Jim Vargas Paper
Title Page
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1 Introduction

BACKGROUND ON PROBLEM

$$\Delta u(z) = 0, \quad z \in \Omega \qquad\qquad u(z) = h(z), \quad z \in \Gamma$$

where Δ is the Laplacian operator, $\Delta = \nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$.

$$u(z) \approx \mathrm{Re}[r(z)]$$
$$r(z) = \underbrace{\sum_{j=1}^{N_1} \frac{a_j}{z - z_j}}_{\text{"Newman"}} + \underbrace{\sum_{j=0}^{N_2} b_j (z - z_*)^j}_{\text{"Runge"}}$$

WHY RATIONAL FUNCTIONS

USES FOR THE LAPLACE OPERATOR

BARYCENTRIC COORDINATES?

2 Preliminaries

We start with a few definitions and theorems from Complex Analysis. These will help us understand and analyze our approximation r.

Definition 1. A function f is called *holomorphic* if, when it is defined in some ϵ-neighborhood of z_0, then it is differentiable in some ϵ-neighborhood of z_0.

We note that this definition requires complex-differentiation, which is very similar to but not identical to real-differentiability. The real and imaginary parts of any holomorphic function are harmonic, meaning they satisfy Laplace's equation, which can be seen in the following theorem.

Theorem 1. Cauchy-Riemann Conditions for Differentiability. A function $f = u + i\,v$ is differentiable at a point $z_0 = x_0 + i\,y_0$ if and only if the partial derivatives of u and v satisfy

$$u_x(x_0, y_0) = v_y(x_0, y_0) \qquad\qquad u_y(x_0, y_0) = -v_x(x_0, y_0),$$

where we use the standard notation $\frac{\partial u}{\partial x} = u_x$, and so on. Furthermore, $f'(z_0) = u_x(x_0, y_0) + i\,v_x(x_0, y_0) = u_y(x_0, y_0) - i\,v_y(x_0, y_0)$.

This tells us that $u_x = v_y$ and $u_y = -v_x$. If we assume we can differentiate once more and that the second partial derivatives of u and v exist and are continuous, then we can obtain the following identity by taking derivatives of these equalities and adding:

$$\Delta u = u_{xx} + u_{yy} = v_{yx} - v_{yx} = 0.$$

The same can be said for v in a similar way. Holomorphic and harmonic functions have many nice properties, so establishing a connection between our problem and these concepts is advantageous. One such property is the *Maximum Principle*, which states that a harmonic function on a closed domain achieves a maximum on the boundary of the domain, and this will guarantee us an identifiable error bound on our approximation.

HOLOMORPHIC=ANALYTIC

ROOT EXPONENTIAL CONVERGENCE

INTEGRAL FORMULA

3 Main Theorems

Theorems from Gopal and Trefethen.

Theorem 3. Let f be a bounded analytic function in the slit disk A_π that satisfies $f(z) = O(|z|^\delta)$ as $z \to 0$ for some $\delta > 0$, and let $\theta \in (0, \pi/2)$ be fixed. Then for some $0 < \rho < 1$ depending on θ but not on f, there exist type $(n-1, n)$ rational functions $\{r_n\}$, $1 \leq n < \infty$, such that

$$||f - r_n||_\Omega = O(e^{-C\sqrt{n}})$$

as $n \to \infty$ for some $C > 0$, where $\Omega = \rho A_\theta$. Moreover, each r_n can be taken to have simple poles only at

$$\beta_j = -e^{-\sigma j/\sqrt{n}}, \ \ 0 \leq j \leq n-1,$$

where $\sigma > 0$ is arbitrary.

Theorem 4. Let Ω be a convex polygon with corners $w_1, \ldots, w_m$, and let f be an analytic function in Ω that is analytic on the interior of each side segment and can be analytically continued to a disk near each w_k with a slit along the exterior bisector there. Assume f satisfies $f(z) - f(w_k) = O(|z - w_k|^\delta)$ as $z \to w_k$ for each k for some $\delta > 0$. There exist degree n rational functions $\{r_n\}$, $1 \leq n < \infty$ such that

$$||f - r_n||_\Omega = O(e^{-C\sqrt{n}})$$

as $n \to \infty$ for some $C > 0$. Moreover, each r_n can be taken to have finite poles only at points exponentially clustered along the exterior bisectors at the corners, with arbitrary clustering parameter σ, as long as the number of poles near each w_k grows at least in proportion to n as $n \to \infty$.

4 Numerical Applications and Experiments

Present the algorithm probably, lots of figures...