Electric Dipole in a Circularly Rotating Electric Field

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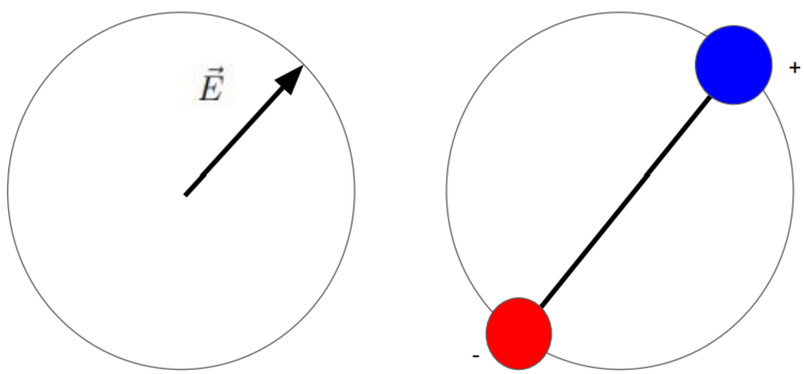

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Abstract

This project studies the behavior of the electric dipole in a circularly rotating field. The two physical effects that will be considered will be the torque from the electric field, and the electric dipole radiation from the angular rotation of the charges. The dipole was constructed as a positive and negative charge located a fixed distance apart, constrained to move on the opposite sides of a circle. The dipole rotates in 2 dimensions, or in the xy plane, and the electric field also rotates in the xy plane. This project used an VPython simulation of the dipole system to give a numerical result and comparison of the solutions. At low frequencies, it was found that the dipole vector follows the electric field in phase, with the positive charge pointing in the same direction as the electric field. For low field frequencies, the dipole was found to oscillate simple harmonically about the field axis, in phase. Overall, the behavior of the system is simple harmonic for low driving field frequencies, but moves towards non-linear behavior for high frequencies. The frequency of the secondary oscillation about the field axis was found experimentally and theoretically, with results matching to about 1 percent error. Additionally, at high frequencies, it was found that the dipole would experience phase-slips, losing alignment with the field.

1 Dipole Construction

The dipole is created normally as a positive and negative charge, separated by some distance. This system is identical to the commonly known "Rigid Rotor" system, but with charges instead of masses. In this system, the positive and negative charges will be fixed to the end of a rigid rod, held at the center. This will restrict the motion of the charges to the circle, always remaining on exactly opposite sides of the circle. The system will be allowed to rotate in the xy plane, corresponding to a torque in the $\hat{z}$ direction. The dipole vector will point towards the positive charge, tracing along the circle as it rotates. From this construction, the dipole $\vec{p}$ is the normal dipole vector,

$$\vec{p} = \sum_{i=0}^{N} q_i \vec{r}_i = q\frac{L}{2}(\sin(\phi)\hat{x} + \cos(\phi)\hat{y}) - q(\frac{L}{2})(-\sin(\phi)\hat{x} - \cos(\phi)\hat{y}) = qL(\sin(\phi)\hat{x} + \cos(\phi)\hat{y}).$$

Where ϕ is an angle measured from the y axis arbitrarily. So, the dipole is the sum of the product of the charge with the vector position of the charge, for all charges. What is physical important about the dipole is this, the product of charge with position. This allows one to create a real, linearly oscillating dipole two different ways.

Firstly, you could oscillate the positions of two charges, by changing their locations every so often, moving them back and forth. Secondly, you could oscillate the value of their charges, changing the positive charge to the negative every so often.

Both of these constructions are identical, and will produce identical results, as the dipole's in both configurations are the same. Now, this system is NOT a linearly oscillating dipole, but rather a circularly rotating one. As mentioned in Jackson E&M, this circularly rotating dipole can be formed as 2 perpendicular linearly oscillating dipoles, out of phase. This makes sense, as rotating circularly can be seen as oscillating linearly in two perpendicular directions. So, this circularly rotating electric dipole is analogous to two linearly oscillating electric dipoles, exactly out of phase and perpendicular to one another. (Out of phase and perpendicular ensures no cross terms in the power emission, from Poynting vector). This feature will be helpful in finding the total power radiated by this rotating dipole.

2 Equations of Motion

The dipole will experience both a torque from the external field, and a loss in energy from radiation. Both these effects will end up producing an angular acceleration on the dipole, so the angle is the only coordinate required to find the full equations of motion. Just the angle ϕ will give all the information needed about the dipole. The Electric field is oscillating circularly in the xy plane, so

$$\vec{E} = E_0(\sin(w_D t)\hat{x} + \cos(w_D t)\hat{y}).$$

Where w_D is the oscillation angular frequency of the External Electric field, and E_0 is the amplitude of the Electric field. w_D can be considered a driving frequency for the system, as it is preset and will never change throughout the motion. The torque onto the dipole is the normal,

$$\tau = \vec{p} \times \vec{E}.$$

Where $\vec{p}$ is the dipole vector from above. The torque is then:

$$\tau = (p_x E_y - p_y E_x)\hat{z} = qLE_0(\sin\phi \cos w_D t - \cos\phi \sin w_D t)\hat{z}.$$

Using a trigonometric identity, we can combine these two terms. This trig identity makes a lot of sense because the cross product magnitude is equal to the sin of the angular difference, which is what the expression has exactly. Also, replacing the dipole term with the dipole amplitude we have,

$$\tau = p_0 E_0 \sin(\phi - w_D t)\hat{z}.$$

This torque produces angular acceleration in the counter clockwise direction by the Right hand rule. The main angle ϕ is defined as increasing in the clockwise direction, so a minus sign must be placed on the angular acceleration term to ensure consistency. This torque produces angular accel in the xy plane only by the RHR, which the angle along this plane is ϕ, and the vector form will be dropped. So,

$$-\tau = -p_0 E_0 \sin(\phi - w_D t) = I\alpha = I\frac{d^2\phi}{dt^2},$$
$$\frac{d^2\phi}{dt^2} = \ddot{\phi} = \frac{-p_0 E_0}{I}\sin(\phi - w_D t).$$

This is the final angular acceleration term from the torque of the electric field, where I is the moment of inertia of the dipole. This equation looks very similar to a real pendulum, but the angle gravity is pointing in is changing directions. This is only one of the terms for the total angular acceleration of the dipole, the other term needed is from the emitted electromagnetic Radiation. The dipole system is a system of accelerated charges, so it must Radiate, decreasing the overall kinetic energy of the system.

Aforementioned, this system is analogous to two linearly oscillating dipoles. Therefore, the power radiated from oscillation, is exactly equal to the power radiated by two linearly oscillating dipoles. So, I will need to multiply the power emission for a linearly oscillating dipole by two. Classical E&M theory must be done to derive the radiated power from a linearly oscillating dipole, but that is beyond the scope of this report. Using the classical formula for time averaged power emission for a single linearly oscillating dipole,

$$P^{rad}_{tot} = 2P^{rad}_1 = 2\frac{p_0^2 w^4}{12\pi\epsilon_0 c^3} = -\frac{dE}{dt}.$$

Where p_0 is the amplitude of the dipole oscillation, and w is the angular frequency of the dipole. Stated above, this emitted radiation power decreases the overall kinetic energy of the system. There is only rotational kinetic energy, $E = \frac{1}{2}Iw^2$, so

$$\frac{dE}{dt} = \frac{d}{dt}(\frac{1}{2}Iw^2) = Iw\frac{dw}{dt}.$$

Solving for the angular acceleration $\frac{dw}{dt}$ and plugging in the radiated power gives us

$$\frac{dw}{dt} = \ddot{\phi} = \frac{1}{Iw}\frac{dE}{dt} = -\frac{p_0^2 w^3}{6\pi I\epsilon_0 c^3}.$$

This is the second and last term for the angular acceleration of this system (Note $w = \dot{\phi}$). Additionally, it is very convenient that the power scales as the cube of the angular velocity, being an odd power it will always maintain the direction of the angular velocity. This term is the minus the cube of the angular velocity, ensuring it will always be a loss term, as this radiation always is a loss of energy. So, we write the full angular acceleration of this dipole in a circularly rotating field as the sum of the two terms,

$$\ddot{\phi} = \frac{-p_0 E_0}{I} \sin(\phi - w_D t) - \frac{p_0^2 \dot{\phi}^3}{6\pi I \epsilon_0 c^3}. \tag{1}$$

This differential equation is highly non-linear, with very interesting behavior that is most likely impossible to predict in general. The first term is the torque from the external field, bringing the dipole angle towards the angle of the Electric Field. The second term is the Radiative loss, always decreasing the angular velocity and overall rotational kinetic energy. Above is the full equation of motion, as we are only considering the behavior of ϕ, everything else is restricted.

3 Simulation

In order to look at this complex behavior, numerical methods must be used to solve this differential equation. I simulated this system using Python, along with a handy module called VPython. This environment allows for extremely easy animations. Additionally, I did this on a website called Trinket with glowscript. This website can run VPython scripts locally in your browser, with zero setup and installation required.

In this simulation, the dipole is constructed exactly as mentioned above. It is a rigid rotor with two opposite charges, free to rotate in the xy plane. For this simulation, the parameters became increasingly important on the result. In other words, this simulation is extremely sensitive to the parameters I give it. These parameters or values include the dipole, External Electric field, Moment of Inertia (mass), Length, and finally the Driving frequency of the external field. There are lot of parameters, and getting them right was quite literally the hardest part of this entire project.

To stay more generalized, the Simulation uses the earlier definition of both the angular acceleration terms. Both equations are equally valid, this way is simply sticking to the more fundamental Physics. The Angular Acceleration used by the simulation is then,

$$\ddot{\phi} = \frac{-(\vec{p} \times \vec{E})_z}{I} - \frac{p_0^2 \dot{\phi}^3}{6\pi I \epsilon_0 c^3}.$$

The Simulation first calculates the dipole vector, then the full angular acceleration using the equation above. It then uses the Euler method to compute the corresponding change in Angular velocity, and change in Angle, over a small time step. The simulation can be ran Here, I suggest the reader run the code a few times for a better intuitive understanding. The parameters used were as follows,

Parameter list		
Parameter	Variable	Value
Electric Charge	q	$1.60 * 10^{-8}$ C
Mass	m	$1.67 * 10^{-17}$ kg
Dipole Length	L	$1.0 * 10^{-2}$ m
Moment of Inertia	I	$8.35 * 10^{-22} kg * m^2$
Electric Field Amplitude	E_0	$1 * 10^1$ N/C
Dipole Vector Amplitude	p_0	$1.60 * 10^{-10}$ C*m
Electric Field Angular Freq.	w_D	varies

The other main initial parameters are the starting angle and starting angular velocity of the dipole. The dipole is started at the same angular frequency of the field, to better track it, while the initial angle is a small angle with respect to the field. The driving frequency of the external field is varied for different simulations, to give different behaviors. The blue ball is the positive charge and orients itself with the field, while the red ball is the negative charge and does the opposite.

4 Solutions

After using the Simulation thoroughly, I conclude that the system is very sensitive to the initial conditions. In order to get computational results that can look similar to analytic ones, approximations for equation 1 are required. The first approximation will be to ignore the non-linear radiation term. The entire constant that multiplies the $\dot{\phi}^3$ term, is equal to $6.8 * 10^{-15}$ for the listed parameters. So this term does not contribute anything to the angular acceleration for small $\dot{\phi}$, which corresponds to small w_D for the external field. So equation 1 becomes

$$\ddot{\phi} = \frac{-p_0 E_0}{I} \sin{(\phi - w_D t)}.$$

This equation looks very familiar, it is almost the exact differential equation for the angle of a real pendulum. The only difference is that the angle the system is restored too, changes over time, which would be analogous to a pendulum with gravity changing direction with respect to the driving frequency.

Additionally, we expect that the dipole seek to orient itself in the same direction as the electric field. For low frequencies, there is nothing preventing it from fully tracking the electric field as it rotates circularly. So, it can be assumed that $\phi - w_D t$ is small. Using a first order expansion of sin, the angular acceleration becomes

$$\ddot{\phi} = \frac{-p_0 E_0}{I} (\phi - w_D t).$$

Substituting $\theta = \phi - w_D t$, $\ddot{\theta} = \ddot{\phi}$, the above equation reduces to

$$\ddot{\theta} = \frac{-p_0 E_0}{I} \theta = -k^2 \theta.$$

Which is the normal differential equation for simple harmonic motion. This angular difference will oscillate simple harmonically in time, $\theta(t) = \theta_0 cos(kt)$, where

$$k = w_{diff} = \sqrt{\frac{p_0 E_0}{I}}. \tag{2}$$

This is the angular frequency of oscillation for the difference in angle, between the dipole and the circularly rotating electric field. This angular frequency is independent of the driving frequency, as it ends up not contributing because the dipole is free to track the electric field for low frequencies. This is not the angular velocity of the dipole, which is going to be roughly equal to the field's angular velocity plus the oscillation about the field axis. To be clear, this is simple harmonic motion for the angular difference, not necessarily the angle of the dipole. This can be seen as a secondary oscillation, while rotating circularly, the dipole oscillates about the field axis. So the dipole tracks the field circularly, but oscillates simple harmonically about that circular rotation. Un-substituting allows us to see the solutions for the original angle ϕ,

$$\phi(t) = \theta_0 cos(w_{diff} t) + w_D t, \tag{3}$$

$$\dot{\phi}(t) = -\theta_0 w_{diff} sin(w_{diff} t) + w_D. \tag{4}$$

From these equations, ϕ being the dipole angle, it can be seen that the dipole oscillates simple harmonically at w_{diff}, while rotating circularly at constant w_D. The angular velocity term also represents the oscillation about a constant frequency w_D. These solutions are only valid for small driving frequency w_D.

Next, the angular frequency at which the Radiation term begins to contribute will be found. Returning to equation 1 and setting the two terms equal to each other, will allow us to find the angular frequency when the radiation term is equal to the torque term.

$$\frac{p_0 E_0}{I}\sin(\phi - w_D t) = \frac{p_0^2 \dot{\phi}^3}{6\pi I \epsilon_0 c^3}.$$

The simple harmonic motion for the angular difference is broken when the radiation term exceeds the torque term. The maximum angle for simple harmonic motion to be accurate is around 0.6 rad, using this for the angular difference $\phi - w_D t$ gives,

$$\dot{\phi}_{max} = (\frac{6\pi\epsilon_0 c^3 E_0(0.6)}{p_0})^{1/3}$$

which is the angular velocity of the dipole at which the radiation term is larger than the torque term. It scales inversely with the dipole moment amplitude, which makes physical sense, the larger the dipole, the more radiation emitted. The factor of 0.6 can be changed to 1 if one wishes to compute the maximum angular velocity at which an harmonic motion in general can exist, not necessarily simple. Using the factor at 0.6, this can be considered a maximum angular velocity for simple harmonic motion to exist. Plugging in our parameters for the simulation gives $\dot{\phi}_{max} = 5.53 * 10^8$ rad/s.

5 Results

Low Driving Frequency

The behavior of the system will be simulated and compared to the analytic result in both cases, small frequencies and large frequencies for the driving external field. In the first case, the system is simulated for $w_D = 5.5 * 10^7$, roughly a tenth of the maximum angular velocity before radiation dominates. Here is the simulation of the system for the conditions mentioned above, low frequency of the external field. This is technically "low frequency" when compared to the frequencies needed for radiation to matter, but it is still in the tens of millions of radians per second. So, there is a trade off on the simulation, between a slow simulation to watch the dipole complete revolutions, vs a fast simulation to complete the secondary oscillations about the field axis. On line 10 you can change the rate of the animation, I suggest a value of 100 to view dipole rotations, but a value of 10,000 to view the plots. Here is the plot of the angular difference between the dipole vector and electric field.

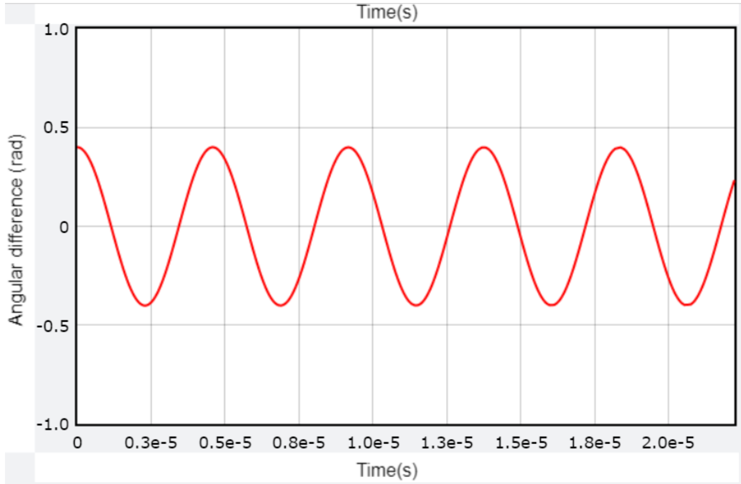

Figure 1: Dipole and Electric field phase difference vs time, low field frequency. Simple harmonic motion exhibited for small initial angular displacement from field axis.

As predicted, the angular difference between the field and dipole is simple harmonic, for low driving frequencies. The simulation found a constant period of $T = 4.58491 * 10^{-6}$ s, which equates to an angular frequency of $w_{diff} = 1.370 * 10^{6}$ rad/s. Using equation 2 and plugging in all the parameters, the theoretical angular frequency of the oscillation is $w_{diff} = 1.384 * 10^{6}$ rad/s. This gives an experimental error of about 1%. These values are extremely close, and give merit to the model. The simulation angular freq. being smaller than the theoretical is also consistent with the model, as the theory neglects radiation, while the simulation always has the radiation term present. The radiation term ends up reducing the angular frequency of the simulation, which is why it is slightly smaller than the theoretical value. Additionally, the overall angular velocity of the dipole (EQ.4) matches the behavior in the simulation,

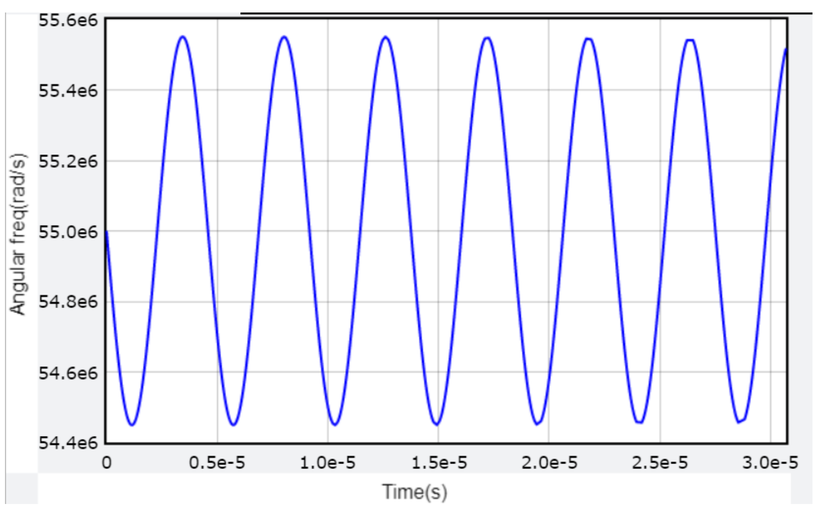

Figure 2: Dipole Angular velocity vs time, low field frequency. Simple harmonic motion about the field frequency w_D.

This is the true angular velocity of the dipole. It's midpoint is exactly the driving frequency $w_D = 5.5 * 10^{7}$, and it oscillates at w_{diff} as well, as predicted by EQ 4. Additionally, it has amplitude $\theta_0 w_{diff}$. Plugging in these values gives a theoretical amplitude of $5.49 * 10^{5}$ rad/s, which looks to be extremely close to the amplitude above. Overall, for low frequency the behavior matches the theoretical prediction exactly, the dipole oscillates simple harmonically about the electric field, which is rotating circularly.

High Driving Frequency

Frequencies considered high will be at or larger than the maximum angular velocity where simple harmonic motion is supported. So, $w_D > \dot{\phi}_{max}$, here the driving frequency will be $w_D = 5.8 * 10^8$, just slightly larger than the maximum angular frequency for SHO. The high frequency simulation can be accessed Here. For high frequencies, the dipole cannot track the electric field as it rotates circularly, the radiation term will overpower the torque from the electric field. So, the dipole begins to lag behind the electric field, as the radiation term continues to slow it down. Eventually, the dipole has lagged behind the quickly rotating electric field enough that it will experience a phase slip, moving out of phase from the electric field. Essentially the field rotates faster than the dipole can, so the dipole begins to experience a torque in the opposite direction, after it gets lapped. This phenomenon of phase slipping leads to very non-linear and chaotic behavior, seen in figure 3.

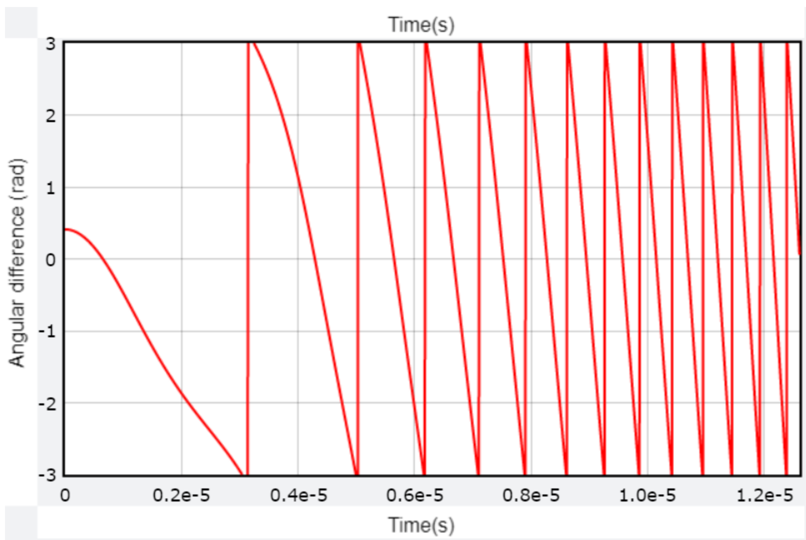

Figure 3: Dipole and Electric field phase difference vs time, High field frequency. Phase-slipping is exhibited, dipole is lapped by external field giving sharp discontinuities in the angular difference.

It can be seen initially how the dipole attempts to maintain the field's angle, but quickly lags behind and is eventually lapped by the electric field. So this plot of the angular difference has sharp discontinuities, as the dipole is completing revolutions against the electric field. The dipole loses the phase of the electric field, falling behind and eventually rotating the other direction to catch up with the field.

Transition Frequency

Additionally, it is interesting to see the transition from simple harmonic oscillation to the more complicated phase-slipping with radiation. The simulation will be ran with w_D very close to, but just below the maximum frequency where SHO is still supported. So, I used $w_D = 5.5 * 10^8$ rad/s, slightly under the maximum angular frequency. The transition frequency simulation can be found Here.

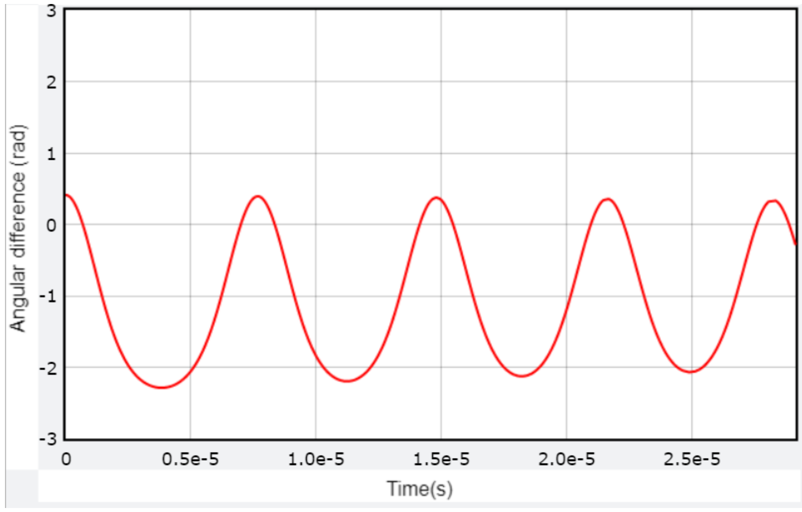

Figure 4: Dipole and Electric field phase difference vs time, transition frequency. It is harmonic motion (NOT simple harmonic) with a longer period than the low field frequency case.

This is the plot of the angular difference at a transition frequency, as you can see it is not simple harmonic anymore, and extends significantly further below 0 than it does above. The shape is no longer symmetric about the midpoint angle, and is significantly shifted. This is because the dipole oscillations in front of the field are cut shorter by the field approaching the dipole, and oscillations behind the field are made longer by the field moving away from the dipole. The contributing radiation term also has the same effect, pulling the dipole back when ahead of the field, but also pulling it back when behind the field, increasing the size of oscillation behind the field. Additionally, comparing this to the Figure 1, the angular frequency of the secondary oscillations about the field has just about doubled. This also makes sense, when radiation is now a considerable factor, the oscillations are damped and made significantly slower. This is very similar to the damped linear oscillator, as adding a friction force decreases the overall angular frequency.

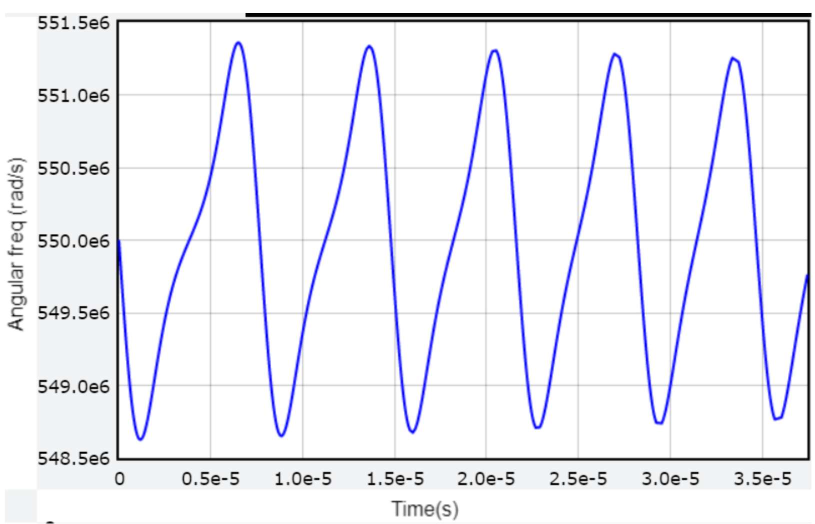

Figure 5: Dipole Angular velocity vs time, transition frequency. Harmonic motion (NOT simple harmonic) with a longer period of oscillation than the low field frequency case.

This is the plot of the Angular frequency of the dipole for this transition field frequency. Clearly the curve is no longer a sin curve, and has shifted towards being more triangular. It is still harmonic though, and oscillates about the driving frequency w_D. The oscillation period is also a lot larger than previously, as the radiation term is contributing significantly.

6 Conclusion

Overall, the dipole in a circularly rotating electric field has very complicated behavior. For low frequencies, it is analogous to the equations of motion for a pendulum, only the axis it is restored towards rotates. So, it is roughly simple harmonic about the electric field, which rotates circularly. This creates a nested oscillation, the dipole is oscillating about a circularly rotating field. Additionally, the radiation term has negligible contribution until very high angular frequencies, (like 10^8 rad/s for a lab dipole). For these high field frequencies where the radiation is significant, the behavior is highly non-linear and difficult to predict. The dipole will track the field as best it can, but occasionally it will get lapped and experience a phase-slip when compared to the field.

For future work, I believe that there is necessarily an equilibrium position, where the dipole will not oscillate or change angular velocity. It occurs when the electric field torque and dipole radiation exactly cancel out. This equilibrium angle will always be behind the field angle, and only exists for small frequencies. This equilibrium angle only exists for small frequencies because there is a maximum torque the electric field can apply, when the vectors are perpendicular, but there is not a maximum "torque" from the radiation term, it can scale to infinity with the cube of the angular frequency. Additionally, the exact period of secondary oscillation be computed using numerical methods for the integral over a period, finding the oscillation period even in the transition frequencies. Lastly, it is possible that at High frequencies, when there is phase-slipping between the dipole and electric field, this happens periodically at evenly spaced time intervals, so its frequency could be predicted. The plot of the phase-slips at high frequencies (fig 3) displays sharp shark tooth wave like patterns, potentially at even spaced intervals. This would most likely require more advanced analysis, and potentially require a new method.