401 project notes

Rough explanation of the paper

Introduction

The setting: solving 2-D Laplace problems using rational function approximations, plus numerical experiments and examples. It turns out using rational function approximations with exponentially clustered points near singularities gets root-exponential convergence.

Their problem:

$$\Delta u(z) = \nabla^2 u(z) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u(z) = 0, \;\; z \in \Omega$$
$$u(z) = h(z), \;\; z \in \Gamma$$

in a domain Ω bounded piece-wise smoothly (with corners) by Γ, with specified boundary data h. This sort of problem comes up a lot in physics: electrostatics, fluid dynamics, heat conduction...

The approach:

$$u(z) \approx \mathrm{Re}[r(z)]$$
$$r(z) = \sum_{j=1}^{N_1} \frac{a_j}{z - z_j} + \sum_{j=0}^{N_2} b_j (z - z_*)^j$$

with poles z_j. The crux of the method is using exponentially clustered sample points on the boundary with h near corners, along with exponentially clustered poles outside the boundary near the corners.

The structure of the paper consists of: theorems establishing root-exponential convergence with rational approximations, then an algorithm using linear least-squares fitting on the boundary to find coefficients a_j and b_j.

Two Theorems

The theorems involve an analytic function f, while the applications involve a harmonic function u. However, and harmonic function u can be the real part of an analytic function, i.e. $f = u + iv$.

(***pictures, define A_θ)

(1) Let f be a bounded analytic function in the slit disk A_π that satisfies $f(z) = O(|z|^\delta)$ as $z \to 0$ for some $\delta > 0$, and let $\theta \in (0, \pi/2)$ be fixed. Then for some $0 < \rho < 1$ depending on θ but not on f, there exist type $(n-1, n)$ rational functions $\{r_n\}$, $1 \le n < \infty$, such that

$$||f - r_n||_\Omega = O(e^{-C\sqrt{n}})$$

as $n \to \infty$ for some $C > 0$, where $\Omega = \rho A_\theta$. Moreover, each r_n can be taken to have simple poles only at

$$\beta_j = -e^{-\sigma j/\sqrt{n}}, \;\; 0 \le j \le n-1,$$

where $\sigma > 0$ is arbitrary.

(2) Let Ω be a convex polygon with corners $w_1, \ldots, w_m$, and let f be an analytic function in Ω that is analytic on the interior of each side segment and can be analytically continued to a disk near each w_k with a slit along the exterior bisector there. Assume f satisfies $f(z) - f(w_k) = O(|z - w_k|^\delta)$ as $z \to w_k$ for each k for some $\delta > 0$. There exist degree n rational functions $\{r_n\}$, $1 \le n < \infty$ such that

$$||f - r_n||_\Omega = O(e^{-C\sqrt{n}})$$

as $n \to \infty$ for some $C > 0$. Moreover, each r_n can be taken to have finite poles only at points exponentially clustered along the exterior bisectors at the corners, with arbitrary clustering parameter σ, as long as the number of poles near each w_k grows at least in proportion to n as $n \to \infty$.

Some extensions: These same results hold for Ω bounded by analytic arcs meeting at corners. Additionally, the authors believe these results are valid also for non-convex domains and $\theta < \pi/2$. Experiments show that placing poles along exterior bisector is also not necessary.

Algorithm and Examples

Probably examples and pictures first.