

# Lambda Calculus

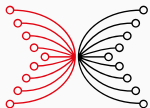
## Haskell and Cryptocurrencies

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INPUT | OUTPUT

# Goals

- Untyped Lambda Calculus
- $\alpha$ -Equivalence
- $\beta$ -Equivalence
- Reduction Strategies
- Church Encodings
- General Recursion & Y-Combinator

# Untyped Lambda Calculus

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# High-level properties

- Provides simple semantics and a formal model for *computation*.
- Turing complete.
- Makes two simplifications:
  - only *anonymous* functions
  - only functions of one argument (*curried* functions)
- *Haskell* is based upon and compiles to a (typed!) version of the Lambda Calculus (as first intermediate compiler target, *Core*).

# History

- Based upon work by *Frege* from 1893 and *Schönfinkel* from the 1920s.
- Introduced by *Alonzo Church* in the 1930s.
- Shown to be logically inconsistent in 1935 by *Stephen Kleene* and *J. B. Rosser*.
- Fixed by Church in 1936 – *Untyped Lambda Calculus*.
- Relation to programming languages clarified in the 1960s.

# Lambda expressions

Lambda expressions (or lambda terms) are composed of

- variables  $v_1, v_2, \dots, v_n, \dots$ ,
- the abstraction symbols  $\lambda$  and  $.$ ,
- parentheses  $( )$ .

The set of lambda expressions  $\Lambda$  is inductively defined as:

- If  $x$  is a variable, then  $x \in \Lambda$ . (variable)
- If  $x$  is a variable and  $M \in \Lambda$ , then  $(\lambda x.M) \in \Lambda$ . (lambda abstraction)
- If  $M, N \in \Lambda$ , then  $(MN) \in \Lambda$ . (application)

# Notation

To keep the notation of lambda expressions uncluttered, the following conventions are normally applied:

- Outermost parentheses are dropped:  $MN$  instead of  $(MN)$ .
- Applications are assumed to be left associative:  $MNP$  instead of  $(MN)P$ .
- The body of an abstraction extends as far right as possible:  $\lambda x.MN$  means  $\lambda x.(MN)$ , *not*  $(\lambda x.M)N$ .
- A sequence of abstractions is contracted:  $\lambda x.\lambda y.\lambda z.M$  is abbreviated as  $\lambda xyz.M$ .

## Notation (contd.)

This is just as in Haskell:

- `f x = (f x)`
- `f x y = (f x) y`
- `\ x -> \ y -> \ z -> f = \ x y z -> f`



- As “syntactic sugar”, we can also introduce **let bindings**:
- For a variable  $x$  and lambda expressions  $M$  and  $N$ , we can define **let**  $x = N$  **in**  $M$  as  $(\lambda x.M) N$ .
- Later, once we learn about  $\beta$ -reduction, we will see that this “means” substituting  $N$  for  $x$  in  $M$ , which coincides with the intuitive idea we have of how **let** “should” behave.
- This technique is actually used frequently in JavaScript ...

# Free variables

- Let  $V$  be the set of variables. For each lambda expression  $M \in \Lambda$ , we define the set of **free variables**  $FV(M) \subset V$  as follows:
  - For a variable  $x \in V$ ,  $FV(x) = \{x\}$ .
  - For an abstraction,  $FV(\lambda x.M) = FV(M) \setminus \{x\}$ .
  - For an application,  $FV(MN) = FV(M) \cup FV(N)$ .
- Given a lambda expression  $M \in \Lambda$ , we call a variable  $x \in V$  **free** (in  $M$ ) if  $x \in FV(M)$ .
- In an abstraction  $\lambda x.M$ , we call the variable  $x$  **bound**.

# Subexpressions

We define the notion of **subexpression** of a lambda expression inductively as follows:

- Each term is a subexpression of itself. Sub-expressions other than the term itself are called **proper** subexpressions.
- A variable has no proper subexpressions.
- The proper subexpressions of an application  $MN$  are the subexpressions of  $M$  and the subexpressions of  $N$ .
- The proper subexpression of an abstraction  $\lambda x.M$  are the subexpressions of the body  $M$ .

# Substitution

Let  $x \in V$  be a variable, and let  $M, N \in \Lambda$  be lambda expressions. We define  $M[x := N]$ , the substitution of  $x$  by  $N$  in  $M$  recursively as follows:

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- $x[x := N] = N$ .
- For a variable  $y \neq x$ ,  $y[x := N] = y$ .

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- $x[x := N] = N$ .
- For a variable  $y \neq x$ ,  $y[x := N] = y$ .
- $(P Q)[x := N] = P[x := N] Q[x := N]$
- $(\lambda x.P)[x := N] := \lambda x.P$ .
- For a variable  $y \neq x$  with  $y \notin FV(N)$ ,  
 $(\lambda y.P)[x := N] := \lambda y.(P[x := N])$ .
- For an abstraction  $\lambda y.P$  with  $y \neq x$  and  $y \in FV(N)$ , **pick** a variable  $z \notin FV(N) \cup FV(P)$ , then  
 $(\lambda y.P)[x := N] := \lambda z.(P[y := z][x := N])$ .

## Substitution (contd.)

- It looks as if substitution was not well defined, because the result depends on a *choice* of variable in the last case.
- In the next section, we will define an equivalence relation on lambda expressions,  *$\alpha$ -equivalence*, which will make the choice of variable name irrelevant.



## Example substitutions

$$\bullet (\lambda x.x)[x := z (\lambda u.u)] = \lambda x.x$$

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$$\begin{aligned} & (\lambda z.x)[x := z (\lambda u.u)] \\ &= \lambda z'.(x[z := z'] [x := z (\lambda u.u)]) \\ &= \lambda z'.(x[x := z (\lambda u.u)]) \\ &= \lambda z'.z (\lambda u.u) \end{aligned}$$

# $\alpha$ -Equivalence

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We define the notion  $M \sim_{\alpha} N$  of  $\alpha$ -equivalence between two lambda expressions  $M$  and  $N$  inductively as follows:

- For two variables  $x$  and  $y$ ,  $x \sim_{\alpha} y$  iff  $x = y$ .
- $(MN) \sim_{\alpha} (PQ)$  iff  $M \sim_{\alpha} P$  and  $N \sim_{\alpha} Q$ .
- For abstractions  $\lambda x.M$  and  $\lambda y.N$ , choose a variable  $z$  with  $z \notin FV(M) \cup FV(N)$ . Then  $\lambda x.M \sim_{\alpha} \lambda y.N$  iff  $M[x := z] \sim_{\alpha} N[y := z]$ .
- All other expressions are *not*  $\alpha$ -equivalent.

Intuitively,  $\alpha$ -equivalence means that two expressions are equal except for the naming of bound variables.

## $\alpha$ -Equivalence (contd.)

### Equality

From now on, we will always consider  $\Lambda / \sim_\alpha$  instead of  $\Lambda$ , i.e. we will consider two  $\alpha$ -equivalent lambda expressions as *equal*.

### Substitution

As soon as  $\alpha$ -equivalence becomes equality, the problem in our definition of substitution goes away, because different “picks” of variable name in the last case clearly lead to  $\alpha$ -equivalent results.

## $\alpha$ -Equivalence (contd.)

- The necessity to deal with  $\alpha$ -equivalence makes implementation of substitution and equality checking quite tricky.
- There are ways (for example *De Bruijn indices*) of handling variable names differently in the definition of lambda expressions that lead to *syntactically identical* terms for  $\alpha$ -equivalent expressions.
- However, the presentation we chose seems to be the most “human readable”.

# $\beta$ -Equivalence

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## $\beta$ -Equivalence captures “computation”

- As explained above,  $\alpha$ -equivalence is more of a technical nuisance, capturing the intuitive idea that the names of bound variables should not matter.
- $\beta$ -equivalence, on the other hand, lies at the very heart of Lambda Calculus and captures the notion of computation.
- Intuitively, the *act of computation* preserves  $\beta$ -equivalence.
- So what does *computation* mean in the context of Lambda Calculus?

# $\beta$ -Reduction

- Consider a lambda expression of the form  $(\lambda x.M) N$ , i.e. an application where the first argument is an abstraction.
- By definition, this term  $\beta$ -reduces to  $M[x := N]$ .
- This act of “plugging in” an expression for the bound variable in an abstraction is what constitutes the idea of computation in Lambda Calculus.

# Redex, $\beta$ -equivalence & normal form

- A **redex** of a lambda expression  $P$  is a subexpression of  $P$  of the form  $(\lambda x.M) N$ .
- Let  $P'$  denote the lambda expression obtained by replacing a redex  $(\lambda x.M) N$  with  $M[x := N]$ . We say that  $P$   **$\beta$ -reduces** to  $P'$  (in one step).
- We say that  $P$   **$\beta$ -reduces** to  $P'$  (or that  $P'$  is a  **$\beta$ -reduct** of  $P$ ) if reducing zero or more redexes transforms  $P$  into  $P'$ .
- Two lambda expressions  $P$  and  $P'$  are  **$\beta$ -equivalent** ( $P \sim_{\beta} P'$ ) if one can be transformed into the other by a chain of  $\beta$ -reductions (or their inverses).
- A lambda expression without redex is said to be in **normal form**.
- We say a lambda expression  $P$  **has normal form**  $P'$  if  $P$   $\beta$ -reduces to  $P'$  and  $P'$  is in normal form.

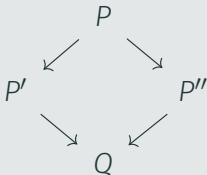
# Questions

- Can lambda expression have more than one normal form?
- Does any lambda expression have a normal form?
- If a lambda expression *does* have a normal form, how can I find it/one?

# The Church-Rosser theorem

## Theorem

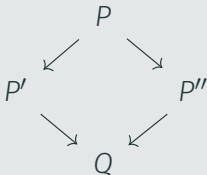
Let  $P$  be a lambda expression, and let  $P'$  and  $P''$  be two  $\beta$ -reducts of  $P$ . Then there is a lambda expression  $Q$  such that both  $P'$  and  $P''$   $\beta$ -reduce to  $Q$ .



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## At most one

Church-Rosser immediately(!) implies that a lambda expression has *at most one* normal form.

## No normal form ...

- Consider the lambda expression  $\omega := (\lambda x.xx)(\lambda x.xx)$ .

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## No normal form ...

- Consider the lambda expression  $\omega := (\lambda x.xx)(\lambda x.xx)$ .
- There is only one redex,  $\omega$  itself.
- Let's reduce it!
- $(xx)[x := \lambda x.xx] = (\lambda x.xx)(\lambda x.xx) = \omega$ .
- $\omega$  only has one redex, and  $\beta$ -reducing that one redex gives  $\omega$  as a reduct!
- By definition,  $\omega$  is not(!) in normal form. So  $\omega$  **has no normal form!**

# Open question

- So we know from Church-Rosser that a lambda expression's normal form is unique if it exists.
- We have seen an example of a lambda expression that does *not* have a normal form.
- Open question: How to find the normal form if it exists?
- To be more precise: If there is more than one redex, which one do we reduce first?

# Reduction Strategies

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# Reduction strategies

## Reduction Strategy

A **reduction strategy** is an algorithm that, given a lambda expression, decides which redex to reduce (if at least one exists).

# Call by value

- Consider the strategy of always reducing the *leftmost innermost* redex first.
- This strategy is called **call by value** or **eager evaluation**.
- Intuitively, it means evaluating the arguments to a function first, then applying the function.
- Example:

$$\begin{aligned} & ((\lambda xy.x) (\lambda x.x)) ((\lambda x.x) y) \\ \sim_{\beta} & (\lambda yx.x) ((\lambda x.x) y) \\ \sim_{\beta} & (\lambda yx.x) y \\ \sim_{\beta} & \lambda x.x \end{aligned}$$

# Call by name

- Now consider the strategy of always reducing the *leftmost outermost* redex first.
- This strategy is called **call by name**.
- Intuitively, it means applying a function before evaluating its arguments.
- Example:

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## Comparison

With both strategies, we arrive at the same normal form (Church-Rosser!). But call by name needs one step less.



# Call by need

- There is an optimization of *call by name*, called **call by need** (or **lazy evaluation**).
- The problem with call by name is that when substituting, we may duplicate function arguments and then maybe will have to evaluate them several times.
- In call by need, instead of substituting arguments by copying them, a pointer to the argument is substituted,
- If an argument needs to be evaluated once, all copies of it profit from the evaluation.

# Theorem on call by name

## Theorem

Let  $P$  be a lambda expression which has a normal form. Then the *call by name/need* strategy will reduce to this normal form.

## Counterexample

- Consider the lambda expression  $(\lambda xy.x) (\lambda x.x) \omega$ .
- It has a normal form, which call by name/need reduces to in two steps:

$$\begin{aligned} & (\lambda xy.x) (\lambda x.x) \omega \\ \sim_{\beta} & (\lambda yx.x) \omega \\ \sim_{\beta} & (\lambda x.x) \end{aligned}$$

- Call by value, on the other hand, enters an infinite loop, because it tries to reduce  $\omega$  in the second step.
- So we see that call by name finds strictly more normal forms than call by value.
- This example corresponds to the Haskell expression `const id undefined`.

# The reduction strategy of Haskell

- The Haskell standard does **not** dictate the use of one particular reduction strategy.
- Instead, it prescribes that Haskell has **non-strict semantics**, which essentially means that reduction must lead to the same results as via call by name.
- In practice, lazy evaluation (call by need) is mostly used by the Haskell compiler, with semantics-preserving optimisations in many places.

# Church Encodings

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- It is possible to encode an amazing range of datatypes in the Untyped Lambda Calculus.
- Examples are natural numbers, booleans, pairs, lists and sums.

# Church encoding of booleans

- Booleans are encoded as functions that take two arguments.
- If the boolean is **True**, the first argument is returned, if it is **False**, the second is returned.
- `true :=  $\lambda xy.x$`
- `false :=  $\lambda xy.y$`
- With this, we can define `ifThenElse`:  
`ifThenElse :=  $\lambda bxy.bxy$`
- And logical functions like `not`:  
`not :=  $\lambda b.\text{ifThenElse } b \text{ false true}$` .

# Church encoding of natural numbers

- Natural numbers are also encoded as functions taking two arguments, where the result of applying  $f$  and  $x$  to a natural number is applying  $f$   $n$ -times to  $x$ :
- **zero**  $:= \lambda fx.x$ .
- **succ**  $:= \lambda nfx.f(nfx)$ .
- We can define addition: **add**  $:= \lambda mnfx.mf(nfx)$
- and multiplication: **mul**  $:= \lambda mnfx.m(nf)x$
- and test for zero: **isZero**  $:= \lambda n.n(\lambda xzy.y)(\lambda xy.x)$
- and predecessor (more complicated!):  
**pred**  $:= \lambda nfx.n(\lambda gh.h(gf))(\lambda u.x)(\lambda u.u)$ .
- and many more...



# Church encoding of pairs

- Pairs are encoded as functions that take a function of two arguments. The idea is to supply the argument function with the two components of the pair as arguments:
- **pair** :=  $\lambda xyf.fxy$ .
- **fst** :=  $\lambda p.p(\lambda xy.x)$ .
- **snd** :=  $\lambda p.p(\lambda xy.y)$ .

# General Recursion & Y-Combinator

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# Fixpoint operators

- A **fixpoint operator**  $F$  is a lambda expression  $F$  with the property that for all lambda expressions  $g$ , we have  $g(Fg) \sim_{\beta} Fg$ .
- In Haskell, we have the function `fix` in `Control.Monad.Fix`:

```
fix :: (a -> a) -> a
fix f = let x = f x in x
```

- `fix` can be used to implement recursive functions:

```
factorial :: Int -> Int
factorial = fix $ \ f n ->
  if n == 0 then 1 else n * f (n - 1)
```

# The Y-combinator

- Consider the lambda expression  
 $Y := \lambda f. (\lambda x. f(xx)) (\lambda x. f(xx)).$
- We claim that  $Y$  is a fixpoint operator:

$$\begin{aligned} & Yg \\ &= (\lambda f. (\lambda x. f(xx)) (\lambda x. f(xx))) g \\ &\sim_{\beta} (\lambda x. g(xx)) (\lambda x. g(xx)) \\ &\sim_{\beta} g((\lambda x. g(xx)) (\lambda x. g(xx))) \\ &\sim_{\beta} g(Yg) \end{aligned}$$

- Using call by name, we can use  $Y$  to define recursive functions. For call by value, this won't work, but there are (more complicated) fixpoint operators that work for call by value as well.

```
factorial :=  $Y \lambda f n.$ ifThenElse(isZero  $n$ )  
                                (succ zero)  
                                (mul  $n$  ( $f$  (pred  $n$ )))
```