8.6, 8.7, 8.8, 9.1 Write Up
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1 8.6 (8, 12)

(8): Find a power series representation for the function $f(x) = \dfrac{x}{2x^2+1}$ and determine the interval of convergence.

$$\begin{aligned} f(x) &= \frac{x}{2x^2+1} && (1)\\ &= x\left(\frac{1}{1-(-2x^2)}\right) && (2)\\ & && (3)\end{aligned}$$

Now, I will use the fact that $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$, for $|x| < 1$ to write $f(x)$ as a sum $\Rightarrow$

$$\begin{aligned} f(x) &= x\sum_{n=0}^{\infty}(-2x^2)^n && (4)\\ &= \sum_{n=0}^{\infty}(-1)^n\, 2^n\, x^{2n+1} && (5)\end{aligned}$$

I will use the Ratio Test to determine the interval of convergence. Let $a_n = (-1)^n\, 2^n\, x^{2n+1}$:

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}2^{n+1}x^{2n+3}}{(-1)^n\, 2^n\, x^{2n+1}}\right| && (6)\\ &= |-2x^2| < 1 && (7)\\ &= 2|x|^2 < 1 && (8)\\ &= |x| < \sqrt{\frac{1}{2}} && (9)\end{aligned}$$

The radius of convergence is $\sqrt{\frac{1}{2}}$. Now I will test the endpoints:

$$\begin{aligned} \text{When } x = \sqrt{\frac{1}{2}}: \qquad & \sum_{n=0}^{\infty}(-1)^n\, 2^n\left[\left(\frac{1}{2}\right)^{(1/2)}\right]^{2n+1} && (10)\\ =\ & \sum_{n=0}^{\infty}(-1)^n\, 2^n\,(2^{-1})^{n+(1/2)} && (11)\\ =\ & \sum_{n=0}^{\infty}(-1)^n\, 2^{-(1/2)} && (12)\end{aligned}$$

This can be shown to diverge using the Test for Divergence. let $a_n = (-1)^n 2^{-(1/2)}$:
$\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}(-1)^n 2^{-(1/2)} \neq 0$, therefore divergent.

$$\begin{aligned} \text{When } x = -\sqrt{\frac{1}{2}}: \qquad & \sum_{n=0}^{\infty}(-1)^n\, 2^n\left[(-1)\left(\frac{1}{2}\right)^{(1/2)}\right]^{2n+1} && (13)\\ =\ & \sum_{n=0}^{\infty}(-1)^n\,(-1)^{2n+1}\, 2^n\,(2^{-1})^{n+(1/2)} && (14)\\ =\ & \sum_{n=0}^{\infty}(-1)^{3n+1}\, 2^{-(1/2)} && (15)\end{aligned}$$

This can be shown to diverge using the same method above, so both endpoints diverge.
$\therefore\ f(x) = \sum_{n=0}^{\infty}(-1)^n\, 2^n\, x^{2n+1}$ on $I = (-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}})$.

(12): (a) Use the equation $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^x$, $|x| < 1$ **to find a power series representation for** $f(x) = \ln(1-x)$**. What is the radius of convergence?**

$$\begin{aligned} f(x) &= \ln(1-x) && (16)\\ &= -\int \frac{1}{1-x}\, dx && (17) \end{aligned}$$

By the given equation,

$$\begin{aligned} f(x) &= \int (-1 - x - x^2 - \cdots)\, dx && (18)\\ &= C + -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots && (19)\\ &= C + \sum_{n=1}^{\infty} \frac{-(x)^n}{n} && (20) \end{aligned}$$

To determine C, $x = 0$ can be put into the equation $\ln(1-0) = C$, so $C = 0$. To determine the radius of convergence, the Ratio test can be used. Let $a_n = \frac{-(x)^n}{n}$:

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{-(x)^{n+1}}{n+1}\cdot\frac{n}{-(x)^n}\right| && (21)\\ &= \lim_{n\to\infty}\left|x\frac{n}{n+1}\right| && (22)\\ &= |x| < 1 && (23) \end{aligned}$$

$\therefore\ f(x) = \sum_{n=1}^{\infty} \frac{-(x)^n}{n}$, $R = 1$.

(12): (b) Use part (a) to find a power series for $f(x) = x\,\ln(1-x)$**.**

$$\begin{aligned} f(x) &= x\,\ln(1-x) && (24)\\ &= x\sum_{n=1}^{\infty} \frac{-(x)^n}{n} && (25)\\ &= \sum_{n=1}^{\infty} \frac{-(x)^{n+1}}{n} && (26) \end{aligned}$$

(12): (c) By putting $x = \frac{1}{2}$ **in the result from part (a), express** $\ln(2)$ **as an infinite series.**

$$\begin{aligned} f\left(\frac{1}{2}\right) &= \sum_{n=1}^{\infty} \frac{-(\frac{1}{2})^n}{n} && (27)\\ &= \frac{-1}{2} - \frac{1}{8} - \frac{1}{24} - \cdots && (28) \end{aligned}$$

2 8.7 (32)

(32): Use a Maclaurin series to obtain the Maclaurin series for the function $f(x) = \dfrac{x^2}{\sqrt{2+x}}$

$$\begin{aligned} f(x) &= \frac{x^2}{\sqrt{2+x}} && (29) \\ &= x^2(2+x)^{-1/2} && (30) \\ &= \frac{x^2}{\sqrt{2}}(1+\frac{x}{2})^{-1/2} && (31) \end{aligned}$$

Using the Binomial Maclaurin Series $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$, $f(x)$ can be written as:

$$\begin{aligned} f(x) &= \frac{x^2}{\sqrt{2}} \sum_{n=0}^{\infty} \binom{-1/2}{n} \left(\frac{x}{2}\right)^n && (32) \\ &= \frac{x^2}{\sqrt{2}} \left[1 + (-1/2)(x/2) + \cdots + \frac{(-1/2)(-3/2)(-5/2)\cdots(-1/2-n+1)}{n!}(x/2)^n\right] && (33) \\ &= \frac{x^2}{\sqrt{2}} \left[1 - \frac{x}{2^2} + \frac{(-1)^2(3)x^2}{2^4 \cdot 2!} - \cdots + \frac{(-1)^n(3)(5)\cdots(2n-1)x^n}{2^{2n} \cdot n!}\right] && (34) \\ &= \frac{x^2}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{(-1)^n[(3)(5)\cdots(2n-1)]x^n}{2^{2n} \cdot n!} && (35) \\ &= \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{(-1)^n \left[\prod_{k=0}^{n-1} 2k+1\right] x^{n+2}}{2^{2n} \cdot n!} && (36) \end{aligned}$$

It is important to note here that an empty product $\prod_{k=t+1}^{t} b_k$, that is, an infinite product where the upper limit is less than the lower limit, is equal to 1.

3 8.8 (18)

(18): (a) Approximate $f(x) = x \ln(x)$ **by a Taylor Polynomial with degree 3 around** $a = 1$**.**

A Taylor Polynomial of degree three centered around a is written as

$$f(x) \approx T_3(x) = f(a) + f^{(1)}(a)\frac{(x-a)}{1!} + f^{(2)}(a)\frac{(x-a)^2}{2!} + f^{(3)}(a)\frac{(x-a)^3}{3!},$$

So the first three derivatives of $f(x)$ are needed.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$f^{(0)}(x) = x \ln(x)$	0
1	$f^{(1)}(x) = \frac{1}{x} \cdot x + \ln(x)$	1
2	$f^{(2)}(x) = x^{-1}$	1
3	$f^{(3)}(x) = -x^{-2}$	-1
4	$f^{(4)}(x) = 2x^{-3}$	

$f^{(4)}(x)$ is shown as well; it will be used later. So, The third degree Taylor Polynomial for $f(x)$ is

$$f(x) \approx T_3(x) = 0 + \frac{(x-1)}{1!} + \frac{(x-1)^2}{2!} - \frac{(x-1)^3}{3!}$$

(18): (b) Use Taylor's Inequality to estimate the accuracy of the approximation $f(x) \approx T_3(x)$ when $.5 \le x \le 1.5$.

Taylor's Inequality states that if $|f^{(n+1)}(x)| \le M$ for $|x-a| \le d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}, \text{ for } |x-a| \le d$$

For this problem, $n = 3$, $a = 1$. Since $.5 \le x \le 1.5$, it follows that $-.5 \le x - 1 \le .5$, so $|x-1| \le .5$. In order to maximize M, and therefore $f^{(4)}(x)$, a small, positive number < 1 is needed, so $x = .5$ can be used to find M:

$$M \ge f^{(4)}(.5) \tag{37}$$

$$\ge \frac{2}{.5^3} \tag{38}$$

$$\ge 16 \tag{39}$$

Finally,

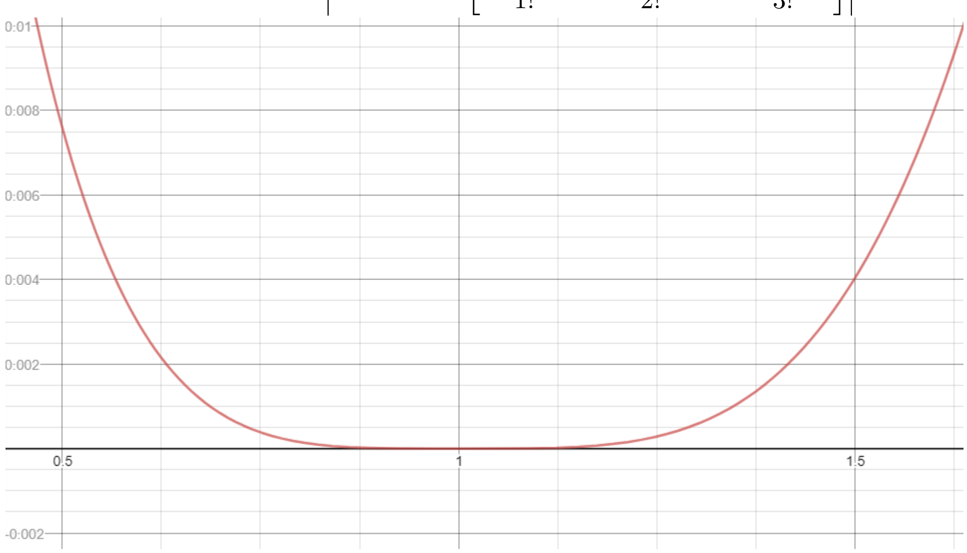
$$|R_3(x)| \le \frac{16}{4!} \cdot .5^4 \tag{40}$$

$$\le \frac{1}{24} \tag{41}$$

$\therefore R_3(x)$ of $[f(x) - T_3(x)]$ will not exceed $.041\bar{6}$ for $.5 \le x \le 1.5$.

(18): (c) Check the result in (b) by graphing $|R_3(x)|$.

The graph of $|R_3(x)| = \left|[x \ln(x)] - \left[\frac{(x-1)}{1!} + \frac{(x-1)^2}{2!} - \frac{(x-1)^3}{3!}\right]\right|$:



From this graph, it seems that the error is less than about .008 for $.5 \le x \le 1.5$

4 9.1 (16)

(16): Show that the equation $3x^2 + 3y^2 + 3z^2 = 10 + 6y + 12z$ represents a sphere, and find its center and radius.

The equation of a sphere is $(x-h)^2+(y-k)^2+(z-l)^2=r^2$, where r is the radius and (h,k,l) is the center of a circle. So,

$$\begin{aligned} 3x^2+3y^2+3z^2 &= 10+6y+12z && (42)\\ 3x^2+3y^2-6y+3z^2-12z &= 10 && (43)\\ x^2+(y^2-2y)+(z^2-4z) &= \frac{10}{3} && (44) \end{aligned}$$

I will complete the squares:

$$\begin{aligned} y^2-2y &= y^2-2y+1-1 && (45)\\ &= (y-1)^2-1 && (46) \end{aligned}$$

and

$$\begin{aligned} z^2-4z &= z^2-4z+4-4 && (47)\\ &= (z-2)^2-4 && (48) \end{aligned}$$

Now I substitute the values:

$$\begin{aligned} x^2+(y^2-2y)+(z^2-4z) &= \frac{10}{3} && (49)\\ x^2+((y-1)^2-1)+((z-2)^2-4) &= \frac{10}{3} && (50)\\ (x-0)^2+(y-1)^2+(z-2)^2 &= \frac{10}{3}+5 && (51)\\ (x-0)^2+(y-1)^2+(z-2)^2 &= \frac{25}{3} && (52) \end{aligned}$$

This matches the equation for a sphere. The radius is $\sqrt{\frac{25}{3}}$ and the center is at $(0,1,2)$.