Institute of Physics

Electromagnetism I

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Chapter N° 2

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All the following problems were taken from Griffiths (2017).

Problem 2.2

(a) Find the electric field (magnitude and direction) a distance z above the midpoint between two equal charges, q, a distance d apart (Fig. 1.1). Check that your result is consistent with what you'd expect when $z > d$.

(b) Repeat part (a), only this time make the right-hand charge $-q$ instead of $+q$.
Resource: Walter Scott (2019)

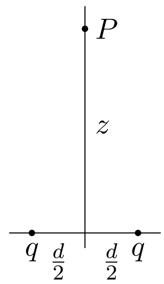


Figure 1.1

Solution:

(a) The distance between a charge and the point P is given by:

$$r = \sqrt{\frac{d^2}{4} + z^2}$$

Assuming that the point P is located in $(0, z)$ and the charges $(-\frac{d}{2}, 0)$ and $(\frac{d}{2}, 0)$, we have two electrical forces being applied in P, therefore for each charge:

$$\boldsymbol{E}_{\text{left}} = \frac{1}{4\pi\varepsilon_o}\frac{q}{r^2}\boldsymbol{e}_{r1} \qquad \& \qquad \boldsymbol{E}_{\text{right}} = \frac{1}{4\pi\varepsilon_o}\frac{q}{r^2}\boldsymbol{e}_{r2}$$

Defining θ as the angle between $q\hat{P}O$, where $O = (0, 0)$, we can write the unitary vectors by:

$$\boldsymbol{e}_{r1} = \cos\theta\boldsymbol{e}_x + \sin\theta\boldsymbol{e}_z \qquad \& \qquad \boldsymbol{e}_{r2} = -\cos\theta\boldsymbol{e}_x + \sin\theta\boldsymbol{e}_z$$

Knowing that $\boldsymbol{E}$ satisfies the superposition principle:

$$\boldsymbol{E} = \boldsymbol{E}_{\text{left}} + \boldsymbol{E}_{\text{right}} = \frac{1}{4\pi\varepsilon_o}\frac{2q}{\frac{d^2}{4}+z^2}\sin\theta\boldsymbol{e}_z$$

Note that we can rewrite $\sin\theta$ as:

$$\sin\theta = \frac{\frac{d}{2}}{r}$$

Concluding that:

$$\boxed{\boldsymbol{E} = \frac{qd}{4\pi\varepsilon_o}\left(\frac{d^2}{4}+z^2\right)^{-3/2}\boldsymbol{e}_z} \tag{1.1}$$

When $z \gg d$, we can disregarded the term $\frac{d^2}{4}$ on denominator and obtain:

$$\boxed{\boldsymbol{E} = \frac{1}{4\pi\varepsilon_o}\frac{qd}{z^3}\boldsymbol{e}_z} \tag{1.2}$$

This mean that the further away is the point P, smaller the electric field $\boldsymbol{E}$. But if $d \gg z$, we disregarded the z^2 and obtain:

$$\boxed{\boldsymbol{E} = \frac{1}{4\pi\varepsilon_o}\frac{4q}{d^2}\boldsymbol{e}_z} \tag{1.3}$$

Again, for $d \to \infty$, the electric field in the point P tends to zero, because the electric field vector generated by the left-hand side will be canceled with the vector generated by the right-hand side charge. Note that if $z \gg d$, the electric field decreases more faster than $d \gg z$.

(b) Now, if we change the sign of the charge on the right-hand side, the only change will be in the versor $\boldsymbol{e}_{r2}$, such that with the negative sign, the force will be directed inward, therefore:

$$\boldsymbol{e}_{r2} = \cos\theta\boldsymbol{e}_x - \sin\theta\boldsymbol{e}_y$$

Therefore the summation between $\boldsymbol{E}_{\text{left}}$ and $\boldsymbol{E}_{\text{right}}$ is:

$$\boldsymbol{E} = \frac{1}{4\pi\varepsilon_o}\frac{2q}{\frac{d^2}{4}+z^2}\cos\theta\boldsymbol{e}_x$$

Rewriting $\cos\theta$ as $\frac{z}{r}$, we obtain:

$$\boxed{\boldsymbol{E} = \frac{2qz}{4\pi\varepsilon_o}\left(\frac{d^2}{4}+z^2\right)^{-3/2}\boldsymbol{e}_x} \tag{1.4}$$

Problem 2.6

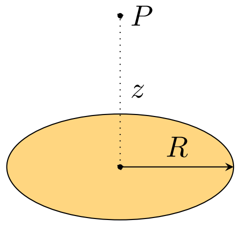


Figure 1.2

Find the electric field a distance z above the center of a flat circular disk of radius R (1.2) that carries a uniform surface charge σ. What does your formula give in the limit $R \to \infty$? Also check the case $z \gg R$.
Resource: Walter Scott (2019)

Solution:

To find the electric field in a circular disk that carries a uniform surface charge σ, we can use:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_o} \int_0^R \frac{\sigma(\boldsymbol{r}')}{\mathscr{r}^2} \hat{\boldsymbol{\mathscr{r}}} \, \mathrm{d}\tau'$$

Firstly we need do see that $\hat{\boldsymbol{\mathscr{r}}}$ isn't a constant, such that to find the Electric field, we need to describe this versor. So, the vector $\boldsymbol{\mathscr{r}}$ can be written by:

$$\boldsymbol{\mathscr{r}} = \|\boldsymbol{\mathscr{r}}\| \hat{\boldsymbol{\mathscr{r}}} \Rightarrow \hat{\boldsymbol{\mathscr{r}}} = \frac{\boldsymbol{\mathscr{r}}}{\sqrt{r^2 + z^2}}$$

By the problem symmetry, we can conclude that the radial vectors will be annulled one-by-one, such that just the azimuth component will survive. These component can be written by:

$$\boldsymbol{\mathscr{r}} = r \sin\theta \hat{z} = \frac{rz}{\sqrt{r^2 + z^2}} \hat{z}$$

Given that σ and z are constant, we conclude that:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_o} \sigma z \int_0^R \frac{r}{(r^2 + z^2)^{\frac{3}{2}}} \, \mathrm{d}r \int_0^{2\pi} \mathrm{d}\phi \, \hat{z}$$

The integral in ϕ appears because we have a circular disk, and if we don't integrate in $\phi \in [0, 2\pi]$, we just consider a line, which is not correct to this problem. So, evaluating the integral in r, we can define $u = r^2 + z^2$, which imply $\mathrm{d}u = 2r \, \mathrm{d}r$. This gives us:

$$\begin{aligned} \int_0^R \frac{r}{(r^2 + z^2)^{\frac{3}{2}}} \, \mathrm{d}r &= \frac{1}{2} \int_{z^2}^{R^2 + z^2} \frac{1}{u^{\frac{3}{2}}} \, \mathrm{d}u = \frac{1}{2} \left(-\frac{2}{\sqrt{u}} \right) \Bigg|_z^{R^2 + z^2} \\ &= \frac{1}{z} - \frac{1}{\sqrt{R^2 + z^2}} \end{aligned}$$

Therefore, the electrical field can be written by:

$$\boxed{\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_o} 2\pi\sigma z \left(\frac{1}{z} - \frac{1}{\sqrt{R^2 + z^2}} \right) \hat{z}} \tag{1.5}$$

In the limit $R \to \infty$, the second term tends to zero, which implies that:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_o} 2\pi\sigma \frac{z}{z} \hat{z}$$

$$\boxed{\boldsymbol{E}(\boldsymbol{r}) = \frac{\sigma}{2\varepsilon_o} \hat{z}} \tag{1.6}$$

Now, if $z \gg R$, the second term can be expand in Taylor series, such that:

$$\frac{1}{\sqrt{R^2 + z^2}} = \frac{1}{z} - \frac{R^2}{2z^3} + \frac{3R^4}{8z^5} - \frac{5R^6}{16z^7} + \mathcal{O}\left(\frac{1}{z^9}\right)$$

Ignoring terms which have power over than $1/z^3$, we can write the electric field by:

$$\boxed{\boldsymbol{E}(\boldsymbol{r}) = \frac{\sigma}{4\varepsilon_o} \frac{R^2}{z^2} \hat{z}} \tag{1.7}$$

Recalling that $Q = \pi R^2 \sigma$ for a uniform circular charge distribution, in other word, we can write $\sigma = \dfrac{Q}{\text{Area}}$, therefore, multiplying and dividing by π:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_o} \frac{\pi\sigma R^2}{z^2} = \frac{1}{4\pi\varepsilon_o} \frac{Q}{z^2} \hat{z}$$

This is the electric field for a punctual charge Q.

Problem 2.10

A charge q sits at the back corner of a cube, as shown in Fig (1.3). What is the flux of $\boldsymbol{E}$ through the shaded side?
Resource: Walter Scott (2019)

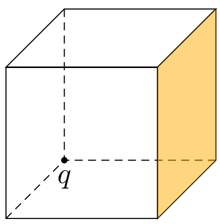


Figure 1.3

Solution:

Here, we can construct a Gaussian surface to the charge q be in the center of them, i.e.

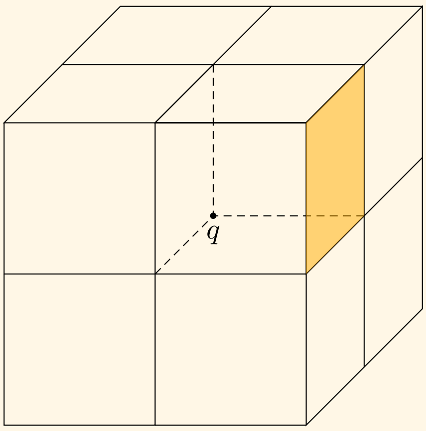


Figure 1.4

With this construction, the flux in the shaded side is 1 of the 24 total sides, i.e.

$$\Phi_{\text{shaded}} = \frac{1}{24}\Phi_{\text{cube}}$$

By the Gauss's Law, we have:

$$\Phi_{\text{cube}} = \oint_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{s} = \frac{q}{\varepsilon_o}$$

Therefore, we can conclude that the flux in the shaded side is:

$$\boxed{\Phi_{\text{shaded}} = \frac{q}{24\varepsilon_o}} \tag{1.8}$$

Problem 2.15

A thick spherical shell carries charge density

$$\rho = \frac{k}{r^2} \qquad (a \leqslant r \leqslant b)$$

Fig. (1.5). Find the electric field in the tree regions:

(i) $r < a$

(ii) $a < r < b$

(iii) $r > b$

Plot $|\boldsymbol{E}|$ as a function of r, for the case $b = 2a$.
Resource: Walter Scott (2019)

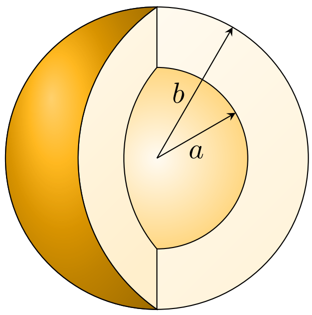


Figure 1.5

Solution:

(i) In the case of $r < a$, we don't have charge density, for the fact that $r \in [a, b]$, which implies that don't have any charge inside the shell, i.e. $q_{\text{inside}} = 0$, so, if we use the Gauss's Law:

$$\oint_{r<a} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{s} = 0$$

$$\boxed{\boldsymbol{E} = 0} \tag{1.9}$$

(ii) Now, for $a < r < b$, we have charge density, and using the Gauss's Law:

$$\oint_{a<r<b} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{s} = \frac{q_{\text{inside}}}{\varepsilon_o}$$

Knowing that for $r < a$ the electric field is zero, we can extrapolate the limits of this integral to $0 < r < b$, because this will give us the total charge of the shell, therefore:

$$\oint_{0<r<b} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{s} = \oint_{0<r<b} E \, \mathrm{d}s \qquad \text{(By the spherical symmetry)}$$

So, $\mathrm{d}s = r'^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$ and E is the norm of $\boldsymbol{E}$, where $a < r' < b$. We use r' to evaluate the surface area of each layer of the shell. This give us:

$$\oint_{0<r<b} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{s} = E r'^2 \int_0^{\pi} \int_0^{2\pi} \sin\theta \, \mathrm{d}\phi \, \mathrm{d}\theta = E(4\pi r'^2)$$

Therefore:

$$E(4\pi r'^2) = \frac{q_{\text{inside}}}{\varepsilon_o}$$

Now we can calculate the q_{inside} using the charge density, i.e.

$$q_{\text{inside}} = \oint_V \rho \, \mathrm{d}\tau = \oint_V \frac{k}{r'^2} r'^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi = k \int_a^{r'} \mathrm{d}r \int_0^{\pi} \sin\theta \int_0^{2\pi} \mathrm{d}\phi = k(r' - a)4\pi$$

Using this result:

$$E(4\pi r'^2) = \frac{k4\pi(r' - a)}{\varepsilon_o} \Rightarrow E = \frac{k(r' - a)}{r'^2}$$

As we said previously, the spherical symmetry it allow us to conclude that $\boldsymbol{E} \parallel \hat{n}$, where $\hat{n} = \hat{r}$ is the versor of the sphere, therefore:

$$\boxed{\boldsymbol{E} = \frac{k(r' - a)}{r'^2} \hat{r}} \tag{1.10}$$

(iii) For the last case, $r > b$, we can simply use that the charge inside a Gaussian surface with radius b is given by:

$$q_{\text{inside}} = k(b - a)4\pi$$

This is because if $r = b$, we obtain the upper limit of $r' < b$, which means that the total charge that fills the spherical shell is given by this expression. Moreover, for $r > b$, there is no charge density, implying that if the radius of the spherical Gaussian surface is, for example, $r = 2b$, the total charge inside it remains the same. Hence, we need only substitute r' for b in the previous expression for the electric charge inside the shell, i.e.,

$$\boxed{\boldsymbol{E} = \frac{k(b - a)}{r'^2} \hat{r}} \tag{1.11}$$

To confirm the result of q_{inside}, the calculations are the same of the previous item, but considering the fact that all charge of the shell are enclosed in the volume of the sphere. Note that we don't change the denominator, this is because the electric field needs to satisfy the Gauss's Law, and the radius dependence in necessary for this.

To compute this, is very simple, just for clarify, I will write what is r' in dependence of r. Basically, r' is the r with an application of a Heaviside function, i.e.,

$$r' = \Theta(b - r)r = \begin{cases} r, & r < b \\ 0, & r \geqslant b \end{cases}$$

We just need to consider $r < b$ and not $a < r < b$ because for $a < r$ we have $\boldsymbol{E} = 0$, as we shown in item (i). With this, we can plot the graphic of $E(r)$ with $b = 2a$:

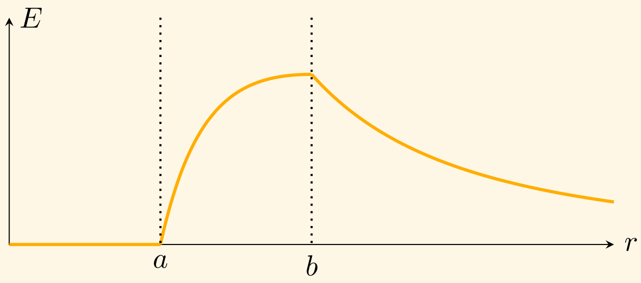


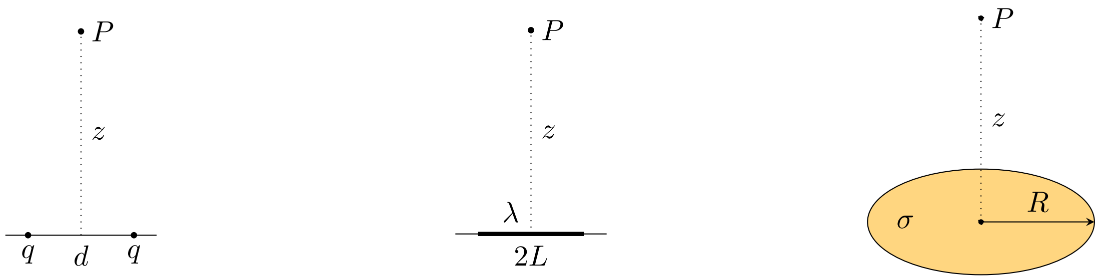
Figure 1.6

Problem 2.25

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_o} \sum_{i=1}^{n} \frac{q_i}{\mathscr{r}_i} \tag{1.12}$$

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_o} \int \frac{\lambda(\boldsymbol{r}')}{\mathscr{r}} \, \mathrm{d}\ell' \qquad \text{and} \qquad V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_o} \int \frac{\sigma(\boldsymbol{r}')}{\mathscr{r}} \, \mathrm{d}a' \tag{1.13}$$

Using equations (1.12) and (1.13), find the potential at a distance z above the center of the charge distributions in Fig. (1.7). In each case, compute, $\boldsymbol{E} = -\boldsymbol{\nabla} V$, and compare your answers with Ex. 2.1, Ex. 2.2, and Prob. 2.6, respectively. Suppose that we changed the right-hand charge in Fig. (1.7a) to $-q$; what the is the potential at P? What does that suggest? Compare your answer to Prob. 2.2, and explain carefully any discrepancy.



(a) Two point charges (b) Uniform line charge (c) Uniform surface charge

Figure 1.7

Resource: Walter Scott (2019)

Solution:

(a) To find the potential at a distance z, we can use the discrete expression, given in (1.12):

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_o}\frac{q_1}{\sqrt{\frac{d^2}{4}+z^2}} + \frac{q_2}{\sqrt{\frac{d^2}{4}+z^2}} = \frac{1}{4\pi\varepsilon_o}\frac{2q}{\sqrt{\frac{d^2}{4}+z^2}}$$

Therefore the potential is given by:

$$\boxed{V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_o}\frac{2q}{\sqrt{\frac{d^2}{4}+z^2}}} \tag{1.14}$$

Now, we can use $\boldsymbol{E} = -\boldsymbol{\nabla} V$. Note that $V(\boldsymbol{r})$ depends only of z, therefore:

$$\boldsymbol{E} = -\frac{\partial V}{\partial z}\hat{z} = -\frac{2q}{4\pi\varepsilon_o}\frac{\partial}{\partial z}\left(\frac{1}{\sqrt{\frac{d^2}{4}+z^2}}\right)\hat{z} = -\frac{2q}{4\pi\varepsilon_o}\frac{\partial}{\partial z}\left[\left(\frac{d^2}{4}+z^2\right)^{-\frac{1}{2}}\right]\hat{z} = -\frac{4qz}{4\pi\varepsilon_o}\left(-\frac{1}{2}\right)\left(\frac{d^2}{4}+z^2\right)^{-\frac{3}{2}}\hat{z}$$

$$\boxed{\boldsymbol{E} = \frac{1}{4\pi\varepsilon_o}\frac{2qz}{\left(\frac{d^2}{4}+z^2\right)^{\frac{3}{2}}}\hat{z}} \tag{1.15}$$

(b) Using the expression for the uniform line charge in (1.13), we need to integrate in $\mathrm{d}\ell$. Given that the line is one-dimensional, we can change $\mathrm{d}\ell \mapsto \mathrm{d}x$, where $x \in [-L, L]$. With this in mind, we can write:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_o}\int_{-L}^{L}\frac{\lambda}{\sqrt{x^2+z^2}}\,\mathrm{d}x$$

Given that z is constant

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_o}\int_{-L}^{L}\frac{\lambda}{z\sqrt{\left(\frac{x}{z}\right)^2+1}}\,\mathrm{d}x$$

We can define $\frac{x}{z} := \tan\theta$, therefore in the limits $x = -L$ and $x = L$, we have:

$$\tan\theta = -\frac{L}{z} \Rightarrow \theta = \arctan\left(-\frac{L}{z}\right) := \theta_1 \qquad \text{or} \qquad \tan\theta = \frac{L}{z} \Rightarrow \theta = \arctan\left(\frac{L}{z}\right) := \theta_2$$

So, the integral is given by:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_o}\int_{\theta_1}^{\theta_2}\frac{\lambda}{z\sqrt{\tan^2\theta+1}}\,\mathrm{d}x$$

We can write $\mathrm{d}x = z\sec^2\theta$ and $\tan^2\theta + 1 = \sec^2\theta$, therefore:

$$\begin{aligned}
V(\boldsymbol{r}) &= \frac{1}{4\pi\varepsilon_o}\int_{\theta_1}^{\theta_2}\lambda\sec\theta\,\mathrm{d}\theta = \frac{\lambda}{4\pi\varepsilon_o}\ln|\tan\theta+\sec\theta|\Big|_{\theta_1}^{\theta_2}\\
&= \frac{\lambda}{4\pi\varepsilon_o}\left(\ln\left|\frac{L}{z}+\sqrt{\frac{L^2}{z^2}+1}\right| - \ln\left|-\frac{L}{z}+\sqrt{\frac{L^2}{z^2}+1}\right|\right)\\
&= \frac{\lambda}{4\pi\varepsilon_o}\ln\left|\frac{\frac{L}{z}+\sqrt{\frac{L^2}{z^2}+1}}{-\frac{L}{z}+\sqrt{\frac{L^2}{z^2}+1}}\right| = \frac{\lambda}{4\pi\varepsilon_o}\ln\left|\frac{L+\sqrt{L^2+z^2}}{-L+\sqrt{L^2+z^2}}\right|
\end{aligned}$$

Given that $L > 0$ and $z > 0$, we can disregarded the absolute value, which implies:

$$\begin{aligned}
V(\boldsymbol{r}) &= \frac{\lambda}{4\pi\varepsilon_o}\ln\left(\frac{L+\sqrt{L^2+z^2}}{-L+\sqrt{L^2+z^2}}\right) = \frac{\lambda}{4\pi\varepsilon_o}\ln\left[\frac{(L+\sqrt{L^2+z^2})(L+\sqrt{L^2+z^2})}{z}\right]\\
&= \frac{\lambda}{4\pi\varepsilon_o}\ln\left[\frac{(L+\sqrt{L^2+z^2})^2}{z}\right] = \frac{2\lambda}{4\pi\varepsilon_o}\ln\left(\frac{L+\sqrt{L^2+z^2}}{z}\right)
\end{aligned}$$

Concluding that:

$$\boxed{V(\boldsymbol{r}) = \frac{\lambda}{2\pi\varepsilon_o}\ln\left(\frac{L+\sqrt{L^2+z^2}}{z}\right)} \tag{1.16}$$

Using $\boldsymbol{E} = -\boldsymbol{\nabla}V$, we just need to consider the derivative in respect to z, therefore:

$$\boldsymbol{E} = -\frac{\lambda}{2\pi\varepsilon_o}\frac{\partial}{\partial z}\left[\ln\left(\frac{L+\sqrt{L^2+z^2}}{z}\right)\right]\hat{z} = \frac{\lambda}{2\pi\varepsilon_o}\frac{L}{z\sqrt{L^2+z^2}}\hat{z}$$

$$\boxed{\boldsymbol{E} = \frac{\lambda}{2\pi\varepsilon_o}\frac{L}{z\sqrt{L^2+z^2}}\hat{z}} \tag{1.17}$$

(c) Finally, for the circular disk, we have:

$$\begin{aligned}
V(\boldsymbol{r}) &= \frac{1}{4\pi\varepsilon_o}\int\frac{\sigma}{\sqrt{x^2+y^2+z^2}}\,\mathrm{d}s\\
&= \frac{1}{4\pi\varepsilon_o}\int_{-R}^{R}\int_{-R}^{R}\frac{\sigma}{\sqrt{x^2+y^2+z^2}}\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

Define $x = r\sin\theta$ and $y = r\cos\theta$, in this way:

$$\begin{aligned}
V(\boldsymbol{r}) &= \frac{1}{4\pi\varepsilon_o}\int_0^R\int_0^{2\pi}\frac{\sigma r}{\sqrt{r^2+z^2}}\,\mathrm{d}\theta\,\mathrm{d}r\\
&= \frac{1}{4\pi\varepsilon_o}\int_0^R\frac{\sigma 2\pi r}{\sqrt{r^2+z^2}}\,\mathrm{d}r\\
&= \frac{\sigma}{2\varepsilon_o}\int_0^R\frac{r}{\sqrt{r^2+z^2}}\,\mathrm{d}r\\
&= \frac{\sigma}{2\varepsilon_o}\sqrt{r^2+z^2}\Big|_0^R\\
&= \frac{\sigma}{2\varepsilon_o}\left(\sqrt{R^2+z^2}-\sqrt{z^2}\right)
\end{aligned}$$

$$V(\boldsymbol{r}) = \frac{\sigma}{2\varepsilon_o}\left(\sqrt{R^2+z^2} - z\right) \tag{1.18}$$

Deriving in z by $\boldsymbol{E} = -\boldsymbol{\nabla} V$, we get:

$$\boldsymbol{E} = \frac{\sigma}{2\varepsilon_o}\left(1 - \frac{z}{\sqrt{R^2+z^2}}\right)\hat{z} \tag{1.19}$$

Problem 2.35

Here is a fourth way of computing the energy of a uniformly charged solid sphere: Assemble it like a snowball, layer by layer, each time bringing in an infinitesimal charge dq form far away and smearing it uniformly over the surface, thereby increasing the radius. How much work dW does it take to build up the radius by an amount dr? Integrate this to find the work necessary to create the entire sphere of radius R and total charge q.
Resource: Walter Scott (2019)

Solution:

We can write the work in function of the potential as $\mathrm{d}W = V\,\mathrm{d}q$. So, if we consider the charge density ρ constant, by increasing the radius by an amount dr, the charge inside will be change to keep the ρ constant, therefore, we have:

$$\mathrm{d}W = V'\,\mathrm{d}q'$$

where V' is the potential considering dq', i.e.

$$V' = \frac{1}{4\pi\varepsilon_o}\frac{q'}{r} \Rightarrow \mathrm{d}W = \frac{1}{4\pi\varepsilon_o}\frac{q'}{r}\,\mathrm{d}q'$$

So, given that ρ is constant, we can write it by $\rho = \dfrac{q}{V}$, where V is the volume of the sphere, written by $V = \dfrac{4}{3}\pi R^3$. In this way

$$q' = \rho\frac{4}{3}\pi r^3 = \frac{q}{V}\frac{4}{3}\pi r^3 = \frac{q}{R^3}r^3$$

Differentiating this in r, we get:

$$\mathrm{d}q' = \frac{3q}{R^3}r^2\,\mathrm{d}r$$

Using these relations:

$$\mathrm{d}W = \frac{1}{4\pi\varepsilon_o}\frac{q}{R^3}r^2\frac{3q}{R^3}r^2\,\mathrm{d}r = \frac{3}{4\pi\varepsilon_o}\frac{q^2}{R^6}r^4\,\mathrm{d}r$$

Integrating in the interval $[0, R]$ (because we need the work inside the sphere), we have:

$$W = \int \mathrm{d}W = \frac{3}{4\pi\varepsilon_o}\frac{q^2}{R^6}\int_0^R r^4\,\mathrm{d}r = \frac{3}{4\pi\varepsilon_o}\frac{q^2}{R^6}\frac{R^5}{5}$$

$$\boxed{W = \frac{3}{5}\left(\frac{1}{4\pi\varepsilon_o}\frac{q^2}{R}\right)} \tag{1.20}$$

Problem 2.36

$$W = \frac{\varepsilon}{2}\int_V E^2\,\mathrm{d}\tau \tag{1.21}$$

$$\begin{aligned} W_{\text{tot}} &= \frac{\varepsilon_o}{2}\int E^2\,\mathrm{d}\tau = \frac{\varepsilon_o}{2}\int (\boldsymbol{E}_1 + \boldsymbol{E}_2)^2\,\mathrm{d}\tau \\ &= \frac{\varepsilon_o}{2}\int (E_1^2 + E_2^2 + 2\boldsymbol{E}_1\cdot\boldsymbol{E}_2)\,\mathrm{d}\tau \\ &= W_1 + W_2 + \varepsilon_o\int \boldsymbol{E}_1\cdot\boldsymbol{E}_2\,\mathrm{d}\tau \end{aligned} \tag{1.22}$$

Consider two concentric spherical shells, of radius a and b. Suppose the inner one carries a charge q, and the outer one a charge $-q$ (both of them uniformly distributed over the surface). Calculate the energy of this configuration:

(a) Using eq. (1.21)

(b) Using eq. (1.22) and the results of Ex. 2.9

Resource: Walter Scott (2019)

Solution:

(a) Firstly, we need to determine $\boldsymbol{E}$ between $a < r < b$. Using that the charge density is constant, we have that:

$$\oint_V \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{s} = E\oint_{a<r<b}\mathrm{d}s = \frac{q_{\text{inside}}}{\varepsilon_o} \Rightarrow E = \frac{q}{4\pi\varepsilon_o}\frac{1}{r^2}$$

So, applying this in (1.21), we get:

$$\begin{aligned}
W &= \frac{\varepsilon_o}{2} \int\limits_V \left(\frac{q}{4\pi\varepsilon_o} \frac{1}{r^2} \right)^2 \mathrm{d}\tau \\
&= \frac{\varepsilon_o}{2} \left(\frac{q}{4\pi\varepsilon_o} \right)^2 \int\limits_V \frac{1}{r^4} r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\phi \, \mathrm{d}\theta \\
&= \frac{q^2}{32\pi^2\varepsilon_o} 4\pi \int_a^b \frac{1}{r^2} \, \mathrm{d}r = -\frac{q^2}{8\pi\varepsilon_o} \frac{1}{r} \bigg|_a^b
\end{aligned}$$

$$\boxed{W = \frac{q^2}{8\pi\varepsilon_o} \left(\frac{1}{a} - \frac{1}{b} \right)} \tag{1.23}$$

(b) Using the eq. (1.21) to calculate W_1 and W_2, we just need to imply that $r > a$ for W_1 and $r > b$ for W_2 in the expression for E in the last item, therefore:

$$E_1 = \frac{q}{4\pi\varepsilon_o} \frac{1}{r^2}, \quad r > a \qquad \& \qquad E_2 = -\frac{q}{4\pi\varepsilon_o} \frac{1}{r^2} \quad r > b$$

which implies that:

$$W_1 = \frac{q^2}{8\pi\varepsilon_o} \frac{1}{a} \qquad \& \qquad W_2 = \frac{q^2}{8\pi\varepsilon_o} \frac{1}{b}$$

So, to obtain the last integral, we just need to know the versors of each one electric field. Since the electrical field are spherical symmetric, we have that $\boldsymbol{E}_1 \cdot \boldsymbol{E}_2 = E_1 * E_2$, i.e.

$$\boldsymbol{E}_1 \cdot \boldsymbol{E}_2 = -\left(\frac{q}{4\pi\varepsilon_o} \right)^2 \frac{1}{r^4}$$

The infinitesimal volume element can be written by

$$\mathrm{d}\tau = r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi$$

Therefore:

$$\int\limits_V \boldsymbol{E}_1 \cdot \boldsymbol{E}_2 \, \mathrm{d}\tau = -\left(\frac{q}{4\pi\varepsilon_o} \right)^2 \int_b^\infty \int_0^\pi \int_0^{2\pi} \frac{1}{r^4} r^2 \sin\theta \, \mathrm{d}\phi \, \mathrm{d}\theta \, \mathrm{d}r = -\left(\frac{q}{4\pi\varepsilon_o} \right)^2 \frac{4\pi}{b}$$

Now, using eq. (1.22):

$$\begin{aligned}
W_{\text{tot}} &= W_1 + W_2 + \varepsilon_o \int\limits_V \boldsymbol{E}_1 \cdot \boldsymbol{E}_2 \, \mathrm{d}\tau \\
&= \frac{q^2}{8\pi\varepsilon_o} \frac{1}{a} + \frac{q^2}{8\pi\varepsilon_o} \frac{1}{b} - \frac{q^2}{4\pi\varepsilon_o} \frac{1}{b} \\
&= \frac{q^2}{8\pi\varepsilon_o} \left(\frac{1}{a} + \frac{1}{b} - \frac{2}{b} \right)
\end{aligned}$$

Concluding that

$$\boxed{W_{\text{tot}} = \frac{q^2}{8\pi\varepsilon_o}\left(\frac{1}{a} - \frac{1}{b}\right)} \tag{1.24}$$

which is equal to the last answer.

A little comment about the integration of $\boldsymbol{E}_1 \cdot \boldsymbol{E}_2$: Knowing that $\boldsymbol{E}_1$ are valid only for $r > a$ and $\boldsymbol{E}_2$ just for $r > b$, we can write two sets $\mathbb{E}_1 = [a, \infty)$ and $\mathbb{E}_2 = [b, \infty)$, which $a < b$, such that:

$$\mathbb{E}_1 \cap \mathbb{E}_2 = [b, \infty)$$

This implies that the scalar product will be defined just in the interval $[b, \infty)$, this is why the integral is calculated in $[b, \infty)$.

Problem 2.39

Two spherical cavities, of radius a and b, are hollowed out from the interior of a (neutral) conducting sphere of radius R Fig. (1.8). At the center of each cavity a point charge is placed – call these charges q_a and q_b

(a) Find the surface charge densities σ_a, σ_b and σ_R.

(b) What is the field outside the conductor?

(c) What is the field within each cavity?

(d) What is the force on q_a and q_b?

(e) Which of these answers would change if a third charge, q_c, were brought near the conductor?

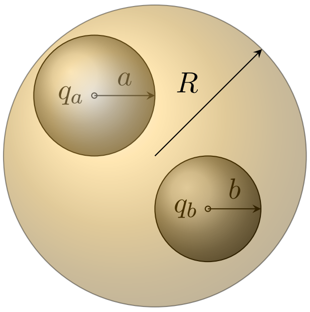


Figure 1.8

Resource: Walter Scott (2019)

Solution:

(a) To find an arbitrary surface charge density σ, we can use:

$$\oint_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{s} = \frac{q_{\text{inside}}}{\varepsilon_o} = \frac{\sigma S}{\varepsilon_o} \Rightarrow \sigma = \frac{q_{\text{inside}}}{S}$$

Therefore, applying $q_{\text{inside}} = a$, b or R, and using that $S = 4\pi r^2$, we conclude that:

$$\boxed{\sigma_a = -\frac{q_a}{4\pi a^2} \quad \& \quad \sigma_b = -\frac{q_b}{4\pi b^2} \quad \& \quad \sigma_R = \frac{q_a + q_b}{4\pi R^2}} \tag{1.25}$$

(b) Now, to find the electric field **outside** the conductor, we just need to considerate the charges inside the conductor, i.e., we consider $q_{\text{inside}} = q_a + q_b$. Applying the Gauss's Law, it is trivial proof that:

$$\boxed{\boldsymbol{E}_{\text{out}} = \frac{1}{4\pi\varepsilon_o}\frac{q_a + q_b}{r^2}\hat{r}, \quad r > R} \tag{1.26}$$

(c) Considering each sphere contribution, the electric field of each one will be independent of the other, therefore, we can use the Gauss's Law again to proof with simple counting that:

$$\boxed{\boldsymbol{E}_a = \frac{1}{4\pi\varepsilon_o}\frac{q_a}{r_a^2}\hat{r}_a, \quad r_a > a \qquad \& \qquad \boldsymbol{E}_b = \frac{1}{4\pi\varepsilon_o}\frac{q_b}{r_b^2}\hat{r}_b, \quad r_b > b} \tag{1.27}$$

where $\hat{r}_a$ is the normal versor of the surface which contain q_a and $\hat{r}_b$ the normal versor of the surface which contain q_b.

(d) Being a conductor, the total electrical charge needs to be zero in all points *inside* of the volume. This implies that the neutral part needs to cancel the sum of q_a and q_b, in other words, the interior of the volume will "create" a resultant charge $q_{Ra} = -q_a$ on the surface of the sphere which contains q_a and a resultant charge $q_{Rb} = -q_b$ on the surface of the sphere which contains q_B, such that:

$$\boldsymbol{F} = \boldsymbol{F}_a + \boldsymbol{F}_b + \boldsymbol{F}_{aR} + \boldsymbol{F}_{bR} = q_a\boldsymbol{E}_a + q_b\boldsymbol{E}_b + q_{Ra}\boldsymbol{E}_{Ra} + q_{Rb}\boldsymbol{E}_{Rb}$$

where we can decompose $\boldsymbol{E}_R = \boldsymbol{E}_{Ra} + \boldsymbol{E}_{Rb}$. In this way, we have:

$$\boldsymbol{F} = \frac{1}{4\pi\varepsilon_o}\left(\frac{q_a^2}{r_a^2}\hat{r}_a + \frac{q_b^2}{r_b^2}\hat{r}_b - \frac{q_a^2}{r_a^2}\hat{r}_a -\right)$$

Resulting in

$$\boxed{\boldsymbol{F} = \boldsymbol{0}} \tag{1.28}$$

(e) Including a third charge q_c *inside* the conductor, the charge surface density and the electrical field outside the sphere will change, because the q_{inside} will be $q_a + q_b + q_c$, therefore

$$\boxed{\sigma_R = \frac{q_a + q_b + q_c}{4\pi R^2} \qquad \& \qquad \boldsymbol{E}_{\text{out}} = \frac{1}{4\pi\varepsilon_o}\frac{q_a + q_b + q_c}{r^2}, \quad r > R} \tag{1.29}$$

Problem 2.46

If the electric field in some region is given (in spherical coordinates) by the expression

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{k}{r}\left[3\boldsymbol{e}_r + 2\sin\theta\cos\theta\sin\phi\boldsymbol{e}_\theta + \sin\theta\cos\phi\boldsymbol{e}_\phi\right]$$

for some constant k, what is the charge density?
Resource: Walter Scott (2019)

Solution:

Using the differential form of the Gauss's Law $\boldsymbol{\nabla}\cdot\boldsymbol{E} = \dfrac{\rho}{\varepsilon_o}$, we just need to evaluate this divergent to obtain (I'm tired to do that pass-by-pass)

$$\boxed{\rho(r,\theta,\phi) = 3k\varepsilon_o\frac{1+\cos(2\theta)\sin\phi}{r^2}} \tag{1.30}$$

References

Griffiths, David J. (2017). *Introduction to Electrodynamics*. Cambridge: Cambridge University Press.