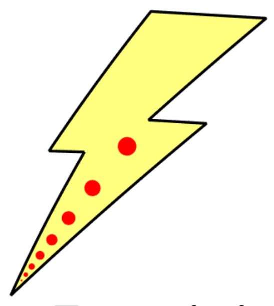
Jim's yucky talk
wowee

This talk is based on...



Solving Laplace Problems with Corner Singularities via Rational Functions

- ...A paper written by Gopal and Trefethen, published in SIAM Journal on Numerical Analysis September 2019
- The Lightning Laplace code, based on the paper, yields accurate approximations quickly (on nice problems)
- https://epubs.siam.org/doi/pdf/10.1137/19M125947X
- https://people.maths.ox.ac.uk/trefethen/lightning.html

Here's the problem

We wish to find a (real) function u over a domain Ω (the complex 2-D plane) which satisfies

$$\Delta u(z) = 0, \quad z \in \Omega \qquad\qquad u(z) = h(z), \quad z \in \Gamma.$$

In particular, we want to be able to handle a domain with sharp corners, curves etc.

We will find r, and approximation of u ($u \approx \mathrm{Re}[r]$).

$$r(z) = \sum_{j=1}^{N_1} \frac{a_j}{z - z_j} + \sum_{j=0}^{N_2} b_j (z - z_*)^j$$

Here's the problem

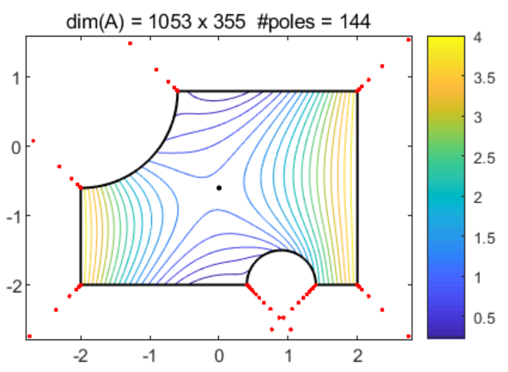
We wish to find a (real) function u over a domain Ω (the complex 2-D plane) which satisfies

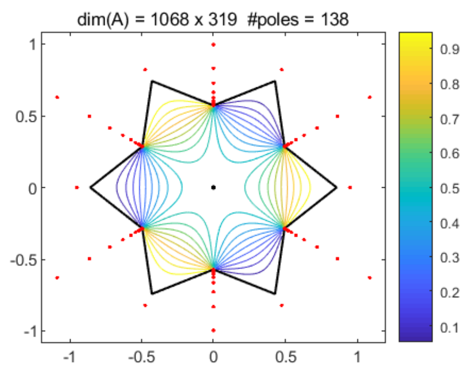
$$\Delta u(z) = 0, \quad z \in \Omega \qquad\qquad u(z) = h(z), \quad z \in \Gamma.$$

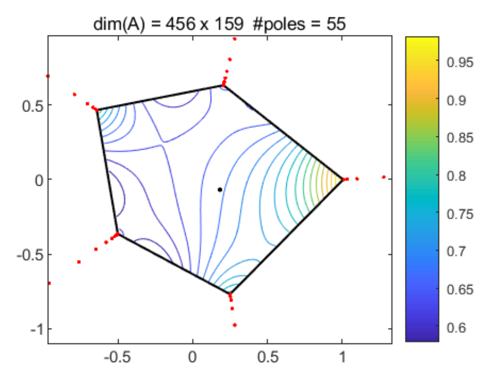
In particular, we want to be able to handle a domain with sharp corners, curves etc.

We will find r, and approximation of u ($u \approx \mathrm{Re}[r]$).

$$r(z) = \sum_{j=1}^{N_1} \frac{a_j}{z - z_j} + \sum_{j=0}^{N_2} b_j (z - z_*)^j$$







Why this problem?

- Problems involving the Laplace operator $\Delta = \nabla^2$ frequently appear in physical equations:
  - Heat Equation $\alpha \nabla^2 u = \partial_t u$
  - Schrodinger Equation $\left[\frac{-\hbar^2}{2m}\nabla^2 + V\right]\Psi = i\hbar\,\partial_t \Psi$
  - Wave Equation $c^2 \nabla^2 u = \partial_t^2 u$
  - And more...
- Functions which satisfy Laplace's Equation have very nice properties, and are called harmonic.

Some nice properties of functions of interest

- The real and imaginary parts of a holomorphic (and thus also an analytic) function $f = u + iv$ are harmonic;
- f is also smooth (infinitely differentiable); by extension this applies to u and v as well.
- Maximum Principle: a harmonic function on a compact domain attains a max. (and min.) on the boundary.

Some nice properties of functions of interest

- The real and imaginary parts of a holomorphic (and thus also an analytic) function $f = u + iv$ are harmonic;
- f is also smooth (infinitely differentiable); by extension this applies to u and v as well.
- Maximum Principle: a harmonic function on a compact domain attains a max. (and min.) on the boundary.

On a simply connected domain we can construct a holomorphic function from a harmonic one: given u, define $g = u - iu$. The theory will work with harmonic functions, which will trickle down to our problem.

If r approximates f, having real part u, the worst we'll do over the whole domain in approximating u is $||u(z) - \mathrm{Re}[r(z)]||$, $z \in \Gamma$.

Back to the problem

$$r(z) = \underbrace{\sum_{j=1}^{N_1} \frac{a_j}{z - z_j}}_{\text{"Newman"}} + \underbrace{\sum_{j=0}^{N_2} b_j (z - z_*)^j}_{\text{"Runge"}}$$

- Using the scheme in the paper, we can have root exponentially good approximations for u. The task at hand is finding the coefficients a_j, b_j.

$$||f - r_n||_\Omega = O(e^{-C\sqrt{n}})$$

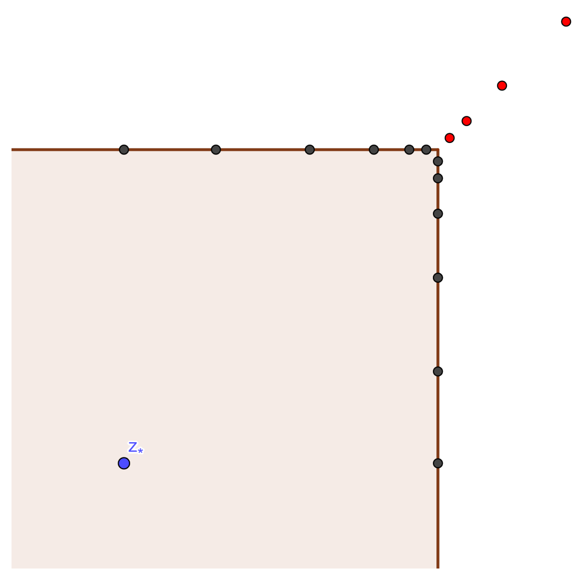
- The theorems in the paper are based on interpolation, showing existence.
- In the code, the problem is solved via a least squares approach using QR factorization. Code is written in MATLAB.

Describing r

$$r(z) = \sum_{j=1}^{N_1} \frac{a_j}{z - z_j} + \sum_{j=0}^{N_2} b_j (z - z_*)^j$$

The Newman Part: built to handle corners.

- The terms z_j are poles, exponentially clustered near a corner on the exterior of Ω (works for spacing scaled at least $O(n^{-1/2})$).
- "Rational functions are more powerful than polynomials for approximating functions near singularities..." [a]



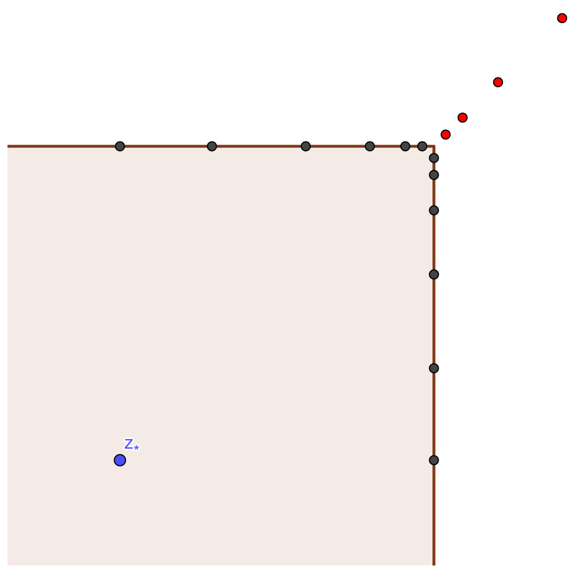
[a] Lloyd N. Trefethen. 2013. Approximation theory and approximation practice, Society for Industrial and Applied Mathematics.

Describing r

$$r(z) = \sum_{j=1}^{N_1} \frac{a_j}{z - z_j} + \sum_{j=0}^{N_2} b_j (z - z_*)^j$$

The Runge part: built to handle the interior.

- The term z_* is an expansion point, near the middle of Ω.
- Polynomials can approximate root exponentially well on a nice domain (going back to Runge).



The function r is harmonic

$$r(z) = \sum_{j=1}^{N_1} \frac{a_j}{z - z_j} + \sum_{j=0}^{N_2} b_j (z - z_*)^j$$

To prove r is harmonic, consider $f(z) = 1/z$ and $g(z) = z^k$. The function f can be decomposed as $f = u + iv$, where

$$u(x, y) = \frac{x}{x^2 + y^2} \qquad\qquad v(x, y) = \frac{-y}{x^2 + y^2}.$$

Taking derivatives will show that u and v satisfy the Cauchy-Riemann equations, $\partial_x u = \partial_y v$, $\partial_y u = -\partial_x v$.

The function r is harmonic

$$r(z) = \sum_{j=1}^{N_1} \frac{a_j}{z - z_j} + \sum_{j=0}^{N_2} b_j (z - z_*)^j$$

To prove r is harmonic, consider $f(z) = 1/z$ and $g(z) = z^k$. The function f can be decomposed as $f = u + iv$, where

$$u(x, y) = \frac{x}{x^2 + y^2} \qquad v(x, y) = \frac{-y}{x^2 + y^2}.$$

Taking derivatives will show that u and v satisfy the Cauchy-Riemann equations, $\partial_x u = \partial_y v$, $\partial_y u = -\partial_x v$, meaning f is harmonic.

Writing g in polar form, then in terms of sines and cosines is enough to see g is harmonic:

$$g(z) = \rho e^{ik\theta} = \rho \cos{(k\theta)} + i \sin k\theta.$$

Adding these templates, applying translations and scaling as necessary give us our result.

An important lemma

Hermite integral formula for rational interpolation.

Let Ω be a simply connected domain in $\mathbb{C}$ bounded by a closed curve Γ, and let f be analytic in that domain and extend continuously to the boundary. Let interpolation points $\alpha_0, \ldots, \alpha_{n-1} \in \Omega$ and poles $\beta_0, \ldots, \beta_{n-1}$ anywhere in the complex plane be given. Let r be the unique type $(n-1, n)$ rational function with simple poles at $\{\beta_j\}$ that interpolate f at $\{\alpha_j\}$. Then for any $z \in \Omega$,

$$f(z) - r(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\phi(z)}{\phi(t)} \frac{f(t)}{t-z} dt,$$

$$\phi(z) = \prod_{j=0}^{n-1} (z - \alpha_j) \Bigg/ \prod_{j=0}^{n-1} (z - \beta_j).$$

First Theorem

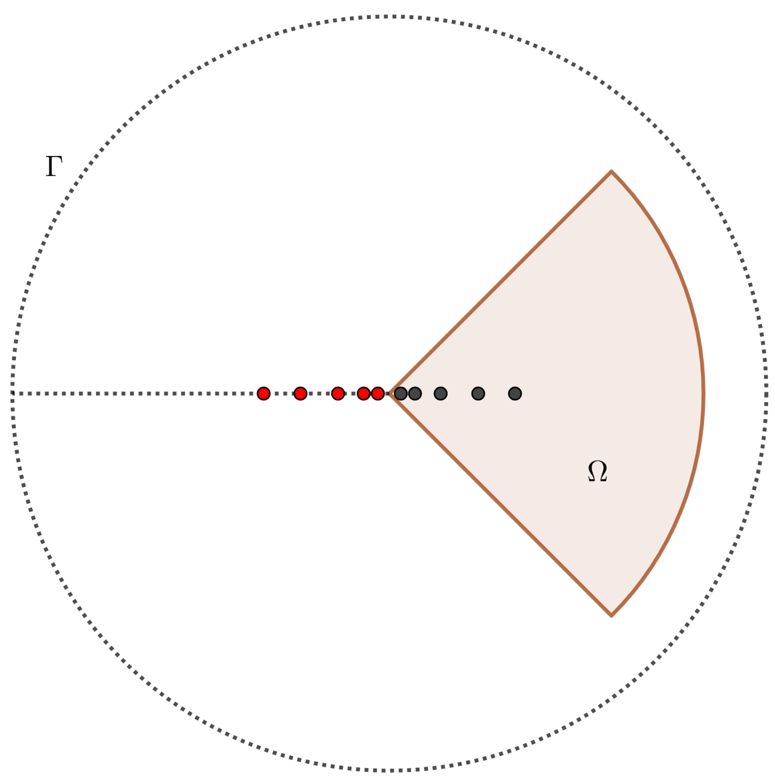
Let f be a bounded analytic function in the slit disk A_π that satisfies $f(z) = O(|z|^\delta)$ as $z \to 0$ for some $\delta > 0$, and let $\theta \in (0, \pi/2)$ be fixed. Then for some $0 < \rho < 1$ depending on θ but not on f, there exist type $(n-1, n)$ rational functions $\{r_n\}$, $1 \le n < \infty$, such that

$$||f - r_n||_\Omega = O(e^{-C\sqrt{n}})$$

as $n \to \infty$ for some $C > 0$, where $\Omega = \rho A_\theta$. Moreover, each r_n can be taken to have simple poles only at

$$\beta_j = -e^{-\sigma j/\sqrt{n}}, \quad 0 \le j \le n-1,$$

where $\sigma > 0$ is arbitrary.



$A_\theta = \{z \in \mathbb{C} : |z| < 1, \quad |\arg(z)| < \theta\}$ $\qquad \Omega = \rho A_\theta$

Second Theorem

Let Ω be a convex polygon with corners $w_1, \ldots, w_m$, and let f be an analytic function in Ω that is analytic on the interior of each side segment and can be analytically continued to a disk near each w_k with a slit along the exterior bisector there. Assume f satisfies $f(z) - f(w_k) = O(|z - w_k|^\delta)$ as $z \to w_k$ for each k for some $\delta > 0$. There exist degree n rational functions $\{r_n\}$, $1 \leq n < \infty$ such that

$$||f - r_n||_\Omega = O(e^{-C\sqrt{n}})$$

as $n \to \infty$ for some $C > 0$. Moreover, each r_n can be taken to have finite poles only at points exponentially clustered along the exterior bisectors at the corners, with arbitrary clustering parameter σ, as long as the number of poles near each w_k grows at least in proportion to n as $n \to \infty$.