01

Lista de Exercícios

Introdução a Física de Hádrons

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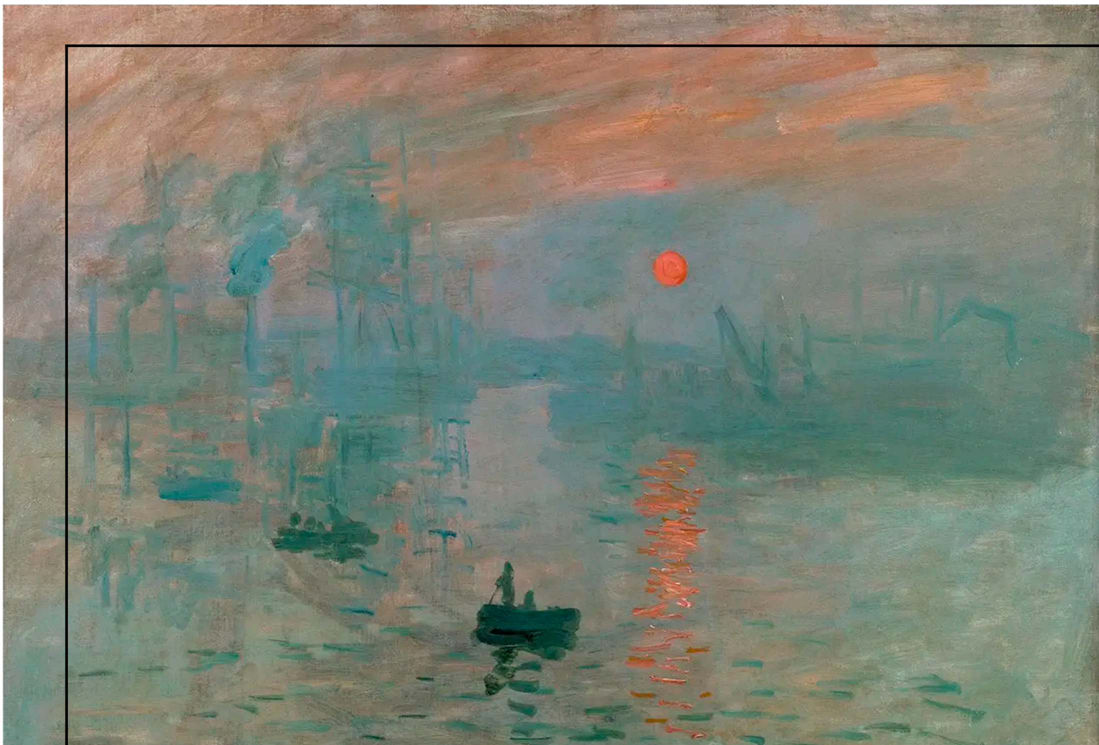

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Q. 01 Campo de Klein-Gordon

A (densidade de) lagrangiana livre de uma partícula escalar neutra de massa m é dada por

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu \phi_0 \partial^\mu \phi_0 - \frac{1}{2}m^2\phi_0^2$$

Usando o truque

$$\partial_\mu \phi_0 \partial^\mu \phi_0 = g^{\alpha\beta}\partial_\alpha \phi_0 \partial_\beta \phi_0$$

e o princípio da mínima ação, $\delta S = 0$, mostre que o campo livre ϕ_0 satisfaz a equação de Klein-Gordon.

Para determinar a ação, integramos a densidade de lagrangiana em d^4x, de modo que

$$S = \int \mathcal{L}_0 \, \mathrm{d}^4x$$

Fazendo uma variação na ação δS, como a densidade de lagrangiana $\mathcal{L}_0 \equiv \mathcal{L}_0(\phi_0, \partial_\mu \phi_0)$, a variação fica

$$\delta S = \int \left[\frac{\delta \mathcal{L}_0}{\delta \phi_0}\delta\phi_0 + \frac{\delta \mathcal{L}_0}{\delta(\partial_\mu \phi_0)}\delta(\partial_\mu \phi_0)\right] \mathrm{d}^4x$$

Note que

$$\partial_\mu \left[\frac{\delta \mathcal{L}_0}{\delta(\partial_\mu \phi_0)}\delta\phi_0\right] = \partial_\mu \left[\frac{\delta \mathcal{L}_0}{\delta(\partial_\mu \phi_0)}\right]\delta\phi_0 + \frac{\delta \mathcal{L}_0}{\delta(\partial_\mu \phi_0)}\partial_\mu(\delta\phi_0)$$

Como uma variação δ pode permutar com uma derivada parcial ∂_μ, temos

$$\partial_\mu \left[\frac{\delta \mathcal{L}_0}{\delta(\partial_\mu \phi_0)}\delta\phi_0\right] = \partial_\mu \left[\frac{\delta \mathcal{L}_0}{\delta(\partial_\mu \phi_0)}\right]\delta\phi_0 + \frac{\delta \mathcal{L}_0}{\delta(\partial_\mu \phi_0)}\delta(\partial_\mu\phi_0)$$

Portanto podemos substituir a variação da ação por

$$\begin{aligned}\delta S &= \int \left\{\frac{\delta \mathcal{L}_0}{\delta \phi_0}\delta\phi_0 - \partial_\mu \left[\frac{\delta \mathcal{L}_0}{\delta(\partial_\mu \phi_0)}\right]\delta\phi_0 + \partial_\mu \left[\frac{\delta \mathcal{L}_0}{\delta(\partial_\mu \phi_0)}\delta\phi_0\right]\right\} \mathrm{d}^4x \\ &= \int \left\{\frac{\delta \mathcal{L}_0}{\delta \phi_0} - \partial_\mu \left[\frac{\delta \mathcal{L}_0}{\delta(\partial_\mu \phi_0)}\right]\right\}\delta\phi_0 \, \mathrm{d}^4x + \int \partial_\mu \left[\frac{\delta \mathcal{L}_0}{\delta(\partial_\mu \phi_0)}\delta\phi_0\right] \mathrm{d}^4x\end{aligned}$$

O último termo desta expressão é uma derivada total em todo o espaço-tempo, de modo que ao impormos que nos limites assintóticos o campo desaparece, concluímos que a integração vai dar zero, restando apenas

$$\delta S = \int \left\{ \frac{\delta \mathcal{L}_0}{\delta \phi_0} - \partial_\mu \left[\frac{\delta \mathcal{L}_0}{\delta(\partial_\mu \phi_0)} \right] \right\} \delta\phi_0 \, \mathrm{d}^4 x$$

O princípio de mínima ação $\delta S = 0$ fornece

$$\delta S = \int \left\{ \frac{\delta \mathcal{L}_0}{\delta \phi_0} - \partial_\mu \left[\frac{\delta \mathcal{L}_0}{\delta(\partial_\mu \phi_0)} \right] \right\} \delta\phi_0 \, \mathrm{d}^4 x = 0$$

que deve ser satisfeito para qualquer variação $\delta\phi_0$ do campo, logo impomos que o argumento dentro das chaves $\{\cdots\}$ é igual a zero, gerando a equação de Euler-Lagrange:

$$\frac{\delta \mathcal{L}_0}{\delta \phi_0} - \partial_\mu \left[\frac{\delta \mathcal{L}_0}{\delta(\partial_\mu \phi_0)} \right] = 0$$

Calculando o primeiro termo com base na densidade de lagrangiana, temos

$$\frac{\delta \mathcal{L}_0}{\delta \phi_0} = \frac{\delta}{\delta \phi_0} \left(-\frac{1}{2} m^2 \phi_0^2 \right) = -\frac{1}{2} m^2 (2\phi_0) = -m^2 \phi_0$$

em que apenas o segundo termo da lagrangiana vai ser relevante, pois o primeiro depende apenas das derivadas do campo. Para o segundo termo da equação de Euler-Lagrange, apenas o primeiro termo da lagrangiana vai ser relevante, de modo que

$$\begin{aligned}\partial_\mu \left[\frac{\delta \mathcal{L}_0}{\delta(\partial_\mu \phi_0)} \right] &= \partial_\mu \left[\frac{\delta}{\delta(\partial_\mu \phi_0)} \left(\frac{1}{2} \partial_\mu \phi_0 \partial^\mu \phi_0 \right) \right] = \partial_\mu \left[\frac{1}{2} \frac{\delta}{\delta(\partial_\mu \phi_0)} \left(g^{\alpha\beta} \partial_\alpha \phi_0 \partial_\beta \phi_0 \right) \right] \\ &= \partial_\mu \left\{ \frac{1}{2} g^{\alpha\beta} \left[\frac{\delta(\partial_\alpha \phi_0)}{\delta(\partial_\mu \phi_0)} \partial_\beta \phi_0 + \partial_\alpha \phi_0 \frac{\delta(\partial_\beta \phi_0)}{\delta(\partial_\mu \phi_0)} \right] \right\}\end{aligned}$$

Sabendo então que

$$\frac{\delta(\partial_\alpha \phi_0)}{\delta(\partial_\mu \phi_0)} = \delta^\mu_\alpha \qquad \& \qquad \frac{\delta(\partial_\beta \phi_0)}{\delta(\partial_\mu \phi_0)} = \delta^\mu_\beta$$

obtemos

$$\begin{aligned}\partial_\mu \left[\frac{\delta \mathcal{L}_0}{\delta(\partial_\mu \phi_0)} \right] &= \partial_\mu \left[\frac{1}{2} g^{\alpha\beta} \left(\delta^\mu_\alpha \partial_\beta \phi_0 + \delta^\mu_\beta \partial_\alpha \phi_0 \right) \right] = \partial_\mu \left[\frac{1}{2} \left(g^{\mu\beta} \partial_\beta \phi_0 + g^{\alpha\mu} \partial_\alpha \phi_0 \right) \right] \\ &= \partial_\mu \left[\frac{1}{2} (\partial^\mu \phi_0 + \partial^\mu \phi_0) \right] = \partial_\mu \partial^\mu \phi_0\end{aligned}$$

Juntando então os resultados na equação de Euler-Lagrange:

$$-m^2 \phi_0 - \partial_\mu \partial^\mu \phi_0 = 0$$

Concluindo que o campo livre ϕ_0 satisfaz a equação de Klein-Gordon

$$(\partial_\mu \partial^\mu + m^2)\phi_0 = 0 \tag{1.1}$$

Q. 02 Operadores de criação e aniquilação de um campo livre

Em termos de $f_k(x) = \exp(-ikx)/\sqrt{2\omega_{\mathbf{k}}}$ em que $\omega_{\mathbf{k}} = \sqrt{|\mathbf{k}|^2 + m^2}$, a quantização de um campo escalar neutro pode ser expressa em termos dos operadores de criação e aniquilação via

$$\phi_0(x) = \int \frac{1}{(2\pi)^3}\left[a(\mathbf{k})f_k(x) + a^\dagger(\mathbf{k})f_k^*(x)\right] \mathrm{d}^3k$$

(a) Partindo da equação acima, obtenha as relações

$$a_0(\mathbf{k}) = i\int f_k^*(x)\overleftrightarrow{\partial_0}\phi_0(x)\,\mathrm{d}^3x \qquad \& \qquad a_0^\dagger(\mathbf{k}) = -i\int f_k(x)\overleftrightarrow{\partial_0}\phi_0(x)\,\mathrm{d}^3x$$

(b) Demonstre que tanto $a_0(\mathbf{k})$ quanto $a_0^\dagger(\mathbf{k})$ são independentes do tempo.

Antes de determinar os operadores de criação e aniquilação, podemos calcular o momento canonicamente conjugado $\pi_0(x)$ do campo livre $\phi_0(x)$:

$$\pi_0 = \frac{\delta\mathcal{L}_0}{\delta\dot{\phi_0}} = \frac{\delta}{\delta\dot{\phi_0}}\left(\frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0 - \frac{1}{2}m^2\phi_0^2\right) = \frac{1}{2}\frac{\delta}{\delta\dot{\phi_0}}\left[\dot{\phi}_0^2 - \nabla\phi_0\cdot\nabla\phi_0 - m^2\phi_0^2\right] = \dot{\phi}_0$$

Portanto

$$\pi_0(x) = \int \frac{1}{(2\pi)^3}\left[a_0(\mathbf{k})\partial_0 f_k(x) + a_0^\dagger(\mathbf{k})\partial_0 f_k^*(x)\right] \mathrm{d}^3k$$

onde

$$\frac{\partial f_k(x)}{\partial t} = \sqrt{2\omega_{\mathbf{k}}}\frac{\partial}{\partial t}\left(e^{-i\omega_{\mathbf{k}}t}\right)e^{i\mathbf{k}\cdot\mathbf{x}} = -i\omega_{\mathbf{k}}\sqrt{2\omega_{\mathbf{k}}}e^{-i\omega_{\mathbf{k}}t+i\mathbf{k}\cdot\mathbf{x}} = -i\omega_{\mathbf{k}}f_k(x) \tag{2.1}$$

$$\frac{\partial f_k^*(x)}{\partial t} = \sqrt{2\omega_{\mathbf{k}}}\frac{\partial}{\partial t}\left(e^{+i\omega_{\mathbf{k}}t}\right)e^{-i\mathbf{k}\cdot\mathbf{x}} = +i\omega_{\mathbf{k}}\sqrt{2\omega_{\mathbf{k}}}e^{i\omega_{\mathbf{k}}t-i\mathbf{k}\cdot\mathbf{x}} = +i\omega_{\mathbf{k}}f_k^*(x) \tag{2.2}$$

Concluindo que

$$\pi_0(x) = \int \frac{i\omega_{\mathbf{k}}}{(2\pi)^3}\left[-a_0(\mathbf{k})f_k(x) + a_0^\dagger(\mathbf{k})f_k^*(x)\right] \mathrm{d}^3k$$

(a) Para determinar os operadores, podemos pensar em escrevê-los em função dos campos livres $\phi_0(x)$ e $\pi_0(x)$, onde para isso fazemos uma transformada de Fourier inversa no espaço nestes campos. No caso do campo $\phi_0(x)$, temos

$$\int \phi_0(x)e^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathrm{d}^3x = \int\left\{\int \frac{1}{(2\pi)^3}\left[a_0(\mathbf{k}')f_{k'}(x) + a_0^\dagger(\mathbf{k}')f_{k'}^*(x)\right]\mathrm{d}^3k'\right\}e^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathrm{d}^3x$$

Expandindo $f_k(x)$, a expressão fica

$$\begin{aligned}\int \phi_0(x)e^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathrm{d}^3x &= \iint \frac{1}{(2\pi)^3\sqrt{2\omega_{\mathbf{k}'}}}\Big[a_0(\mathbf{k}')e^{-i\omega_{\mathbf{k}'}t+i\mathbf{k}'\cdot\mathbf{x}} + a_0^\dagger(\mathbf{k}')e^{i\omega_{\mathbf{k}'}t-i\mathbf{k}\cdot\mathbf{x}}\Big]e^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathrm{d}^3k'\,\mathrm{d}^3x \\ &= \iint \frac{e^{-i\omega_{\mathbf{k}'}t}}{(2\pi)^3\sqrt{2\omega_{\mathbf{k}'}}}\Big[a_0(\mathbf{k}')e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}} + a_0^\dagger(\mathbf{k}')e^{2i\omega_{\mathbf{k}'}t}e^{-i(\mathbf{k}'+\mathbf{k})\cdot\mathbf{x}}\Big]\,\mathrm{d}^3x\,\mathrm{d}^3k'\end{aligned}$$

Podemos separar as integrais em d^3x, de modo que

$$\int \phi_0(x)e^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathrm{d}^3x = \int \frac{e^{-i\omega_{\mathbf{k}'}t}}{(2\pi)^3\sqrt{2\omega_{\mathbf{k}'}}}\Big[a_0(\mathbf{k}')\int e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}}\,\mathrm{d}^3x + a_0^\dagger(\mathbf{k}')e^{2i\omega_{\mathbf{k}'}t}\int e^{-i(\mathbf{k}'+\mathbf{k})\cdot\mathbf{x}}\,\mathrm{d}^3x\Big]\,\mathrm{d}^3k'$$

Note então que as integrais em d^3x são, juntamente com o fator $1/(2\pi)^3$, distribuições delta de Dirac em 3 dimensões, tal que

$$\begin{aligned}\frac{1}{(2\pi)^3}\int e^{\pm i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}}\,\mathrm{d}^3x &= \delta^3(\mathbf{k}'-\mathbf{k}) \\ \frac{1}{(2\pi)^3}\int e^{\pm i(\mathbf{k}'+\mathbf{k})\cdot\mathbf{x}}\,\mathrm{d}^3x &= \delta^3(\mathbf{k}'+\mathbf{k})\end{aligned}$$

Sendo assim, obtemos

$$\int \phi_0(x)e^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathrm{d}^3x = \int \frac{e^{-i\omega_{\mathbf{k}'}t}}{\sqrt{2\omega_{\mathbf{k}'}}}\Big[a_0(\mathbf{k}')\delta^3(\mathbf{k}'-\mathbf{k}) + a_0^\dagger(\mathbf{k}')e^{2i\omega_{\mathbf{k}'}t}\delta^3(\mathbf{k}'+\mathbf{k})\Big]\,\mathrm{d}^3k'$$

Podemos notar também que $\omega_{-\mathbf{k}} = \sqrt{|-\mathbf{k}|^2+m^2} = \omega_{\mathbf{k}}$, logo ao performar a integral obtemos

$$\begin{aligned}\int \phi_0(x)e^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathrm{d}^3x &= \frac{e^{-i\omega_{\mathbf{k}}t}}{\sqrt{2\omega_{\mathbf{k}}}}\Big[a_0(\mathbf{k}) + a_0^\dagger(-\mathbf{k})e^{2i\omega_{-\mathbf{k}}t}\Big] \\ &= \frac{e^{-i\omega_{\mathbf{k}}t}}{\sqrt{2\omega_{\mathbf{k}}}}\Big[a_0(\mathbf{k}) + a_0^\dagger(\mathbf{k})e^{2i\omega_{\mathbf{k}}t}\Big]\end{aligned}$$

No caso do momento conjugado, temos

$$\begin{aligned}\int \pi_0(x)e^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathrm{d}^3x &= \int\left\{\int \frac{i\omega_{\mathbf{k}'}}{(2\pi)^3}\Big[-a_0(\mathbf{k}')f_k(x) + a_0^\dagger(\mathbf{k}')f_{k'}^*(x)\Big]\,\mathrm{d}^3k'\right\}e^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathrm{d}^3x \\ &= \iint \frac{i\omega_{\mathbf{k}'}e^{-i\omega_{\mathbf{k}'}t}}{(2\pi)^3\sqrt{2\omega_{\mathbf{k}'}}}\Big[-a_0(\mathbf{k}')e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}} + a_0^\dagger(\mathbf{k}')e^{2i\omega_{\mathbf{k}'}t}e^{-i(\mathbf{k}'+\mathbf{k})\cdot\mathbf{x}}\Big]\,\mathrm{d}^3x\,\mathrm{d}^3k' \\ &= -i\int\sqrt{\frac{\omega_{\mathbf{k}'}}{2}}e^{-i\omega_{\mathbf{k}'}t}\Big[a_0(\mathbf{k}')\delta^3(\mathbf{k}'-\mathbf{k}) - a_0^\dagger(\mathbf{k}')e^{2i\omega_{\mathbf{k}'}t}\delta^3(\mathbf{k}'+\mathbf{k})\Big]\,\mathrm{d}^3k' \\ &= -i\sqrt{\frac{\omega_{\mathbf{k}}}{2}}e^{-i\omega_{\mathbf{k}}t}\Big[a_0(\mathbf{k}) - a_0^\dagger(-\mathbf{k})e^{2i\omega_{-\mathbf{k}}t}\Big] \\ &= -i\sqrt{\frac{\omega_{\mathbf{k}}}{2}}e^{-i\omega_{\mathbf{k}}t}\Big[a_0(\mathbf{k}) - a_0^\dagger(-\mathbf{k})e^{2i\omega_{\mathbf{k}}t}\Big]\end{aligned}$$

Com estes resultados, podemos obter as formas

$$\sqrt{2\omega_{\mathbf{k}}}e^{i\omega_{\mathbf{k}}t}\int \phi_0(x)e^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathrm{d}^3x = a_0(\mathbf{k}) + a_0^\dagger(-\mathbf{k})e^{2i\omega_{\mathbf{k}}t}$$

$$i\sqrt{\frac{2}{\omega_{\mathbf{k}}}}e^{i\omega_{\mathbf{k}}t}\int \pi_0(x)e^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathrm{d}^3x = a_0(\mathbf{k}) - a_0^\dagger(-\mathbf{k})e^{2i\omega_{\mathbf{k}}t}$$

A exponencial $e^{i\omega_{\mathbf{k}}t}$ pode se juntar à $e^{-i\mathbf{k}\cdot\mathbf{x}}$ sob a forma e^{ikx}, pois a integral está sendo feita apenas em d^3x, de modo que as equações ficam

$$\sqrt{2\omega_{\mathbf{k}}}\int \phi_0(x)e^{ikx}\,\mathrm{d}^3x = a_0(\mathbf{k}) + a_0^\dagger(-\mathbf{k})e^{2i\omega_{\mathbf{k}}t}$$

$$i\sqrt{\frac{2}{\omega_{\mathbf{k}}}}\int \pi_0(x)e^{ikx}\,\mathrm{d}^3x = a_0(\mathbf{k}) - a_0^\dagger(-\mathbf{k})e^{2i\omega_{\mathbf{k}}t}$$

Somando as duas equações:

$$2a_0(\mathbf{k}) = i\sqrt{\frac{2}{\omega_{\mathbf{k}}}}\int \pi_0(x)e^{ikx}\,\mathrm{d}^3x + \sqrt{2\omega_{\mathbf{k}}}\int \phi_0(x)e^{ikx}\,\mathrm{d}^3x$$

Temos então

$$\begin{aligned}
a_0(\mathbf{k}) &= \frac{i}{\sqrt{2\omega_{\mathbf{k}}}}\int \pi_0(x)e^{ikx}\,\mathrm{d}^3x + \sqrt{\frac{\omega_{\mathbf{k}}}{2}}\int \phi_0(x)e^{ikx}\,\mathrm{d}^3x \\
&= \frac{i}{\sqrt{2\omega_{\mathbf{k}}}}\int \partial_0\phi_0(x)e^{ikx}\,\mathrm{d}^3x + \frac{\omega_{\mathbf{k}}}{\sqrt{2\omega_{\mathbf{k}}}}\int \phi_0(x)e^{ikx}\,\mathrm{d}^3x \\
&= i\int \partial_0\phi_0(x)f_k^*(x)\,\mathrm{d}^3x + \omega_{\mathbf{k}}\int \phi_0(x)f_k^*(x)\,\mathrm{d}^3x
\end{aligned}$$

Usando a (2.2),

$$-i\partial_0 f_k^*(x) = \omega_{\mathbf{k}} f_k^*(x)$$

O que modifica a forma do operador na forma

$$\begin{aligned}
a_0(\mathbf{k}) &= i\int \partial_0\phi_0(x)f_k^*(x)\,\mathrm{d}^3x - i\int \phi_0(x)\partial_0 f_k^*(x)\,\mathrm{d}^3x \\
&= i\int \Big[f_k^*(x)\partial_0\phi_0(x) - \partial_0 f_k^*(x)\phi_0(x)\Big]\,\mathrm{d}^3x
\end{aligned}$$

Pela notação

$$A\overleftrightarrow{\partial_0}B = A(\partial_0 B) - (\partial_0 A)B$$

Podemos concluir que o operador de aniquilação $a_0(\mathbf{k})$ admite ser escrito sob a forma

$$a_0(\mathbf{k}) = i\int f_k^*(x)\overleftrightarrow{\partial_0}\phi_0(x)\,\mathrm{d}^3x \tag{2.3}$$

Como estamos considerando um campo $\phi_0(x)$ real, temos $\phi_0^\dagger(x) = \phi_0(x)$, portanto ao calcular $a_0^\dagger(\mathbf{k})$, trocaremos $i \mapsto -i$ e $f_k^*(x) \mapsto f_k(x)$, concluindo que

$$a_0^\dagger(\mathbf{k}) = -i\int f_k(x)\overleftrightarrow{\partial_0}\phi(x)\,\mathrm{d}^3x \tag{2.4}$$

(b) Calculando a derivada de $a_0(\mathbf{k})$ em relação ao tempo, temos que como a integral é feita apenas nas coordenadas espaciais, a derivação pode comutar com a integral, de modo que

$$\begin{aligned}
\partial_0 a_0(\mathbf{k}) &= i\int \partial_0\left[f_k^*(x)\overleftrightarrow{\partial_0}\phi_0(x)\right]\mathrm{d}^3x \\
&= i\int \partial_0\left[f_k^*(x)\partial_0\phi_0(x) - \partial_0 f_k^*(x)\phi_0(x)\right]\mathrm{d}^3x \\
&= i\int \left[\partial_0 f_k^*(x)\partial_0\phi_0(x) + f_k^*(x)\partial_0^2\phi_0(x) - \partial_0^2 f_k^*(x)\phi_0(x) - \partial_0 f_k^*(x)\partial_0\phi_0(x)\right]\mathrm{d}^3x \\
&= i\int \left[f_k^*(x)\partial_0^2\phi_0(x) - \partial_0^2 f_k^*(x)\phi_0(x)\right]\mathrm{d}^3x
\end{aligned}$$

A derivada temporal de segunda ordem em f_k^* explicitamente fica

$$\partial_0^2 f_k^*(x) = \frac{1}{\sqrt{2\omega_\mathbf{k}}}\partial_0^2 e^{ikx} = \frac{1}{\sqrt{2\omega_\mathbf{k}}}(i\omega_\mathbf{k})^2 e^{ikx} = -\omega_\mathbf{k}^2 f_k^*(x) \tag{2.5}$$

ou seja

$$\partial_0 a_0(\mathbf{k}) = i\int f_k^*(x)\left(\partial_0^2 + \omega_\mathbf{k}^2\right)\phi_0(x)\,\mathrm{d}^3x$$

Como $\omega_\mathbf{k}^2 = |\mathbf{k}|^2 + m^2$, reescrevemos

$$\partial_0 a_0(\mathbf{k}) = i\int f_k^*(x)\left(\partial_0^2 + |\mathbf{k}|^2 + m^2\right)\phi_0(x)\,\mathrm{d}^3x$$

Note que

$$\begin{aligned}
\nabla\cdot\left[f_k^*(x)\overleftrightarrow{\nabla}\phi_0(x)\right] &= \nabla\cdot\left[f_k^*(x)\nabla\phi_0(x) - \nabla f_k^*(x)\phi_0(x)\right] \\
&= \nabla f_k^*(x)\cdot\nabla\phi_0(x) + f_k^*(x)\nabla^2\phi_0(x) - \nabla^2 f_k^*(x)\phi_0(x) - \nabla f_k^*(x)\cdot\nabla\phi_0(x) \\
&= f_k^*(x)\nabla^2\phi_0(x) - \nabla^2 f_k^*(x)\phi_0(x)
\end{aligned}$$

onde

$$\nabla^2 f_k^*(x) = \frac{1}{\sqrt{2\omega_\mathbf{k}}}\nabla^2 e^{ikx} = \frac{1}{\sqrt{2\omega_\mathbf{k}}}(-i\mathbf{k})\cdot(-i\mathbf{k})e^{ikx} = -|\mathbf{k}|^2 f_k^*(x)$$

Portanto

$$\nabla \cdot \left[f_k^*(x) \overleftrightarrow{\nabla} \phi_0(x) \right] = f_k^*(x) \nabla^2 \phi_0(x) + |\mathbf{k}|^2 f_k^*(x) \phi_0(x)$$

Como o lado esquerdo desta equação é uma derivada total no espaço, temos que dentro de uma integral em d^3x esse termo vai pra zero pela hipótese do campo desaparecer nos limites asssintóticos, de modo que podemos fazer a substituição

$$|\mathbf{k}|^2 f_k^*(x) \phi_0(x) \mapsto -f_k^*(x) \nabla^2 \phi_0(x)$$

dentro da integral em $\partial_0 a_0(\mathbf{k})$, logo

$$\begin{aligned} \partial_0 a_0(\mathbf{k}) &= i \int f_k^*(x) \big(\partial_0^2 - \nabla^2 + m^2 \big) \phi_0(x) \, \mathrm{d}^3x \\ &= i \int f_k^*(x) \big(\partial_\mu \partial^\mu + m^2 \big) \phi_0(x) \, \mathrm{d}^3x \end{aligned}$$

E como $\phi_0(x)$ é um campo livre, equação de Klein-Gordon é satisfeita, $(\partial_\mu \partial^\mu + m^2)\phi_0(x) = 0$, concluindo que

$$\partial_0 a_0(\mathbf{k}) = 0 \Rightarrow a_0(\mathbf{k}) \text{ é independente do tempo} \tag{2.6}$$

O raciocínio para $a_0^\dagger(\mathbf{k})$ é idêntico, de modo que a alteração pode ser visualizada por

$$\begin{aligned} \partial_0 a_0(\mathbf{k}) &\mapsto \partial_0 a_0^\dagger(\mathbf{k}) \\ i \int f_k^*(x) \big(\partial_\mu \partial^\mu + m^2 \big) \phi_0(x) \, \mathrm{d}^3x &\mapsto -i \int f_k(x) \big(\partial_\mu \partial^\mu + m^2 \big) \phi_0(x) \, \mathrm{d}^3x \end{aligned}$$

Portanto

$$\partial_0 a_0^\dagger(\mathbf{k}) = 0 \Rightarrow a_0^\dagger(\mathbf{k}) \text{ é independente do tempo} \tag{2.7}$$

Q. 03 Propagador de um campo livre

O propagador de uma partícula escalar neutra pode ser obtida via a seguinte receita:

$$\mathcal{L}_0 \Rightarrow \frac{1}{2}\phi_0 M \phi_0 + \partial_\mu[\cdots], \qquad \tilde{\Delta}(k) \overset{\partial_\mu \to -ik_\mu}{=} [M]^{-1}$$

Mostre que:

(a) $\tilde{\Delta}(k) = 1/(k^2 - m^2 + i\varepsilon)$, em que ε é um parâmetro infinitesimal que previne a singularidade do propagador em $k^2 = m^2$.

(b) No espaço de coordenadas o propagador é escrito como

$$i\Delta(x-y) = i\int \frac{1}{(2\pi)^4}\frac{e^{-ik(x-y)}}{k^2 - m^2 + i\varepsilon}\,\mathrm{d}^4k$$

Mostre que a integral abaixo define a função degrau de Heaviside

$$\int_{-\infty}^{\infty} \frac{1}{2\pi}\frac{e^{-ik_0 z_0}}{k_0^2 - \omega_{\mathbf{k}}^2 + i\varepsilon}\,\mathrm{d}k_0 = -i\Theta(z_0)\frac{e^{-i\omega_{\mathbf{k}} z_0}}{2\omega_{\mathbf{k}}} - i\Theta(-z_0)\frac{e^{i\omega_{\mathbf{k}} z_0}}{2\omega_{\mathbf{k}}}$$

(c) A partir do resultado acima, obtenha a relação

$$i\Delta(x-y) = \int \frac{1}{(2\pi)^3}\left[\Theta(x_0 - y_0) f_k(x) f_k^*(y) + \Theta(y_0 - x_0) f_k(y) f_k^*(x)\right]\mathrm{d}^3k$$

em que fixamos $k_0 = \omega_{\mathbf{k}}$.

Nota: Este resultado corresponde ao propagador clássico, com os campos ainda não quantizados. Após a segunda quantização o propagador assume a forma

$$i\Delta(x-y) = \langle 0|\, T\{\phi_0(x)\phi_0(y)\}\,|0\rangle$$

em que

$$T\{\phi_0(x)\phi_0(y)\} = \begin{cases} \phi_0(x)\phi_0(y) & \text{, se } x_0 > y_0 \\ \phi_0(y)\phi_0(x) & \text{, se } y_0 > x_0 \end{cases}$$

(a) Considerando a lagrangiana livre de uma partícula escalar neutra de massa m, podemos escrever que

$$\partial_\mu(\phi_0 \partial^\mu \phi_0) = \partial_\mu \phi_0 \partial^\mu \phi_0 + \phi_0 \partial_\mu \partial^\mu \phi_0$$

ou seja

$$\begin{aligned}\mathcal{L}_0 &= \frac{1}{2}\partial_\mu(\phi_0\partial^\mu\phi_0) - \frac{1}{2}\phi_0\partial_\mu\partial^\mu\phi_0 - \frac{1}{2}\phi_0 m^2\phi_0 \\ &= \frac{1}{2}\partial_\mu(\phi_0\partial^\mu\phi_0) + \frac{1}{2}\phi_0(-\partial_\mu\partial^\mu - m^2)\phi_0\end{aligned}$$

Como o primeiro termo é uma derivada total, ao calcularmos a ação com $\mathcal{L}_0$, vamos ter a integral em todo o espaço-tempo de uma derivada total, e como sempre assumimos que um campo desaparece nos infinitos, esta integração vai se anular, de tal forma que

$$S_0 = \int \frac{1}{2}\phi_0(-\partial_\mu\partial^\mu - m^2)\phi_0 \,\mathrm{d}^4x$$

Temos então uma lagrangiana equivalente à $\mathcal{L}_0$ da forma

$$\mathcal{L}_0' = \frac{1}{2}\phi_0(-\partial_\mu\partial^\mu - m^2)\phi_0$$

Sendo $(-\partial_\mu\partial^\mu - m^2)$ um operador dentro da lagrangiana, o propagador associado a ele, $\Delta(x-y)$, vai ser uma função de Green de dois pontos desse operador, ou seja, satisfaz a equação

$$(-\partial_\mu\partial^\mu - m^2)\Delta(x-y) = \delta^4(x-y)$$

Considerando uma transformada de Fourier no propagador, temos

$$\Delta(x-y) = \int \frac{1}{(2\pi)^4}\tilde{\Delta}(k)e^{-ik(x-y)}\,\mathrm{d}^4k$$

Então

$$\begin{aligned}(-\partial_\mu\partial^\mu - m^2)\int \frac{1}{(2\pi)^4}\tilde{\Delta}(k)e^{-ik(x-y)}\,\mathrm{d}^4k &= \delta^4(x-y) \\ \int \frac{1}{(2\pi)^4}\tilde{\Delta}(k)[\partial_\mu\partial^\mu e^{-ik(x-y)} - m^2 e^{-ik(x-y)}]\,\mathrm{d}^4k &= \int \frac{1}{(2\pi)^4}e^{-ik(x-y)}\,\mathrm{d}^4k \\ \int \frac{1}{(2\pi)^4}\tilde{\Delta}(k)(k^2 - m^2)e^{-ik(x-y)}\,\mathrm{d}^4k &= \end{aligned}$$

Comparando os dois lados da equação, obtemos que

$$\tilde{\Delta}(k) = \frac{1}{k^2 - m^2}$$

Porém, possuimos 2 singularidades nesta expressão, que ocorrem quando $k^2 = m^2$, que equivale a $k_0^2 - |\mathbf{k}|^2 = m^2$, ou seja $k_0 = \pm\sqrt{|\mathbf{k}|^2 + m^2} = \pm\omega_{\mathbf{k}}$. Pela existência dessas singularidades que ocorrem no eixo real, adicionamos ao denominador uma quantidade no eixo imaginário $+i\varepsilon$ para regularizar a função, em que $\varepsilon > 0$ e $\epsilon \to 0$, concluindo que

$$\tilde{\Delta}(k) = \frac{1}{k^2 - m^2 + i\varepsilon} \tag{3.1}$$

(b) Sabendo a forma do propagador no espaço de coordenadas, podemos separar a parte espacial da temporal de modo que

$$i\Delta(x-y) = i\int \frac{1}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\,\mathrm{d}^3k \int_{-\infty}^{\infty} \frac{1}{2\pi} \frac{e^{-ik_0(x_0-y_0)}}{k^2 - m^2 + i\varepsilon}\,\mathrm{d}k_0$$

Podemos modificar o denominador da segunda integral de modo que $k^2 - m^2 = k_0^2 - |\mathbf{k}|^2 - m^2 = k_0^2 - \omega_{\mathbf{k}}^2$ e denotar por $z_0 := x_0 - y_0$, resultando então em

$$i\Delta(x-y) = i\int \frac{1}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\,\mathrm{d}^3k \int_{-\infty}^{\infty} \frac{1}{2\pi} \frac{e^{-ik_0z_0}}{k_0^2 - \omega_{\mathbf{k}}^2 + i\varepsilon}\,\mathrm{d}k_0$$

Nesta forma, temos que os polos se encontram em $k_0 = \pm\sqrt{\omega_{\mathbf{k}}^2 - i\varepsilon}$, que como ε é muito pequeno, podemos aproximar

$$k_0 \approx \pm\left(\omega_{\mathbf{k}} - \frac{i\varepsilon}{2\omega_{\mathbf{k}}}\right)$$

No plano complexo, o polo em $k_0 = \omega_{\mathbf{k}} - \dfrac{i\varepsilon}{2\omega_{\mathbf{k}}} = \kappa_1$ pertence ao semiplano inferior $\mathfrak{Im}[k_0] < 0$ e o polo $k_0 = -\omega_{\mathbf{k}} + \dfrac{i\varepsilon}{2\omega_{\mathbf{k}}} = \kappa_2$ pertence ao semiplano superior $\mathfrak{Im}[k_0] > 0$. Ao considerarmos que a integração está sendo feita no plano complexo, temos que k_0 possui parte real e imaginária, portanto

$$k_0 = \mathfrak{Re}[k_0] + i\mathfrak{Im}[k_0]$$

Com isso, a exponencial $e^{-ik_0z_0}$ é da forma

$$e^{-ik_0z_0} = e^{-i(\mathfrak{Re}[k_0] - i\mathfrak{Im}[k_0])z_0}$$

implicando que para $z_0 > 0$ a exponencial decai no semiplano inferior e para $z_0 < 0$ decai no semiplano superior. Definindo a função

$$f(k_0) = \frac{e^{-ik_0z_0}}{k_0^2 - \omega_{\mathbf{k}}^2 + i\varepsilon} = \frac{e^{-ik_0z_0}}{(k_0 - \kappa_1)(k_0 - \kappa_2)}$$

temos que o resíduo de $f(k_0)$ no polo $k_0 = \kappa_1$ ($z_0 > 0$)

$$\mathrm{Res}(f, \kappa_1) = \lim_{k_0 \to \kappa_1} (k_0 - \kappa_1) f(k_0) = \frac{e^{-i\kappa_1 z_0}}{\kappa_1 - \kappa_2} = \frac{e^{-i\left(\omega_{\mathbf{k}} - \frac{i\varepsilon}{2\omega_{\mathbf{k}}}\right)z_0}}{2\omega_{\mathbf{k}}} = \frac{e^{-i\omega_{\mathbf{k}}z_0}}{2\omega_{\mathbf{k}}} e^{-\frac{\varepsilon z_0}{2\omega_{\mathbf{k}}}}$$

Como $\varepsilon \to 0$, podemos aproximar a segunda exponencial para 1, de modo que

$$\mathrm{Res}(f, \kappa_1) \approx \frac{e^{-i\omega_{\mathbf{k}}z_0}}{2\omega_{\mathbf{k}}}$$

Calculado o resíduo, temos para $z_0 > 0$, ao orientar a integral no sentido horário, que

$$\int_{-\infty}^{\infty} \frac{e^{-ik_0 z_0}}{k_0^2 - \omega_{\mathbf{k}}^2 + i\varepsilon}\, \mathrm{d}k_0 = -2\pi i \cdot \mathrm{Res}(f, \kappa_1) = -\frac{2\pi i}{2\omega_{\mathbf{k}}} e^{-i\omega_{\mathbf{k}} z_0}$$

Já no polo $k_0 = \kappa_2$ ($z_0 < 0$):

$$\mathrm{Res}(f, \kappa_2) = \lim_{k_0 \to \kappa_2} (k_0 - \kappa_2) f(k_0) = \frac{e^{-i\kappa_2 z_0}}{\kappa_2 - \kappa_1} = \frac{e^{-i\left(-\omega_{\mathbf{k}} + \frac{i\varepsilon}{2\omega_{\mathbf{k}}}\right) z_0}}{-2\omega_{\mathbf{k}}} = -\frac{e^{i\omega_{\mathbf{k}} z_0}}{2\omega_{\mathbf{k}}} e^{\frac{\varepsilon z_0}{2\omega_{\mathbf{k}}}}$$

em que podemos tomar novamente $\varepsilon \to 0$ e aproximar

$$\mathrm{Res}(f, \kappa_2) \approx -\frac{e^{i\omega_{\mathbf{k}} z_0}}{2\omega_{\mathbf{k}}}$$

Então para $z_0 < 0$

$$\int_{-\infty}^{\infty} \frac{e^{-ik_0 z_0}}{k_0^2 - \omega_{\mathbf{k}}^2 + i\varepsilon}\, \mathrm{d}k_0 = 2\pi i \cdot \mathrm{Res}(f, \kappa_2) = -\frac{2\pi i}{2\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}} z_0}$$

Para considerar tanto $z_0 > 0$ quanto $z_0 < 0$ na integração, adicionamos a cada parte uma função de Heaviside: $\Theta(z_0)$ para $z_0 > 0$ e $\Theta(-z_0)$ para $z_0 < 0$, concluindo que

$$\int \frac{1}{2\pi} \frac{e^{-ik_0 z_0}}{k_0^2 - \omega_{\mathbf{k}}^2 + i\varepsilon}\, \mathrm{d}k_0 = -i\Theta(z_0) \frac{e^{-i\omega_{\mathbf{k}} z_0}}{2\omega_{\mathbf{k}}} - i\Theta(-z_0) \frac{e^{i\omega_{\mathbf{k}} z_0}}{2\omega_{\mathbf{k}}} \tag{3.2}$$

(c) Com o resultado (3.2), podemos escrever o propagador sob a forma

$$i\Delta(x - y) = i \int \frac{1}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\, \mathrm{d}^3k \left[-i\Theta(z_0) \frac{e^{-i\omega_{\mathbf{k}} z_0}}{2\omega_{\mathbf{k}}} - i\Theta(-z_0) \frac{e^{i\omega_{\mathbf{k}} z_0}}{2\omega_{\mathbf{k}}} \right]$$

em que como $z_0 = x_0 - y_0$ e não há dependência em $\mathbf{k}$, podemos inserir o termo entre colchetes $[\cdots]$ dentro da integral tal que

$$\begin{aligned} i\Delta(x - y) &= \int \frac{1}{(2\pi)^3} \left[\Theta(x_0 - y_0) \frac{e^{-i\omega_{\mathbf{k}}(x_0 - y_0) + i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{2\omega_{\mathbf{k}}} + \Theta(y_0 - x_0) \frac{e^{i\omega_{\mathbf{k}}(x_0 - y_0) + i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{2\omega_{\mathbf{k}}} \right] \mathrm{d}^3k \\ &= \int \frac{1}{(2\pi)^3} \left[\Theta(x_0 - y_0) \frac{e^{-i\omega_{\mathbf{k}}(x_0 - y_0) + i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{2\omega_{\mathbf{k}}} + \Theta(y_0 - x_0) \frac{e^{-i\omega_{\mathbf{k}}(y_0 - x_0) + i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{2\omega_{\mathbf{k}}} \right] \mathrm{d}^3k \end{aligned}$$

Fazendo a mudança $\mathbf{k} \mapsto -\mathbf{k}$ no segundo termo (essa transformação mantém a integral invariante), temos

$$\begin{aligned} i\Delta(x - y) &= \int \frac{1}{(2\pi)^3} \left[\Theta(x_0 - y_0) \frac{e^{-i\omega_{\mathbf{k}}(x_0 - y_0) + i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{2\omega_{\mathbf{k}}} + \Theta(y_0 - x_0) \frac{e^{-i\omega_{\mathbf{k}}(y_0 - x_0) + i\mathbf{k}\cdot(\mathbf{y}-\mathbf{x})}}{2\omega_{\mathbf{k}}} \right] \mathrm{d}^3k \\ &= \int \frac{1}{(2\pi)^3} \left[\Theta(x_0 - y_0) \frac{e^{-i\omega_{\mathbf{k}} x_0 + i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{2\omega_{\mathbf{k}}}} \frac{e^{i\omega_{\mathbf{k}} y_0 - i\mathbf{k}\cdot\mathbf{y}}}{\sqrt{2\omega_{\mathbf{k}}}} + \Theta(y_0 - x_0) \frac{e^{-i\omega_{\mathbf{k}} y_0 + i\mathbf{k}\cdot\mathbf{y}}}{\sqrt{2\omega_{\mathbf{k}}}} \frac{e^{i\omega_{\mathbf{k}} x_0 - i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{2\omega_{\mathbf{k}}}} \right] \mathrm{d}^3k \end{aligned}$$

Concluindo que

$$i\Delta(x-y) = \int \frac{1}{(2\pi)^3}[\Theta(x_0-y_0)f_k(x)f_k^*(y) + \Theta(y_0-x_0)f_k(y)f_k^*(x)]\,\mathrm{d}^3k \tag{3.3}$$

Q. 04 Ordenamento normal de um campo livre

O ordenamento normal dos campos no mesmo ponto do espaço-tempo evita o surgimento de certas singularidades, como a chamada "energia do ponto zero" da hamiltoniana de um dado sistema. Define-se $\phi_0(x) = \phi_0^+(x) + \phi_0^-(x)$ tais que

$$\phi_0^+(x) = \int \frac{1}{(2\pi)^3} a(\mathbf{k}) f_k(x)\, \mathrm{d}^3k \qquad \text{(freq. positiva)},$$

$$\phi_0^-(x) = \int \frac{1}{(2\pi)^3} a^\dagger(\mathbf{k}) f_k^*(x)\, \mathrm{d}^3k \qquad \text{(freq. negativa)}.$$

O produto normal de dois campos move os termos de frequência positiva para a direita, e os termos de frequência negativa, à esquerda. Partindo da hamiltoniana de uma partícula livre,

$$H = \int \frac{1}{2}\Big[\pi_0^2 + (\nabla\phi_0)^2 + m^2\phi_0^2\Big]\, \mathrm{d}^3x$$

mostre que seu ordenamento normal elimina a energia de ponto zero, ou seja,

$$:H: = \int \frac{1}{(2\pi)^3} \omega_{\mathbf{k}} a^\dagger(\mathbf{k}) a(\mathbf{k})\, \mathrm{d}^3k$$

Partindo da hamiltoniana de uma partícula livre, podemos separá-la em 3 integrais, de modo que

$$H = \frac{1}{2}\left[\int \pi_0^2\, \mathrm{d}^3x + \int (\nabla\phi_0)^2\, \mathrm{d}^3x + \int m^2\phi^2\, \mathrm{d}^3x\right] = \frac{1}{2}(I_1 + I_2 + I_3)$$

O primeiro termo vai ficar

$$\begin{aligned}
I_1 &= -\iiint \frac{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}{(2\pi)^6}\Big[a_0(\mathbf{k}) f_k(x) - a_0^\dagger(\mathbf{k}) f_k^*(x)\Big]\Big[a_0(\mathbf{k}') f_{k'}(x) - a_0^\dagger(\mathbf{k}') f_{k'}^*(x)\Big]\, \mathrm{d}^3k\, \mathrm{d}^3k'\, \mathrm{d}^3x \\
&= -\iiint \frac{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}{(2\pi)^6}\Big[a_0(\mathbf{k}) a_0(\mathbf{k}') f_k(x) f_{k'}(x) - a_0(\mathbf{k}) a_0^\dagger(\mathbf{k}') f_k(x) f_{k'}^*(x) - \\
&\quad - a_0^\dagger(\mathbf{k}) a_0(\mathbf{k}') f_k^*(x) f_{k'}(x) + a_0^\dagger(\mathbf{k}) a_0^\dagger(\mathbf{k}') f_k^*(x) f_{k'}^*(x)\Big]\, \mathrm{d}^3k\, \mathrm{d}^3k'\, \mathrm{d}^3x
\end{aligned}$$

Lembrando da forma de $f_k(x)$, temos que os produtos entre estas funções são

$$f_k(x) f_{k'}(x) = \frac{1}{\sqrt{2\omega_{\mathbf{k}}}\sqrt{2\omega_{\mathbf{k}'}}} e^{-i\omega_{\mathbf{k}}t + i\mathbf{k}\cdot\mathbf{x}} e^{-i\omega_{\mathbf{k}'}t + i\mathbf{k}'\cdot\mathbf{x}} = \frac{1}{\sqrt{2\omega_{\mathbf{k}}}\sqrt{2\omega_{\mathbf{k}'}}} e^{-i(\omega_{\mathbf{k}} + \omega_{\mathbf{k}'})t} e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}}$$

$$
\begin{aligned}
f_k(x)f^*_{k'}(x) &= \frac{1}{\sqrt{2\omega_{\mathbf{k}}}\sqrt{2\omega_{\mathbf{k}'}}}e^{-i\omega_{\mathbf{k}}t+i\mathbf{k}\cdot\mathbf{x}}e^{i\omega_{\mathbf{k}'}t-i\mathbf{k}'\cdot\mathbf{x}} = \frac{1}{\sqrt{2\omega_{\mathbf{k}}}\sqrt{2\omega_{\mathbf{k}'}}}e^{-i(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})t}e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}\\
f^*_k(x)f_{k'}(x) &= \frac{1}{\sqrt{2\omega_{\mathbf{k}}}\sqrt{2\omega_{\mathbf{k}'}}}e^{i\omega_{\mathbf{k}}t-i\mathbf{k}\cdot\mathbf{x}}e^{-i\omega_{\mathbf{k}'}t+i\mathbf{k}'\cdot\mathbf{x}} = \frac{1}{\sqrt{2\omega_{\mathbf{k}}}\sqrt{2\omega_{\mathbf{k}'}}}e^{i(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})t}e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}\\
f^*_k(x)f^*_{k'}(x) &= \frac{1}{\sqrt{2\omega_{\mathbf{k}}}\sqrt{2\omega_{\mathbf{k}'}}}e^{i\omega_{\mathbf{k}}t-i\mathbf{k}\cdot\mathbf{x}}e^{i\omega_{\mathbf{k}'}t-i\mathbf{k}'\cdot\mathbf{x}} = \frac{1}{\sqrt{2\omega_{\mathbf{k}}}\sqrt{2\omega_{\mathbf{k}'}}}e^{i(\omega_{\mathbf{k}}+\omega_{\mathbf{k}'})t}e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}}
\end{aligned}
$$

Note então que as integrais em d^3x podem ser feitas apenas nos produtos de $f_k(x)$, pois os operadores de criação e aniquilação independem de $\mathbf{x}$, logo, junto com um fator $1/(2\pi)^3$, temos

$$
\begin{aligned}
\frac{1}{(2\pi)^3}\int f_k(x)f_{k'}(x)\,\mathrm{d}^3x &= \frac{e^{-i(\omega_{\mathbf{k}}+\omega_{\mathbf{k}'})t}}{\sqrt{2\omega_{\mathbf{k}}}\sqrt{2\omega_{\mathbf{k}'}}}\int\frac{1}{(2\pi)^3}e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}}\,\mathrm{d}^3x = \frac{e^{-i(\omega_{\mathbf{k}}+\omega_{\mathbf{k}'})t}}{\sqrt{2\omega_{\mathbf{k}}}\sqrt{2\omega_{\mathbf{k}'}}}\delta^3(\mathbf{k}+\mathbf{k}')\\
\frac{1}{(2\pi)^3}\int f_k(x)f^*_{k'}(x)\,\mathrm{d}^3x &= \frac{e^{-i(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})t}}{\sqrt{2\omega_{\mathbf{k}}}\sqrt{2\omega_{\mathbf{k}'}}}\int\frac{1}{(2\pi)^3}e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}\,\mathrm{d}^3x = \frac{e^{-i(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})t}}{\sqrt{2\omega_{\mathbf{k}}}\sqrt{2\omega_{\mathbf{k}'}}}\delta^3(\mathbf{k}-\mathbf{k}')\\
\frac{1}{(2\pi)^3}\int f^*_k(x)f_{k'}(x)\,\mathrm{d}^3x &= \frac{e^{i(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})t}}{\sqrt{2\omega_{\mathbf{k}}}\sqrt{2\omega_{\mathbf{k}'}}}\int\frac{1}{(2\pi)^3}e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}\,\mathrm{d}^3x = \frac{e^{i(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})t}}{\sqrt{2\omega_{\mathbf{k}}}\sqrt{2\omega_{\mathbf{k}'}}}\delta^3(\mathbf{k}-\mathbf{k}')\\
\frac{1}{(2\pi)^3}\int f^*_k(x)f^*_{k'}(x)\,\mathrm{d}^3x &= \frac{e^{i(\omega_{\mathbf{k}}+\omega_{\mathbf{k}'})t}}{\sqrt{2\omega_{\mathbf{k}}}\sqrt{2\omega_{\mathbf{k}'}}}\int\frac{1}{(2\pi)^3}e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}}\,\mathrm{d}^3x = \frac{e^{i(\omega_{\mathbf{k}}+\omega_{\mathbf{k}'})t}}{\sqrt{2\omega_{\mathbf{k}}}\sqrt{2\omega_{\mathbf{k}'}}}\delta^3(\mathbf{k}+\mathbf{k}')
\end{aligned}
$$

Então a integral de π_0^2 fica

$$
\begin{aligned}
I_1 = -\iint \frac{1}{(2\pi)^3}\frac{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}{\sqrt{2\omega_{\mathbf{k}}}\sqrt{2\omega_{\mathbf{k}'}}}\Big[&a_0(\mathbf{k})a_0(\mathbf{k}')e^{-i(\omega_{\mathbf{k}}+\omega_{\mathbf{k}'})t}\delta^3(\mathbf{k}+\mathbf{k}')-\\
&- a_0(\mathbf{k})a_0^\dagger(\mathbf{k}')e^{-i(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})t}\delta^3(\mathbf{k}-\mathbf{k}') - a_0^\dagger(\mathbf{k})a_0(\mathbf{k}')e^{i(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})t}\delta^3(\mathbf{k}-\mathbf{k}')+\\
&+ a_0^\dagger(\mathbf{k})a_0^\dagger(\mathbf{k}')e^{i(\omega_{\mathbf{k}}+\omega_{\mathbf{k}'})t}\delta^3(\mathbf{k}+\mathbf{k}')\Big]\,\mathrm{d}^3k\,\mathrm{d}^3k'
\end{aligned}
$$

Realizando a integração em d^3k', e usando que $\omega_{\mathbf{k}} = \omega_{-\mathbf{k}}$ por construção, temos

$$
I_1 = -\int\frac{1}{(2\pi)^3}\frac{\omega_{\mathbf{k}}}{2}\Big[a_0(\mathbf{k})a_0(-\mathbf{k})e^{-2i\omega_{\mathbf{k}}t} - a_0(\mathbf{k})a_0^\dagger(\mathbf{k}) - a_0^\dagger(\mathbf{k})a_0(\mathbf{k}) + a_0^\dagger(\mathbf{k})a_0^\dagger(-\mathbf{k})e^{2i\omega_{\mathbf{k}}t}\Big]\,\mathrm{d}^3k
$$

Antes de performar a segunda integral da hamiltoniana, temos

$$
\begin{aligned}
\nabla\phi_0 &= \int\frac{1}{(2\pi)^3}\left[a_0(\mathbf{k})\frac{e^{-i\omega_{\mathbf{k}}}}{\sqrt{2\omega_{\mathbf{k}}}}\nabla(e^{i\mathbf{k}\cdot\mathbf{x}}) + a_0^\dagger(\mathbf{k})\frac{e^{i\omega_{\mathbf{k}}t}}{\sqrt{2\omega_{\mathbf{k}}}}\nabla(e^{-i\mathbf{k}\cdot\mathbf{x}})\right]\mathrm{d}^3k\\
&= \int\frac{i\mathbf{k}}{(2\pi)^3}\left[a_0(\mathbf{k})f_k(x) - a_0^\dagger(\mathbf{k})f_k^*(x)\right]\mathrm{d}^3k
\end{aligned}
$$

Sendo assim

$$\begin{aligned}
I_2 &= \iiint \frac{-\mathbf{k}\cdot\mathbf{k}'}{(2\pi)^6}\Big[a_0(\mathbf{k})f_k(x) - a_0^\dagger(\mathbf{k})f_k^*(x)\Big]\Big[a_0(\mathbf{k}')f_{k'}(x) - a_0^\dagger(\mathbf{k}')f_{k'}^*(x)\Big]\,\mathrm{d}^3k\,\mathrm{d}^3k'\,\mathrm{d}^3x \\
&= \iiint \frac{-\mathbf{k}\cdot\mathbf{k}'}{(2\pi)^6}\Big[a_0(\mathbf{k})a_0(\mathbf{k}')f_k(x)f_{k'}(x) - a_0(\mathbf{k})a_0^\dagger(\mathbf{k}')f_k(x)f_{k'}^*(x) - \\
&\quad - a_0^\dagger(\mathbf{k})a_0(\mathbf{k}')f_k^*(x)f_{k'}(x) + a_0^\dagger(\mathbf{k})a_0^\dagger(\mathbf{k}')f_k^*(x)f_{k'}^*(x)\Big]\,\mathrm{d}^3k\,\mathrm{d}^3k'\,\mathrm{d}^3x \\
&= \iint \frac{1}{(2\pi)^3}\frac{-\mathbf{k}\cdot\mathbf{k}'}{\sqrt{2\omega_\mathbf{k}}\sqrt{2\omega_{\mathbf{k}'}}}\Big[a_0(\mathbf{k})a_0(\mathbf{k}')e^{-i(\omega_\mathbf{k}+\omega_{\mathbf{k}'})t}\delta^3(\mathbf{k}+\mathbf{k}') - \\
&\quad - a_0(\mathbf{k})a_0^\dagger(\mathbf{k}')e^{-i(\omega_\mathbf{k}-\omega_{\mathbf{k}'})t}\delta^3(\mathbf{k}-\mathbf{k}') - a_0^\dagger(\mathbf{k})a_0(\mathbf{k}')e^{i(\omega_\mathbf{k}-\omega_{\mathbf{k}'})t}\delta^3(\mathbf{k}-\mathbf{k}') + \\
&\quad + a_0^\dagger(\mathbf{k})a_0^\dagger(\mathbf{k}')e^{i(\omega_\mathbf{k}+\omega_{\mathbf{k}'})t}\delta^3(\mathbf{k}+\mathbf{k}')\Big]\,\mathrm{d}^3k\,\mathrm{d}^3k'
\end{aligned}$$

Fazendo então a integral em d^3k', temos que as duas distribuições $\delta^3(\mathbf{k}+\mathbf{k}')$ que ocorrem vão mudar o sinal do produto escalar $\mathbf{k}\cdot\mathbf{k}'$, deixando todos os termos dentro dos colchetes $[\cdots]$ negativos e com isso cancelando o sinal negativo fora dos colchetes, tal que

$$I_2 = \int \frac{1}{(2\pi)^3}\frac{|\mathbf{k}|^2}{2\omega_\mathbf{k}}\Big[a_0(\mathbf{k})a_0(-\mathbf{k})e^{-2i\omega_\mathbf{k}t} + a_0(\mathbf{k})a_0^\dagger(\mathbf{k}) + a_0^\dagger(\mathbf{k})a_0(\mathbf{k}) + a_0^\dagger(\mathbf{k})a_0^\dagger(-\mathbf{k})e^{2i\omega_\mathbf{k}t}\Big]\,\mathrm{d}^3k$$

Por fim, o último termo da hamiltoniana vai se desenvolver por

$$\begin{aligned}
I_3 &= \iiint \frac{m^2}{(2\pi)^6}\Big[a_0(\mathbf{k})f_k(x) + a_0^\dagger(\mathbf{k})f_k^*(x)\Big]\Big[a_0(\mathbf{k}')f_{k'}(x) + a_0^\dagger(\mathbf{k}')f_{k'}^*(x)\Big]\,\mathrm{d}^3k\,\mathrm{d}^3k'\,\mathrm{d}^3x \\
&= \iiint \frac{m^2}{(2\pi)^6}\Big[a_0(\mathbf{k})a_0(\mathbf{k}')f_k(x)f_{k'}(x) + a_0(\mathbf{k})a_0^\dagger(\mathbf{k}')f_k(x)f_{k'}^*(x) + \\
&\quad + a_0^\dagger(\mathbf{k})a_0(\mathbf{k}')f_k^*(x)f_{k'}(x) + a_0^\dagger(\mathbf{k})a_0^\dagger(\mathbf{k}')f_k^*(x)f_{k'}^*(x)\Big]\,\mathrm{d}^3k\,\mathrm{d}^3k'\,\mathrm{d}^3x \\
&= \iint \frac{1}{(2\pi)^3}\frac{m^2}{\sqrt{2\omega_\mathbf{k}}\sqrt{2\omega_{\mathbf{k}'}}}\Big[a_0(\mathbf{k})a_0(\mathbf{k}')e^{-i(\omega_\mathbf{k}+\omega_{\mathbf{k}'})t}\delta^3(\mathbf{k}+\mathbf{k}') + \\
&\quad + a_0(\mathbf{k})a_0^\dagger(\mathbf{k}')e^{-i(\omega_\mathbf{k}-\omega_{\mathbf{k}'})t}\delta^3(\mathbf{k}-\mathbf{k}') + a_0^\dagger(\mathbf{k})a_0(\mathbf{k}')e^{i(\omega_\mathbf{k}-\omega_{\mathbf{k}'})t}\delta^3(\mathbf{k}-\mathbf{k}') + \\
&\quad + a_0^\dagger(\mathbf{k})a_0^\dagger(\mathbf{k}')e^{i(\omega_\mathbf{k}+\omega_{\mathbf{k}'})t}\delta^3(\mathbf{k}+\mathbf{k}')\Big]\,\mathrm{d}^3k\,\mathrm{d}^3k'
\end{aligned}$$

Concluindo que

$$I_3 = \int \frac{1}{(2\pi)^3}\frac{m^2}{2\omega_\mathbf{k}}\Big[a_0(\mathbf{k})a_0(-\mathbf{k})e^{-2i\omega_\mathbf{k}t} + a_0(\mathbf{k})a_0^\dagger(\mathbf{k}) + a_0^\dagger(\mathbf{k})a_0(\mathbf{k}) + a_0^\dagger(\mathbf{k})a_0^\dagger(-\mathbf{k})e^{2i\omega_\mathbf{k}t}\Big]\,\mathrm{d}^3k$$

Podemos perceber que em I_2 e I_3 a única diferença está nas constantes que acompanham os operadores, de modo que

$$I_2 + I_3 = \int \frac{1}{(2\pi)^3}\frac{|\mathbf{k}|^2+m^2}{2\omega_\mathbf{k}}[\cdots]\,\mathrm{d}^3k$$

Mas sabemos que $\omega_{\mathbf{k}}^2 = |\mathbf{k}|^2 + m^2$, portanto

$$I_2 + I_3 = \int \frac{1}{(2\pi)^3} \frac{\omega_{\mathbf{k}}}{2} [\cdots] \, \mathrm{d}^3 k$$

Logo a hamiltoniana pode ser reescrita por

$$\begin{aligned} H &= \frac{1}{2} \int \frac{1}{(2\pi)^3} \frac{\omega_{\mathbf{k}}}{2} \Big[- a_0(\mathbf{k}) a_0(-\mathbf{k}) e^{-2i\omega_{\mathbf{k}} t} + a_0(\mathbf{k}) a_0^\dagger(\mathbf{k}) + a_0^\dagger(\mathbf{k}) a_0(\mathbf{k}) - a_0^\dagger(\mathbf{k}) a_0^\dagger(-\mathbf{k}) e^{2i\omega_{\mathbf{k}} t} + \\ &\quad + a_0(\mathbf{k}) a_0(-\mathbf{k}) e^{-2i\omega_{\mathbf{k}} t} + a_0(\mathbf{k}) a_0^\dagger(\mathbf{k}) + a_0^\dagger(\mathbf{k}) a_0(\mathbf{k}) + a_0^\dagger(\mathbf{k}) a_0^\dagger(-\mathbf{k}) e^{2i\omega_{\mathbf{k}} t} \Big] \, \mathrm{d}^3 k \\ &= \frac{1}{2} \int \frac{1}{(2\pi)^3} \omega_{\mathbf{k}} \Big[a_0(\mathbf{k}) a_0^\dagger(\mathbf{k}) + a_0^\dagger(\mathbf{k}) a_0(\mathbf{k}) \Big] \, \mathrm{d}^3 k \end{aligned}$$

O que nos dá a forma

$$H = \int \frac{1}{(2\pi)^3} \frac{\omega_{\mathbf{k}}}{2} \Big[a_0(\mathbf{k}) a_0^\dagger(\mathbf{k}) + a_0^\dagger(\mathbf{k}) a_0(\mathbf{k}) \Big] \, \mathrm{d}^3 k$$

Sabendo que o comutador $[a_0(\mathbf{k}), a_0^\dagger(\mathbf{k}')] = (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}')$, usamos isso para reescrever a hamiltoniana sob a forma

$$\begin{aligned} H &= \int \frac{1}{(2\pi)^3} \frac{\omega_{\mathbf{k}}}{2} \Big[(2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}) + 2 a_0^\dagger(\mathbf{k}) a_0(\mathbf{k}) \Big] \, \mathrm{d}^3 k \\ &= \int \frac{1}{(2\pi)^3} \omega_{\mathbf{k}} a_0^\dagger(\mathbf{k}) a_0(\mathbf{k}) \, \mathrm{d}^3 k + \int \frac{\omega_{\mathbf{k}}}{2} \delta^3(0) \, \mathrm{d}^3 k \end{aligned}$$

O segundo termo é infinito, de modo que para contornar este problema e manter apenas o termo convergente, impomos na hamiltoniana o ordenamento normal que vai fazer com que todos os operadores de criação de partículas estejam à esquerda dos operadores de aniquilação, eliminando por consequência as divergências indesejadas, ou seja

$$:H: = \int \frac{1}{(2\pi)^3} \omega_{\mathbf{k}} a_0^\dagger(\mathbf{k}) a_0(\mathbf{k}) \, \mathrm{d}^3 k \tag{4.1}$$

Q. 05 Espinores de Dirac

Partindo das formas explícitas dos espinores de Dirac abaixo transcritos,

$$u(\mathbf{p}, s) = \frac{1}{\sqrt{E_p + m}} \begin{bmatrix} (E_p + m)\chi_s \\ \boldsymbol{\sigma} \cdot \mathbf{p}\chi_s \end{bmatrix} \qquad \& \qquad v(\mathbf{p}, s) = \frac{1}{\sqrt{E_p + m}} \begin{bmatrix} \boldsymbol{\sigma} \cdot \mathbf{p}\chi_s \\ (E_p + m)\chi_s \end{bmatrix}$$

obtenha as seguintes relações:

(a) $\bar{u}(\mathbf{p}, s)u(\mathbf{p}, s) = -\bar{v}(\mathbf{p}, s)v(\mathbf{p}, s) = 2m$

(b) $\left[\Lambda^+(\mathbf{p})\right]_{\alpha\beta} = \frac{1}{2m}\sum_s u_\alpha(\mathbf{p}, s)\bar{u}_\beta(\mathbf{p}, s) = \frac{1}{2m}(\not{p} + m)_{\alpha\beta}$

(c) $\left[\Lambda^-(\mathbf{p})\right]_{\alpha\beta} = -\frac{1}{2m}\sum_s v_\alpha(\mathbf{p}, s)\bar{v}_\beta(\mathbf{p}, s) = \frac{1}{2m}(-\not{p} + m)_{\alpha\beta}$

(a) Expandindo o produto $\boldsymbol{\sigma} \cdot \mathbf{p}$, temos

$$(\boldsymbol{\sigma} \cdot \mathbf{p}) = \sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3 = \begin{bmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{bmatrix}$$

O que nos dá também

$$(\boldsymbol{\sigma} \cdot \mathbf{p})^\dagger = \begin{bmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{bmatrix} = (\boldsymbol{\sigma} \cdot \mathbf{p})$$

A quantidade χ_s é um vetor coluna tal que $\chi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ e $\chi_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Com isso, impomos que o produto $\chi_s^\dagger \chi_{s'} = \delta_{ss'}$. A forma explicita do espinor $\bar{u}(\mathbf{p}, s)$ é

$$\bar{u}(\mathbf{p}, s) = u^\dagger(\mathbf{p}, s)\gamma^0 = \frac{1}{\sqrt{E_\mathbf{p} + m}} \begin{bmatrix} (E_\mathbf{p} + m)\chi_s^\dagger & -\chi_s^\dagger(\boldsymbol{\sigma} \cdot \mathbf{p})^\dagger \end{bmatrix}$$

Portanto,

$$\bar{u}(\mathbf{p}, s)u(\mathbf{p}, s') = \frac{1}{E_\mathbf{p} + m}\Big[(E_\mathbf{p} + m)^2 \chi_s^\dagger \chi_s - \chi_s^\dagger (\boldsymbol{\sigma} \cdot \mathbf{p})^\dagger (\boldsymbol{\sigma} \cdot \mathbf{p})\chi_s\Big]$$

O produto

$$
\begin{aligned}
(\boldsymbol{\sigma}\cdot\mathbf{p})^\dagger(\boldsymbol{\sigma}\cdot\mathbf{p}) &= \begin{bmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{bmatrix}\begin{bmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{bmatrix} \\
&= \begin{bmatrix} p_1^2 + p_2^2 + p_3^2 & 0 \\ 0 & p_1^2 + p_2^2 + p_3^2 \end{bmatrix} \\
&= \begin{bmatrix} |\mathbf{p}|^2 & 0 \\ 0 & |\mathbf{p}|^2 \end{bmatrix} \\
&= |\mathbf{p}|^2 \mathbb{1}_{2\times 2}
\end{aligned}
$$

E como $E_{\mathbf{p}}^2 = |\mathbf{p}|^2 + m^2$, temos

$$
(\boldsymbol{\sigma}\cdot\mathbf{p})^\dagger(\boldsymbol{\sigma}\cdot\mathbf{p}) = (E_{\mathbf{p}}^2 - m^2)\mathbb{1}_{2\times 2} = (E_{\mathbf{p}} + m)(E_{\mathbf{p}} - m)\mathbb{1}_{2\times 2}
$$

Segue que

$$
\begin{aligned}
\bar{u}(\mathbf{p}, s)u(\mathbf{p}, s') &= \frac{1}{E_{\mathbf{p}} + m}\left[(E_{\mathbf{p}} + m)^2\delta_{ss'} - (E_{\mathbf{p}} + m)(E_{\mathbf{p}} - m)\chi_s^\dagger \mathbb{1}_{2\times 2}\chi_s\right] \\
&= (E_{\mathbf{p}} + m)\delta_{ss'} - (E_{\mathbf{p}} - m)\delta_{ss'} \\
&= 2m\delta_{ss'}
\end{aligned}
$$

Concluindo que

$$
\bar{u}(\mathbf{p}, s)u(\mathbf{p}, s) = 2m \tag{5.1}
$$

Agora a forma explicita de $\bar{v}(\mathbf{p}, s)$ é

$$
\bar{v}(\mathbf{p}, s) = v^\dagger(\mathbf{p}, s)\gamma^0 = \frac{1}{\sqrt{E_{\mathbf{p}} + m}}\begin{bmatrix}\chi_s^\dagger(\boldsymbol{\sigma}\cdot\mathbf{p})^\dagger & -(E_{\mathbf{p}} + m)\chi_s^\dagger\end{bmatrix}
$$

Portanto

$$
\begin{aligned}
\bar{v}(\mathbf{p}, s)v(\mathbf{p}, s') &= \frac{1}{E_{\mathbf{p}} + m}\left[\chi_s^\dagger(\boldsymbol{\sigma}\cdot\mathbf{p})^\dagger(\boldsymbol{\sigma}\cdot\mathbf{p})\chi_{s'} - (E_{\mathbf{p}} + m)^2\chi_s^\dagger\chi_{s'}\right] \\
&= \frac{1}{E_{\mathbf{p}} + m}\left[(E_{\mathbf{p}} + m)(E_{\mathbf{p}} - m)\delta_{ss'} - (E_{\mathbf{p}} + m)^2\delta_{ss'}\right] \\
&= (E_{\mathbf{p}} - m)\delta_{ss'} - (E_{\mathbf{p}} + m)\delta_{ss'} \\
&= -2m\delta_{ss'}
\end{aligned}
$$

Concluindo que

$$
-\bar{v}(\mathbf{p}, s)v(\mathbf{p}, s) = 2m \tag{5.2}
$$

(b) Usando a definição do projetor, temos

$$
\begin{aligned}
\left[\Lambda^+(\mathbf{p})\right]_{\alpha\beta} &= \frac{1}{2m}\sum_s u_\alpha(\mathbf{p},s)\bar{u}_\beta(\mathbf{p},s) \\
&= \frac{1}{2m(E_\mathbf{p}+m)}\sum_s \begin{bmatrix}(E_\mathbf{p}+m)\chi_s \\ (\boldsymbol{\sigma}\cdot\mathbf{p})\chi_s\end{bmatrix}_\alpha \begin{bmatrix}(E_\mathbf{p}+m)\chi_s^\dagger & -\chi_s^\dagger(\boldsymbol{\sigma}\cdot\mathbf{p})^\dagger\end{bmatrix}_\beta \\
&= \frac{1}{2m(E_\mathbf{p}+m)}\sum_s \begin{bmatrix}(E_\mathbf{p}+m)^2\chi_s\chi_s^\dagger & -(E_\mathbf{p}+m)\chi_s\chi_s^\dagger(\boldsymbol{\sigma}\cdot\mathbf{p})^\dagger \\ (E_\mathbf{p}+m)(\boldsymbol{\sigma}\cdot\mathbf{p})\chi_s\chi_s^\dagger & -(\boldsymbol{\sigma}\cdot\mathbf{p})\chi_s\chi_s^\dagger(\boldsymbol{\sigma}\cdot\mathbf{p})^\dagger\end{bmatrix}_{\alpha\beta}
\end{aligned}
$$

Note que

$$
\chi_1\chi_1^\dagger = \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} \qquad \& \qquad \chi_2\chi_2^\dagger = \begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}
$$

Implicando que quando somarmos em s, estaremos somando estas duas matrizes, gerando a identidade $\mathbb{1}_{2\times2}$, portanto o projetor fica

$$
\begin{aligned}
\left[\Lambda^+(\mathbf{p})\right]_{\alpha\beta} &= \frac{1}{2m(E_\mathbf{p}+m)}\begin{bmatrix}(E_\mathbf{p}+m)^2\mathbb{1}_{2\times2} & -(E_\mathbf{p}+m)\mathbb{1}_{2\times2}(\boldsymbol{\sigma}\cdot\mathbf{p})^\dagger \\ (E_\mathbf{p}+m)(\boldsymbol{\sigma}\cdot\mathbf{p})\mathbb{1}_{2\times2} & -(\boldsymbol{\sigma}\cdot\mathbf{p})\mathbb{1}_{2\times2}(\boldsymbol{\sigma}\cdot\mathbf{p})^\dagger\end{bmatrix}_{\alpha\beta} \\
&= \frac{1}{2m}\begin{bmatrix}(E_\mathbf{p}+m)\mathbb{1}_{2\times2} & -\mathbb{1}_{2\times2}(\boldsymbol{\sigma}\cdot\mathbf{p})^\dagger \\ (\boldsymbol{\sigma}\cdot\mathbf{p})\mathbb{1}_{2\times2} & -\mathbb{1}_{2\times2}(E_\mathbf{p}-m)\end{bmatrix}_{\alpha\beta} \\
&= \frac{1}{2m}\begin{bmatrix}E_\mathbf{p}\mathbb{1}_{2\times2} & -(\boldsymbol{\sigma}\cdot\mathbf{p})\mathbb{1}_{2\times2} \\ (\boldsymbol{\sigma}\cdot\mathbf{p})\mathbb{1}_{2\times2} & -E_\mathbf{p}\mathbb{1}_{2\times2}\end{bmatrix}_{\alpha\beta} + \frac{1}{2m}\begin{bmatrix}m\mathbb{1}_{2\times2} & 0 \\ 0 & m\mathbb{1}_{2\times2}\end{bmatrix}_{\alpha\beta}
\end{aligned}
$$

Como $p_0 = E_\mathbf{p}$, a forma explicita deste projetor é

$$
\begin{aligned}
\left[\Lambda^+(\mathbf{p})\right]_{\alpha\beta} &= \frac{1}{2m}\begin{bmatrix}p_0 & 0 & -p_3 & -p_1+ip_2 \\ 0 & p_0 & -p_1-ip_2 & p_3 \\ p_3 & p_1-ip_2 & -p_0 & 0 \\ p_1+ip_2 & -p_3 & 0 & -p_0\end{bmatrix}_{\alpha\beta} + \frac{1}{2m}\begin{bmatrix}m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & m\end{bmatrix}_{\alpha\beta} \\
&= \frac{1}{2m}(\gamma^0 p_0 + \gamma^1 p_1 + \gamma^2 p_2 + \gamma^3 p_3)_{\alpha\beta} + \frac{1}{2m}(m)_{\alpha\beta} \\
&= \frac{1}{2m}(\gamma^\mu p_\mu + m)_{\alpha\beta}
\end{aligned}
$$

Concluindo que

$$
\left[\Lambda^+(\mathbf{p})\right]_{\alpha\beta} = \frac{1}{2m}\sum_s u_\alpha(\mathbf{p},s)\bar{u}_\beta(\mathbf{p},s) = \frac{1}{2m}(\not{p}+m)_{\alpha\beta} \tag{5.3}
$$

(c) Novamente, usando a forma explícita do projetor, temos

$$
\begin{aligned}
\left[\Lambda^-(\mathbf{p})\right]_{\alpha\beta} &= -\frac{1}{2m}\sum_s v_\alpha(\mathbf{p},s)\bar{v}_\beta(\mathbf{p},s) \\
&= \frac{1}{2m(E_\mathbf{p}+m)}\sum_s \begin{bmatrix} (\boldsymbol{\sigma}\cdot\mathbf{p})\chi_s \\ (E_\mathbf{p}+m)\chi_s \end{bmatrix}_\alpha \begin{bmatrix} \chi_s^\dagger(\boldsymbol{\sigma}\cdot\mathbf{p})^\dagger_\beta & -(E_\mathbf{p}+m)\chi_s^\dagger \end{bmatrix}_\beta \\
&= -\frac{1}{2m(E_\mathbf{p}+m)}\sum_s \begin{bmatrix} (\boldsymbol{\sigma}\cdot\mathbf{p})\chi_s\chi_s^\dagger(\boldsymbol{\sigma}\cdot\mathbf{p})^\dagger & -(E_\mathbf{p}+m)(\boldsymbol{\sigma}\cdot\mathbf{p})\chi_s\chi_s^\dagger \\ (E_\mathbf{p}+m)\chi_s\chi_s^\dagger(\boldsymbol{\sigma}\cdot\mathbf{p})^\dagger & -(E_\mathbf{p}+m)^2\chi_s\chi_s^\dagger \end{bmatrix}_{\alpha\beta} \\
&= -\frac{1}{2m}\begin{bmatrix} \mathbb{1}_{2\times2}(E_\mathbf{p}-m) & -(\boldsymbol{\sigma}\cdot\mathbf{p})\mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2}(\boldsymbol{\sigma}\cdot\mathbf{p}) & -(E_\mathbf{p}+m)\mathbb{1}_{2\times2} \end{bmatrix}_{\alpha\beta} \\
&= -\frac{1}{2m}\begin{bmatrix} E_\mathbf{p}\mathbb{1}_{2\times2} & -(\boldsymbol{\sigma}\cdot\mathbf{p})\mathbb{1}_{2\times2} \\ (\boldsymbol{\sigma}\cdot\mathbf{p})\mathbb{1}_{2\times2} & -E_\mathbf{p}\mathbb{1}_{2\times2} \end{bmatrix}_{\alpha\beta} - \frac{1}{2m}\begin{bmatrix} -m\mathbb{1}_{2\times2} & 0 \\ 0 & -m\mathbb{1}_{2\times2} \end{bmatrix}_{\alpha\beta} \\
&= -\frac{1}{2m}\begin{bmatrix} p_0 & 0 & -p_3 & -p_1+ip_2 \\ 0 & p_0 & -p_1-ip_2 & p_3 \\ p_3 & p_1-ip_2 & -p_0 & 0 \\ p_1+ip_2 & -p_3 & 0 & -p_0 \end{bmatrix}_{\alpha\beta} + \frac{1}{2m}\begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \end{bmatrix}_{\alpha\beta} \\
&= \frac{1}{2m}(-\gamma^0 p_0 + \gamma^1 p_1 + \gamma^2 p_2 + \gamma^3 p_3)_{\alpha\beta} + \frac{1}{2m}(m)_{\alpha\beta} \\
&= \frac{1}{2m}(-\gamma^\mu p_\mu + m)_{\alpha\beta}
\end{aligned}
$$

Concluindo que

$$
\left[\Lambda^-(\mathbf{p})\right]_{\alpha\beta} = -\frac{1}{2m}\sum_s v_\alpha(\mathbf{p},s)\bar{v}_\beta(\mathbf{p},s) = \frac{1}{2m}(-\not{p}+m)_{\alpha\beta} \tag{5.4}
$$

Q. 06 Operadores de um campo espinorial

Usando as definições

$$U_k^s(x) = \frac{\tilde{U}_k^s(x)}{\sqrt{2E_{\mathbf{k}}}} = u(\mathbf{k}, s)\frac{e^{-ikx}}{\sqrt{2E_{\mathbf{k}}}} \qquad V_k^s(x) = \frac{\tilde{V}_k^s(x)}{\sqrt{2E_{\mathbf{k}}}} = v(\mathbf{k}, s)\frac{e^{ikx}}{\sqrt{2E_{\mathbf{k}}}}$$

$$\psi(x) = \int \frac{1}{(2\pi)^3} \sum_s \left[b_s(\mathbf{k}) U_k^s(x) + d_s^\dagger(\mathbf{k}) V_k^s(x) \right] \mathrm{d}^3 k$$

obtenha

$$b_s(\mathbf{k}) = \int \bar{U}_k^s(x)\gamma^0\psi(x)\,\mathrm{d}^3x \qquad \& \qquad d_s(\mathbf{k}) = \int \bar{\psi}(x)\gamma^0 V_k^s(x)\,\mathrm{d}^3x$$

sendo $\bar{\psi} = \psi^\dagger\gamma^0$ e $E_{\mathbf{k}} = \sqrt{|\mathbf{k}|^2 + m^2}$. (Obs.: em unidades naturais, $E_{\mathbf{k}} = \omega_{\mathbf{k}}$).

Com as definições fornecidas, podemos escrever o campo $\psi(x)$ sob a forma

$$\psi(x) = \int \frac{1}{(2\pi)^3} \sum_{s'} \left[b_{s'}(\mathbf{k}') u(\mathbf{k}', s') \frac{e^{-ik'x}}{\sqrt{2E_{\mathbf{k}'}}} + d_{s'}^\dagger(\mathbf{k}') v(\mathbf{k}', s') \frac{e^{ik'x}}{\sqrt{2E_{\mathbf{k}'}}} \right] \mathrm{d}^3 k'$$

e portanto

$$\psi^\dagger(x) = \int \frac{1}{(2\pi)^3} \sum_{s'} \left[b_{s'}^\dagger(\mathbf{k}') u^\dagger(\mathbf{k}', s') \frac{e^{ik'x}}{\sqrt{2E_{\mathbf{k}'}}} + d_{s'}(\mathbf{k}') v^\dagger(\mathbf{k}', s') \frac{e^{-ik'x}}{\sqrt{2E_{\mathbf{k}'}}} \right] \mathrm{d}^3 k'$$

Tendo $\psi(x)$ e $\psi^\dagger(x)$, vamos aplicar $u^\dagger(\mathbf{k}, s)$ pela esquerda do campo $\psi(x)$ (fazemos isso, pois $\psi(x)$ é um vetor devidos à presença dos espinores, então ao aplicar $u^\dagger(\mathbf{k}, s)$, teremos um "número", o que faz com que as contas possam ser feitas de forma padrão) de modo que

$$u^\dagger(\mathbf{k}, s)\psi(x) = \int \frac{1}{(2\pi)^3} \sum_{s'} \left[b_{s'}(\mathbf{k}') u^\dagger(\mathbf{k}, s) u(\mathbf{k}', s') \frac{e^{-ik'x}}{\sqrt{2E_{\mathbf{k}'}}} + d_{s'}^\dagger(\mathbf{k}) u^\dagger(\mathbf{k}, s) v(\mathbf{k}', s') \frac{e^{ik'x}}{\sqrt{2E_{\mathbf{k}'}}} \right] \mathrm{d}^3 k'$$

Calculando uma transformada de Fourier inversa no espaço desta quantidade, temos

$$\begin{aligned}\int u^\dagger(\mathbf{k},s)\psi(x)e^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathrm{d}^3x &= \iint \frac{1}{(2\pi)^3}\sum_{s'}\Bigg[b_{s'}(\mathbf{k}')u^\dagger(\mathbf{k},s)u(\mathbf{k}',s')\frac{e^{-iE_{\mathbf{k}'}t+i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}}}{\sqrt{2E_{\mathbf{k}'}}}+\\ &\quad + d^\dagger_{s'}(\mathbf{k}')u^\dagger(\mathbf{k},s)v(\mathbf{k}',s')\frac{e^{iE_{\mathbf{k}'}t-i(\mathbf{k}'+\mathbf{k})\cdot\mathbf{x}}}{\sqrt{2E_{\mathbf{k}'}}}\Bigg]\,\mathrm{d}^3k'\,\mathrm{d}^3x\\ &= \iint \frac{e^{-iE_{\mathbf{k}'}t}}{(2\pi)^3\sqrt{2E_{\mathbf{k}'}}}\sum_{s'}\Big[b_{s'}(\mathbf{k}')u^\dagger(\mathbf{k},s)u(\mathbf{k}',s')e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}}+\\ &\quad + d^\dagger_{s'}(\mathbf{k}')u^\dagger(\mathbf{k},s)v(\mathbf{k}',s')e^{2iE_{\mathbf{k}'}t}e^{-i(\mathbf{k}'+\mathbf{k})\cdot\mathbf{x}}\Big]\,\mathrm{d}^3k'\,\mathrm{d}^3x\end{aligned}$$

As integrações em d^3x vão nos dar distribuições delta de Dirac, de modo que

$$\begin{aligned}\int u^\dagger(\mathbf{k},s)\psi(x)e^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathrm{d}^3x = \int \frac{e^{-iE_{\mathbf{k}'}t}}{\sqrt{2E_{\mathbf{k}'}}}\sum_{s'}\Big[&b_{s'}(\mathbf{k}')u^\dagger(\mathbf{k},x)u(\mathbf{k}',s')\delta^3(\mathbf{k}'-\mathbf{k})+\\ &+ d^\dagger_{s'}(\mathbf{k}')u^\dagger(\mathbf{k},s)v(\mathbf{k}',s')e^{2iE_{\mathbf{k}'}t}\delta^3(\mathbf{k}'+\mathbf{k})\Big]\,\mathrm{d}^3k'\end{aligned}$$

Como $E_\mathbf{k} = E_{-\mathbf{k}}$ por construção, temos pela integração em d^3k':

$$\int u^\dagger(\mathbf{k},s)\psi(x)e^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathrm{d}^3x = \frac{e^{-iE_\mathbf{k}t}}{\sqrt{2E_\mathbf{k}}}\sum_{s'}\Big[b_{s'}(\mathbf{k})u^\dagger(\mathbf{k},s)u(\mathbf{k},s') + d^\dagger_{s'}(-\mathbf{k})u^\dagger(\mathbf{k},s)v(-\mathbf{k},s')e^{2iE_\mathbf{k}t}\Big]$$

Considerando a forma explicita dos espinores, o produto entre $u^\dagger(\mathbf{k},s)$ e $u(\mathbf{k},s')$ vai resultar em

$$\begin{aligned}u^\dagger(\mathbf{k},s)u(\mathbf{k},s') &= \frac{1}{E_\mathbf{k}+m}\Big[(E_\mathbf{k}+m)^2\chi^\dagger_s\chi_{s'} + \chi^\dagger_s(\boldsymbol{\sigma}\cdot\mathbf{p})^\dagger(\boldsymbol{\sigma}\cdot\mathbf{p})\chi_{s'}\Big]\\ &= \frac{1}{E_\mathbf{k}+m}\big[(E_\mathbf{k}+m)^2\delta_{ss'} + (E_\mathbf{k}+m)(E_\mathbf{k}-m)\delta_{ss'}\big]\\ &= (E_\mathbf{k}+m)\delta_{ss'} + (E_\mathbf{k}-m)\delta_{ss'}\\ &= 2E_\mathbf{k}\delta_{ss'}\end{aligned}$$

Usando agora a forma explicita para $v(-\mathbf{k},s')$, teremos

$$\begin{aligned}u^\dagger(\mathbf{k},s)v(-\mathbf{k},s') &= \frac{1}{E_\mathbf{k}+m}\Big\{(E_\mathbf{k}+m)\chi^\dagger_s[\boldsymbol{\sigma}\cdot(-\mathbf{k})]\chi_{s'} + (E_{-\mathbf{k}}+m)(\boldsymbol{\sigma}\cdot\mathbf{k})\chi^\dagger_s\chi_{s'}\Big\}\\ &= -(\boldsymbol{\sigma}\cdot\mathbf{k})\chi^\dagger_s\chi_{s'} + (\boldsymbol{\sigma}\cdot\mathbf{k})\chi^\dagger_s\chi_{s'}\\ &= 0\end{aligned}$$

Portanto, a integral que estávamos calculando resulta em

$$\int u^\dagger(\mathbf{k},s)\psi(x)e^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathrm{d}^3x = \frac{e^{-iE_\mathbf{k}t}}{\sqrt{2E_\mathbf{k}}}\sum_{s'}b_{s'}(\mathbf{k})2E_\mathbf{k}\delta_{ss'} = \sqrt{2E_\mathbf{k}}e^{-iE_\mathbf{k}t}b_s(\mathbf{k})$$

Isolando $b_s(\mathbf{k})$, obtemos

$$\begin{aligned} b_s(\mathbf{k}) &= \frac{1}{\sqrt{2E_\mathbf{k}}} e^{iE_\mathbf{k}t} \int u^\dagger(\mathbf{k}, s)\psi(x) e^{-i\mathbf{k}\cdot\mathbf{x}} \,\mathrm{d}^3x \\ &= \int u^\dagger(\mathbf{k}, s) \frac{e^{ikx}}{\sqrt{2E_\mathbf{k}}} \psi(x) \,\mathrm{d}^3x \end{aligned}$$

Podemos ainda manipular o argumento da integral adicionando uma matriz identidade entre o espinor $u^\dagger(\mathbf{k}, s)$ e o campo $\psi(x)$, onde essa matriz identidade pode ser escrita por $I = (\gamma^0)^2$, de modo que

$$u^\dagger(\mathbf{k}, s)\psi(x) = u^\dagger(\mathbf{k}, s)\gamma^0\gamma^0\psi(x) = \bar{u}(\mathbf{k}, s)\gamma^0\psi(x)$$

Ou seja, o argumento da integral fica, com base na definição de $U_k^s(x)$ dada no enunciado:

$$\bar{u}(\mathbf{k}, s)\frac{e^{ikx}}{\sqrt{2E_\mathbf{k}}}\gamma^0\psi(x) = \bar{U}_k^s(x)\gamma^0\psi(x)$$

Concluindo que

$$b_s(\mathbf{k}) = \int \bar{U}_k^s(x)\gamma^0\psi(x) \,\mathrm{d}^3x \tag{6.1}$$

Aplicando agora $v(\mathbf{k}, s)$ pela direita do campo $\psi^\dagger(x)$, pelo mesmo motivo do caso anterior, temos

$$\psi^\dagger(x)v(\mathbf{k}, s) = \int \frac{1}{(2\pi)^3} \sum_{s'} \left[b_{s'}^\dagger(\mathbf{k}')u^\dagger(\mathbf{k}', s')v(\mathbf{k}, s)\frac{e^{ik'x}}{\sqrt{2E_{\mathbf{k}'}}} + d_{s'}(\mathbf{k}')v^\dagger(\mathbf{k}', s')v(\mathbf{k}, s)\frac{e^{-ik'x}}{\sqrt{2E_{\mathbf{k}'}}} \right] \mathrm{d}^3k'$$

Fazendo também uma transformada de Fourier inversa no espaço, temos

$$\begin{aligned} \int \psi^\dagger(x)v(\mathbf{k}, s)e^{-i\mathbf{k}\cdot\mathbf{x}} \,\mathrm{d}^3x &= \iint \frac{1}{(2\pi)^3} \sum_{s'} \left[b_{s'}^\dagger(\mathbf{k}')u^\dagger(\mathbf{k}', s')v(\mathbf{k}, s)\frac{e^{iE_\mathbf{k}t - i(\mathbf{k}'+\mathbf{k})\cdot\mathbf{x}}}{\sqrt{2E_{\mathbf{k}'}}} + \right. \\ &\quad \left. + d_{s'}(\mathbf{k}')v^\dagger(\mathbf{k}', s')v(\mathbf{k}, s)\frac{e^{-iE_{\mathbf{k}'}t + i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}}}{\sqrt{2E_{\mathbf{k}'}}} \right] \mathrm{d}^3k' \,\mathrm{d}^3x \\ &= \iint \frac{e^{-iE_\mathbf{k}'t}}{(2\pi)^3\sqrt{2E_{\mathbf{k}'}}} \sum_{s'} \left[b_{s'}^\dagger(\mathbf{k}')u^\dagger(\mathbf{k}', s')v(\mathbf{k}, s)e^{2iE_{\mathbf{k}'}t}e^{-i(\mathbf{k}'+\mathbf{k})\cdot\mathbf{x}} + \right. \\ &\quad \left. + d_{s'}(\mathbf{k}')v^\dagger(\mathbf{k}', s')v(\mathbf{k}, s)e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}} \right] \mathrm{d}^3k' \,\mathrm{d}^3x \end{aligned}$$

Integrando em d^3x teremos o surgimento das deltas de Dirac tal que

$$\begin{aligned} \int \psi^\dagger(x)v(\mathbf{k}, s)e^{-i\mathbf{k}\cdot\mathbf{x}} \,\mathrm{d}^3x = \int \frac{e^{-iE_\mathbf{k}'t}}{(2\pi)^3\sqrt{2E_{\mathbf{k}'}}} \sum_{s'} &\left[b_{s'}^\dagger(\mathbf{k}')u^\dagger(\mathbf{k}', s')v(\mathbf{k}, s)e^{2iE_{\mathbf{k}'}t}\delta^3(\mathbf{k}'+\mathbf{k}) \right. \\ &\left. + d_{s'}(\mathbf{k}')v^\dagger(\mathbf{k}', s')v(\mathbf{k}, s)\delta^3(\mathbf{k}'-\mathbf{k}) \right] \mathrm{d}^3k' \,\mathrm{d}^3x \end{aligned}$$

E levando novamente em consideração que $E_{\mathbf{k}} = E_{-\mathbf{k}}$, ao realizarmos a integração em d^3k', temos

$$\int \psi^\dagger(x)v(\mathbf{k},s)e^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathrm{d}^3x = \frac{e^{-iE_{\mathbf{k}}t}}{\sqrt{2E_{\mathbf{k}}}}\sum_{s'}\left[b^\dagger_{s'}(-\mathbf{k})u^\dagger(-\mathbf{k},s')v(\mathbf{k},s)e^{2iE_{\mathbf{k}}t} + d_{s'}(\mathbf{k})v^\dagger(\mathbf{k},s')v(\mathbf{k},s)\right]$$

Dado que encontramos $u^\dagger(\mathbf{k},s)v(-\mathbf{k},s')$, podemos simplesmente trocar $\mathbf{k} \leftrightarrow -\mathbf{k}$ e $s \leftrightarrow s'$, de modo que o resultado se mantém, ou seja

$$u^\dagger(-\mathbf{k},s')v(\mathbf{k},s) = 0$$

Já para o segundo produto de espinores, basta considerar a forma explícita, tal que

$$\begin{aligned} v^\dagger(\mathbf{k},s')v(\mathbf{k},s) &= \frac{1}{E_{\mathbf{k}}+m}\left[(\boldsymbol{\sigma}\cdot\mathbf{k})^\dagger\chi^\dagger_{s'}\chi_s(\boldsymbol{\sigma}\cdot\mathbf{k}) + (E_{\mathbf{k}}+m)^2\chi^\dagger_{s'}\chi_s\right] \\ &= \frac{1}{E_{\mathbf{k}}+m}\left[(E_{\mathbf{k}}+m)(E_{\mathbf{k}}-m)\delta_{s's} + (E_{\mathbf{k}}+m)^2\delta_{s's}\right] \\ &= (E_{\mathbf{k}}-m)\delta_{s's} + (E_{\mathbf{k}}+m)\delta_{s's} \\ &= 2E_{\mathbf{k}}\delta_{s's} \end{aligned}$$

Portanto

$$\int \psi^\dagger(x)v(\mathbf{k},s)e^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathrm{d}^3x = \frac{e^{-iE_{\mathbf{k}}t}}{\sqrt{2E_{\mathbf{k}}}}\sum_{s'} d_{s'}(\mathbf{k})2E_{\mathbf{k}}\delta_{s's} = \sqrt{2E_{\mathbf{k}}}e^{-iE_{\mathbf{k}}t}d_s(\mathbf{k})$$

Isolando $d_s(\mathbf{k})$, temos

$$\begin{aligned} d_s(\mathbf{k}) &= \frac{e^{iE_{\mathbf{k}}t}}{\sqrt{2E_{\mathbf{k}}}}\int \psi^\dagger(x)v(\mathbf{k},s)e^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathrm{d}^3x \\ &= \int \psi^\dagger(x)v(\mathbf{k},s)\frac{e^{ikx}}{\sqrt{2E_{\mathbf{k}}}}\,\mathrm{d}^3x \end{aligned}$$

Inserindo a identidade $I = (\gamma^0)^2$ entre $\psi^\dagger(x)$ e $v(\mathbf{k},s)$, podemos reescrever o integrando por

$$\psi^\dagger(x)v(\mathbf{k},s)\frac{e^{ikx}}{\sqrt{2E_{\mathbf{k}}}} = \psi^\dagger(x)\gamma^0\gamma^0v(\mathbf{k},s)\frac{e^{ikx}}{\sqrt{2E_{\mathbf{k}}}} = \bar{\psi}(x)\gamma^0V^s_k(x)$$

Concluindo que

$$d_s(\mathbf{k}) = \int \bar{\psi}(x)\gamma^0V^s_k(x)\,\mathrm{d}^3x \tag{6.2}$$

Q. 07 Píons

Píons são mésons escalares com uma simetria de isospin do grupo SU(2), o mesmo grupo da álgebra de momento angular. Seus campos podem ser escritos na chamada **representação cartesiana**, tal que $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$, ou na chamada **representação esférica**, em que $\pi_0 = \pi_3$ e $\pi^{\pm} = (\pi_1 \mp i\pi_2)/\sqrt{2}$. Os geradores do grupo são as matrizes de Pauli agrupadas como componentes de um isovetor $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$. Considere a lagrangiana

$$\mathcal{L} = \frac{1}{4}\left\langle \partial_\mu \Phi^\dagger \partial^\mu \Phi \right\rangle - \frac{m^2}{4}\left\langle \Phi^\dagger \Phi \right\rangle$$

em que $\Phi = \boldsymbol{\tau} \cdot \boldsymbol{\pi}$ e $\langle A \rangle$ denota o traço da matriz A no espaço (2×2) de isospin.

(a) Na representação esférica, mostre que a lagrangiana acima reproduz as lagrangianas livres de um campo escalar neutro (π_0) e de um campo escalar carregado ($\pi^{\pm}$).

(b) Na representação cartesiana, obtenha a corrente de Noether associada à simetria de isospin. Dica: considere a transformação $U(\boldsymbol{\theta}) = \exp(-i\boldsymbol{\tau} \cdot \boldsymbol{\theta})$, com $\boldsymbol{\theta}$ um vetor de componentes infinitesimais no espaço de isospin, aplicada à matriz Φ: $\Phi' = U\Phi$.

(a) Podemos representar as componentes de $\boldsymbol{\pi}$ por

$$\pi_1 = \frac{\pi^+ + \pi^-}{\sqrt{2}} \qquad \& \qquad \pi_2 = \frac{i(\pi^- - \pi^+)}{\sqrt{2}} \qquad \& \qquad \pi_3 = \pi_0$$

Dessa forma, o campo $\Phi = \boldsymbol{\tau} \cdot \boldsymbol{\sigma}$ pode ser representado matricialmente

$$\begin{aligned}
\Phi &= \tau_1\pi_1 + \tau_2\pi_2 + \tau_3\pi_3 \\
&= \tau_1\left(\frac{\pi^+ + \pi^-}{\sqrt{2}}\right) + \tau_2\left[\frac{i(\pi^- - \pi^+)}{\sqrt{2}}\right] + \tau_3\pi_0 \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\left(\frac{\pi^+ + \pi^-}{\sqrt{2}}\right) + \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\left[\frac{i(\pi^- - \pi^+)}{\sqrt{2}}\right] + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\pi_0 \\
&= \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & \pi^+ + \pi^- \\ \pi^+ + \pi^- & 0 \end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & \pi^- - \pi^+ \\ -\pi^- + \pi^+ & 0 \end{bmatrix} + \begin{bmatrix} \pi_0 & 0 \\ 0 & -\pi_0 \end{bmatrix} \\
&= \begin{bmatrix} \pi_0 & \sqrt{2}\pi^- \\ \sqrt{2}\pi^+ & -\pi_0 \end{bmatrix} = \begin{bmatrix} \pi_3 & \pi_1 + i\pi_2 \\ \pi_1 - i\pi_2 & \pi_3 \end{bmatrix}
\end{aligned}$$

Portanto

$$\Phi^\dagger = \begin{bmatrix} \pi_0^* & \sqrt{2}(\pi^+)^* \\ \sqrt{2}(\pi^-)^* & -\pi_0^* \end{bmatrix}$$

Pela forma definida de $\pi^{\pm}$, é fácil ver que $(\pi^-)^* = \pi^+$ e $(\pi^+)^* = \pi^-$, logo

$$\Phi^\dagger = \begin{bmatrix} \pi_0^* & \sqrt{2}\pi^- \\ \sqrt{2}\pi^+ & -\pi_0^* \end{bmatrix}$$

Segue que

$$\begin{aligned}
\partial_\mu \Phi^\dagger \partial^\mu \Phi &= \begin{bmatrix} \partial_\mu \pi_0^* & \sqrt{2}\partial_\mu \pi^- \\ \sqrt{2}\partial_\mu \pi^+ & -\partial_\mu \pi_0^* \end{bmatrix} \begin{bmatrix} \partial^\mu \pi_0 & \sqrt{2}\partial^\mu \pi^- \\ \sqrt{2}\partial^\mu \pi^+ & -\partial^\mu \pi_0 \end{bmatrix} \\
&= \begin{bmatrix} \partial_\mu \pi_0^* \partial^\mu \pi_0 + 2\partial_\mu \pi^- \partial^\mu \pi^+ & \sqrt{2}\partial_\mu \pi_0^* \partial^\mu \pi^- - \sqrt{2}\partial_\mu \pi^- \partial^\mu \pi_0 \\ \sqrt{2}\partial_\mu \pi^+ \partial^\mu \pi_0 - \sqrt{2}\partial_\mu \pi_0^* \partial^\mu \pi^2 & 2\partial_\mu \pi^+ \partial^\mu \pi^- + \partial_\mu \pi_0^* \partial^\mu \pi_0 \end{bmatrix}
\end{aligned}$$

Calculando o traço desta matriz:

$$\begin{aligned}
\left\langle \partial_\mu \Phi^\dagger \partial^\mu \Phi \right\rangle &= \partial_\mu \pi_0^* \partial^\mu \pi_0 + 2\partial_\mu \pi^- \partial^\mu \pi^+ + 2\partial_\mu \pi^+ \partial^\mu \pi^- + \partial_\mu \pi_0^* \partial^\mu \pi_0 \\
&= 2\partial_\mu \pi_0^* \partial^\mu \pi_0 + 2\partial_\mu \pi^- \partial^\mu \pi^+ + 2\partial_\mu \pi^+ \partial^\mu \pi^-
\end{aligned}$$

Fazendo o mesmo procedimento para $\Phi^\dagger \Phi$, temos

$$\begin{aligned}
\Phi^\dagger \Phi &= \begin{bmatrix} \pi_0^* & \sqrt{2}\pi^- \\ \sqrt{2}\pi^+ & -\pi_0^* \end{bmatrix} \begin{bmatrix} \pi_0 & \sqrt{2}\pi^- \\ \sqrt{2}\pi^+ & -\pi_0 \end{bmatrix} \\
&= \begin{bmatrix} \pi_0^* \pi_0 + 2\pi^- \pi^+ & \sqrt{2}\pi_0^* \pi^- - \sqrt{2}\pi^- \pi_0 \\ \sqrt{2}\pi^+ \pi_0 - \sqrt{2}\pi_0^* \pi^+ & 2\pi^+ \pi^- + \pi_0^* \pi_0 \end{bmatrix}
\end{aligned}$$

Então

$$\begin{aligned}
\left\langle \Phi^\dagger \Phi \right\rangle &= \pi_0^* \pi_0 + 2\pi^- \pi^+ + 2\pi^+ \pi^- + \pi_0^* \pi_0 \\
&= 2\pi_0^* \pi_0 + 2\pi^- \pi^+ + 2\pi^+ \pi^-
\end{aligned}$$

A lagrangiana fica então

$$\begin{aligned}
\mathcal{L} &= \frac{1}{4}\left(2\partial_\mu \pi_0^* \partial^\mu \pi_0 + 2\partial_\mu \pi^- \partial^\mu \pi^+ + 2\partial_\mu \pi^+ \partial^\mu \pi^-\right) - \frac{m^2}{4}\left(2\pi_0^* \pi_0 + 2\pi^- \pi^+ + 2\pi^+ \pi^-\right) \\
&= \left[\frac{1}{2}\partial_\mu \pi_0^* \partial^\mu \pi_0 - \frac{m^2}{2}\pi_0^* \pi_0\right] + \left[\left(\frac{1}{2}\partial_\mu \pi^- \partial^\mu \pi^+ + \frac{1}{2}\partial_\mu \pi^+ \partial^\mu \pi^-\right) - \frac{m^2}{2}\left(\pi^- \pi^+ + \pi^+ \pi^-\right)\right]
\end{aligned}$$

Sabendo que $\pi_0(x)$ é um campo real, temos que $\pi_0^* = \pi_0$, o que nos permite escrever o primeiro $[\cdots]$ da lagrangiana sob a forma de uma lagrangiana livre de um campo escalar neutro para π_0:

$$\mathcal{L}_{\pi_0} = \frac{1}{2}\partial_\mu \pi_0 \partial^\mu \pi_0 - \frac{1}{2}m^2 \pi_0^2 \tag{7.1}$$

No segundo $[\cdots]$, identificamos que $\partial_\mu \pi^- \partial^\mu \pi^+ = \partial_\mu \pi^+ \partial^\mu \pi^-$ e $\pi^- \pi^+ = \pi^+ \pi^-$, nos permitindo escrever uma lagrangiana de campo escalar carregado

$$\mathcal{L}_{\pi^\pm} = \partial_\mu \pi^+ \partial^\mu \pi^- - m^2 \pi^+ \pi^- \tag{7.2}$$

(b) A lagrangiana original pode ser reescrita após calcularmos os traços. O primeiro traço da lagrangiana fica, lembrando que $\Phi^\dagger = \Phi = \boldsymbol{\tau} \cdot \boldsymbol{\pi}$:

$$\begin{aligned}\left\langle \partial_\mu \Phi^\dagger \partial^\mu \Phi \right\rangle &= \langle \partial_\mu (\boldsymbol{\tau} \cdot \boldsymbol{\pi}) \partial^\mu (\boldsymbol{\tau} \cdot \boldsymbol{\pi}) \rangle = \left\langle \sum_{a,b} \partial_\mu (\tau_a \pi_a) \partial^\mu (\tau_b \pi_b) \right\rangle \\ &= \left\langle \sum_{a,b} \tau_a \tau_b (\partial_\mu \pi_a)(\partial^\mu \pi_b) \right\rangle = \sum_{a,b} \langle \tau_a \tau_b \rangle (\partial_\mu \pi_a)(\partial^\mu \pi_b)\end{aligned}$$

As matrizes de Pauli satisfazem a propriedade $\langle \tau_a \tau_b \rangle = 2\delta_{ab}$, portanto

$$\left\langle \partial_\mu \Phi^\dagger \partial^\mu \Phi \right\rangle = 2 \sum_{a,b} \delta_{ab} (\partial_\mu \pi_a)(\partial^\mu \pi_b) = 2(\partial_\mu \pi_a)(\partial^\mu \pi_a) = 2(\partial_\mu \boldsymbol{\pi}) \cdot (\partial^\mu \boldsymbol{\pi})$$

O segundo traço da lagrangiana fica

$$\begin{aligned}\left\langle \Phi^\dagger \Phi \right\rangle &= \langle (\boldsymbol{\tau} \cdot \boldsymbol{\pi})(\boldsymbol{\tau} \cdot \boldsymbol{\pi}) \rangle = \left\langle \sum_{a,b} \tau_a \pi_a \tau_b \pi_b \right\rangle = \sum_{a,b} \pi_a \pi_b \langle \tau_a \tau_b \rangle \\ &= 2 \sum_{a,b} \pi_a \pi_b \delta_{ab} = 2\pi_a \pi_a = 2(\boldsymbol{\pi} \cdot \boldsymbol{\pi})\end{aligned}$$

A lagrangiana na representação cartesiana pode ser escrita então por

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \boldsymbol{\pi}) \cdot (\partial^\mu \boldsymbol{\pi}) - \frac{m^2}{2} (\boldsymbol{\pi} \cdot \boldsymbol{\pi})$$

Considerando uma transformação $U(\boldsymbol{\theta}) = \exp(-i\boldsymbol{\tau} \cdot \boldsymbol{\theta})$, com $\boldsymbol{\theta}$ um vetor de componentes infinitesimais no espaço de isospin, podemos aproximar

$$U(\boldsymbol{\theta}) \approx \mathbb{1}_{2\times 2} - i\boldsymbol{\tau} \cdot \boldsymbol{\theta}$$

Sendo Φ uma matriz 2×2 hermitiana e de traço nulo, ele é um elemento da álgebra de Lie $\mathfrak{su}(2)$, cujos elementos se transformam como $A' = UAU^{-1}$, para uma transformação unitária U em SU(2), então como $U^\dagger(\boldsymbol{\theta}) = U^{-1}(\boldsymbol{\theta})$ e é um elemento de SU(2), uma transformação equivalente à do enunciado que mantém o campo invariante é $\Phi' = U(\boldsymbol{\theta}) \Phi U^\dagger(\boldsymbol{\theta})$. Usar esta transformação ao invés de apenas $\Phi' = U(\boldsymbol{\theta})\Phi$ é possível, pois ambas vão gerar a mesma variação dos campos π_j (basta utilizar a forma $\Phi = \boldsymbol{\tau} \cdot \boldsymbol{\pi}$ em $\delta\Phi$ e identificar a parte que representa a variação nos campos π_j) , a única diferença seria o fato de $U(\boldsymbol{\theta})\Phi$ não manter a hermiticidade e o traço nulo ao determinar $\delta\Phi$, logo, utilizo a transformação padrão de elementos de uma álgebra de Lie $\mathfrak{su}(2)$ apenas por buscar manter essas duas propriedades em $\delta\Phi$. Portanto

$$\begin{aligned}\Phi' = U(\boldsymbol{\theta}) \Phi U^\dagger(\boldsymbol{\theta}) &= [\Phi - i(\boldsymbol{\tau} \cdot \boldsymbol{\theta})\Phi][\mathbb{1}_{2\times 2} + i(\boldsymbol{\tau} \cdot \boldsymbol{\theta})] \\ &= \Phi + i\Phi(\boldsymbol{\tau} \cdot \boldsymbol{\theta}) - i(\boldsymbol{\tau} \cdot \boldsymbol{\theta})\Phi + \mathcal{O}(\boldsymbol{\theta}^2) \\ &= \Phi + i[\Phi, (\boldsymbol{\tau} \cdot \boldsymbol{\theta})] + \mathcal{O}(\boldsymbol{\theta}^2)\end{aligned}$$

Ignorando os termos quadráticos em $\boldsymbol{\theta}$ por ele ter componentes infinitesimais, temos que a variação do campo nos dá

$$\delta\Phi = i\sum_a [\Phi, \tau_a]\theta_a = i\sum_{a,b}[\tau_b, \tau_a]\theta_a\pi_b$$

As matrizes de Pauli por serem geradores de uma álgebra de Lie satisfazem a relação de comutação

$$[\tau_a, \tau_b] = \sum_c 2i\epsilon_{abc}\tau_c$$

ou seja

$$\delta\Phi = -2\sum_{a,b,c}\epsilon_{bac}\tau_c\theta_a\pi_b = 2\sum_{a,b,c}\epsilon_{abc}\tau_c\theta_a\pi_b$$

Podemos considerar que

$$\delta\Phi = \delta\left(\sum_c \tau_c\pi_c\right) = \sum_c \tau_c\delta\pi_c = 2\sum_{a,b,c}\epsilon_{abc}\tau_c\theta_a\pi_b$$

Então a variação do campo fica

$$\delta\pi_c = 2\sum_{a,b}\epsilon_{abc}\theta_a\pi_b$$

Para obter a corrente de Noether, consideramos a simetria global em θ_k, de tal forma que para $k = 1, 2, 3$, temos

$$J_k^\mu = \sum_c \frac{\delta\mathcal{L}}{\delta(\partial_\mu\pi_c)}\frac{\delta\pi_c}{\delta\theta_k}$$

O segundo termo é facilmente obtido, tal que

$$\frac{\delta\pi_c}{\delta\theta_k} = 2\sum_{a,b}\epsilon_{abc}\delta_{ak}\pi_b = 2\sum_b \epsilon_{kbc}\pi_b$$

Já o primeiro

$$\begin{aligned}
\frac{\delta\mathcal{L}}{\delta(\partial_\mu\pi_c)} &= \frac{1}{2}\frac{\delta}{\delta(\partial_\mu\pi_c)}\left[\sum_d \partial_\nu\pi_d\partial^\nu\pi_d\right] = \frac{1}{2}\sum_d\left[\frac{\delta(\partial_\nu\pi_d)}{\delta(\partial_\mu\pi_c)}\partial^\nu\pi_d + \partial_\nu\pi_d g^{\nu\beta}\frac{\delta(\partial_\beta\pi_d)}{\delta(\partial_\mu\pi_c)}\right] \\
&= \frac{1}{2}\sum_d\left[\delta_\nu^\mu\delta_{cd}\partial^\nu\pi_d + \partial_\nu\pi_d g^{\nu\beta}\partial_\beta^\mu\delta_{cd}\right] = \frac{1}{2}\sum_d[\delta_{cd}\partial^\mu\pi_d + \partial^\mu\pi_d\delta_{cd}] \\
&= \frac{1}{2}[\partial^\mu\pi_c + \partial^\mu\pi_c] = \partial^\mu\pi_c
\end{aligned}$$

Logo

$$J_k^\mu = 2\sum_c \partial^\mu\pi_c\sum_b \epsilon_{kbc}\pi_b = 2\sum_{b,c}\epsilon_{kbc}\pi_b\partial^\mu\pi_c$$

Juntando todas as componentes k, concluímos que a corrente de Noether é

$$J^\mu = 2\boldsymbol{\pi}\times\partial^\mu\boldsymbol{\pi} \tag{7.3}$$

Q. 08 Redução LSZ (bóson–bóson)

Considere a redução LSZ de um espalhamento com dois bósons escalares, tanto no estado inicial, quanto no estado final. A primeira redução fornece

$$ {}_{\text{out}}\langle \mathbf{p}_1'\mathbf{p}_2'|\mathbf{p}_1\mathbf{p}_2\rangle_{\text{in}} = \text{“1”} + (iZ^{-1/2}) \int {}_{\text{out}}\langle \mathbf{p}_1'\mathbf{p}_2'|\, \phi(x_1)\, |\mathbf{p}_2\rangle_{\text{in}} \overleftarrow{K}_{x_1} \tilde{f}_{p_1}(x_1)\, \mathrm{d}^4x_1 $$

Conclua a segunda redução (em quaisquer estados bosônicos remanescentes), demonstrando a necessidade de se introduzir o operador de ordenamento temporal.

Continuando a redução LSZ para o estado bosônico $|\mathbf{p}_2\rangle_{\text{in}}$, temos

$$ |\mathbf{p}_2\rangle_{\text{in}} = \sqrt{2\omega_{\mathbf{p}_2}}\, a^\dagger(\mathbf{p}_2)\, |\mathbf{0}\rangle_{\text{in}} $$

que pela convergência fraca nos permite escrever

$$ |\mathbf{p}_2\rangle_{\text{in}} = Z^{-1/2}\sqrt{2\omega_{\mathbf{p}_2}} \lim_{t_2\to-\infty} a^\dagger(\mathbf{p}_2, t_2)\, |\mathbf{0}\rangle_{\text{in}} $$

Logo, definindo $A \coloneqq {}_{\text{out}}\langle \mathbf{p}_1'\mathbf{p}_2'|\mathbf{p}_1\mathbf{p}_2\rangle_{\text{in}}$, obtemos

$$ A = \text{“1”} + i(Z^{-1/2})^2 \lim_{t_2\to-\infty} \int {}_{\text{out}}\langle \mathbf{p}_1'\mathbf{p}_2'|\, \phi(x_1)\sqrt{2\omega_{\mathbf{p}_2}}\, a^\dagger(\mathbf{p}_2, t_2)\, |\mathbf{0}\rangle_{\text{in}} \overleftarrow{K}_{x_1} \tilde{f}_{p_1}(x_1)\, \mathrm{d}^4x_1 $$

Podemos fazer uma substituição considerando uma integral de $-\infty$ a $+\infty$ de uma derivada total no tempo t_2, isto é

$$ \begin{aligned} \int_{-\infty}^{\infty} \partial_0 \Big[{}_{\text{out}}\langle \mathbf{p}_1'\mathbf{p}_2'| \cdots a^\dagger(\mathbf{p}_2, t_2)\, |\mathbf{0}\rangle_{\text{in}} \Big] \mathrm{d}t_2 = \lim_{t_2\to+\infty} \Big[{}_{\text{out}}\langle \mathbf{p}_1'\mathbf{p}_2'| \cdots a^\dagger(\mathbf{p}_2, t_2)\, |\mathbf{0}\rangle_{\text{in}} \Big] - \\ - \lim_{t_2\to-\infty} \Big[{}_{\text{out}}\langle \mathbf{p}_1'\mathbf{p}_2'| \cdots a^\dagger(\mathbf{p}_2, t_2)\, |\mathbf{0}\rangle_{\text{in}} \Big] \end{aligned} $$

Notemos então que

$$ Z^{-1/2} \lim_{t_2\to+\infty} {}_{\text{out}}\langle \mathbf{p}_1'\mathbf{p}_2'|\, \phi(x_1)\sqrt{2\omega_{\mathbf{p}_2}}\, a^\dagger(\mathbf{p}_2, t_2)\, |\mathbf{0}\rangle_{\text{in}} = {}_{\text{out}}\langle \mathbf{p}_1'\mathbf{p}_2'|\, \phi(x_1)\sqrt{2\omega_{\mathbf{p}_2}}\, a^\dagger(\mathbf{p}_2)\, |\mathbf{0}\rangle_{\text{in}} $$

e isto vai ser diferente de zero apenas se $|\mathbf{p}_2\rangle_{\text{in}}$ for igual a pelo menos um dos estados ${}_{\text{out}}\langle \mathbf{p}_1'|$ ou ${}_{\text{out}}\langle \mathbf{p}_2'|$ de modo que isso vai contribuir apenas para o processo de 1 partícula inicial transformando-se em 1 partícula final, o que torna-se irrelevante para o processo onde temos 2 partículas iniciais transformando-se em 2 partículas finais que estamos interessados, de modo que podemos fazer

$$ Z^{-1/2} \lim_{t_2\to+\infty} {}_{\text{out}}\langle \mathbf{p}_1'\mathbf{p}_2'|\, \phi(x_1)\sqrt{2\omega_{\mathbf{p}_2}}\, a^\dagger(\mathbf{p}_2, t_2)\, |\mathbf{0}\rangle_{\text{in}} \mapsto \text{“1”} $$

Fazendo a substituição dentro do que estamos interessados, ficamos com

$$A = \text{“1”} - i(Z^{-1/2})^2 \iint_{-\infty}^{\infty} \partial_0 \Big[{}_{\text{out}}\langle \mathbf{p}_1'\mathbf{p}_2' |\, \phi(x_1) \sqrt{2\omega_{\mathbf{p}_2}} a^\dagger(\mathbf{p}_2, t_2) \,|\mathbf{0}\rangle_{\text{in}} \Big] \overleftarrow{K}_{x_1} \tilde{f}_{p_1}(x_1)\, \mathrm{d}t_2\, \mathrm{d}^4x_1$$

Como a derivação é feita apenas em t_2, o único termo dependente desta variável é $a^\dagger(\mathbf{p}_2, t_2)$, portanto, expandindo explicitamente o operador interpolante, temos

$$\begin{aligned} \partial_0 a^\dagger(\mathbf{p}_2, t_2) &= \partial_0 \left[-i \int f_{p_2}(x_2) \overleftrightarrow{\partial_0} \phi(x_2)\, \mathrm{d}^3x_2 \right] \\ &= -i \int \partial_0 \left[f_{p_2}(x_2) \overleftrightarrow{\partial_0} \phi(x_2) \right] \mathrm{d}^3x_2 \end{aligned}$$

No exercício **2.(b)**, calculamos esta mesma derivada, porém consideramos o campo $\phi_0(x)$, que é um campo livre, porém $\phi(x_2)$ é um campo interpolante e não necessariamente satisfaz a equação de Klein-Gordon, de modo que

$$\partial_0 a^\dagger(\mathbf{p}_2, t_2) = -i \int f_{p_2}(x_2) \big(\partial_\mu \partial^\mu + m^2\big)_{x_2} \phi(x_2)\, \mathrm{d}^3x_2$$

Sendo então

$$\phi(x_2) \overleftarrow{K}_{x_2} = \big(\partial_\mu \partial^\mu + m^2\big)_{x_2} \phi(x_2)$$

Podemos escrever que

$$A = \text{“1”} + (iZ^{-1/2})^2 \iiint_{-\infty}^{\infty} {}_{\text{out}}\langle \mathbf{p}_1'\mathbf{p}_2' |\, \phi(x_1)\phi(x_2) \,|\mathbf{0}\rangle_{\text{in}}\, \overleftarrow{K}_{x_1} \tilde{f}_{p_1}(x_1) \overleftarrow{K}_{x_2} \tilde{f}_{p_2}(x_2)\, \mathrm{d}t_2\, \mathrm{d}^3x_2\, \mathrm{d}^4x_1$$

Ou seja

$$A = \text{“1”} + (iZ^{-1/2})^2 \int {}_{\text{out}}\langle \mathbf{p}_1'\mathbf{p}_2' |\, \phi(x_1)\phi(x_2) \,|\mathbf{0}\rangle_{\text{in}}\, \overleftarrow{K}_{x_1} \overleftarrow{K}_{x_2} \tilde{f}_{p_1}(x_1) \tilde{f}_{p_2}(x_2)\, \mathrm{d}^4x_1\, \mathrm{d}^4x_2$$

Continuando agora a redução para ${}_{\text{out}}\langle \mathbf{p}_1'\mathbf{p}_2' |$, temos pela convergência fraca

$${}_{\text{out}}\langle \mathbf{p}_1'\mathbf{p}_2' | = Z^{-1/2} \lim_{\tau_2 \to +\infty} {}_{\text{out}}\langle \mathbf{p}_1' |\, a(\mathbf{p}_2', \tau_2) \sqrt{2\omega_{\mathbf{p}_2'}}$$

O que fornece a expressão

$$\begin{aligned} A = \text{“1”} + i^2 (Z^{-1/2})^3 \lim_{\tau_2 \to \infty} \int {}_{\text{out}}\langle \mathbf{p}_1' |\, a(\mathbf{p}_2', \tau_2) \sqrt{2\omega_{\mathbf{p}_2'}} \phi(x_1)\phi(x_2) \,|\mathbf{0}\rangle_{\text{in}} \times \\ \times \overleftarrow{K}_{x_1} \overleftarrow{K}_{x_2} \tilde{f}_{p_1}(x_1) \tilde{f}_{p_2}(x_2)\, \mathrm{d}^4x_1\, \mathrm{d}^4x_2 \end{aligned}$$

Usando novamente uma integração de $-\infty$ a $+\infty$ em uma derivada total no tempo, mas agora em relação à τ_2, teremos

$$\begin{aligned} \int_{-\infty}^{\infty} \partial_0 \Big[{}_{\text{out}}\langle \mathbf{p}_1' |\, a(\mathbf{p}_2', \tau_2) \cdots |\mathbf{0}\rangle_{\text{in}} \Big] \mathrm{d}\tau_2 = \lim_{\tau_2 \to +\infty} \Big[{}_{\text{out}}\langle \mathbf{p}_1' |\, a(\mathbf{p}_2', \tau_2) \cdots |\mathbf{0}\rangle_{\text{in}} \Big] - \\ - \lim_{\tau_2 \to -\infty} \Big[{}_{\text{out}}\langle \mathbf{p}_1' |\, a(\mathbf{p}_2', \tau_2) \cdots |\mathbf{0}\rangle_{\text{in}} \Big] \end{aligned}$$

Neste caso, o termo de interação que não nos interessa vai ser

$$Z^{-1/2} \lim_{\tau_2\to-\infty} {}_{\text{out}}\langle \mathbf{p}_1' |\, a(\mathbf{p}_2', \tau_2)\sqrt{2\omega_{\mathbf{p}_2'}}\phi(x_1)\phi(x_2)\,|\mathbf{0}\rangle_{\text{in}} = Z^{-1/2}\; {}_{\text{out}}\langle \mathbf{p}_1' |\, a(\mathbf{p}_2')\sqrt{2\omega_{\mathbf{p}_2'}}\phi(x_1)\phi(x_2)\,|\mathbf{0}\rangle_{\text{in}}$$

por um motivo análogo ao anterior, portanto inserimos este termo em "1". Com a substituição pela integral, temos então

$$A = \text{“1”} + i^2(Z^{-1/2})^3 \iint_{-\infty}^{\infty} \partial_0 \Big[{}_{\text{out}}\langle \mathbf{p}_1' |\, a(\mathbf{p}_2', \tau_2)\sqrt{2\omega_{\mathbf{p}_2'}}\phi(x_1)\phi(x_2)\,|\mathbf{0}\rangle_{\text{in}} \Big] \times$$
$$\times \overleftarrow{K}_{x_1}\overleftarrow{K}_{x_2}\tilde{f}_{p_1}(x_1)\tilde{f}_{p_2}(x_2)\, \mathrm{d}\tau_2\, \mathrm{d}^4x_1\, \mathrm{d}^4x_2$$

Dado que a derivada ∂_0 é feita em τ_2, ela vai ser aplicada apenas em $a(\mathbf{p}_2', \tau_2)$, que explicitamente se escreve

$$\partial_0 a(\mathbf{p}_2', \tau_2) = i \int \partial_0 \Big[f^*_{p_2'}(y_2)\overleftrightarrow{\partial_0}\phi(y_2) \Big]\, \mathrm{d}^3y_2$$

que levando em conta que $\phi(y_2)$ é um campo interpolante e não satisfaz necessariamente a equação de Klein-Gordon, o desenvolvimento do exercício **2.(b)** nos fornece

$$\partial_0 a(\mathbf{p}_2', \tau_2) = i \int f^*_{p_2'}(y_2)\big(\partial_\mu\partial^\mu + m^2\big)_{y_2}\phi(y_2)\, \mathrm{d}^3y_2$$

Sendo então

$$\overrightarrow{K}_{y_2}\phi(y_2) = \big(\partial_\mu\partial^\mu + m^2\big)_{y_2}\phi(y_2)$$

Escrevemos

$$A = \text{“1”} + (iZ^{-1/2})^3 \iiint_{-\infty}^{\infty} \tilde{f}^*_{p_2'}(y_2)\overrightarrow{K}_{y_2}\; {}_{\text{out}}\langle \mathbf{p}_1' |\, \phi(y_2)\phi(x_1)\phi(x_2)\,|\mathbf{0}\rangle_{\text{in}} \times$$
$$\times \overleftarrow{K}_{x_1}\overleftarrow{K}_{x_2}\tilde{f}_{p_1}(x_1)\tilde{f}_{p_2}(x_2)\, \mathrm{d}\tau_2\, \mathrm{d}^3y_2\, \mathrm{d}^4x_1\, \mathrm{d}^4x_2$$

ou seja

$$A =; \text{“1”} + (iZ^{-1/2})^3 \int \tilde{f}^*_{p_2'}(y_2)\overrightarrow{K}_{y_2}\; {}_{\text{out}}\langle \mathbf{p}_1' |\, \phi(y_2)\phi(x_1)\phi(x_2)\,|\mathbf{0}\rangle_{\text{in}} \times$$
$$\times \overleftarrow{K}_{x_1}\overleftarrow{K}_{x_2}\tilde{f}_{p_1}(x_1)\tilde{f}_{p_2}(x_2)\, \mathrm{d}^4y_2\, \mathrm{d}^4x_1\, \mathrm{d}^4x_2$$

Tendo feito o procedimento para ${}_{\text{out}}\langle \mathbf{p}_2' |$, a extensão para ${}_{\text{out}}\langle \mathbf{p}_1' |$ torna-se imediata, tal que

$$A = \text{“1”} + (iZ^{-1/2})^4 \int \tilde{f}^*_{p_1'}(y_1)\tilde{f}^*_{p_2'}(y_2)\overrightarrow{K}_{y_1}\overrightarrow{K}_{y_2}\; {}_{\text{out}}\langle \mathbf{0} |\, \phi(y_1)\phi(y_2)\phi(x_1)\phi(x_2)\,|\mathbf{0}\rangle_{\text{in}} \times$$
$$\times \overleftarrow{K}_{x_1}\overleftarrow{K}_{x_2}\tilde{f}_{p_1}(x_1)\tilde{f}_{p_2}(x_2)\, \mathrm{d}^4y_1\, \mathrm{d}^4y_2\, \mathrm{d}^4x_1\, \mathrm{d}^4x_2$$

Analisando então apenas ${}_{\text{out}}\langle \mathbf{0} |\, \phi(y_1)\phi(y_2)\phi(x_1)\phi(x_2)\,|\mathbf{0}\rangle_{\text{in}}$, temos que pensar do fato de que todos os campos interpolantes são combinações de operadores de criação e aniquilação interpolantes, de modo que a ocorrência de operadores $a^\dagger(\mathbf{p}, \infty)$ podem ser problemáticos quando estiverem à direita, pois indicaria uma aniquilação do vácuo inicial $|\mathbf{0}\rangle_{\text{in}}$, de modo que queremos colocar estes operadores à esquerda a fim de ser aplicado a ${}_{\text{out}}\langle\mathbf{0}|$ e criar um estado do tipo ${}_{\text{out}}\langle\mathbf{p}|$, um argumento análogo ocorre ao pensarmos na ocorrência de operadores $a(\mathbf{p}, -\infty)$, pois se estes estiverem à esquerda, haveria uma indicação de aniquilação do vácuo final ${}_{\text{out}}\langle\mathbf{0}|$, então queremos que esse tipo de operador fique à direita para criar estados do tipo $|\mathbf{p}\rangle_{\text{in}}$ ao serem aplicados ao vácuo inicial $|\mathbf{0}\rangle_{\text{in}}$. Estas ocorrências aparecem com o fato de trocarmos os limites por integrais em derivadas totais e ignorarmos os termos desconexos. Para evitar estes problemas, temos a necessidade de utilizar o operador de ordenamento temporal $T\{\cdots\}$ nos campos, concluindo que a redução LSZ completa é dada por

$$\begin{aligned} {}_{\text{out}}\langle\mathbf{p}_1'\mathbf{p}_2'|\mathbf{p}_1\mathbf{p}_2\rangle_{\text{in}} = \text{“1”} + (iZ^{-1/2})^4 \int \tilde{f}^*_{p_1'}(y_1)\tilde{f}^*_{p_2'}(y_2)\overrightarrow{K}_{y_1}\overrightarrow{K}_{y_2}\times \\ \times\, {}_{\text{out}}\langle\mathbf{0}|\, T\{\phi(y_1)\phi(y_2)\phi(x_1)\phi(x_2)\}\, |\mathbf{0}\rangle_{\text{in}}\, \overleftarrow{K}_{x_1}\overleftarrow{K}_{x_2}\tilde{f}_{p_1}(x_1)\tilde{f}_{p_2}(x_2)\, \mathrm{d}^4y_1\, \mathrm{d}^4y_2\, \mathrm{d}^4x_1\, \mathrm{d}^4x_2 \end{aligned} \tag{8.1}$$

Q. 09 Redução LSZ (férmion–anti-férmion)

Considere o espalhamento entre um férmion e um anti-férmion. A normalização do campo interpolante com relação ao campo livre inicial é dada por

$$\lim_{t\to-\infty} {}_{\text{out}}\langle\beta|\,\psi_\mu(t)\,|\alpha\rangle_{\text{in}} = Z^{1/2}\;{}_{\text{out}}\langle\beta|\,\psi_\mu^{\text{in}}(t)\,|\alpha\rangle_{\text{in}}$$

Com as mesmas convenções do exercício 6, derive a redução LSZ para este processo:

$$\begin{aligned} {}_{\text{out}}\big\langle\mathbf{q}_1\bar{\mathbf{q}}_1\big|\mathbf{p}_1\bar{\mathbf{p}}_1\big\rangle_{\text{in}} = \text{“1”} + (-iZ^{-1/2})^2(iZ^{-1/2})^2 \int \bar{\tilde{U}}^s_{q_1}(y_1)\overrightarrow{F}_{y_1}\bar{\tilde{V}}^s_{\bar{p}_1}(\bar{x}_1)\overrightarrow{F}_{\bar{x}_1}\times \\ \times\;{}_{\text{out}}\langle 0|\,T\{\bar{\psi}(\bar{y}_1)\psi(y_1)\bar{\psi}(x_1)\psi(\bar{x}_1)\}\,|0\rangle_{\text{in}}\,\overleftarrow{F}_{x_1}\tilde{U}^s_{p_1}(x_1)\overleftarrow{F}_{\bar{y}_1}\tilde{V}^s_{\bar{q}_1}(\bar{y}_1)\,\mathrm{d}^4x_1\,\mathrm{d}^4\bar{x}_1\,\mathrm{d}^4y_1\,\mathrm{d}^4\bar{y}_1 \end{aligned}$$

sendo $\overrightarrow{F}_{y_1} = (i\partial\!\!\!/ - m)_{y_1}$.

A convergência fraca nos dá a normalização do campo interpolante em relação ao campo livre inicial, de modo que podemos escrever

$$|\mathbf{p}_1\bar{\mathbf{p}}_1\rangle_{\text{in}} = Z_2^{-1/2}\lim_{t_1\to-\infty}\sqrt{2E_{\mathbf{p}_1}}\,b_s^\dagger(\mathbf{p}_1,t_1)\,|\bar{\mathbf{p}}_1\rangle_{\text{in}}$$

Definindo então $A \coloneqq {}_{\text{out}}\big\langle\mathbf{q}_1\bar{\mathbf{q}}_1\big|\mathbf{p}_1\bar{\mathbf{p}}_1\big\rangle_{\text{in}}$, temos

$$A = Z_2^{-1/2}\lim_{t_1\to-\infty}\;{}_{\text{out}}\big\langle\mathbf{q}_1\bar{\mathbf{q}}_1\big|\,\sqrt{2E_{\mathbf{p}_1}}\,b_s^\dagger(\mathbf{p}_1,t_1)\,|\bar{\mathbf{p}}_1\rangle_{\text{in}}$$

Considerando então uma integral em $\mathrm{d}t_1$ de uma derivada total temporal também em t_1, escrevemos

$$\begin{aligned}\int_{-\infty}^{\infty}\partial_0\Big[\,{}_{\text{out}}\big\langle\mathbf{q}_1\bar{\mathbf{q}}_1\big|\,\sqrt{2E_{\mathbf{p}_1}}\,b_s^\dagger(\mathbf{p}_1,t_1)\,|\bar{\mathbf{p}}_1\rangle_{\text{in}}\Big]\,\mathrm{d}t_1 = \lim_{t_1\to\infty}\Big[\,{}_{\text{out}}\big\langle\mathbf{q}_1\bar{\mathbf{q}}_1\big|\,\sqrt{2E_{\mathbf{p}_1}}\,b_s^\dagger(\mathbf{p}_1,t_1)\,|\bar{\mathbf{p}}_1\rangle_{\text{in}}\Big] - \\ - \lim_{t_1\to-\infty}\Big[\,{}_{\text{out}}\big\langle\mathbf{q}_1\bar{\mathbf{q}}_1\big|\,\sqrt{2E_{\mathbf{p}_1}}\,b_s^\dagger(\mathbf{p}_1,t_1)\,|\bar{\mathbf{p}}_1\rangle_{\text{in}}\Big]\end{aligned}$$

Note então que

$$Z_2^{-1/2}\lim_{t_1\to\infty}\Big[\,{}_{\text{out}}\big\langle\mathbf{q}_1\bar{\mathbf{q}}_1\big|\,\sqrt{2E_{\mathbf{p}_1}}\,b_s^\dagger(\mathbf{p}_1,t_1)\,|\bar{\mathbf{p}}_1\rangle_{\text{in}}\Big]$$

será não nulo apenas para casos irrelevantes, onde 1 férmion de momento $\mathbf{p}_1$ se transforma em um outro férmion de momento $\mathbf{q}_1$ ou um anti-férmion de momento $\bar{\mathbf{q}}_1$, que não é o caso de interesse, logo fazemos este termo ser “1”. Portanto

$$A = \text{“1”} - Z_2^{-1/2}\int_{-\infty}^{\infty}\partial_0\Big[\,{}_{\text{out}}\big\langle\mathbf{q}_1\bar{\mathbf{q}}_1\big|\,\sqrt{2E_{\mathbf{p}_1}}\,b_s^\dagger(\mathbf{p}_1,t_1)\,|\bar{\mathbf{p}}_1\rangle_{\text{in}}\Big]\,\mathrm{d}t_1$$

onde essa derivada só se aplica no operador $b_s^\dagger(\mathbf{p}_1, t_1)$, sendo então a quantidade que precisamos calcular. Sabendo a forma de $b_s(\mathbf{k})$, conforme (6.1), fica claro que

$$b_s^\dagger(\mathbf{p}_1, t_1) = \int \bar{\psi}(x_1)\gamma^0 U_{p_1}^s(x_1)\, \mathrm{d}^3 x_1$$

Segue que a derivada em relação à t_1 nos dá

$$\partial_0 b_s^\dagger(\mathbf{p}_1, t_1) = \int \left[\partial_0\bar{\psi}(x_1)\right]\gamma^0 U_{p_1}^s(x_1)\, \mathrm{d}^3 x_1 + \int \bar{\psi}(x_1)\gamma^0\left[\partial_0 U_{p_1}^s(x_1)\right] \mathrm{d}^3 x_1$$

Notemos que $U_{p_1}^s(x_1)$ satisfaz a equação de Dirac, tal que

$$(i\partial\!\!\!/ - m)U_{p_1}^s(x_1) = 0$$

portanto

$$i\gamma^0\partial_0 U_{p_1}^s(x_1) + i\gamma^k\partial_k U_{p_1}^s(x_1) - mU_{p_1}^s(x_1) = 0$$

o que nos dá, multiplicando tudo por $-i$ e isolando o termo de interesse

$$\begin{aligned}\gamma^0\partial_0 U_{p_1}^s(x_1) &= -imU_{p_1}^s(x_1) - \gamma^k\partial_k U_{p_1}^s(x_1) \\ &= -imU_{p_1}^s(x_1) - \gamma^k(ip_k)U_{p_1}^s(x_1)\end{aligned}$$

Então a segunda integral pode ser substituída, de modo que

$$\partial_0 b_s^\dagger(\mathbf{p}_1, t_1) = \int \left[\partial_0\bar{\psi}(x_1)\gamma^0\right]U_{p_1}^s(x_1)\, \mathrm{d}^3 x_1 + \int \bar{\psi}(x_1)\Big[-im - i\gamma^k p_k\Big]U_{p_1}^s(x_1)\, \mathrm{d}^3 x$$

Em uma integração por partes no segundo termo, consideramos que o campo desaparece nos limites assintóticos, e obtemos

$$\begin{aligned}\partial_0 b_s^\dagger(\mathbf{p}_1, t_1) &= \int \Big[\partial_0\bar{\psi}(x_1)\gamma^0 + \partial_k\bar{\psi}(x_1)\gamma^k - i\bar{\psi}(x_1)m\Big]U_{p_1}^s(x_1)\, \mathrm{d}^3 x_1 \\ &= i\int \Big[-i\partial_0\bar{\psi}(x_1)\gamma^0 - i\partial_k\bar{\psi}(x_1)\gamma^k - \bar{\psi}(x_1)m\Big]U_{p_1}^s(x_1)\, \mathrm{d}^3 x_1 \\ &= i\int \left[-i\partial_\mu\bar{\psi}(x_1)\gamma^\mu - \bar{\psi}(x_1)m\right]U_{p_1}^s(x_1)\, \mathrm{d}^3 x_1\end{aligned}$$

Sendo então

$$-i\partial_\mu\bar{\psi}(x_1)\gamma^\mu - \bar{\psi}(x_1)m = \bar{\psi}(x_1)\big(-i\overleftarrow{\partial\!\!\!/} - m\big)_{x_1} = \bar{\psi}(x_1)\overleftarrow{F}_{x_1}$$

obtemos que

$$\partial_0 b_s^\dagger(\mathbf{p}_1, t_1) = i\int \bar{\psi}(x_1)\overleftarrow{F}_{x_1} U_{p_1}^s(x_1)\, \mathrm{d}^3 x_1$$

Portanto

$$A = \text{“1”} - iZ_2^{-1/2}\iint_{-\infty}^{\infty} {}_{\text{out}}\langle\mathbf{q}_1\bar{\mathbf{q}}_1|\,\bar{\psi}(x_1)\,|\bar{\mathbf{p}}_1\rangle_{\text{in}}\,\overleftarrow{F}_{x_1}\sqrt{2E_{\mathbf{p}_1}}U_{p_1}^s(x_1)\, \mathrm{d}^3 x_1\, \mathrm{d}t_1$$

Como $\tilde{U}^s_{p_1}(x_1) = \sqrt{2E_{\mathbf{p}_1}}U^s_{p_1}(x_1)$ (exercício 6), concluindo que a primeira redução fica

$$A = \text{“1”} - iZ_2^{-1/2}\int {}_{\text{out}}\langle \mathbf{q}_1\bar{\mathbf{q}}_1|\, \bar{\psi}(x_1)\, |\bar{\mathbf{p}}_1\rangle_{\text{in}}\, \overleftarrow{F}_{x_1}\tilde{U}^s_{p_1}(x_1)\, \mathrm{d}^4x_1$$

Seguindo com a redução, podemos escrever

$$|\bar{\mathbf{p}}_1\rangle_{\text{in}} = Z_2^{-1/2}\lim_{\bar{t}_1\to-\infty}\sqrt{2E_{\bar{\mathbf{p}}_1}}d_s^\dagger(\bar{\mathbf{p}}_1,\bar{t}_1)\,|\mathbf{0}\rangle_{\text{in}}$$

Portanto

$$A = \text{“1”} - i(Z_2^{-1/2})^2\lim_{\bar{t}_1\to-\infty}\int {}_{\text{out}}\langle \mathbf{q}_1\bar{\mathbf{q}}_1|\, \bar{\psi}(x_1)\sqrt{2E_{\bar{\mathbf{p}}_1}}d_s^\dagger(\bar{\mathbf{p}}_1,\bar{t}_1)\,|\mathbf{0}\rangle_{\text{in}}\, \overleftarrow{F}_{x_1}\tilde{U}^s_{p_1}(x_1)\, \mathrm{d}^4x_1$$

Usando novamente o argumento de uma integração em uma derivada total, mas agora $\bar{t}_1$, temos

$$\int_{-\infty}^{\infty}\partial_0\Big[{}_{\text{out}}\langle \mathbf{q}_1\bar{\mathbf{q}}_1|\cdots d_s^\dagger(\bar{\mathbf{p}}_1,\bar{t}_1)\,|\mathbf{0}\rangle_{\text{in}}\Big]\,\mathrm{d}\bar{t}_1 = \lim_{\bar{t}_1\to\infty}{}_{\text{out}}\langle \mathbf{q}_1\bar{\mathbf{q}}_1|\cdots d_s^\dagger(\bar{\mathbf{p}}_1,\bar{t}_1)\,|\mathbf{0}\rangle_{\text{in}} - \\ - \lim_{\bar{t}_1\to-\infty}{}_{\text{out}}\langle \mathbf{q}_1\bar{\mathbf{q}}_1|\cdots d_s^\dagger(\bar{\mathbf{p}}_1,\bar{t}_1)\,|\mathbf{0}\rangle_{\text{in}}$$

O limite de $\bar{t}_1 \to \infty$ vai gerar um termo desconexo (não relevante para interação que estamos interessados), tal que o adicionamos a “1” e ficamos com

$$A = \text{“1”} - i(Z_2^{-1/2})^2\iint_{-\infty}^{\infty} {}_{\text{out}}\langle \mathbf{q}_1\bar{\mathbf{q}}_1|\, \bar{\psi}(x_1)\sqrt{2E_{\bar{\mathbf{p}}_1}}\partial_0 d_s^\dagger(\bar{\mathbf{p}}_1,\bar{t}_1)\,|\mathbf{0}\rangle_{\text{in}}\, \overleftarrow{F}_{x_1}\tilde{U}^s_{p_1}(x_1)\, \mathrm{d}\bar{t}_1\,\mathrm{d}^4x_1$$

Sabendo a forma explícita de $d_s(\mathbf{k})$, conforme (6.2), podemos escrever seu hermitiano conjugado interpolante sob a forma

$$d_s^\dagger(\bar{\mathbf{p}}_1,\bar{t}_1) = \int \bar{V}^s_{\bar{p}_1}(\bar{x}_1)\gamma^0\psi(\bar{x}_1)\,\mathrm{d}^3\bar{x}_1$$

Logo a derivada em $\mathrm{d}\bar{t}_1$ se expande na forma

$$\partial_0 d_s^\dagger(\bar{\mathbf{p}}_1,\bar{t}_1) = \int [\partial_0\bar{V}^s_{\bar{p}_1}(\bar{x}_1)]\gamma^0\psi(\bar{x}_1)\,\mathrm{d}^3\bar{x}_1 + \int \bar{V}^s_{\bar{p}_1}(\bar{x}_1)\gamma^0[\partial_0\psi(\bar{x}_1)]\,\mathrm{d}^3\bar{x}_1$$

Notemos que $\bar{V}^s_{\bar{p}_1}(\bar{x}_1)$ satisfaz a equação de Dirac adjunta, isto é

$$\bar{V}^s_{\bar{p}_1}(\bar{x}_1)(-i\overleftarrow{\not{\partial}} - m)_{\bar{x}_1} = 0$$

que expandindo assume a forma

$$-i\partial_0\bar{V}^s_{\bar{p}_1}(\bar{x}_1)\gamma^0 - i\partial_k\bar{V}^s_{\bar{p}_1}(\bar{x}_1)\gamma^k - m\bar{V}^s_{\bar{p}_1}(\bar{x}_1) = 0$$

Multiplicando tudo por i e isolando um termo de interesse, temos

$$\begin{aligned}\partial_0\bar{V}^s_{\bar{p}_1}(\bar{x}_1)\gamma^0 &= im\bar{V}^s_{\bar{p}_1}(\bar{x}_1) - \partial_k\bar{V}^s_{\bar{p}_1}(\bar{x}_1)\gamma^k \\ &= im\bar{V}^s_{\bar{p}_1}(\bar{x}_1) - ip_k\bar{V}^s_{\bar{p}_1}(\bar{x}_1)\gamma^k \\ &= \bar{V}^s_{\bar{p}_1}(\bar{x}_1)\Big(im - ip_k\gamma^k\Big)\end{aligned}$$

Portanto

$$\partial_0 d_s^\dagger(\bar{\mathbf{p}}_1,\bar{t}_1) = \int \bar{V}^s_{\bar{p}_1}(\bar{x}_1)\Big(im - ip_k\gamma^k\Big)\psi(\bar{x}_1)\,\mathrm{d}^3\bar{x}_1 + \int \bar{V}^s_{\bar{p}_1}(\bar{x}_1)\gamma^0\partial_0\psi(\bar{x}_1)\,\mathrm{d}^3\bar{x}_1$$

A partir de uma integração por partes, transformamos o primeiro termo, de modo que

$$\begin{aligned}
\partial_0 d_s^\dagger(\bar{\mathbf{p}}_1,\bar{t}_1) &= \int \bar{V}^s_{\bar{p}_1}(\bar{x}_1)\Big(im + \gamma^k\partial_k\Big)\psi(\bar{x}_1)\,\mathrm{d}^3\bar{x}_1 + \int \bar{V}^s_{\bar{p}_1}(\bar{x}_1)\gamma^0\partial_0\psi(\bar{x}_1)\,\mathrm{d}^3\bar{x}_1 \\
&= \int \bar{V}^s_{\bar{p}_1}(\bar{x}_1)\Big(\gamma^0\partial_0 + \gamma^k\partial_k + im\Big)\psi(\bar{x}_1)\,\mathrm{d}^3\bar{x}_1 \\
&= -i\int \bar{V}^s_{\bar{p}_1}(\bar{x}_1)(i\gamma^\mu\partial_\mu - m)_{\bar{x}_1}\psi(\bar{x}_1)\,\mathrm{d}^3\bar{x}_1
\end{aligned}$$

Sendo então

$$(i\partial\!\!\!/ - m)_{\bar{x}_1}\psi(\bar{x}_1) = \overrightarrow{F}_{\bar{x}_1}\psi(\bar{x}_1)$$

obtemos

$$\partial_0 d_s^\dagger(\bar{\mathbf{p}}_1,\bar{t}_1) = -i\int \bar{V}^s_{\bar{p}_1}(\bar{x}_1)\overrightarrow{F}_{\bar{x}_1}\psi(\bar{x}_1)\,\mathrm{d}^3\bar{x}_1$$

Portanto

$$\begin{aligned}
A = \text{“1”} + (iZ_2^{-1/2})(-iZ_2^{-1/2}) \iint_{-\infty}^{\infty} \sqrt{2E_{\bar{\mathbf{p}}_1}}\bar{V}^s_{\bar{p}_1}(\bar{x}_1)\overrightarrow{F}_{\bar{x}_1}\psi(\bar{x}_1)\;{}_{\text{out}}\big\langle \mathbf{q}_1\bar{\mathbf{q}}_1\big|\,\bar{\psi}(x_1)\psi(\bar{x}_1)\,|\mathbf{0}\rangle_{\text{in}} \times \\
\times \overleftarrow{F}_{x_1}\tilde{U}^s_{p_1}(x_1)\,\mathrm{d}\bar{t}_1\,\mathrm{d}^3\bar{x}_1\,\mathrm{d}^4x_1
\end{aligned}$$

Como $\bar{\tilde{V}}^s_{\bar{p}_1} = \sqrt{2E_{\bar{\mathbf{p}}_1}}\bar{V}^s_{\bar{p}_1}(\bar{x}_1)$, obtemos

$$\begin{aligned}
A = \text{“1”} + (iZ_2^{-1/2})(-iZ_2^{-1/2}) \int \bar{\tilde{V}}^s_{\bar{p}_1}(\bar{x}_1)\overrightarrow{F}_{\bar{x}_1}\psi(\bar{x}_1)\;{}_{\text{out}}\big\langle \mathbf{q}_1\bar{\mathbf{q}}_1\big|\,\bar{\psi}(x_1)\psi(\bar{x}_1)\,|\mathbf{0}\rangle_{\text{in}} \times \\
\times \overleftarrow{F}_{x_1}\tilde{U}^s_{p_1}(x_1)\,\mathrm{d}^4\bar{x}_1\,\mathrm{d}^4x_1
\end{aligned}$$

Considerando agora os estados finais, temos

$${}_{\text{out}}\big\langle \mathbf{q}_1\bar{\mathbf{q}}_1\big| = Z_2^{-1/2}\lim_{\bar{\tau}_1\to\infty}\;{}_{\text{out}}\big\langle \mathbf{q}_1\big|\,d_s(\bar{\mathbf{q}}_1,\bar{\tau}_1)\sqrt{2E_{\bar{q}_1}}$$

Com isso

$$\begin{aligned}
A = \text{“1”} + (iZ_2^{-1/2})(-iZ_2^{-1/2})Z_2^{-1/2}\lim_{\bar{\tau}_1\to\infty}\int \bar{\tilde{V}}^s_{\bar{p}_1}(\bar{x}_1)\overrightarrow{F}_{\bar{x}_1}\psi(\bar{x}_1)\;{}_{\text{out}}\big\langle \mathbf{q}_1\big|\,d_s(\bar{\mathbf{q}}_1,\bar{\tau}_1)\sqrt{2E_{\bar{q}_1}} \times \\
\times \bar{\psi}(x_1)\psi(\bar{x}_1)\,|\mathbf{0}\rangle_{\text{in}}\overleftarrow{F}_{x_1}\tilde{U}^s_{p_1}(x_1)\,\mathrm{d}^4\bar{x}_1\,\mathrm{d}^4x_1
\end{aligned}$$

Fazendo outra vez a substituição por uma integração de uma derivada total no tempo, tal que

$$\begin{aligned}
\int_{-\infty}^{\infty}\partial_0\big[\,{}_{\text{out}}\big\langle \mathbf{q}_1\big|\,d_s(\bar{\mathbf{q}}_1,\bar{\tau}_1)\cdots|\mathbf{0}\rangle_{\text{in}}\big]\,\mathrm{d}\bar{\tau}_1 = \lim_{\bar{\tau}_1\to\infty}\;{}_{\text{out}}\big\langle \mathbf{q}_1\big|\,d_s(\bar{\mathbf{q}}_1,\bar{\tau}_1)\cdots|\mathbf{0}\rangle_{\text{in}} - \\
- \lim_{\bar{\tau}_1\to-\infty}\;{}_{\text{out}}\big\langle \mathbf{q}_1\big|\,d_s(\bar{\mathbf{q}}_1,\bar{\tau}_1)\cdots|\mathbf{0}\rangle_{\text{in}}
\end{aligned}$$

Neste caso, o termo com o limite $\bar{\tau}_1 \to \infty$ será irrelevante para o tipo de interação que estamos interessados, de modo a inserirmos em "1" e portanto

$$A = \text{"1"} + (iZ_2^{-1/2})(-iZ_2^{-1/2})Z_2^{-1/2} \iint_{-\infty}^{\infty} \bar{\tilde{V}}^s_{\bar{p}_1}(\bar{x}_1)\overrightarrow{F}_{\bar{x}_1}\psi(\bar{x}_1)\ {}_{\text{out}}\langle \mathbf{q}_1|\ \partial_0 d_s(\bar{\mathbf{q}}_1, \bar{\tau}_1)\sqrt{2E_{\bar{q}_1}}\times$$
$$\times\ \bar{\psi}(x_1)\psi(\bar{x}_1)\ |\mathbf{0}\rangle_{\text{in}}\ \overleftarrow{F}_{x_1}\tilde{U}^s_{p_1}(x_1)\,\mathrm{d}\bar{\tau}_1\,\mathrm{d}^4\bar{x}_1\,\mathrm{d}^4x_1$$

Calculando então a derivada do operador $d_s(\bar{\mathbf{q}}_1, \bar{\tau}_1)$ com base na forma explícita (6.2), temos

$$\partial_0 d_s(\bar{\mathbf{q}}_1, \bar{\tau}_1) = \int [\partial_0\bar{\psi}(\bar{y}_1)]\gamma^0 V^s_k(\bar{y}_1)\,\mathrm{d}^3\bar{y}_1 + \int \bar{\psi}(\bar{y}_1)\gamma^0[\partial_0 V^s_k(\bar{y}_1)]\,\mathrm{d}^3\bar{y}_1$$

Essa derivada é similar à $\partial_0 b^\dagger_s(\mathbf{p}_1, t_1)$, de modo podemos seguir o mesmo raciocínio e obter

$$\partial_0 d_s(\bar{\mathbf{q}}_1, \bar{\tau}_1) = i\int \bar{\psi}(\bar{y}_1)\overleftarrow{F}_{\bar{y}_1}V_{\bar{q}_1}(\bar{y}_1)\,\mathrm{d}^3\bar{y}_1$$

Considerando então $\tilde{V}^s_{\bar{q}_1}(\bar{y}_1) = \sqrt{2E_{\bar{q}_1}}V^s_{\bar{q}_1}(\bar{y}_1)$, temos

$$A = \text{"1"} + (iZ_2^{-1/2})^2(-iZ_2^{-1/2}) \int \bar{\tilde{V}}^s_{\bar{p}_1}(\bar{x}_1)\overrightarrow{F}_{\bar{x}_1}\ {}_{\text{out}}\langle \mathbf{q}_1|\ \bar{\psi}(\bar{y}_1)\bar{\psi}(x_1)\psi(\bar{x}_1)\ |\mathbf{0}\rangle_{\text{in}}\times$$
$$\times\ \overleftarrow{F}_{x_1}\tilde{U}^s_{p_1}(x_1)\overleftarrow{F}_{\bar{y}_1}\tilde{V}^s_{\bar{q}_1}(\bar{y}_1)\,\mathrm{d}^4\bar{y}_1\,\mathrm{d}^4\bar{x}_1\,\mathrm{d}^4x_1$$

Com todo o procedimento feito até então, fica simples induzir a redução para ${}_{\text{out}}\langle \mathbf{q}_1|$ tal que

$$A = \text{"1"} + (iZ_2^{-1/2})^2(-iZ_2^{-1/2})^2 \int \bar{\tilde{U}}^s_{\bar{q}_1}(y_1)\overrightarrow{F}_{y_1}\bar{\tilde{V}}^s_{\bar{p}_1}(\bar{x}_1)\overrightarrow{F}_{\bar{x}_1}\ {}_{\text{out}}\langle \mathbf{0}|\ \psi(y_1)\bar{\psi}(\bar{y}_1)\bar{\psi}(x_1)\psi(\bar{x}_1)\ |\mathbf{0}\rangle_{\text{in}}\times$$
$$\times\ \overleftarrow{F}_{x_1}\tilde{U}^s_{p_1}(x_1)\overleftarrow{F}_{\bar{y}_1}\tilde{V}^s_{\bar{q}_1}(\bar{y}_1)\,\mathrm{d}^4y_1\,\mathrm{d}^4\bar{y}_1\,\mathrm{d}^4\bar{x}_1\,\mathrm{d}^4x_1$$

Olhando somente para o termo ${}_{\text{out}}\langle \mathbf{0}|\ \psi(y_1)\bar{\psi}(\bar{y}_1)\bar{\psi}(x_1)\psi(\bar{x}_1)\ |\mathbf{0}\rangle_{\text{in}}$ vemos a necessidade da inclusão do operador de ordenamento temporal $T\{\cdots\}$ para evitar que operadores de aniquilação se apliquem no vácuo, argumento equivalente ao feito no exercício 8, portanto a redução LSZ para um espalhamento férmion–anti-férmion é dada por

$$\begin{aligned}{}_{\text{out}}\langle \mathbf{q}_1\bar{\mathbf{q}}_1|\mathbf{p}_1\bar{\mathbf{p}}_1\rangle_{\text{in}} = \text{"1"} + (-iZ^{-1/2})^2(iZ^{-1/2})^2 \int \bar{\tilde{U}}^s_{q_1}(y_1)\overrightarrow{F}_{y_1}\bar{\tilde{V}}^s_{\bar{p}_1}(\bar{x}_1)\overrightarrow{F}_{\bar{x}_1}\times \\ \times\ {}_{\text{out}}\langle 0|\,T\{\bar{\psi}(\bar{y}_1)\psi(y_1)\bar{\psi}(x_1)\psi(\bar{x}_1)\}\,|0\rangle_{\text{in}}\,\overleftarrow{F}_{x_1}\tilde{U}^s_{p_1}(x_1)\overleftarrow{F}_{\bar{y}_1}\tilde{V}^s_{\bar{q}_1}(\bar{y}_1)\,\mathrm{d}^4x_1\,\mathrm{d}^4\bar{x}_1\,\mathrm{d}^4y_1\,\mathrm{d}^4\bar{y}_1\end{aligned} \tag{9.1}$$

Q. 10 Relações úteis com operadores hamiltonianos

A partir dos fatos abaixo,

(a) $\dfrac{\mathrm{d}}{\mathrm{d}t}\left[U(t)U^{-1}(t)\right] = 0$

(b) Campo interpolante $\phi(\mathbf{r},t) = U^{-1}(t)\phi_{\text{in}}(\mathbf{r},t)U(t)$

(c) $\dfrac{\partial}{\partial t}\phi(\mathbf{r},t) = i\Big[\hat{H}(\phi,\pi),\phi(\mathbf{r},t)\Big] \qquad \dfrac{\partial}{\partial t}\phi_{\text{in}}(\mathbf{r},t) = i\Big[\hat{H}_0(\phi_{\text{in}},\pi_{\text{in}}),\phi_{\text{in}}(\mathbf{r},t)\Big]$,

mostre que

$$\dot{U}U^{-1} + i\hat{H}(\phi_{\text{in}},\pi_{\text{in}}) = i\hat{H}_0(\phi_{\text{in}},\pi_{\text{in}}) \qquad \& \qquad i\frac{\partial U(t)}{\partial t} = \hat{H}_{\text{int}}(\phi_{\text{in}},\pi_{\text{in}})U(t)$$

Considere a primeira propriedade do item (c), de modo que utilizando o campo interpolante do item (b) obtemos

$$\begin{aligned}\frac{\partial\phi}{\partial t} &= \frac{\partial}{\partial t}\left[U^{-1}\phi_{\text{in}}U\right] \\ &= \dot{U}^{-1}\phi_{\text{in}}U + U^{-1}\dot{\phi}_{\text{in}}U + U^{-1}\phi_{\text{in}}\dot{U}\end{aligned}$$

A propriedade (a) nos dá

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[UU^{-1}\right] = \dot{U}U^{-1} + U\dot{U}^{-1} = 0$$

Logo

$$\dot{U}U^{-1} = -U\dot{U}^{-1}$$

Como $U^{-1}U = \mathbb{1}$, podemos aplicar U^{-1} pela esquerda, ficando com

$$\dot{U}^{-1} = -U^{-1}\dot{U}U^{-1}$$

Portanto

$$\begin{aligned}\frac{\partial\phi}{\partial t} &= -U^{-1}\dot{U}U^{-1}\phi_{\text{in}}U + U^{-1}\frac{\partial\phi_{\text{in}}}{\partial t}U + U^{-1}\phi_{\text{in}}\dot{U} \\ &= -U^{-1}\dot{U}\phi + U^{-1}\frac{\partial\phi_{\text{in}}}{\partial t}U + U^{-1}U\phi U^{-1}\dot{U} \\ &= -U^{-1}\dot{U}\phi + U^{-1}\frac{\partial\phi_{\text{in}}}{\partial t}U + \phi U^{-1}\dot{U}\end{aligned}$$

Pela propriedade (c), temos

$$
\begin{aligned}
\frac{\partial \phi_{\text{in}}}{\partial t} &= i\Big[\hat{H}_0(\phi_{\text{in}}, \pi_{\text{in}}), \phi_{\text{in}}\Big] \\
&= i\hat{H}_0(\phi_{\text{in}}, \pi_{\text{in}})\phi_{\text{in}} - i\phi_{\text{in}}\hat{H}_0(\phi_{\text{in}}, \pi_{\text{in}}) \\
&= i\hat{H}_0(\phi_{\text{in}}, \pi_{\text{in}})U\phi U^{-1} - iU\phi U^{-1}\hat{H}_0(\phi_{\text{in}}, \pi_{\text{in}})
\end{aligned}
$$

Como precisamos calcular $U^{-1}\dfrac{\partial \phi_{\text{in}}}{\partial t}U$,

$$
\begin{aligned}
U^{-1}\frac{\partial \phi_{\text{in}}}{\partial t}U &= U^{-1}i\hat{H}_0(\phi_{\text{in}}, \pi_{\text{in}})U\phi U^{-1}U - U^{-1}iU\phi U^{-1}\hat{H}_0(\phi_{\text{in}}, \pi_{\text{in}})U \\
&= U^{-1}i\hat{H}_0(\phi_{\text{in}}, \pi_{\text{in}})U\phi - i\phi U^{-1}\hat{H}_0(\phi_{\text{in}}, \pi_{\text{in}})U \\
&= \Big[U^{-1}i\hat{H}(\phi_{\text{in}}, \pi_{\text{in}})U, \phi\Big]
\end{aligned}
$$

Considerando que $\hat{H}(\phi, \pi)$ é polinomial nos campos ϕ e π, podemos assumir que vale a igualdade $\hat{H}(\phi, \pi) = U^{-1}\hat{H}(\phi_{\text{in}}, \pi_{\text{in}})U$, pois os operadores de evolução temporal são aplicados nos campos, tal que

$$
\begin{aligned}
\frac{\partial \phi}{\partial t} &= i\Big[\hat{H}(\phi, \pi), \phi\Big] \\
&= i\hat{H}(\phi, \pi)\phi - i\phi\hat{H}(\phi, \pi) \\
&= U^{-1}i\hat{H}(\phi_{\text{in}}, \pi_{\text{in}})U\phi - i\phi U^{-1}\hat{H}(\phi_{\text{in}}, \pi_{\text{in}})U \\
&= \Big[U^{-1}i\hat{H}(\phi_{\text{in}}, \pi_{\text{in}})U, \phi\Big]
\end{aligned}
$$

Temos então

$$
\Big[U^{-1}i\hat{H}(\phi_{\text{in}}, \pi_{\text{in}})U, \phi\Big] = -U^{-1}\dot{U}\phi + \Big[U^{-1}i\hat{H}_0(\phi_{\text{in}}, \pi_{\text{in}})U, \phi\Big] + \phi U^{-1}\dot{U}
$$

Somando e subtraindo $U^{-1}\phi\dot{U}$, podemos escrever

$$
\begin{aligned}
-U^{-1}\dot{U}\phi + \phi U^{-1}\dot{U} + U^{-1}\phi\dot{U} - U^{-1}\phi\dot{U} &= [\phi, U^{-1}]\dot{U} + U^{-1}\Big[\phi, \dot{U}\Big] = \Big[\phi, U^{-1}\dot{U}\Big] \\
&= -\Big[U^{-1}\dot{U}, \phi\Big]
\end{aligned}
$$

Adicionando $\mathbb{1} = U^{-1}U$ teremos

$$
-\Big[U^{-1}\dot{U}, \phi\Big] = -\Big[U^{-1}\dot{U}U^{-1}U, \phi\Big]
$$

Resultando portanto na expressão

$$
\Big[U^{-1}i\hat{H}(\phi_{\text{in}}, \pi_{\text{in}})U, \phi\Big] = \Big[U^{-1}i\hat{H}_0(\phi_{\text{in}}, \pi_{\text{in}})U, \phi\Big] - \Big[U^{-1}\dot{U}U^{-1}U, \phi\Big]
$$

Reorganizando um pouco, levando em conta que $[A+B,C]=[A,B]+[B,C]$,

$$\left[U^{-1}\left(\dot{U}U^{-1}+i\hat{H}(\phi_{\text{in}},\pi_{\text{in}})\right)U,\phi\right]=\left[U^{-1}i\hat{H}_0(\phi_{\text{in}},\pi_{\text{in}})U,\phi\right]$$

De modo que pela igualdade, podemos afirmar que

$$\dot{U}U^{-1}+i\hat{H}(\phi_{\text{in}},\pi_{\text{in}})=i\hat{H}_0(\phi_{\text{in}},\pi_{\text{in}}) \tag{10.1}$$

Dado que $\hat{H}=\hat{H}_0+\hat{H}_{\text{int}}$, temos que

$$\hat{H}_{\text{int}}(\phi_{\text{in}},\pi_{\text{in}})=\hat{H}(\phi_{\text{in}},\pi_{\text{in}})-\hat{H}_0(\phi_{\text{in}},\pi_{\text{in}})$$

A (10.1) pode ser reescrita multiplicando tudo por i e isolando $\dot{U}U^{-1}$, ou seja

$$i\dot{U}U^{-1}=\hat{H}(\phi_{\text{in}},\pi_{\text{in}})-\hat{H}_0(\phi_{\text{in}},\pi_{\text{in}})=\hat{H}_{\text{int}}(\phi_{\text{in}},\pi_{\text{in}})$$

Multiplicando U pela direita das duas expressões, concluímos que

$$i\frac{\partial U(t)}{\partial t}=\hat{H}_{\text{int}}(\phi_{\text{in}},\pi_{\text{in}})U(t) \tag{10.2}$$