equivalent to identity map:

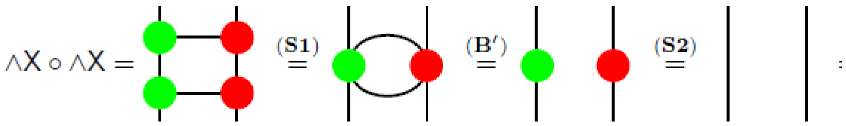
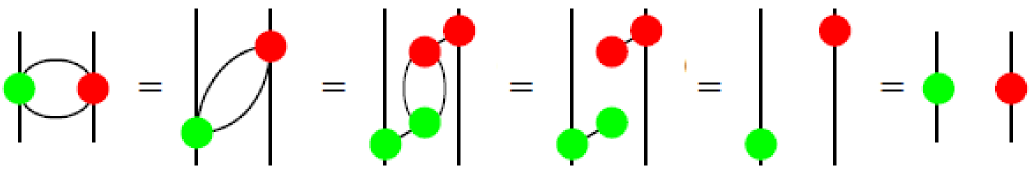


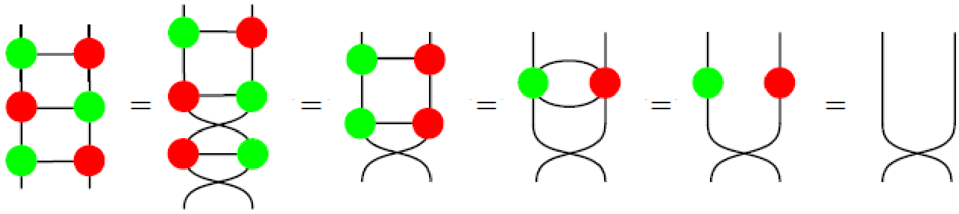

Figure 5.2: Unitary of Controlled NOT Gate

A graphical proof of this fact is as follows.



Nodes of the same color merge into each other while of other show disconnection between them(Hopf law).

One of the potential application for controlled NOT gate is their use for swapping of two qubits using three controlled not gates in altering order :



Second and third diagrammatic equalities follows from the topology and powerful Bialgebra law respectively. Ultimately takes to the swapping operation for two qubits.

**The $\wedge$Z Gate**

Diagrammatic representation of controlled Z gate is shown below:

$$\wedge Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} =$$

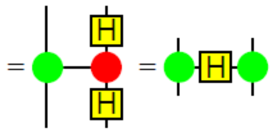



Figure 5.3: Controlled Z Gate

We can immediately follow from its graphical representation that controlled Z-gate is self-adjoint, symmetric in its inputs and also that it is unitary. Unitary is being confirmed by the following diagrammatic derivation:

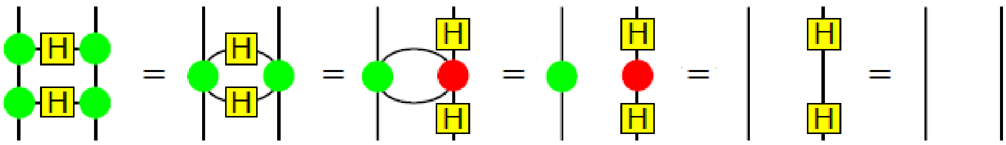



Figure 5.4: Unitary of Controlled Z Gate

**The $\wedge$ Phase Gate**

Phase deformed version of controlled Z gate is show below:

$$\wedge \mathsf{Z}_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\alpha} \end{pmatrix} =$$

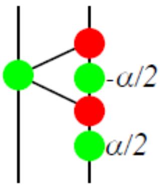



Figure 5.5: Controlled Phase Gate

One of the application of $\wedge$ phase gate is quantum fourier transform shown as:

$$QFT_2 =$$

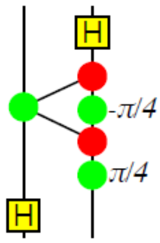
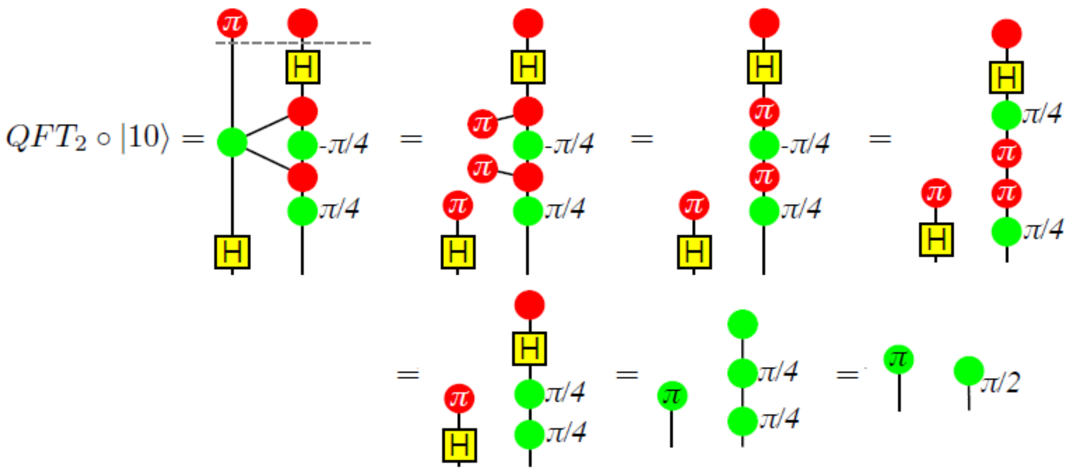



Figure 5.6: Quantum Fourier Transform

For the practical demonstration of quantum fourier transformation we choose an input state and simply concatenate the input to the circuit.

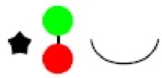



Last diagram is the representation of the separable quantum state indicated by the disconnection of graph.

## 5.3 Measurement Based Quantum Computation

### 5.3.1 Amplitude Derivation through Graphical Representation

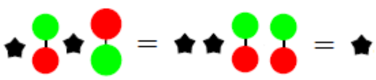
Graphical representation of Bell state interms of a cup and normalising factor:

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) =$$ 

where black star shows the value of 1/2. Computational and Hadamard basis measurements represented as follows:

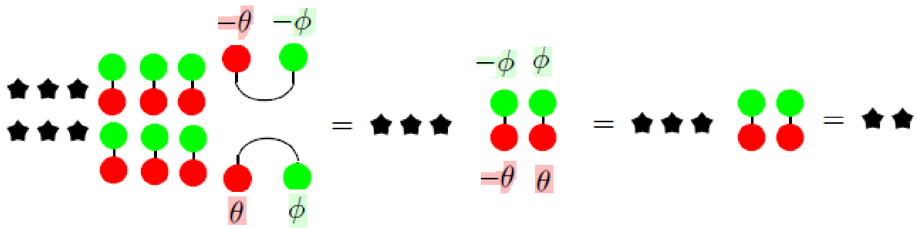

We can use this graphical representation to find amplitude and hence the *probability* of the Bell state in computationl or Hadamard basis [29]. Computing amplitude of Bell state in the state $\langle 00|$ as follows

The probability corresponding to the state obtained by following graphical derivation.



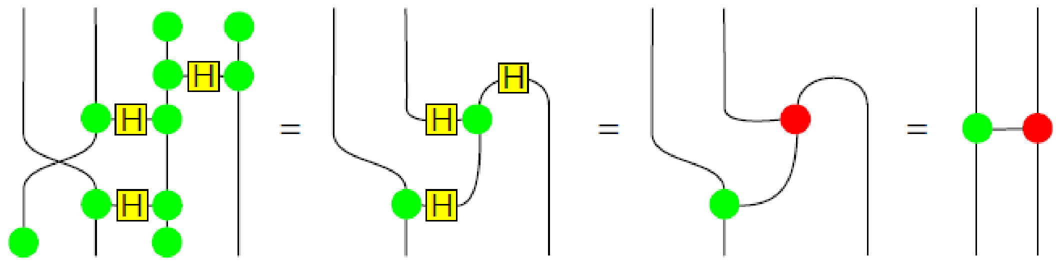
Overlap of Bell state with the measurement $\langle 11|$ shows amplitude derived from graphical representation method as follows:


Similarly for $\theta, \phi \in \{0, \pi\}$, the probability is derived as follows:

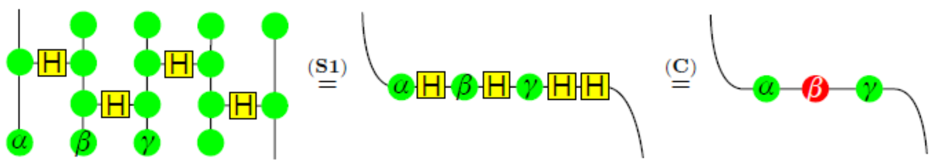



### 5.3.2 Derivation of Unitary Maps

Derivation of controlled NOT from measurement based quantum computing resource as follows

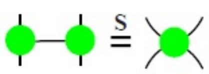



Any single qubit unitary map can be implemented wit the help of the measurements of five qubits as follows:




**The State Transfer Protocol**

We can easily seen that this protocol is self adjoint and also idempotent one, hence we can take it as a projector.




$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathrm{P}_{Z\otimes Z}$$

Figure 5.7: Projector

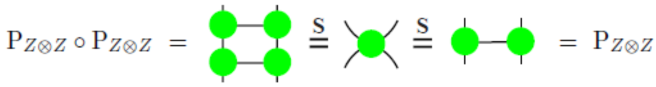
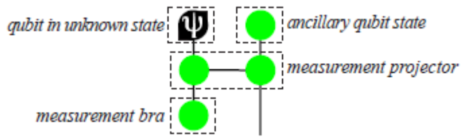
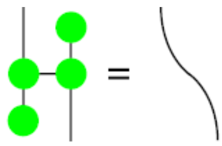


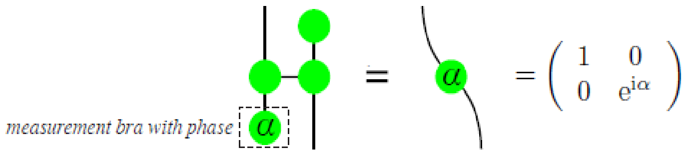

Figure 5.8: Idempotent

Assume a state transfer protocol contain only two qubits of which one of them lie in an unknown state $|\psi\rangle$ while other in a known state $|+\rangle$ respectively. We wish to transfer $|\psi\rangle$ the unknown state from first position to the second of the qubits. It is being implemented in the following way :




We can also measure unknown qubit in phase shifted basis also





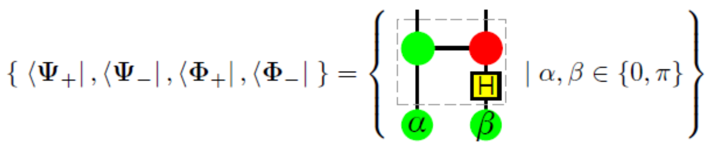

### 5.3.3 Representation of Entangled States

**Bipartite Entangled States**

**Bell states**

We show how this deformed version of quantum mechanics i.e; ZX-calculus leads to representation of the entangled states. First of all, Bell states the most commonly used entangled states shown in a much more intuitive way as follows:

For $\alpha; \beta = 0; \pi$ the following diagram shows 4 possibilities that can comes out of a Bell basis measurement:



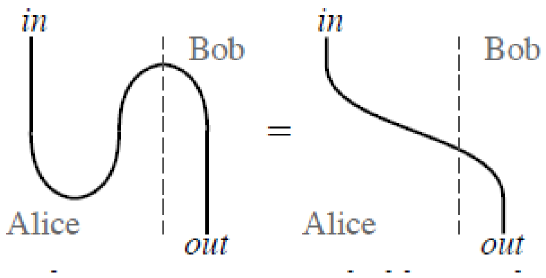
The boxed part indicates the rotation of the Bell basis onto the X-basis. Teleportation between two objects also happened to be via entangled state and measurements explained by the framework of ZX-calculus shown below.

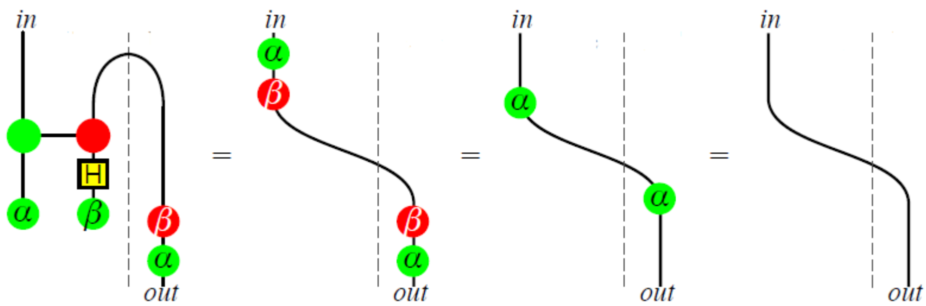
**The Teleportation Protocol**

This protocol specifically made of two components i.e; Bell state, and the Bell basis measurement. Bell state and Bell measurement usually represented by a cap and cup respectively as shown below:

$|00\rangle + |11\rangle =$  , $\langle 00| + \langle 11| =$ 

Combining them together form the following



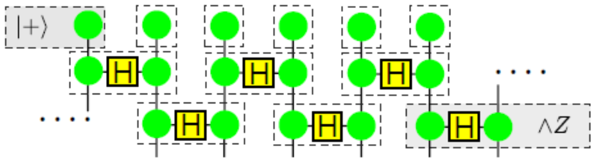

Varying input over the 4 possible pairs correspond to 4 possibilities for the Bell basis measurement. This description display the ranges of the Pauli errors that will arise if the observation of Alice found to be something else. If we include the correction of Bob which are classically correlated to Alice's observations derive a complete description for this protocol. Here the fact that $2\alpha = 2\beta = 0$.

**Multipartite Entangled States**

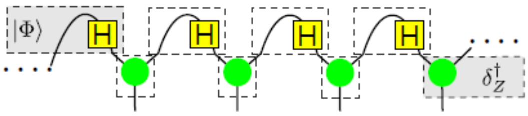
Multipartite entangled states are also shown in a much simpler and easy way using ZX-calculus.

**Cluster States**

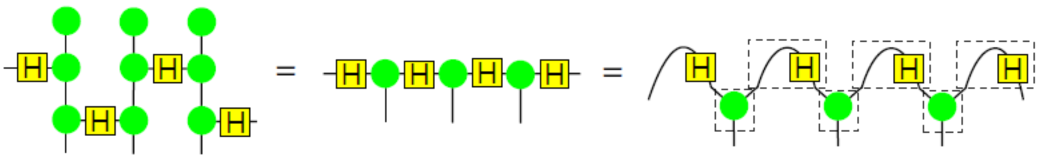
The original scheme looks to be a bit tedious and involved:




Alternative scheme seems to be very easy and approached by the application of Hadamard gate to one segment of a Bell pair to obtain $|\phi\rangle$ than fusing them as shown in figure



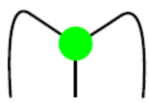

On equal footing both these approaches leads to




**Two distinct classes of genuine three qubit Entangled States**

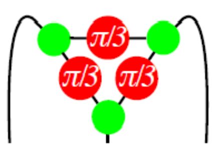
One of them is GHZ state, found to be very simply form of spider with three legs.

$$|\mathrm{GHZ}\rangle = |000\rangle + |111\rangle =$$



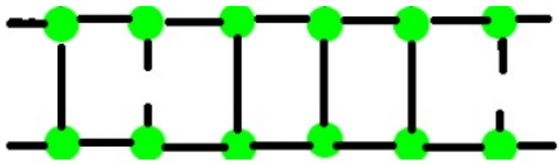
Its importance stem from the fact that simple form takes to the concept of algebric structure whose phase deformed version takes us to the explaination of non-locality [27]. On the other hand W state being found just to be as a pairwise entanglement between each pair of qubits ultimately comes up with the three-partite system.

$$|\mathrm{W}\rangle = |001\rangle + |010\rangle + |100\rangle =$$

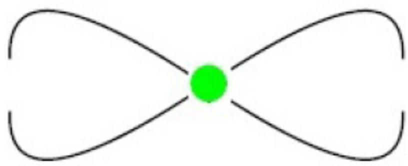


## 5.4 Analysis of the Entangled States

GHZ state is globally entangled state as revealed by the approach of two-site reduced density matrix. Coefficients of this state only be non-zero when we inject any of these components $|00...0\rangle$ and $|11...1\rangle$. We can calculate reduced density matrix for any two-sites by contracting all of the indices except for two of them i.e.

$\rho_{AB} =$ 

It clearly exposes the nature of the entangled state.

$\rho_{AB} =$ 

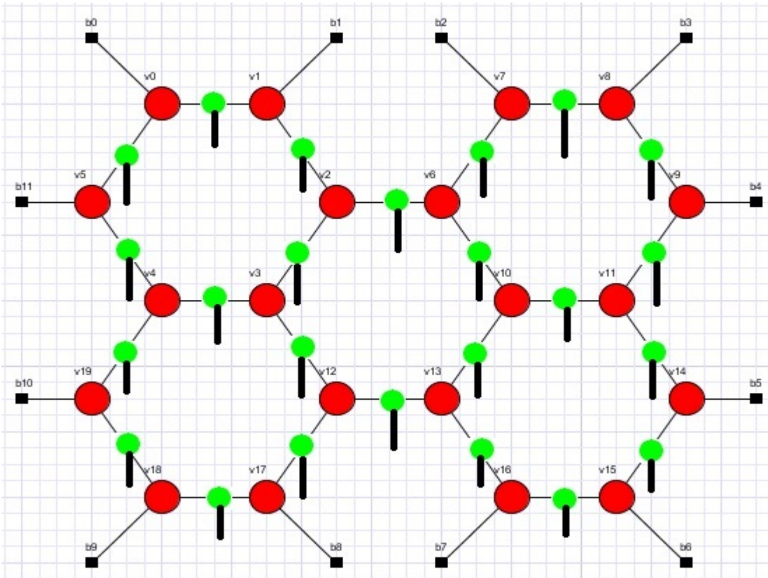
Note that, rewrites rules are only defined up to proportionality,we have normalised to see full picture.

$$= \frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right)$$

Diagrammatically it always comes out in this form which ever sites we have chosen for them, indicates that the correlation length of this state is found to be infinite.

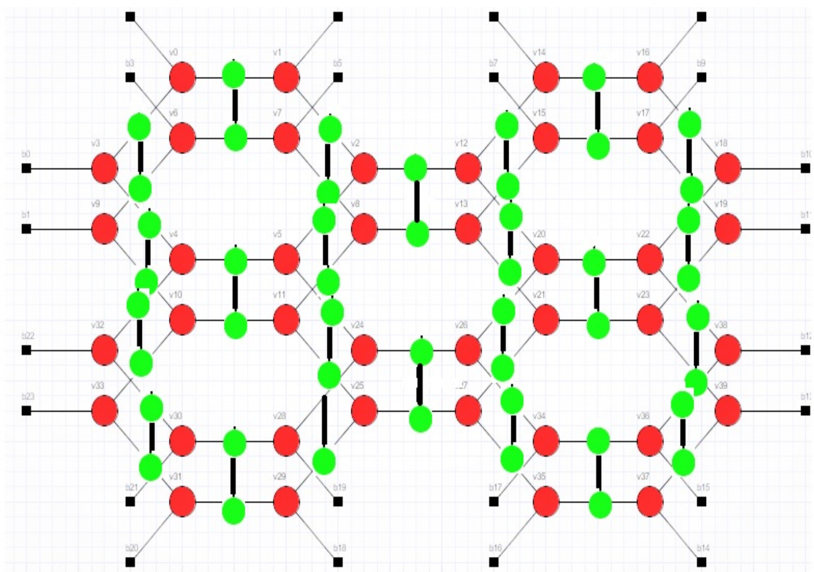
**$Z_2$ state**

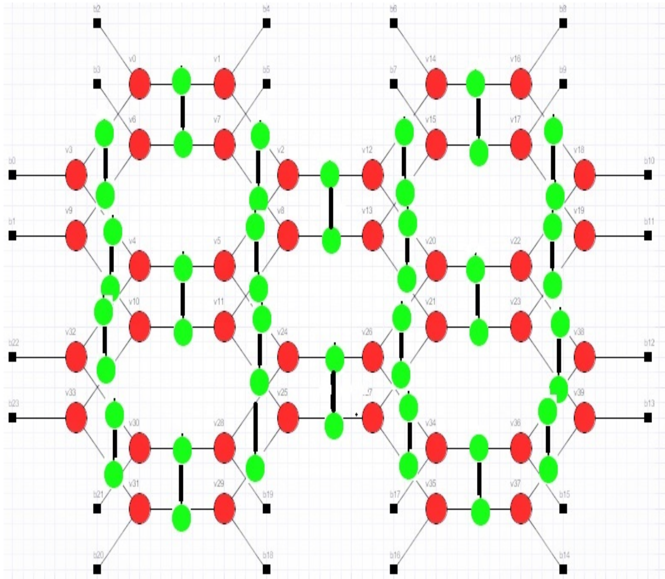
*Tensor network* for the $Z_2$ state diagrammatically shown as follows. "$Z_2$"state can be regard as the ground state of a Hamiltonian having $Z_2$ topological order. Here tensor network for the $Z_2$ state represent a class which is exactly contractible via the interaction structures of complementary observable bialgebra , Hopf and commonly used fusion rules for the same tensors.




Contraction of the tensor networks via the fusion rule for the observable structures flattens to a single layer

$\langle\Psi|\Psi\rangle =$





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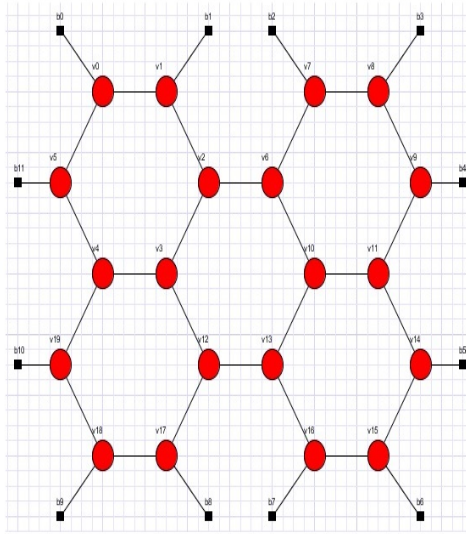
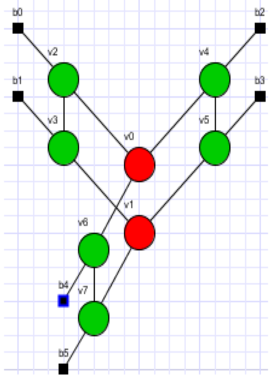


This is explained as follows



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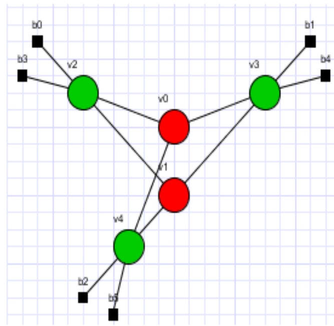

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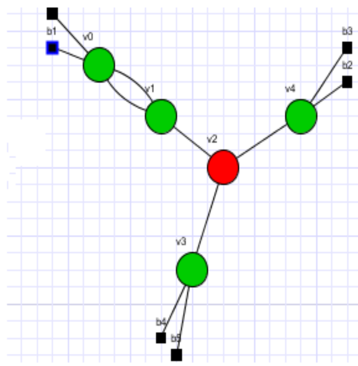

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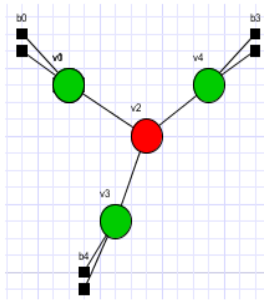
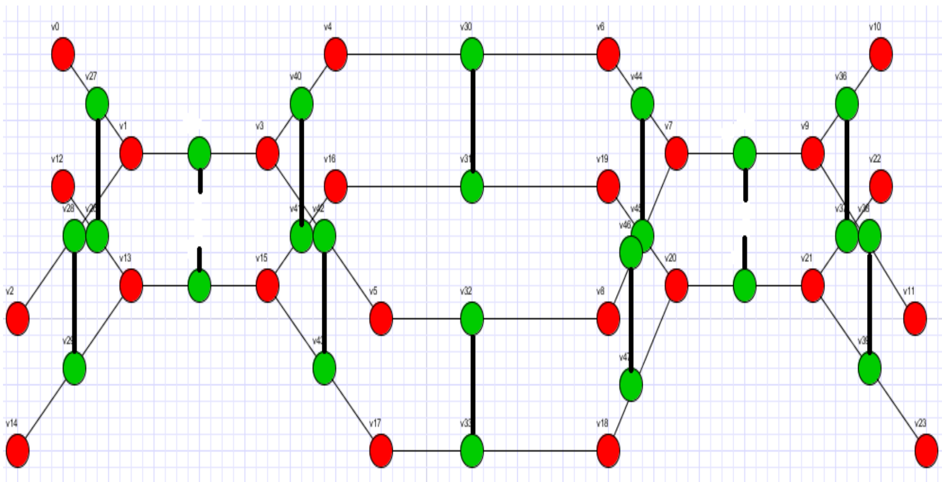
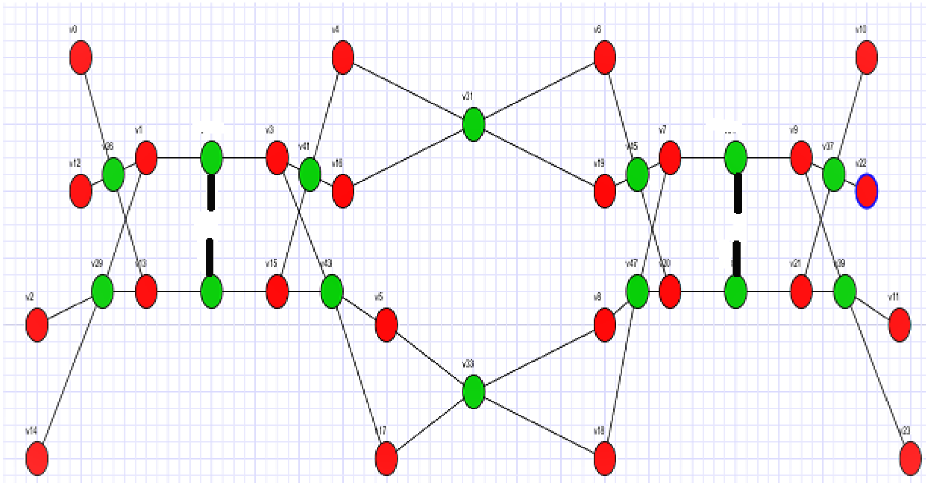
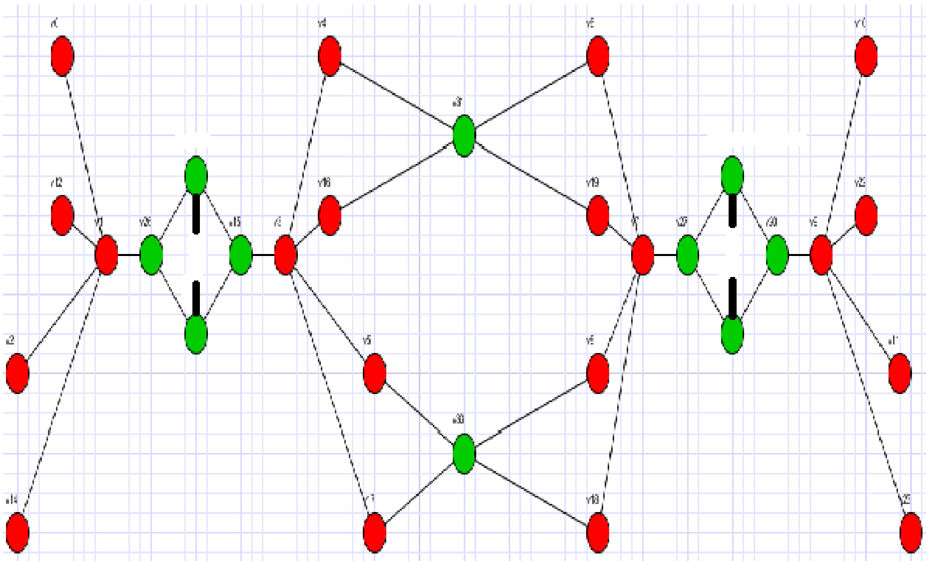



Once again as that of GHZ state the approach we adopt to expose the nature of the "$Z_2$" state is that of reduced density matrix for any two-sites. When we use the same approach to calculate their two-site reduced density matrix, we found different result for that which confirms its product state nature. We use same procedure to calculate correlation functions for any of the two-sites and found that the correlation length is zero as that which will always be for the product states.

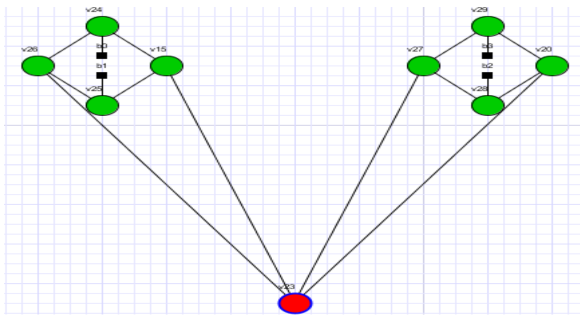


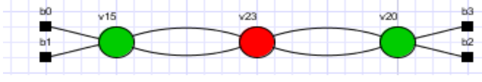

Fusion rule simplifies this to the following diagram. Here we use the powerful Bialgebra law to simplifies things further

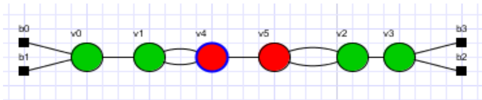


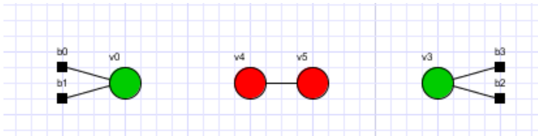


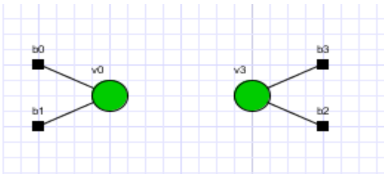
Further simplification made by Hopf law and fusion rule








At the end we came to know that it is tensor network representation of the product states. The correlation length of this state is zero which is also true of product states. All this paid our attention to the fact that we can apply ZX-calculus to the entangled states and more general system as well.

## 5.5 Conclusion

This deformed version of Quantum Mechanics i.e. ZX-calculus have the power to represent not only the product states but also the entangled states in much intuitive and formal way. Powerful graphical equational rules associated with it prove to be an exact choice for exposing the nature of the states in intuitive way. One of them is shown in last section where we exposes the nature of the entangled state in comparison to the product states via their reduced matrices by using these equational rules. We can compute other physical properties of states as well by using these graphical equational rules in an intuitive way. Lastly i want to say that it provide an intuitive way to analyze states and their properties.

# Appendix A

# Stabilizer Graphs

## A.1 Stabilizer States

*For each n-qubit stabilizer state $|\phi\rangle$, subgroup of Abelian $S \subseteq P_n$ exists such that $|\phi\rangle$ being stabilized by all elements of S.*

Proof. First, assume the state $|0\rangle^{\otimes\, n}$ and the set:

$$S_{|0\rangle^{\otimes n}} = \{g_1 \otimes g_2 \otimes g_3 ... \otimes g_n |\, g_k \in \{I, Z\} \text{ for } k = 1...n\}$$

It,s easy to verify that $S_{|0\rangle^{\otimes n}}$ is an Abelian subgroup with each element $\omega \in S_{|0\rangle^{\otimes n}}$ for which $\omega|0\rangle^{\otimes n}$=$|0\rangle^{\otimes n}$ . What we need to show is the uniqueness of the stabilized state $|0\rangle^{\otimes n}$ with respect to the all elements of $S_{|0\rangle^{\otimes n}}$. $Z_k$ shows the Pauli product of a Z-operator and identities i.e;

$$Z_k = \underbrace{I \otimes I \otimes ...I \otimes I}_{Z_{k-1}} \otimes \underbrace{Z}_{k} \underbrace{\otimes I ... \otimes I}_{Z_{k+1}}$$

$Z_k \in S_{|0\rangle^{\otimes n}}$ for k=1,2,3...,n such that

$$|\phi\rangle = \textstyle\sum_{x_1...x_n \in \{0,1\}} \phi_{x_1...x_n} \, |x_1...x_n\rangle$$
$$Z_k \, |\phi\rangle = \textstyle\sum_{x_1...x_n \in \{0,1\}} (-1)^{x_k} \, \phi_{x_1...x_n} \, |x_1...x_n\rangle$$

Thus by comparison $Z_k$ stabilizes $|\phi\rangle$ iff $\phi_{x_1...x_n} = 0$ whenever $x_k = 1$. Therefore, as $Z_k \in S_{|0\rangle^{\otimes n}}$for k = 1... n, the only state stabilized by all elements of $S_{|0\rangle^{\otimes n}}$ is the all-zero state $|0\rangle^{\otimes n}$ .

### A.1.1 Graph States

Graph G = (V;E) with $|V|$= n implies n-qubit graph state $|G\rangle$ prepared as follows:

- vertex v∈V , qubit is prepared in the state $|+\rangle = H\,|0\rangle$,

- for each edge e ∈ E, a controlled-Z operator is used to the suitable qubits.

As Controlled-Z gates does commute, therefore the order in which they applied is not a matter of concern.

### A.1.2 General Graph States

Next, consider some stabilizer state $|\psi\rangle = U\,|0\rangle^{\otimes n}$ where U ∈ $C_n$. It is simple to check that this too is an Abelian subgroup of$P_n$.

$$\mathrm{S}_{|\psi\rangle} = \left\{ U\sigma U^{\dagger} \middle| \, S_{|0\rangle^{\otimes n}} \right.$$

Now for any $U\sigma U^{\dagger} \in S_{|\psi\rangle}$

$$\left(U\sigma U^{\dagger}\right)|\psi\rangle = U\sigma U^{\dagger}U\,|0\rangle^{\otimes n} = U\sigma\,|0\rangle^{\otimes n} = U\,|0\rangle^{\otimes n} = |0\rangle^{\otimes n}$$

so all elements of $S_{|\psi\rangle}$ stabilize $|\psi\rangle$. The group of Pauli products stabilizing a given state is often called its *stabilizer group.*
*Stabilizer scalars* are those complex numbers that can arise as outcomes of a stabilizer computation on all n qubits. stabilizer scalars take values of the form $2^{-r/2}e^{i\psi}$, where r is some non-negative integer and $\psi$ is an integer multiple of $\pi/4$.

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