

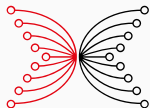
Category Theory

Haskell and Cryptocurrencies

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2018-02-09



INPUT | OUTPUT

Goals

- Categories
- Functors
- Initial- & Final Objects
- Sums & Products
- Natural Transformations

Motivation

Sets

- Let S , T and U be **sets**, and let $f : S \rightarrow T$ and $g : T \rightarrow U$ be **functions**. Then there is a **composition** function $gf = g \circ f : S \rightarrow U$.
- Function composition is **associative**: If $h : U \rightarrow V$ is a third function, then $h(gf) = (hg)f$.
- For each set S , we have the **identity** function $\text{id}_S = 1_S : S \rightarrow S$.
- If $f : S \rightarrow T$ is a function, then $f1_S = f$.
- If $g : R \rightarrow S$ is a function, then $1_Sg = g$.
- So identity functions are **neutral** with respect to composition.
- Two sets X and Y are **isomorphic** if there are functions $f : X \rightarrow Y$ and $g : Y \rightarrow X$ with $fg = 1_X$ and $gf = 1_Y$. Functions f and g are called **isomorphisms** (or bijections). For all intents and purposes, isomorphic sets are “equal”.

Vector spaces

- Let k be a field, let S, T and U be k -vector spaces, and let $f : S \rightarrow T$ and $g : T \rightarrow U$ be k -linear maps. Then the composition $gf = g \circ f : S \rightarrow U$ is also a k -linear map.
- Composition of k -linear maps is **associative**: If $h : U \rightarrow V$ is a third k -linear map, then $h(gf) = (hg)f$.
- For each k -vector space S , the **identity** function $\text{id}_S = 1_S : S \rightarrow S$ is k -linear.
- If $f : S \rightarrow T$ is a k -linear, then $f1_S = f$.
- If $g : R \rightarrow S$ is a k -linear, then $1_Sg = g$.
- So the identity k -linear maps are **neutral** with respect to composition of k -linear maps.
- Two k -vector spaces X and Y are **isomorphic** if there are k -linear maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$ with $fg = 1_X$ and $gf = 1_Y$. The maps f and g are called **isomorphisms**. For all intents and purposes, isomorphic k -vector spaces are “equal”.

Groups

- Let S, T and U be **groups**, and let $f : S \rightarrow T$ and $g : T \rightarrow U$ be **group homomorphisms**. Then the **composition** $gf = g \circ f : S \rightarrow U$ is also a group homomorphism.
- Composition of group homomorphisms is **associative**: If $h : U \rightarrow V$ is a third group homomorphism, then $h(gf) = (hg)f$.
- For each group S , the **identity** function $\text{id}_S = 1_S : S \rightarrow S$ is a group homomorphism.
- If $f : S \rightarrow T$ is a group homomorphism, then $f1_S = f$.
- If $g : R \rightarrow S$ is a group homomorphism, then $1_Sg = g$.
- So the identity group homomorphisms are **neutral** with respect to composition of group homomorphisms.
- Two groups X and Y are **isomorphic** if there are group homomorphisms $f : X \rightarrow Y$ and $g : Y \rightarrow X$ with $fg = 1_X$ and $gf = 1_Y$. The homomorphisms f and g are called (group) **isomorphisms**. For all intents and purposes, isomorphic groups are “equal”.

Topological spaces

- Let S , T and U be **topological spaces**, and let $f : S \rightarrow T$ and $g : T \rightarrow U$ be **continuous**. Then the **composition** $gf = g \circ f : S \rightarrow U$ is also continuous.
- Composition of continuous maps is **associative**: If $h : U \rightarrow V$ is a third continuous map, then $h(gf) = (hg)f$.
- For each topological space S , the **identity** function $\text{id}_S = 1_S : S \rightarrow S$ is continuous.
- If $f : S \rightarrow T$ is a continuous, then $f1_S = f$.
- If $g : R \rightarrow S$ is a continuous, then $1_Sg = g$.
- So the identity continuous maps are **neutral** with respect to composition of continuous maps.
- Two topological spaces X and Y are **isomorphic** if there are continuous maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$ with $fg = 1_X$ and $gf = 1_Y$. The maps f and g are called **isomorphisms** (or homeomorphisms). For all intents and purposes, isomorphic topological spaces are “equal”.

Homotopy

- Let X be a topological space, let $s, t, u \in X$ be points, let $f : s \rightsquigarrow t$ and $g : t \rightsquigarrow u$ be *paths*, and let $[f]$ and $[g]$ be their *homotopy classes*. Then the **composition** $[g][f] = [gf] : s \rightsquigarrow u$ is also a (class of a) path.
- Composition is **associative**: If $h : u \rightsquigarrow v$ is a third path, then $h(gf) \cong (hg)f$.
- For each point $s \in X$, we have the *constant path* $\text{id}_s = 1_s : s \rightsquigarrow s$.
- If $f : s \rightsquigarrow t$ is a path, then $f1_s \cong f$.
- If $g : r \rightarrow s$ is a path, then $1_sg \cong g$.
- So the constant paths are **neutral** with respect to composition of (classes of) paths.
- If $f : s \rightsquigarrow t$ is a path, then there is a path $f^{-1} : t \rightsquigarrow s$, such that $ff^{-1} \cong f^{-1}f \cong 1_s$. For all intent and pupose, two points connected by a path are indistinguishable from the point of view of homotopy theory.

Categories

Category

A **category** \mathcal{C} is given by the following data:

- A class/set $\text{Ob}(\mathcal{C})$ of **objects**.
- For each pair $X, Y \in \text{Ob}(\mathcal{C})$ of objects, a set $\text{Mor}_{\mathcal{C}}(X, Y)$ of **morphisms**. A morphism $f \in \text{Mor}_{\mathcal{C}}(X, Y)$ is often written as $f : X \rightarrow Y$.

- For each triple X, Y, Z of objects a map

$$\circ : \text{Mor}_{\mathcal{C}}(Y, Z) \times \text{Mor}_{\mathcal{C}}(X, Y) \rightarrow \text{Mor}_{\mathcal{C}}(X, Z),$$

called **composition**, which must be **associative** (i.e.

$f(gh) = (fg)h$ for composable morphisms).

- For each object $X \in \text{Ob}(\mathcal{C})$, a morphism $\text{id}_X = 1_X : X \rightarrow X$ in $\text{Mor}_{\mathcal{C}}(X, X)$, the **identity (morphism)** of X , such that the identity morphisms are **neutral** with respect to composition of morphisms (i.e. $f1_X = f$ and $1_Xg = g$ for all suitable f and g).

Isomorphism

- Let \mathcal{C} be a category, let $X, Y \in \text{Ob}(\mathcal{C})$ be objects, and let $f : X \rightarrow Y$ be a morphism.
- f is called an **isomorphism** if there is a morphism $g : Y \rightarrow X$ with $fg = 1_Y$ and $gf = 1_X$.
- X and Y are called **isomorphic** ($X \cong Y$) if there exists an isomorphism $f : X \rightarrow Y$.
- An isomorphism $f : X \rightarrow Y$ is often denoted by $f : X \xrightarrow{\sim} Y$.
- If $f : X \xrightarrow{\sim} Y$ is an isomorphism, then there is *exactly one* $g : Y \rightarrow X$ with $fg = 1_Y$ and $gf = 1_X$:

$$g' = g'1_Y = g'(fg) = (g'f)g = 1_Xg = g.$$

This unique g is called the **inverse** of f and denoted by f^{-1} .

Isomorphic versus equal

- If two objects X and Y of a category are isomorphic, then all *categorical properties* (i.e. properties formulated in the language of category theory) that hold for X also hold for Y and vice versa.
- Therefore isomorphic objects are “as good as equal” – undistinguishable from a categorical point of view.
- Often objects are only known “up to isomorphism”, and that is good enough. Equality does not really make sense in a categorical setting.

Example: Set

- The “mother of all categories” is Set, the **category of sets**.
- Objects in Set are *sets*. (Note that there is no “set of all sets”, hence in general, the objects of a category don’t form a set.)
- Morphisms $\text{Mor}_{\text{Set}}(X, Y)$ are (total) *functions* from X to Y .
- Composition is usual function composition.
- Identities are usual identity functions.

Translating set-theoretical concepts

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$$\forall s, s' \in S : f(s) = f(s') \Rightarrow s = s'.$$

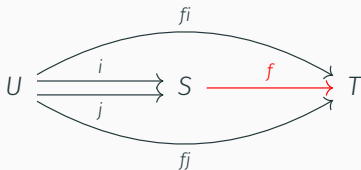
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- However, this is equivalent to the following property, which is formulated purely in terms of categorical notions:

$$\forall U : \text{Ob}(\underline{\text{Set}}) : \forall i, j : \text{Mor}_{\underline{\text{Set}}}(U, S) : fi = fj \Rightarrow i = j.$$



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$$\forall U : \text{Ob}(\underline{\text{Set}}) : \forall i, j : \text{Mor}_{\underline{\text{Set}}}(U, S) : fi = fj \Rightarrow i = j.$$

- This latter definition makes sense in *any* category. A morphism f with this property is called a **monomorphism**.
- Dually (revert all arrows!), surjective maps can be defined in categorical terms and are called **epimorphisms** then.

Example: sets with structure

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- Objects are sets with some extra structure.
- Morphisms are maps that “respect” the structure.

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- Objects are sets with some extra structure.
- Morphisms are maps that “respect” the structure.
- The categories of (Abelian) *groups* with *group homomorphisms* (Ab and Grp).
- For a field k , the category of k -vector spaces and k -linear maps.
- The category of (commutative) *rings* with *ring homomorphisms* Ring.
- The category of *topological spaces* with *continuous maps* Top.
- The category of (real/complex) *manifolds* with *differentiable maps*.
- ...

Example: simplex category

- We define the **simplex category** Δ as follows:
- Objects are non-empty, finite sets $[n] := \{0, 1, \dots, n\}$ for $n \in \mathbb{N}$.
- Morphisms $[m] \rightarrow [n]$ are *monotonically increasing* maps.
- Composition is usual function composition.
- The identity $1_{[n]}$ is the usual identity function $[n] \rightarrow [n]$.

Example: one group

- Let G be a group. We can regard G as a category \underline{G} as follows:
- There is exactly one object, let's call it $*$.
- Morphisms $* \rightarrow *$ are *elements* of G .
- Composition is given by the group operation.
- The (only) identity 1_* is the neutral element.

Example: partially ordered set

- Let (S, \leq) be a *partially ordered set*. We can turn S into a category \underline{S} as follows:
- Objects are the elements of S .
- For $x, y \in S$, we define

$$\text{Mor}(x, y) := \begin{cases} \{*\} & \text{if } x \leq y \\ \emptyset & \text{otherwise} \end{cases}$$

- Composition is given by *transitivity*.
- Identities are given by *reflexivity*.
- Note that $x, y \in S$ are *isomorphic* if $x \leq y \wedge y \leq x$, i.e. iff $x = y$ (by *antisymmetry*).

Example: opposite category

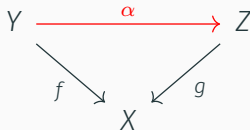
- Let \mathcal{C} be a category, We define the **opposite category** \mathcal{C}^{op} as follows:
- Objects of \mathcal{C}^{op} are just the objects of \mathcal{C} .
- For objects $X, Y \in \text{Ob}(\mathcal{C}^{\text{op}}) = \text{Ob}(\mathcal{C})$, we define $\text{Mor}_{\mathcal{C}^{\text{op}}}(X, Y) := \text{Mor}_{\mathcal{C}}(Y, X)$.
- Composition and identities are given by composition and identities in \mathcal{C} .

Example: product

- Let \mathcal{C} and \mathcal{D} be categories. We define the **product category** $\mathcal{C} \times \mathcal{D}$ as follows:
- Objects of $\mathcal{C} \times \mathcal{D}$ are pairs (X, Y) with $X \in \text{Ob}(\mathcal{C})$ and $Y \in \text{Ob}(\mathcal{D})$.
- For objects $(X, Y), (X', Y') \in \text{Ob}(\mathcal{C} \times \mathcal{D})$, morphisms $(X, Y) \rightarrow (X', Y')$ are pairs of morphisms (f, g) with $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$.
- For an object $(X, Y) \in \text{Ob}(\mathcal{C} \times \mathcal{D})$, $1_{(X, Y)} = (1_X, 1_Y)$, and composition is given componentwise.

Example: slice category

- Let \mathcal{C} be a category, and let $X \in \text{Ob}(\mathcal{C})$ be an object. We define the **slice category** \mathcal{C}/X as follows:
- Objects are morphisms $Y \rightarrow X$ in \mathcal{C} .
- A morphism from $f : Y \rightarrow X$ to $g : Z \rightarrow X$ is a morphism $\alpha : Y \rightarrow Z$ in \mathcal{C} such that $g\alpha = f$.



- Composition and identity are given by composition and identity in \mathcal{C} .

Example: Hask

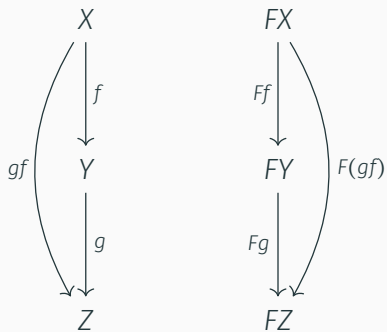
- We define the category Hask of Haskell types and functions as follows:
- Objects are Haskell types (of kind $*$).
- A morphism $f \in \text{Mor}_{\text{Hask}}(a, b)$ is a (total) Haskell function `f :: a -> b`. We identify two such functions `f` and `g` if they “behave” the same, i.e. produce the same output for all inputs: `f x = g x` for all `x`.
- Composition is given by Haskell composition of functions `.`, and identities are given by Haskell’s polymorphic identity function `id`, restricted to the type in question.

Functors

Functor

- Let \mathcal{C} and \mathcal{D} be categories. Then a **Functor** $F : \mathcal{C} \rightarrow \mathcal{D}$ is given by the following data:
- For each object $X \in \text{Ob}(\mathcal{C})$ an object $FX \in \text{Ob}(\mathcal{D})$.
- For each morphism $f : X \rightarrow Y$ in \mathcal{C} , a morphism $Ff : FX \rightarrow FY$ in \mathcal{D} , so that the following conditions hold:
- For each $X \in \text{Ob}(\mathcal{C})$, we have $F1_X = 1_{FX}$, i.e. identities are mapped to identities.
- For objects $X, Y, Z \in \mathcal{C}$ and morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, we have $F(gf) = Fg \circ Ff$ as morphisms in \mathcal{D} , i.e. the functor “respects compositions”.

Functor (contd.)



Example: identity functor

- Let \mathcal{C} be a category. Then the **identity functor** $1_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$ is defined as follows:
- For each object $X \in \text{Ob}(\mathcal{C})$, we have $1_{\mathcal{C}}X = X$.
- For each morphism $f : X \rightarrow Y$, we have $1_{\mathcal{C}}f = f$.
- The functor laws are obviously satisfied!

Example: composition of functors

- Let \mathcal{C} , \mathcal{D} and \mathcal{E} be categories, and let $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{E}$ be functors. Then we define the **composition** $GF : \mathcal{C} \rightarrow \mathcal{E}$ as follows:
- For each object $X \in \text{Ob}(\mathcal{C})$, we have $(GF)X = G(FX)$.
- For each morphism $f : X \rightarrow Y$, we have $(GF)f = G(Ff)$.
- The functor laws for GF follow trivially from the functor laws for F and G .

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- The functor laws for GF follow trivially from the functor laws for F and G .
- We get yet another example for categories, the **category of categories** Cat: Objects are categories, and morphisms are functors!

Example: “Hom”-functors

- Let \mathcal{C} be a category, and let $X \in \text{Ob}(\mathcal{C})$ be an object. Then we have a functor $\text{Mor}_{\mathcal{C}}(X, \cdot) : \mathcal{C} \rightarrow \underline{\text{Set}}$ defined as follows:
- An object $Y \in \text{Ob}(\mathcal{C})$ is sent to the set $\text{Mor}_{\mathcal{C}}(X, Y)$.
- A morphism $f : Y \rightarrow Z$ is sent to the function $\text{Mor}_{\mathcal{C}}(X, Y) \rightarrow \text{Mor}_{\mathcal{C}}(X, Z)$ given by composition with f :
 $g \mapsto fg$.
- Such functors are called “Hom-functors”, because **homomorphism** is often used as a synonym for “morphism”, especially in “algebraic” categories.

Example: forgetful functors

- Let \mathcal{C} be a category of “sets with extra structure”, for example the category Grp of groups.
- Then we have the so-called **forgetful** functor Grp \rightarrow Set which sends a group to its “underlying” set (“forgetting” the group structure) and a group homomorphism to itself, considered as a simple map between sets.
- The functor laws hold trivially.

Example: discrete topology

- According to the previous slide, we have a forgetful functor $\underline{\text{Top}} \rightarrow \underline{\text{Set}}$.
- We also have a functor in the opposite direction $\underline{\text{Set}} \rightarrow \underline{\text{Top}}$:
- This functor sends a set S to S equipped with the *discrete topology* (i.e. every subset is open) and a map $S \rightarrow T$ to itself (which is continuous due to our choice of topology).

Example: polynomial ring

- According to the previous slide, we have a forgetful functor $\underline{\text{Ring}} \rightarrow \underline{\text{Set}}$.
- We also have a functor in the opposite direction $\underline{\text{Set}} \rightarrow \underline{\text{Ring}}$:
- This functor sends a set S to the *polynomial ring* $\mathbb{Z}[S]$ and a map $f : S \rightarrow T$ to the ring homomorphism $\mathbb{Z}[S] \rightarrow \mathbb{Z}[T]$ given by

$$\begin{aligned} (\dots + ns_1s_2 \dots s_k + \dots) \\ \mapsto (\dots + nf(s_1)f(s_2) \dots f(s_k) + \dots) \end{aligned}$$

Example: connected components

- Apart from the forgetful functor, we can define a more interesting functor $\pi_0 : \underline{\mathbf{Top}} \rightarrow \underline{\mathbf{Set}}$ as follows:
- Send a topological space to the set of its *connected components* $\pi_0(X)$.
- Due to the fact that a continuous map $f : X \rightarrow Y$ maps connected subsets to connected subsets, we get an induced map $\pi_0 : \pi_0(X) \rightarrow \pi_0(Y)$.

Example: group homomorphism

- Let $\varphi : G \rightarrow H$ be a group homomorphism, and let \underline{G} and \underline{H} be the categories associated to G and H .
- Then φ induces a functor $\underline{\varphi} : \underline{G} \rightarrow \underline{H}$ by sending $*$ to $*$ and a morphism g (which is just an element of G !) to $\varphi(g)$.
- The functor laws follow immediately from the properties of a group homomorphism.

Example: monotonic maps

- Let $f : (S, \leq) \rightarrow (T, \leq)$ be a monotonic (i.e. order preserving) map between partially ordered sets.
- Then f induces a functor between the associated categories $\underline{f} : \underline{S} \rightarrow \underline{T}$ by sending objects $s \in S$ to $f(s) \in T$.
- Seeing as morphism sets in these categories have at most one element, we have no choice for morphisms.
- The fact that f is monotonic implies that we get a well-defined functor in this way.

Example: Haskell functors

- Let `f :: * -> *` be a *Haskell functor* that obeys the Haskell functor laws.
- Then `f` defines a functor (in the categorical sense) $f : \underline{\text{Hask}} \rightarrow \underline{\text{Hask}}$ by sending a type `a` to `f a` and a Haskell function `g :: a -> b` to `fmap g :: f a -> f b`.
- The Haskell functor laws imply the categorical functor laws!

Functors respect isomorphisms

Lemma

Let \mathcal{C} and \mathcal{D} be categories, let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor, and let $f : X \xrightarrow{\sim} Y$ be an isomorphism in \mathcal{C} . Then $Ff : FX \rightarrow FY$ is an isomorphism in \mathcal{D} .

Proof.

We claim that $F(f^{-1})$ is an/the inverse of Ff :

$$Ff \circ F(f^{-1}) = F(ff^{-1}) = F1_Y = 1_{FY}$$

and

$$Ff^{-1} \circ Ff = F(f^{-1}f) = F1_X = 1_{FX}.$$



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Corollary

Let \mathcal{C} and \mathcal{D} be categories, let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor, and let X and Y be isomorphic objects of \mathcal{C} . Then FX and FY are isomorphic in \mathcal{D} .

Importance of functors

- We have seen examples of functors between categories of quite different branches of mathematics.
- So once we have a functor, we can often transfer problems from one area of mathematics to another.

Example

Are the topological spaces \mathbb{R} and $\mathbb{R} \setminus \{0\}$ homeomorphic? No, because $\pi_0(\mathbb{R}) = \{*\}$, but $\pi_0(\mathbb{R} \setminus \{0\}) = \{-, +\}$. By the previous corollary, the two spaces cannot be isomorphic, because the two sets clearly are not. We have reduced a difficult topological problem to simple counting of elements!

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- So once we have a functor, we can often transfer problems from one area of mathematics to another.

Outlook

This example is only a first glimpse at the power of functors. Using functors to translate geometric problems into algebraic ones has revolutionized 20th century mathematics – with concepts like (co-)homology and (higher) homotopy groups.

Contravariant functors

- Let \mathcal{C} and \mathcal{D} be categories.
- A **contravariant functor** $\mathcal{C} \rightarrow \mathcal{D}$ is a functor $\mathcal{C} \rightarrow \mathcal{D}^{\text{op}}$.
- In concrete terms, that means that a contravariant functor F sends objects $X \in \text{Ob}(\mathcal{C})$ to objects $FX \in \text{Ob}(\mathcal{D})$ and morphisms $f : X \rightarrow Y$ to morphisms $Ff : FY \rightarrow FX$ in \mathcal{D} .
- In case of possible confusion, “normal” functors are also called **covariant** functors.

Example: contravariant “Hom”-functors

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- A special – hopefully familiar – case is given by *dual vector spaces*: If k is a field, then we have a contravariant functor $*$ from k -vector spaces to k -vector spaces, which sends a k -vector space V to its dual $V^* := \text{Hom}_k(V, k)$ and a k -linear map $\varphi : V \rightarrow W$ to the dual map $\varphi^* : W^* \rightarrow V^*$.

Initial- & Final Objects

Initial Object

- Let \mathcal{C} be a category. An object I in \mathcal{C} is called **initial object** if it has the following **universal property**: For all objects X in \mathcal{C} , there is exactly one morphism $i_X : I \rightarrow X$.
- If such an initial object exists, it is *unique up to isomorphism*: Let I' be another initial object. Then

$$i'_I i_I, 1_I : I \rightarrow I \xrightarrow{I \text{ initial}} i'_I i_I = 1_I,$$

and similarly $i_I i'_I = 1_{I'}$, so $i_I : I \xrightarrow{\sim} I'$.

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and similarly $i_{I'} i'_I = 1_{I'}$, so $i_{I'} : I \xrightarrow{\sim} I'$.

Universal Property

Many objects in category theory are defined via a “universal property” like this. If such an object exists, it is always unique up to (unique) isomorphism.

Final Object

- Let \mathcal{C} be a category. An object T in \mathcal{C} is called **final object** or **terminal object** if it is an initial object in \mathcal{C}^{op} .
- Explicitly, T is a final object if it has the *universal property* that for all objects X in \mathcal{C} , there is exactly one morphism $t_X : X \rightarrow T$.
- If such a final object exists, it is *unique up to isomorphism*.

Example: initial- & final object in Set

- The category Set has both an initial and a final object.

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Also note that \emptyset is *not* isomorphic to $\{*\}$.

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Note

Note that in Grp, the initial object is isomorphic to the final object!

Example: final object in Δ

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Example: final object in Δ

- The simplex-category Δ has a final object, but *no initial object*.
- The final object is $[0]$.

Example: initial- & final object in a slice category

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- The final object of \mathcal{C}/X is $1_X : X \rightarrow X$.

Example: initial- & final object in Hask

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- The category Hask has an initial and a final object.
- The initial object is `Void`.

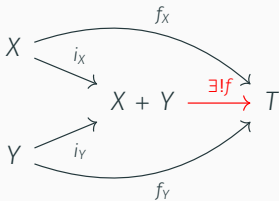
Example: initial- & final object in Hask

- The category Hask has an initial and a final object.
- The initial object is `Void`.
- The final object is `()`.

Sums & Products

Sum

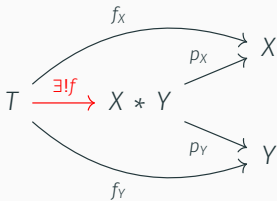
- Let \mathcal{C} be a category, and let $X, Y \in \text{Ob}(\mathcal{C})$ be objects. We say that X and Y have a **sum** (or **coproduct**) if there is an object $X + Y$ and morphisms $i_X : X \rightarrow X + Y$ and $i_Y : Y \rightarrow X + Y$ with the following universal property: For all objects T and morphisms $f_X : X \rightarrow T$ and $f_Y : Y \rightarrow T$, there is exactly one morphism $f : X + Y \rightarrow T$ such that the following diagram **commutes** ($f i_X = f_X \wedge f i_Y = f_Y$):



- If all pairs of objects in \mathcal{C} have a sum, we say that \mathcal{C} **has (finite) sums**.

Product

- Let \mathcal{C} be a category, and let $X, Y \in \text{Ob}(\mathcal{C})$ be objects. We say that X and Y have a **product** if there is an object $X * Y$ and morphisms $p_X : X * Y \rightarrow X$ and $p_Y : X * Y \rightarrow Y$ with the following universal property: For all objects T and morphisms $f_X : T \rightarrow X$ and $f_Y : T \rightarrow Y$, there is exactly one morphism $f : T \rightarrow X * Y$ such that the following diagram **commutes** ($p_X f = f_X \wedge p_Y f = f_Y$):



- If all pairs of objects in \mathcal{C} have a product, we say that \mathcal{C} **has (finite) products**.

Example: sums and products in Set

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Note

Note that for most pairs of sets S and T , $S \uplus T$ and $S \times T$ are *not* isomorphic.

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Example: sums & products in $\underline{\mathbf{Ab}}$

- The category $\underline{\mathbf{Ab}}$ of abelian groups has both sums and products.
- The sum of two groups A and B is the *direct sum* $A \oplus B$ (with injections $a \mapsto (a, 0)$ and $b \mapsto (0, b)$).
- The product of two groups A and B is *also* $A \oplus B$ (with projections $(a, b) \mapsto a$ and $(a, b) \mapsto b$).

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- The product of types `a` and `b` is `(a, b)` (with projections `fst` and `snd`).

Natural Transformations

Natural transformation

- Let $F, G : \mathcal{C} \rightarrow \mathcal{D}$ be functors. A **natural transformation** $\varphi : F \rightarrow G$ is given by the following data:
- For each object $X \in \text{Ob}(\mathcal{C})$, a morphism $\varphi_X : FX \rightarrow GX$ in \mathcal{D} .
- For each morphism $f : X \rightarrow Y$ in \mathcal{C} , the following diagram must commute (i.e. $\varphi_Y Ff = Gf \varphi_X$):

$$\begin{array}{ccccc} X & & FX & \xrightarrow{\varphi_X} & GX \\ f \downarrow & & Ff \downarrow & & \downarrow Gf \\ Y & & FY & \xrightarrow{\varphi_Y} & GY \end{array}$$

Example: Hom-functors

Let \mathcal{C} be a category, and let $g : U \rightarrow V$ be a morphism. Then g induces a natural transformation $f^* : \text{Mor}_{\mathcal{C}}(V, \cdot) \rightarrow \text{Mor}_{\mathcal{C}}(U, \cdot)$, given by composition with g :

$$\begin{array}{ccccc} X & \text{Mor}_{\mathcal{C}}(V, X) & \xrightarrow{f_X^*} & \text{Mor}_{\mathcal{C}}(U, X) & \\ \textcolor{blue}{f} \downarrow \textcolor{blue}{\Downarrow} & \text{Mor}_{\mathcal{C}}(V, \cdot) f \downarrow & & \downarrow \text{Mor}_{\mathcal{C}}(U, \cdot) f & \\ Y & \text{Mor}_{\mathcal{C}}(V, Y) & \xrightarrow{f_Y^*} & \text{Mor}_{\mathcal{C}}(U, Y) & \end{array}$$

Example: determinant

- Let n be a natural number. Sending a commutative ring R to the group $\mathrm{GL}_n(R)$ of invertible $n \times n$ -matrices defines a functor $\mathrm{GL}_n : \underline{\mathrm{Ring}} \rightarrow \underline{\mathrm{Grp}}$ (for a ring homomorphism $\alpha : R \rightarrow S$, we get an induced group homomorphism $\mathrm{GL}_n(\alpha) : \mathrm{GL}_n(R) \rightarrow \mathrm{GL}_n(S)$ by applying α to each matrix element).
- We get another functor $*$: $\underline{\mathrm{Ring}} \rightarrow \underline{\mathrm{Grp}}$ by sending a ring R to its *group of units* (i.e. invertible elements) R^* .
- Then the *determinant* $\det : \mathrm{GL}_n(R) \rightarrow R^*$ defines a natural transformation $\det : \mathrm{GL}_n \rightarrow *$:

$$\begin{array}{ccccc} R & & \mathrm{GL}_n(R) & \xrightarrow{\det} & R^* \\ \alpha \downarrow & & \mathrm{GL}_n(\alpha) \downarrow & & \downarrow \alpha|_{R^*} \\ S & & \mathrm{GL}_n(S) & \xrightarrow{\det} & S^* \end{array}$$

Example: polymorphic Haskell functions

Let `F` and `G` be Haskell functors, and let `g :: F a -> G a` be a polymorphic function. Then `g` defines a natural transformation $F \rightarrow G$ in Hask by type specialization. For example, consider `F = Maybe`, `G = []` and

```
g :: Maybe a -> [a]
g Nothing = []
g (Just x) = [x]
```

```
GHCi> g $ fmap show $ Just True
["True"]
GHCi> fmap show $ g $ Just True
["True"]
```

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