Mth 610 Topics in Optimization: HW #3 due on May 29, in class
Extended Kalman filter for joint state and parameter estimation - application to the SIR model

In this homework you are required to implement the extended Kalman filter (EKF) for joint estimation of the state and a time-varying β parameter in the SIR model

$$S'(t) = -\frac{\beta(t)S(t)I(t)}{N} \tag{1}$$

$$I'(t) = \frac{\beta(t)S(t)I(t)}{N} - \gamma I \tag{2}$$

$$R'(t) = \gamma I(t) \tag{3}$$

A discrete version of the differential equations system is obtained by numerical integration with Euler's method and a time step $h = 1$ (here h=1 is identified with a 1 day time interval), to advance the state from time t_k to time $t_k + h$

$$S_{k+1} = S_k - h\frac{\beta_k S_k I_k}{N} \tag{4}$$

$$I_{k+1} = I_k + h\frac{\beta_k S_k I_k}{N} - h\gamma I_k \tag{5}$$

$$R_{k+1} = R_k + h\gamma I_k \tag{6}$$

over a time period of NT= 200 time steps (200 days). The total size of the population is $N = 1000$. Henceforth, we assume that the time evolution of true state of the system is given by equations (4-6), corrupted by small random noise. The true initial state (S_0^t, I_0^t, R_0^t) at time $t_0 = 0$ is unknown and the time-varying infection rate parameter β_k^t is also unknown. The recovery (or death) rate of the disease parameter $\gamma > 0$ is assumed to be known, and is specified as $\gamma = 0.04$. The true state vector at time $t_k = k * h$ is denoted

$$\mathbf{x}_k^t = \begin{bmatrix} S_k \\ I_k \\ R_k \end{bmatrix} \tag{7}$$

The EKF for joint estimation of the state and model parameter is formulated as follows. An augmented state is defined by appending β to the model state,

$$\hat{\mathbf{x}}_k = \begin{bmatrix} S_k \\ I_k \\ R_k \\ \beta_k \end{bmatrix} \tag{8}$$

and an additional equation is appended to model the parameter time evolution, for simplicity taken as

$$\beta_{k+1} = \beta_k \tag{9}$$

The time evolution of the augmented state vector $\hat{\mathbf{x}}_k \in \mathbb{R}^4$ modeled by equations (4), (5), (6), and (9) is represented as

$$\hat{\mathbf{x}}_{k+1} = \mathcal{M}_k(\hat{\mathbf{x}}_k), \quad k = 0, 1, \ldots, NT - 1 \tag{10}$$

An estimate to the true (unknown) initial state $\hat{\mathbf{x}}_0^t \in \mathbb{R}^4$ is specified as

$$\hat{\mathbf{x}}_0^a = \begin{bmatrix} S_0^a \\ I_0^a \\ R_0^a \\ \beta_0^a \end{bmatrix} = \begin{bmatrix} 997 \\ 3 \\ 0 \\ 0.1 \end{bmatrix} \tag{11}$$

Data is collected daily over the 200-day period (NT = 200)

$$\mathbf{y}_k = \mathbf{H}_k \hat{\mathbf{x}}_k^t + \epsilon_k^o, \quad k = 1, 2, \ldots, NT \tag{12}$$

Here we assume that reliable data is available only on the size of the recovered population R_k and therefore, the observation operator $\mathbf{H}_k$ is a row vector

$$\mathbf{H}_k = [0 \quad 0 \quad 1 \quad 0] \tag{13}$$

The random observation error is assumed to have a Gaussian distribution $\epsilon_k^o \sim N(0, \sigma_o^2)$ with standard deviation $\sigma_o = 0.001$. The time series of data (observations) and the initial model forecast based on (10)-(11) are shown in Fig. 1.

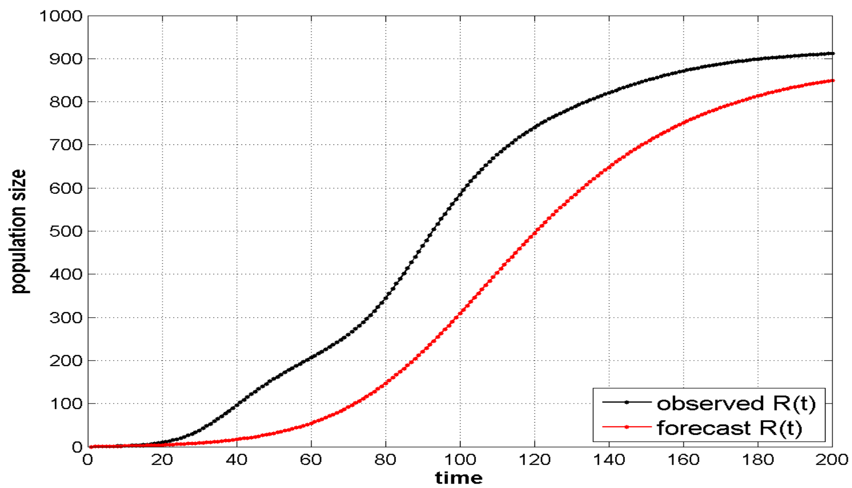

Figure 1: *The time series of data and the initial model forecast of the recovered population $R(t)$*

The EKF algorithm for joint state and parameter estimation is formulated as follows:

Initialization at time t_0:
$\hat{\mathbf{x}}_0^a$ extended state vector as specified in (11)
$\hat{\mathbf{P}}_0^a = diag([10 \;\; 10 \;\; 0.01 \;\; 0.04]) \in \mathbb{R}^{4\times4}$ % analysis error covariance at t_0 is set to a diagonal matrix

Time loop forecast/anaysis cycles:
for $k = 1 : NT$

% *forecast stage*
$\hat{\mathbf{x}}_k^f = \mathcal{M}_{k-1}(\hat{\mathbf{x}}_{k-1}^a)$ forecast state at t_k

$\hat{\mathbf{P}}_k^f = \mathbf{M}_{k-1}^a \hat{\mathbf{P}}_{k-1}^a \mathbf{M}_{k-1}^{a\,\mathrm{T}} + \mathbf{Q}_{k-1}$ % forecast error covariance where $\mathbf{M}_{k-1}^a = \dfrac{\partial \mathcal{M}_{k-1}(\hat{\mathbf{x}}_{k-1}^a)}{\partial \hat{\mathbf{x}}_{k-1}^a} \in \mathbb{R}^{4\times4}$

% use model error covariance $\mathbf{Q}_{k-1} = diag([0.01 \; 0.01 \; 0.01 \; 0.04]) \in \mathbb{R}^{4\times4}$ (diagonal matrix)

% *analysis stage*
$\hat{\mathbf{x}}_k^a = \hat{\mathbf{x}}_k^f + \mathbf{K}(\mathbf{y}_k - \mathbf{H}_k \hat{\mathbf{x}}_k^f)$ % where $\mathbf{K} = \hat{\mathbf{P}}_k^f \mathbf{H}_k^{\mathrm{T}} \left(\mathbf{H}_k \hat{\mathbf{P}}_k^f \mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k \right)^{-1}$ is the Kalman gain matrix[1]

$\hat{\mathbf{P}}_k^a = (\mathbf{I} - \mathbf{K}\mathbf{H}_k)\hat{\mathbf{P}}_k^f$ % analysis error covariance at t_k
end % end time loop

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[1]since there is only one observation available at each time step $\mathbf{K}$ is a column vector, $\mathbf{K} \in \mathbb{R}^{4\times1}$ and $\mathbf{R}_k = \sigma_o^2 = 10^{-6}$

The file obsdata.m contains the 200 data values of the recovered population R_k for $k = 1, 2, \ldots 200$.

Your job:

(60 points) Implement the EKF algorithm, as described above. Provide a listing of your code.

(20 points) Provide a figure with the graph of the estimated time-varying parameter β_k and a figure with the graphs of the analysis estimates for each state component: S, I, R. For your reference, the graph of the true state component $I(t)$ is shown in Fig. 2. Also shown in Fig. 2 is the initial model forecast of the infected population based on (10)-(11).

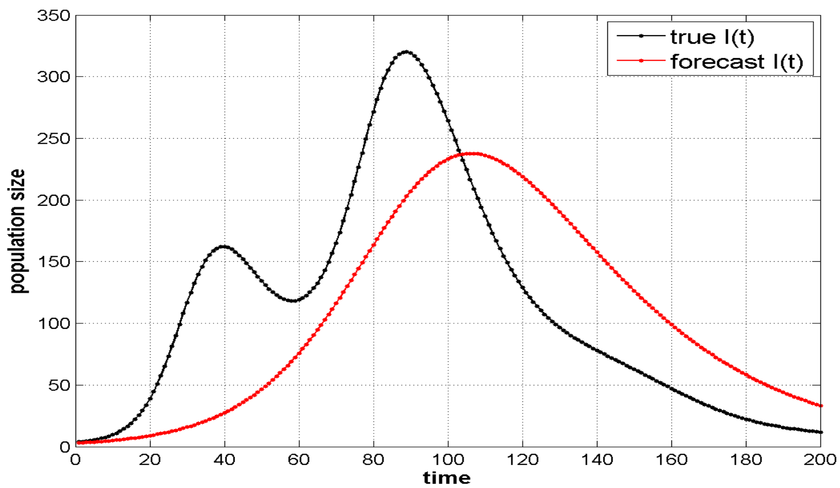

Figure 2: *The time evolution of the true size of the infected population $I(t)$ and the initial model forecast.*