Mth 610: Topics in Model-Constrained Optimization

HW #1 - due on April 20, in class

Problem 1. (25 points) Consider the initial value problem

$$\begin{aligned} x'(t) &= ax(t) + u(t), \quad 0 < t \leq 1 &(1)\\ x(0) &= \bar{x}_0 &(2)\end{aligned}$$

where $a \in \mathbb{R}$ is a given scalar, $u(t)$ is a control function and the initial condition x_0 is specified. Given the cost functional (performance index)

$$J(x,u) = \frac{1}{2}x^2(1) + \frac{1}{2}\int_0^1 \left[x^2(t) + u^2(t)\right]\, dt \tag{3}$$

- Find the expression of control function $u(t)$ that minimizes J subject to the model constraints (1) and with the specified initial condition (2).
- Provide the graphs of the optimal control $u(t)$, state $x(t)$ and the influence function $\lambda(t)$ (Lagrange multiplier) for each of the following:

  (i) $\bar{x}_0 = 2, a = 1$

  (ii) $\bar{x}_0 = 2, a = -1$

Problem 2. (25 points) Consider the multistage system obtained by applying Euler's method to the initial-value problem (1-2)

$$\begin{aligned} x_{i+1} &= x_i + h\left[ax_i + u_i\right], \quad i = 0 : N-1 &(4)\\ x_0 &= \bar{x}_0 &(5)\end{aligned}$$

with the specified initial value $\bar{x}_0$. The time step h is taken as $h = 1/N$, such that x_N is the discrete version of the solution to (1)-(2) at time $t = 1$.

The cost functional is also defined as a discrete version to (3),

$$\tilde{J}(x,u) = \frac{1}{2}x_N^2 + \frac{h}{2}\sum_{i=0}^{N-1}\left[x_i^2 + u_i^2\right] \tag{6}$$

- Find the expression of control vector $\mathbf{u} \in \mathbb{R}^N$ that minimizes $\tilde{J}$ subject to the model constraints (4) and with the specified initial condition $\bar{x}_0$.
- Plot the discrete optimal control u, state x and the Lagrange multipliers λ for each of the following:

  (i) $\bar{x}_0 = 2, a = 1$

  (ii) $\bar{x}_0 = 2, a = -1$

  Compare with the results obtained in the continuous formulation (Problem 1).