Institute of Physics

Quantum Field Theory I

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HOMEWORK N° 1

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1. Relativity Gymnastics

(a) Show that the invariance of the interval between two events $(t^2 - |\boldsymbol{x}|^2)$ under Lorentz transformations is satisfied if we define the position four-vector $x^\mu \equiv (t; \boldsymbol{x})$ such that

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

where we defined the Lorentz transformations as $\Lambda^\mu{}_\nu$. The inverse Lorentz transformation is defined by $x^\nu = \Lambda_\mu{}^\nu x'^\mu$, and it satisfies

$$\Lambda^\mu{}_\nu \Lambda_\rho{}^\nu = \delta^\mu_\rho$$

Solution:

A small comment about this solution: I attempted to solve this item without using the $g_{\mu\nu}$ metric tensor, but all of my attempts failed. As a result, I based my solution on the first chapter of Srednicki (2009).

According to Srednicki (2009) Lorentz transformations have the following property:

$$g_{\rho\sigma} = g_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma$$

where $g_{\mu\nu}$ is an arbitrary metric tensor. In item (b).iii, I will prove why $v^\mu = g^{\mu\nu}v_\nu$ and $v_\mu = g_{\mu\nu}v^\nu$. To do so, I will assume that these relations hold and use them to derive the following:

$$(x_\mu - y_\mu) = g_{\mu\nu}(x^\nu - y^\nu)$$

In item (b).ii I will show that the four-vector scalar product can be written by $v_\mu v^\mu = v^2$. Hence I will assume that is valid too.

Therefore, considering that $x^\mu \equiv (t; \boldsymbol{x})$, we can express the interval Δs^2 as follows:

$$\begin{aligned}
\Delta s'^2 = \Delta t'^2 - \Delta \boldsymbol{x}'^2 &= (x'^\mu - y'^\mu)^2 = (x' - y')^2 \\
&= g_{\mu\nu}(x'^\mu - y'^\mu)(x'^\nu - y'^\nu) \\
&= g_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma (x^\rho - y^\rho)(x^\sigma - y^\sigma) \\
&= g_{\rho\sigma}(x^\rho - y^\rho)(x^\sigma - y^\sigma) \\
&= (x_\sigma - y_\sigma)(x^\sigma - y^\sigma) \\
&= \Delta s^2
\end{aligned}$$

Concluding that:

$$\boxed{\Delta s'^2 = \Delta s^2} \tag{1.1}$$

Which proof that the interval between two events is invariant under Lorentz transformation.

(b) The Minkowski space metric is given by

$$g^{\mu\nu} = g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

(i) Show that the interval is given by the position four-vector squared defined by $x \cdot x \equiv x^\mu g_{\mu\nu} x^\nu$.

(ii) A contra-variant four-vector is denoted as $v^\mu = (t; \boldsymbol{v})$. Show that if we define a covariant four-vector as $v_\mu \equiv (t; -\boldsymbol{v})$, we can write $v \cdot v = v^\mu v_\mu$.

(iii) Verify that $v_\mu = g_{\mu\nu} v^\nu$, and $v^\mu = g^{\mu\nu} v_\nu$.

Solution:

(i) To proof that $\Delta s = x \cdot x$, we simply need use $x^\mu g_{\mu\nu} x^\nu$ in matrix form. To do this, I will suppose that x^μ is a row line vector and $g_{\mu\nu} x^\nu$ a column vector, which implies that x^ν is a column vector, just to satisfy the matrix product. With this in mind, we get:

$$\begin{aligned}
x \cdot x &= x^\mu g_{\mu\nu} x^\nu \\
&= \begin{bmatrix} x^0 & x^1 & x^2 & x^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \\
&= \begin{bmatrix} x^0 & x^1 & x^2 & x^3 \end{bmatrix} \begin{bmatrix} x^0 \\ -x^1 \\ -x^2 \\ -x^3 \end{bmatrix} \\
&= (x^0)^1 - (x^1)^2 - (x^2)^2 - (x^3)^2 \\
&= \Delta s
\end{aligned}$$

Therefore we proof that:

$$\boxed{x \cdot x \equiv x^\mu g_{\mu\nu} x^\nu = \Delta s} \tag{1.2}$$

(ii) Defining $v_\mu = (t; -\boldsymbol{v})$, we can evaluate the scalar product $\boldsymbol{v} \cdot \boldsymbol{v}$:

$$\begin{aligned} v \cdot v &= t^2 - \boldsymbol{v}^2 \\ &= (v^0)^2 - (v^1)^2 - (v^2)^2 - (v^3)^2 \\ &= (v^0)(v^0) - (v^1)(v^1) - (v^2)(v^2) - (v^3)(v^3) \\ &= (v^0)(v^0) + (v^1)(-v^1) + (v^2)(-v^2) + (v^3)(-v^3) \\ &= (v^0; \boldsymbol{v}) \cdot (v^0; -\boldsymbol{v}) \end{aligned}$$

Therefore:

$$\boxed{v \cdot v = v^\mu v_\mu} \tag{1.3}$$

(iii) By the items (a) and (b), we can conclude that v_μ is a column vector and v^ν a row line vector and $g_{\mu\nu}$ can change the orientation of the vector, i.e., $g_{\mu\nu}$ change the signal of the spatial part and transpose the vector. Therefore to verify that $v^\mu = g^{\mu\nu} v_\nu$, we can write the matrix form to write the metric tensor $g^{\mu\nu}$ as follows:

$$g^{\mu\nu} v_\nu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v^0 \\ -v^1 \\ -v^2 \\ -v^3 \end{bmatrix} = \begin{bmatrix} v^0 & v^1 & v^2 & v^3 \end{bmatrix} = v^\mu$$

And to $v_\mu = g_{\mu\nu} v^\nu$:

$$g_{\mu\nu} v^\nu = \begin{bmatrix} v^0 & v^1 & v^2 & v^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} v^0 \\ -v^1 \\ -v^2 \\ -v^3 \end{bmatrix} = v_\mu$$

Therefore:

$$\boxed{v^\mu = g^{\mu\nu} v_\nu \qquad \text{and} \qquad v_\mu = g_{\mu\nu} v^\nu} \tag{1.4}$$

(c) Show that the relativistic dispersion relation $E^2 - |\boldsymbol{p}|^2 = m^2$, where m is the particle's mass, is consistent with defining the momentum four-vector as $P^\mu = (E; \boldsymbol{p})$.

Solution:

The momentum four-vector P^μ is defined as $P^\mu = (E, \boldsymbol{p})$, where E is the energy of the particle and $\boldsymbol{p}$ is its three-momentum. To verify that the relativistic dispersion relation $E^2 - |\boldsymbol{p}|^2 = m^2$ is consistent with this definition, we need to express this relation in terms of the momentum four-vector.

$$\begin{aligned} P^\mu P_\mu &= (E; \boldsymbol{p}) \cdot (E; -\boldsymbol{p}) \\ &= E^2 - \boldsymbol{p} \cdot \boldsymbol{p} \\ &= E^2 - |\boldsymbol{p}|^2 \equiv m^2 \end{aligned}$$

Thus, we've shown that the relativistic dispersion relation is equivalent to the definition of the momentum four-vector P^μ, which is given by:

$$\boxed{P^\mu P_\mu = m^2} \tag{1.5}$$

(d) Define the differential operators

$$\frac{\partial}{\partial x^\mu} \equiv \partial_\mu \equiv \left(\frac{\partial}{\partial t}; \boldsymbol{\nabla}\right) \qquad \frac{\partial}{\partial x_\mu} \equiv \partial^\mu \equiv \left(\frac{\partial}{\partial t}; -\boldsymbol{\nabla}\right)$$

(i) Construct the Klein-Gordon operator using the identifications

$$E \to i\frac{\partial}{\partial t} \qquad \boldsymbol{p} \to -i\boldsymbol{\nabla}$$

(ii) Show the if we define the four-vector current $\mathcal{J}^\mu \equiv (\rho; \boldsymbol{j})$, where ρ and $\boldsymbol{j}$ the charge density and current respectively, the continuity equation in electrodynamics can be written as the conservation of the four-current as

$$\partial_\mu \mathcal{J}^\mu = 0$$

Solution:

(i) Using the relativistic dispersion relation $E^2 - |\boldsymbol{p}|^2 = m^2$:

$$\begin{aligned} \left(i\frac{\partial}{\partial t}\right)^2 - (-i\boldsymbol{\nabla})^2 &= m^2 \\ -\frac{\partial^2}{\partial t^2} + \boldsymbol{\nabla}^2 &= \\ -\left(\frac{\partial}{\partial t}; \boldsymbol{\nabla}\right) \cdot \left(\frac{\partial}{\partial t}; -\boldsymbol{\nabla}\right) &= \\ -\partial_\mu \partial^\mu &= \end{aligned}$$

Therefore we obtain the equation:

$$\boxed{\partial_\mu \partial^\mu + m^2 = 0} \tag{1.6}$$

This is the Klein-Gordon operator. Applying a function $\psi(x)$ in both sides:

$$(\partial_\mu \partial^\mu + m^2)\psi(x) = 0$$

we obtain the Klein-Gordon equation.

(ii) In the classical formulation, the continuity equation can be written by:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0$$

Therefore we can rewrite it as follows:

$$\left(\frac{\partial}{\partial t}; \boldsymbol{\nabla}\right) \cdot (\rho; \boldsymbol{j}) = 0$$

Concluding that:

$$\boxed{\partial_\mu \mathcal{J}^\mu = 0} \tag{1.7}$$

2. About those Delta Functions

Using the defining property of the Delta function

$$\int_{-\infty}^{\infty} f(x)\delta(x)\mathrm{d}x = f(0)$$

for some function $f(x)$, and that $\delta(-x) = \delta(x)$

(a) Show that

$$\int_{-\infty}^{\infty} f(x)\delta(ax)\mathrm{d}x = \frac{1}{|a|}f(0)$$

for an arbitrary constant a.

Solution:

Using the basic definition of Dirac's delta, we just need to proof that $\delta(x)$ can be rewritten by $|a|\delta(ax)$. To start this proof, we can show that:

$$\begin{aligned}
|a|\delta(ax) &= \begin{cases} |a|(+\infty), & \text{when } ax = 0 \\ |a|(0), & \text{otherwise} \end{cases} \\
&= \begin{cases} +\infty, & \text{when } ax = 0 \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} +\infty, & \text{when } x = 0 \\ 0, & \text{otherwise} \end{cases} \\
&= \delta(x)
\end{aligned}$$

The next step is prove that for $f(x) = 1$, we have 1. To do this, exists 2 cases: $a > 0$ or $a < 0$. First I will demonstrate for $a > 0$:

$$\begin{aligned}
\int_{-\infty}^{\infty} |a|\delta(ax)\mathrm{d}x &= \int_{-\infty}^{\infty} \frac{|a|}{a}\delta(t)\mathrm{d}t && (t = ax) \\
&= \frac{|a|}{a}\int_{-\infty}^{\infty} \delta(t)\mathrm{d}t \\
&= \int_{-\infty}^{\infty} \delta(t)\mathrm{d}t && (|a| = a) \\
&= 1
\end{aligned}$$

Now, for $a < 0$:

$$\begin{aligned}
\int_{-\infty}^{\infty} |a|\delta(ax)\mathrm{d}x &= -\int_{-\infty}^{\infty} \frac{|a|}{a}\delta(t)\mathrm{d}t && (t = ax) \\
&= -\frac{|a|}{a}\int_{-\infty}^{\infty} \delta(t)\mathrm{d}t \\
&= \int_{-\infty}^{\infty} \delta(t)\mathrm{d}t && (|a| = -a) \\
&= 1
\end{aligned}$$

This conclude that $|a|\delta(ax) = \delta(x)$, therefore, for any function $f(x)$:

$$\boxed{\int_{-\infty}^{\infty} f(x)\delta(ax)\mathrm{d}x = \frac{1}{|a|}f(0)} \tag{1.8}$$

(b) Using all these show that for a function $g(x)$

$$\int_{-\infty}^{\infty} f(x)\delta[g(x)]\mathrm{d}x = \sum_a \frac{1}{|g'(a)|}\int_{-\infty}^{\infty} f(x)\delta(x-a)\mathrm{d}x$$

where a are the zeroes of $g(x)$ (i.e. $g(a) = 0,\ \forall a$) while $g'(a) \neq 0$.

Solution:

Suppose that $g : \mathbb{R} \to \mathbb{R}$ is a smooth function such that $g(x_i) = 0$ and $g'(x_i) > 0$ for some $x_i \in \mathbb{R}$. Then, there exists an open interval $(x_i - \epsilon, x_i + \epsilon)$ on which g is invertible and $g'(x) > 0$ for all $x \in (x_i - \epsilon, x_i + \epsilon)$. Let $y = g(x)$ be the substitution. Then, by the chain rule, we have $\frac{\mathrm{d}y}{\mathrm{d}x} = g'(x)$, which implies $\mathrm{d}y = g'(x), \mathrm{d}x$.

$$\begin{aligned}\int_{x_i-\epsilon}^{x_i+\epsilon} f(x)\delta[g(x)]\mathrm{d}x &= \int_{g(x_i-\epsilon)}^{g(x_i+\epsilon)} f[g^{-1}(y)]\delta(y)\frac{1}{g'[g^{-1}(y)]}\mathrm{d}y \\ &= f[g^{-1}(0)]\frac{1}{g'[g^{-1}(0)]} = f(x_i)\frac{1}{g'(x_i)}\end{aligned}$$

rewriting $f(x_i)$ in term of Dirac's delta:

$$\begin{aligned}f(x_i)\frac{1}{g'(x_i)} &= \frac{1}{g'(x_i)}\int_{x_i-\epsilon}^{x_i+\epsilon} f(x)\delta(x-x_i)\mathrm{d}x \\ &= \int_{x_i-\epsilon}^{x_i+\epsilon} \frac{\delta(x-x_i)}{g'(x_i)}f(x)\mathrm{d}x\end{aligned}$$

If instead $f'(x_i) < 0$, then $f(x_i - \epsilon) > f(x_i + \epsilon)$ and we get:

$$\begin{aligned}\int_{x_i-\epsilon}^{x_i+\epsilon} f(x)\delta(g(x))\mathrm{d}x &= \int_{g(x_i-\epsilon)}^{g(x_i+\epsilon)} \delta(y)f[g^{-1}(y)]\frac{\mathrm{d}y}{g'[g^{-1}(y)]} \\ (\text{ swapping integration limits })\quad &= -\int_{g(x_i+\epsilon)}^{g(x_i-\epsilon)} \delta(y)f[g^{-1}(y)]\frac{\mathrm{d}y}{g'[g^{-1}(y)]} \\ &= -f[g^{-1}(0)]\frac{1}{g'[g^{-1}(0)]} = -f(x_i)\frac{1}{g'(x_i)} \\ &= -\frac{1}{g'(x_i)}\int_{x_i-\epsilon}^{x_i+\epsilon} f(x)\delta(x-x_i)\mathrm{d} \\ &= -\int_{x_i-\epsilon}^{x_i+\epsilon} f(x)\frac{\delta(x-x_i)}{g'(x_i)}\mathrm{d}x\end{aligned}$$

The two cases can be combined into

$$\int_{x_i-\epsilon}^{x_i+\epsilon} f(x)\delta[g(x)]\mathrm{d}x = \int_{x_i-\epsilon}^{x_i+\epsilon} f(x)\frac{\delta(x-x_i)}{|g'(x_i)|}\mathrm{d}x$$

by the item (a). So, taking all zeroes into account (by summing in all zeroes) we get:

$$\boxed{\int_{-\infty}^{\infty} f(x)\delta[g(x)]\mathrm{d}x = \sum_{x_i} \frac{1}{|g'(x_i)|}\int_{-\infty}^{\infty} f(x)\delta(x-x_i)\mathrm{d}x} \tag{1.9}$$

(c) Now verify that for $\omega_p = \sqrt{(\boldsymbol{p})^2 + m^2} > 0$ we have

$$\begin{aligned}\int \delta(P^2 - m^2) f(P)\mathrm{d}^4 p &= \int \left\{\int \delta[p_o^2 - (\boldsymbol{p})^2 - m^2] f(P)\mathrm{d}p_o\right\}\mathrm{d}^3 p \\ &= \frac{1}{2\omega_p}\int f(\omega_p, \boldsymbol{p})\mathrm{d}^3 p\end{aligned}$$

where you need to use the fact that since the four-momentum is always time-like, the sign of p_o is Lorentz invariant. Thus we need only integrate p_o from 0 to ∞, which in turn means we only need the positive root.

Solution:

Basing on the relativistic dispersion relation, we get:

$$E^2 = |\boldsymbol{p}|^2 + m^2 \Rightarrow \omega_p = \sqrt{E^2} = \sqrt{p_o^2} \Rightarrow p_o^2 = \omega_p^2$$

moreover:

$$P^2 = P^\mu P_\mu = E^2 - |\boldsymbol{p}|^2 = p_o^2 - |\boldsymbol{p}|^2 = \omega_p^2 - |\boldsymbol{p}|^2$$

this implies that $f(P) \equiv f(p_o, \boldsymbol{p})$. Expanding the argument inside the Dirac‘s delta:

$$\begin{aligned}\int \delta(P^2 - m^2) f(P)\mathrm{d}^4 p &= \int \delta(\omega_p^2 - |\boldsymbol{p}|^2 - m^2) f(p_o, \boldsymbol{p})\mathrm{d}^4 p \\ &= \int \delta(p_o^2 - |\boldsymbol{p}|^2 - m^2) f(p_o, \boldsymbol{p})\mathrm{d}^4 p \\ &= \int \left\{\int \delta(p_o^2 - |\boldsymbol{p}|^2 - m^2) f(p_o, \boldsymbol{p})\mathrm{d}p_o\right\}\mathrm{d}^3 p\end{aligned}$$

Using the equation (1.9), we can write $g(p_o) = p_o^2 - |\boldsymbol{p}^2| - m^2$, such that $g'(p_o) = 2p_o = 2\omega_p$, this results in:

$$\begin{aligned}\int \left\{\int \delta(p_o^2 - |\boldsymbol{p}|^2 - m^2) f(p_o, \boldsymbol{p})\mathrm{d}p_o\right\}\mathrm{d}^3 p &= \int \left\{\frac{1}{2\omega_p}\int \delta(p_o - \omega_p) f(p_o, \boldsymbol{p})\mathrm{d}p_o\right\}\mathrm{d}^3 p \\ &= \frac{1}{2\omega_p}\int f(\omega_p, \boldsymbol{p})\mathrm{d}^3 p\end{aligned}$$

Therefore, we prove that:

$$\boxed{\int \delta(P^2 - m^2) f(P)\mathrm{d}^4 p = \frac{1}{2\omega_p}\int f(\omega_p, \boldsymbol{p})\mathrm{d}^3 p} \tag{1.10}$$

3. Electromagnetism

The covariant form of the Lagrangian density for the electromagnetic field is:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu \mathcal{J}^\mu$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor defined in terms of the potential 4-vector A_μ, and $\mathcal{J}^\mu = (\rho, \boldsymbol{j})$ is the current four-vector.

(a) Show that the Lagrangian is gauge invariant, i.e. is invariant under

$$A_\mu(x) \mapsto A_\mu(x) + \partial_\mu \alpha(x)$$

What condition must the current $\mathcal{J}^\mu$ satisfy for this to be the case?

Solution:

By imposing the transformation previously mentioned, we have:

$$\mathcal{L} \mapsto \mathcal{L}' = -\frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} - [A_\mu(x) + \partial_\mu \alpha(x)]\mathcal{J}^\mu$$

We know that $F_{\mu\nu}F^{\mu\nu}$ is gauge invariant, in order that we calculate only the second part of the "*new*" Lagrangian.

$$\begin{aligned}
[A_\mu(x) + \partial_\mu \alpha(x)]\mathcal{J}^\mu &= A_\mu(x)\mathcal{J}^\mu + \partial_\mu[\alpha(x)]\mathcal{J}^\mu \\
&\overset{!}{=} A_\mu(x)\mathcal{J}^\mu + \partial_\mu[\mathcal{J}^\mu \alpha(x)] - \alpha(x)\partial_\mu[\mathcal{J}^\mu]
\end{aligned}$$

Now, note that if we integrate this in d^4x, we will have a portion of the the action $S[A_\mu]$, i.e.

$$\int \partial_\mu[\mathcal{J}^\mu \alpha(x)]\,\mathrm{d}^4x - \int \alpha(x)\underbrace{\partial_\mu[\mathcal{J}^\mu]}_{=0}\,\mathrm{d}^4x$$

Building a Gaussian surface with volume V, we can rewrite the first integral using the divergence theorem by:

$$\int_{t_1}^{t_2}\int_V \partial_\mu[\mathcal{J}^\mu \alpha(x)]\,\mathrm{d}^3x\,\mathrm{d}t = \int_{t_1}^{t_2} \alpha(x)\rho\,\mathrm{d}t + \oint_{\partial V} \alpha(x)\mathcal{J}^i e_i\,\mathrm{d}^2x$$

where e_i are the normal versors of the Gaussian surface. Using the fact that $\mathcal{J}^i$ vanishes in the contour ∂V and $\mathcal{J}^0 = \rho$ vanishes in t_1 and t_2 in all V, the integral is equal to zero, the just term which are non-zero in the transformed action will be $A_\mu(x)\mathcal{J}^\mu(x)$, therefore the only condition that the current $\mathcal{J}^\mu$ need to satisfy in order to show the invariance is:

$$\boxed{A_\mu(x)\mathcal{J}^\mu = 0} \tag{1.11}$$

which is the continuity equation.

(b) Derive the equations of motion for A_μ assuming it is the dynamical field, using Euler-Lagrange. Show that these are equivalent to the homogeneous Maxwell equations. (It is useful to remember that $E^i = -F^{0i}$ and $\epsilon^{ijk}B_k = F^{ij}$.)

Solution:

Euler-Lagrange equation for classical fields, is given by:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right] = 0$$

In our case, we have that $\phi(x) \equiv A_\mu(x)$, i.e.:

$$\frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\mu)}\right] = 0$$

For the first part:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial A_\mu} &= -\frac{1}{4}\underbrace{\frac{\partial}{\partial A_\mu}(F_{\mu\nu}F^{\mu\nu})}_{=0} - \frac{\partial}{\partial A_\mu}(A_\mu \mathcal{J}^\mu) \\ &= -\mathcal{J}^\mu\end{aligned}$$

And for the second term:

$$\begin{aligned}\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\mu)}\right] &= -\frac{1}{4}\frac{\partial}{\partial(\partial_\mu A_\mu)}(F_{\mu\nu}F^{\mu\nu}) - \underbrace{\frac{\partial}{\partial(\partial_\mu A_\mu)}(A_\mu \mathcal{J}^\mu)}_{=0} \\ &= -\frac{1}{4}\frac{\partial}{\partial(\partial_\mu A_\mu)}\left[(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\nu)\right] \\ &= -\frac{1}{4}\frac{\partial}{\partial(\partial_\mu A_\mu)}\left[\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu - \partial_\nu A_\mu \partial^\mu A^\nu + \partial_\nu A_\mu \partial^\nu A^\mu\right]\end{aligned}$$

With this, we need to evaluate four different derivatives. Let's evaluate each of these in turn:

$$\begin{aligned}\frac{\partial}{\partial(\partial_\mu A_\mu)}(\partial_\mu A_\nu \partial^\mu A^\nu) &= \frac{\partial}{\partial(\partial_\mu A_\mu)}(\delta^\mu_\nu \partial_\mu A_\mu \partial^\mu A^\nu) \\ &= \frac{\partial}{\partial(\partial_\mu A_\mu)}(4\partial_\mu A_\mu \partial^\mu A^\nu) \\ &= 4(\partial^\mu A^\nu)\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial(\partial_\mu A_\mu)}(\partial_\mu A_\nu \partial^\nu A^\mu) &= \frac{\partial}{\partial(\partial_\mu A_\mu)}(\delta^\mu_\nu \partial_\mu A_\mu \partial^\nu A^\mu) \\ &= 4(\partial^\nu A^\mu)\end{aligned}$$

For the last two derivatives, we just need to look at the first part of each one. This means that we need to consider $\partial_\nu A_\mu$. This don't depends of $\partial_\mu A_\mu$, we just can change the derivative for $4\partial_\nu A_\nu$, which is constant in relation to $\partial_\mu A_\mu$, therefore the last two derivatives is zero. Thus, we obtain:

$$\frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\mu)}\right] = -\mathcal{J}^\mu + \frac{1}{4}\partial_\mu(4\partial^\mu A^\nu - 4\partial^\nu A^\mu) = 0$$

Therefore, the motion equations can be written by:

$$\boxed{\partial_\mu F^{\mu\nu} = \mathcal{J}^\mu} \tag{1.12}$$

Now, using that $F^{0i} = -E^i$ (or $F^{i0} = E^i$ from the anti-symmetry of $F^{\mu\nu}$):

$$\partial_i F^{i0} = \mathcal{J}^0 \Rightarrow \partial_i E^i = \rho \Rightarrow \boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho$$

$$\boxed{\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho \quad (\text{using } F^{0i} = E^i)} \tag{1.13}$$

Finally, we can use $\epsilon^{ijk} B_k = F^{ij}$:

$$\partial_0 F^{0j} + \partial_i F^{ij} = \mathcal{J}^j \Rightarrow -\partial_0 E^j + \partial_i \epsilon^{ijk} B_k = \mathcal{J}^j \Rightarrow \boldsymbol{\nabla} \times \boldsymbol{B} = \boldsymbol{j} + \frac{\partial \boldsymbol{E}}{\partial t}$$

$$\boxed{\boldsymbol{\nabla} \times \boldsymbol{B} = \boldsymbol{J} + \frac{\partial \boldsymbol{E}}{\partial t} \quad (\text{using } F^{0i} = -E^i \text{ and } F^{ij} = \epsilon^{ijk} B_k)} \tag{1.14}$$

(c) Show that the remaining two Maxwell's equations can be obtained from:

$$\widetilde{F}^{\mu\nu} \equiv \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$$

which (show this explicitly) satisfies:

$$\partial_\mu \widetilde{F}^{\mu\nu} = 0$$

Solution:

Firstly, we can use $\mu = i$ and $\nu = k$, therefore:

$$\widetilde{F}^{ik} = \frac{1}{2} \epsilon^{ik0j} E_j - \frac{1}{2} \underbrace{\epsilon^{ikij}}_{=0} \epsilon_{ijk} B^k = \frac{1}{2} \epsilon^{ikj} E_j$$

Applying the derivative ∂_i:

$$\partial_i \widetilde{F}^{ik} = \frac{1}{2} \partial_i \epsilon^{ikj} E_j = \frac{1}{2} \boldsymbol{\nabla} \times \boldsymbol{E}$$

In a homogeneous space, the curl of $\boldsymbol{E}$ is zero because the electric field doesn't depend on the spatial distribution of charge in this case, therefore:

$$\boxed{\boldsymbol{\nabla} \times \boldsymbol{E} = 0} \tag{1.15}$$

Now, using $\mu = 0$ and $\nu = k$:

$$\widetilde{F}^{0k} = \frac{1}{2} \underbrace{\epsilon^{0k0j}}_{=0} E_j - \frac{1}{2} \epsilon^{0kij} \epsilon_{ijk} B^k$$

We just need to proof what is $\epsilon^{0kij}\epsilon_{ijk}$. To do this, we can use the properties of the Levi-Civita symbol, simplifying the expression as follows:

$$\epsilon^{0kij}\epsilon_{ijk} = \delta_0^k\delta_j^i - \delta_i^k\delta_0^j + \delta_j^k\delta_0^i = \delta_0^i\delta_0^j - \delta_0^i\delta_0^j + \delta_0^i\delta_0^j = \delta_0^i\delta_0^j = 1$$

Applying ∂_k:

$$\partial_k\widetilde{F}^{0k} = -\frac{1}{2}\partial_k B^k = -\frac{1}{2}\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0 \quad \text{(always)}$$

$$\boxed{\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0} \tag{1.16}$$

Note that if we apply the derivative with $k = 0$ or $i = 0$, the Levi-Civita symbols will be annulled. Therefore, we just consider the spatial derivatives and use $\mu = 0$ in $\widetilde{F}^{\mu\nu}$ to calculate all possibilities. Hence, we have shown that:

$$\boxed{\partial_\mu\widetilde{F}^{\mu\nu} = 0} \tag{1.17}$$

References

Srednicki, Mark (2009). *Quantum Field Theory*. Vol. III. New York: Cambrige University Press.