Part 2 Jim Vargas MTH 410 HW4

Let it be assumed that I will be using the method of Lagrange multipliers for all of the following problems.

1) a)

Given $f(x) = x_1^2+2x_1x_2+3x_2^2+4x_1+5x_2+6x_3$ for $x \in \mathbb{R}^3$, the gradient and Hessian of f are, respectively,

$$\nabla f(x) = \begin{pmatrix} 2x_1 + 2x_2 + 4 \\ 2x_1 + 6x_2 + 5 \\ 6 \end{pmatrix}$$

$$\nabla^2 f(x) = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 6 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It follows that $\det \nabla^2 f(x) = 0$ and $\operatorname{tr} \nabla^2 f(x) = 8$, and so f is convex.

1) b)

The Lagrangian function for the problem is

$$\mathscr{L}(x, \lambda) = f(x) - \lambda_1(x_1 + 2x_2 - 3) - \lambda_2(4x_1 + 5x_3 - 6)$$

with $\lambda = (\lambda_1, \lambda_2)$. Differentiating component-wise with respect to x and setting each to zero to find a unique minimizer x, in conjunction with the complimentary slackness conditions, we get the following equations:

$$\begin{aligned} 0 &= 2x_1 + 2x_2 - \lambda_1 - 4\lambda_2 + 4 \\ 0 &= 2x_1 + 6x_2 - 2\lambda_1 + 5 \\ 0 &= -5\lambda_2 + 6 \\ 0 &= \lambda_1(x_1 + 2x_2 - 3) \\ 0 &= \lambda_2(4x_1 + 5x_3 - 6). \end{aligned}$$

The third equation yields $\lambda_2 = \frac{6}{5}$. After substituting this and doing some manipulations, we get $x_1 = \frac{\lambda_1}{4} + \frac{37}{20}$ and $x_2 = \frac{\lambda_1}{4} - \frac{29}{20}$ and $x_3 = \frac{-\lambda_1}{5} - \frac{7}{24}$. Eventually more computation yields $\lambda_1 = \frac{27}{5}$ or $\lambda_1 = 0$, but $\lambda_1 = 0$ does not satisfy the constraints. Finally, the optimal solution is $x = \left(\frac{16}{5}, \frac{-1}{10}, \frac{-34}{24}\right)$ with $f(x) = \frac{1377}{100}$.

2)

The Lagrangian function for this problem can be written as

$$\mathscr{L}(x, \lambda) = \frac{1}{2}\left\langle Q^\top x, x\right\rangle - c^\top x + d - \langle \lambda A^\top, x\rangle + \langle \lambda, b\rangle.$$

Taking the gradient of $\mathscr{L}$ with respect to x and setting it to zero, we get

$$\begin{aligned} 0 &= Q^\top x - c^\top - \lambda A^\top \\ \Rightarrow x &= Q^{-1}(c^\top + \lambda A^\top). \end{aligned}$$

With the domain restriction, this means (at least, I think)

$$\begin{aligned} b &= A(Q^{-1}(c^\top + \lambda A^\top)) \\ \Rightarrow \lambda &= (AQ^{-1}A^\top)^{-1}(b - AQ^{-1}c^\top) \\ \Rightarrow x &= Q^{-1}(c^\top + [(AQ^{-1}A^\top)^{-1}(b - AQ^{-1}c^\top)]A^\top). \end{aligned}$$

3)

The Lagrangian function for this problem is

$$\mathscr{L}(x, y, \lambda) = x + 2y - \lambda(x^2 + y^2 - 1).$$

Differentiating with respect to x and y and setting to zero, along with the complimentary slackness condition, yields

$$\begin{aligned} 0 &= 1 - 2\lambda x \\ 0 &= 2 - 2\lambda y \\ 0 &= \lambda(x^2 + y^2 - 1). \end{aligned}$$

With these, we get $x = \dfrac{1}{2\lambda}$ and $y = \dfrac{1}{\lambda}$ with $\lambda^2 = \dfrac{5}{4}$. Taking the positive root of λ to maximize $f(x, y) = x+2y$, the solution is $(x, y) = \left(\dfrac{1}{\sqrt{5}}, \dfrac{2}{\sqrt{5}}\right)$ with $f(x, y) = \sqrt{5}$.

4)

The Lagrangian function for this problem is

$$\mathscr{L}(x, \lambda) = x_1^2 + x_2^2 - \lambda_1(-x_1) - \lambda_2(-x_2) - \lambda_3(5 - x_1 - x_2)$$

for $x \in \mathbb{R}^2$, $\lambda = (\lambda_1, \lambda_2, \lambda_3)$. Taking the derivative component-wise and setting them to zero, after some computations we get $x_1 = \dfrac{-\lambda_1 - \lambda_3}{2}$ and $x_2 = \dfrac{-\lambda_2 - \lambda_3}{2}$ with the following conditions on λ_1, λ_2, λ_3:

$$\begin{aligned} 0 &= \lambda_1 \left(\frac{-\lambda_1 - \lambda_3}{2}\right) \\ 0 &= \lambda_2 \left(\frac{-\lambda_2 - \lambda_3}{2}\right) \\ 0 &= \lambda_3 \left(5 - \frac{-\lambda_1 - \lambda_3}{2} - \frac{-\lambda_2 - \lambda_3}{2}\right). \end{aligned}$$

There are a few possible combinations of the lambdas which satisfy the above system alone, however, simple methods will show that only $\lambda_3 = -5$, $\lambda_1 = \lambda_2 = 0$ will satisfy the given boundary conditions. Therefore the solution is $x = \left(\dfrac{5}{2}, \dfrac{5}{2}\right)$ with $f(x) = \dfrac{25}{2}$.