# Next Generation Matrix

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In order to better understand how *Wolbachia* impacts the spread of dengue in both mosquito and human populations, we will consider a next generation matrix for the system

1) We will consider dengue as our infection. In this case, to align the systems in order, we will make the assumption that only humans and non infected female mosquitoes can contract dengue, seeing as they are the only vectors of the disease.

$$\frac{dN_{FI}}{dt} = \mu_N N_{FS} I - \alpha_N N_{FI} - \beta_N N_{FI} P$$

$$\frac{dI}{dt} = \mu_H S N_{FI} - \Psi_H I$$

$$\frac{dN_{FS}}{dt} = \rho_N f(N_{FS} + N_{FI})(\frac{N_M}{W_M + N_M}) - \alpha_N N_{FS} - \beta_N N_{FS} P - \mu_N N_{FS} I$$

$$\frac{dN_M}{dt} = \rho_N (1 - f)(N_{FI} + N_{FS})(\frac{N_M}{W_M + N_M}) - \alpha_N N_M - \rho_N N_M P$$

$$\frac{dW_F}{dt} = \rho_W q W_F - \alpha_W W_F - \rho_W W_F P + u_F(t) W_F$$

$$\frac{dW_M}{dt} = \rho_W (1 - q) W_F - \alpha_W W_M - \beta_W W_M P + u_M(t) W_M$$

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syms N_fi N_fs N_m W_f W_m I H alpha_n alpha_w beta_n beta_w  rho_n rho_w
mu_n mu_h u_f u_m f q psi_h
dN_fi_dt = mu_n * N_fs *I -alpha_n *N_fi -beta_n*N_fi *(N_fi+N_fs
+N_m+W_f+W_m);
dI_dt = mu_h * (H-I) * N_fi - psi_h *I;
dN_fs_dt = rho_n *f* (N_fs + N_fi)*(N_m/(W_m+N_m)) -alpha_n * N_fs -beta_n *
N_fs * (N_fi+N_fs +N_m+W_f+W_m) -mu_n *N_fs*I;
dN_m_dt = rho_n * (1-f)*(N_fs+N_fi)*(N_m/(W_m+N_m)) -alpha_n*N_m -beta_w
*N_m*(N_fi+N_fs +N_m+W_f+W_m);
dW_f_dt = rho_w * q * W_f -alpha_w*W_f - beta_w*W_f*(N_fi+N_fs +N_m+W_f+W_m)
+u_f*W_f;
dW_m_dt = rho_w*(1-q)*W_f-alpha_w*W_m - beta_w*W_m*(N_fi+N_fs +N_m+W_f+W_m)
+u_f*W_f;
```

2) Now, we form the infected equations into two different compartments, $\mathscr{F}$ and ⊠. These denote the addition of new infections and all other remaining transitional terms. We claim that

$\mathscr{F} = \begin{bmatrix} \mu_N N_{\text{FS}} I \\ \mu_H \text{S} N_{\text{FI}} \end{bmatrix}$ and $\mathscr{V} = \begin{bmatrix} \alpha_N N_{\text{FI}} + \rho_N N_{\text{FI}}(N_{\text{FS}} + N_{\text{FI}} + N_M + W_F + W_M) \\ \Psi_H I \end{bmatrix}$

```
entry_F_1 = mu_n *N_fs *I;
entry_F_2 = mu_h * (H-I) * N_fi;

entry_V_1 = alpha_n *N_fi + beta_n*N_fi *(N_fi+N_fs +N_m+W_f+W_m);
entry_V_2 = psi_h *I;
```

This is due to the fact that dengue infected mosquitoes are added via the birth of noninfected female mosquitoes, we include their birth as part of ⊠.

3) Now we assume a disease free equilibrium. In this case, we will focusing on the disease free where the noninfected mosquitoes dominate. We establish the entries first, and we will plug in the equilibrium point below.

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Nfs_eq_entry = (-f*(alpha_n-f*rho_n))/beta_n
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Nfs_eq_entry =

$$-\frac{f\ (\alpha_n - f\ \rho_n)}{\beta_n}$$

```
sigma_4 = ((f-1)*(alpha_n-f*rho_n))/beta_n
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sigma_4 =

$$\frac{(f-1)\ (\alpha_n - f\ \rho_n)}{\beta_n}$$

4) Now, we determine the matrices F and V by the components done earlier.

$$F = \left[ \frac{\partial \mathscr{F}(0,0,N_{\text{F0}},N_{\text{M0}})}{\partial N_{\text{FI}}} \quad | \frac{\partial \mathscr{F}(0,0,N_{\text{F0}},N_{\text{M0}})}{\partial I} \right]$$ and

$$V = \left[ \frac{\partial \mathscr{V}(0,0,N_{\text{F0}},N_{\text{M0}})}{\partial N_{\text{FI}}} \quad | \frac{\partial \mathscr{V}(0,0,N_{\text{F0}},N_{\text{M0}})}{\partial I} \right]$$

```
%entries for F
F_A = diff(entry_F_1, N_fi);
F_B = diff(entry_F_1, I);
F_C = diff(entry_F_2, N_fi);
F_D = diff(entry_F_2, I);

%entries for V
V_A = diff(entry_V_1, N_fi);
V_B = diff(entry_V_1, I);
V_C = diff(entry_V_2, N_fi);
V_D = diff(entry_V_2, I);
```

5. Now, we know that the next generation matrix (K) is defined as

$K = \mathrm{FV}^{-1}$. Thus, we can use our F and V to solve for K as follows.

```
%setting up the matrices
F = [F_A F_B; F_C F_D]
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F =

$$\begin{pmatrix} 0 & N_{\mathrm{fs}}\,\mu_n \\ \mu_h\,(H-\mathrm{I}) & -N_{\mathrm{fi}}\,\mu_h \end{pmatrix}$$

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V = [V_A V_B; V_C V_D]
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V =

$$\begin{pmatrix} \alpha_n + N_{\mathrm{fi}}\,\beta_n + \beta_n\,(N_{\mathrm{fi}} + N_{\mathrm{fs}} + N_m + W_f + W_m) & 0 \\ 0 & \psi_h \end{pmatrix}$$

```
f_new = subs(F,[N_fi,I, N_fs,N_m, W_f, W_m],[0,0,Nfs_eq_entry, sigma_4, 0,0])
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f_new =

$$\begin{pmatrix} 0 & -\dfrac{f\,\mu_n\,(\alpha_n - f\,\rho_n)}{\beta_n} \\ H\,\mu_h & 0 \end{pmatrix}$$

```
v_new = subs(V,[N_fi,I, N_fs,N_m, W_f, W_m],[0,0,Nfs_eq_entry, sigma_4, 0,0])
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v_new =

$$\begin{pmatrix} \alpha_n - \beta_n\left(\dfrac{f\,(\alpha_n - f\,\rho_n)}{\beta_n} - \dfrac{(f-1)\,(\alpha_n - f\,\rho_n)}{\beta_n}\right) & 0 \\ 0 & \psi_h \end{pmatrix}$$

```
%inverse of V
V_inv = inv(v_new)
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V_inv =

$$\begin{pmatrix} \dfrac{1}{f\,\rho_n} & 0 \\ 0 & \dfrac{1}{\psi_h} \end{pmatrix}$$

```
%Next Generation Matrix
K = f_new*V_inv
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K =

$$\begin{pmatrix} 0 & -\frac{f\,\mu_n\,(\alpha_n - f\,\rho_n)}{\beta_n\,\psi_h} \\ \frac{H\,\mu_h}{f\,\rho_n} & 0 \end{pmatrix}$$

Now, to find $R_0$ , we can solve for the eigenvalues of K.

```
%solving eigenvalues of K
eigenvalues = eig(K)
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eigenvalues =

$$\begin{pmatrix} \frac{\sqrt{-H}\,\sqrt{\mu_h}\,\sqrt{\mu_n}\,\sqrt{\alpha_n - f\,\rho_n}}{\sqrt{\beta_n}\,\sqrt{\psi_h}\,\sqrt{\rho_n}} \\ -\frac{\sqrt{-H}\,\sqrt{\mu_h}\,\sqrt{\mu_n}\,\sqrt{\alpha_n - f\,\rho_n}}{\sqrt{\beta_n}\,\sqrt{\psi_h}\,\sqrt{\rho_n}} \end{pmatrix}$$

```
%testing values
test = subs(eigenvalues,[rho_n, rho_w,alpha_n, alpha_w,
beta_n, beta_w,mu_n, mu_h, psi_h, H,f,q], [4.55,2.275,0.03333,
0.06666,0.00261258,0.00312792,0.2,0.2,0.1,100,0.5,0.5])
```

test =

$$\begin{pmatrix} -\frac{\sqrt{10}\,\sqrt{20}\,\sqrt{91}\,\sqrt{100000}\,\sqrt{224167}\,\sqrt{6024199329011513}\,\sqrt{2305843009213693952}}{27410106947002384150000} \\ \frac{\sqrt{10}\,\sqrt{20}\,\sqrt{91}\,\sqrt{100000}\,\sqrt{224167}\,\sqrt{6024199329011513}\,\sqrt{2305843009213693952}}{27410106947002384150000} \end{pmatrix}$$

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test_2 =simplify(test)
```

test_2 =

$$\begin{pmatrix} -\frac{1073741824\,\sqrt{7271528336081282186690}}{10542348825770147\,75} \\ \frac{1073741824\,\sqrt{7271528336081282186690}}{1054234882577014775} \end{pmatrix}$$

We can see that both eigenvalues are the same, thus our $R_0$ must fufill the following to be an epidemic (in respect to the noninfected mosquitoes dominating):

$$R_0 = \frac{\sqrt{-H}\,\sqrt{f}\,\sqrt{\mu_h}\,\sqrt{\mu_n}\,\sqrt{\alpha_n - f\,\rho_n}}{\sqrt{\psi_h}\,\sqrt{f\,{\rho_n}^2 - \alpha_n\,\rho_n + \alpha_n\,\beta_n}} > 1$$

Recall that I = 0 so S is the entire population, so they are the same!

$$\left(\begin{array}{c} \frac{\sqrt{-S}\ \sqrt{f}\ \sqrt{\mu_h}\ \sqrt{\mu_n}\ \sqrt{\alpha_n - f\,\rho_n}}{\sqrt{\psi_h}\ \sqrt{f\,\rho_n{}^2 - \alpha_n\,\rho_n + \alpha_n\,\beta_n}} \\ -\frac{\sqrt{-S}\ \sqrt{f}\ \sqrt{\mu_h}\ \sqrt{\mu_n}\ \sqrt{\alpha_n - f\,\rho_n}}{\sqrt{\psi_h}\ \sqrt{f\,\rho_n{}^2 - \alpha_n\,\rho_n + \alpha_n\,\beta_n}} \end{array}\right) \left(\begin{array}{c} \frac{\sqrt{-H}\ \sqrt{f}\ \sqrt{\mu_h}\ \sqrt{\mu_n}\ \sqrt{\alpha_n - f\,\rho_n}}{\sqrt{\psi_h}\ \sqrt{f\,\rho_n{}^2 - \alpha_n\,\rho_n + \alpha_n\,\beta_n}} \\ -\frac{\sqrt{-H}\ \sqrt{f}\ \sqrt{\mu_h}\ \sqrt{\mu_n}\ \sqrt{\alpha_n - f\,\rho_n}}{\sqrt{\psi_h}\ \sqrt{f\,\rho_n{}^2 - \alpha_n\,\rho_n + \alpha_n\,\beta_n}} \end{array}\right)$$