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E.g. 1 $\forall x\, P(x) \rightarrow \exists x\, Q(x)$

sol: $\exists x\, (\neg P(x) \vee Q(x))$

E.g. 2 $\forall x\, \forall y\, \exists z\, (P(x,z) \wedge P(y,z)) \rightarrow \exists u Q(x,y,u)$

sol: $\forall x\, \forall y\, \forall z\, \exists u\, (\neg P(x,z) \vee \neg P(x,z) \vee Q(x,y,u))$

From: file:///C:/Users/leslie/Downloads/Lecture3(1.7-1.9).pdf

-1-

Which of the following expressions are in prenex normal form?

$$\forall x P(x) \vee \forall x Q(x)$$
$$\forall x \forall y \neg (P(x) \rightarrow Q(y))$$
$$\forall x \exists y R(x, y)$$
$$R(x, y)$$
$$\neg \forall x R(x, y)$$

$$\neg\exists y(\forall x P(x) \to \forall x Q(x, y))$$

$\equiv \neg\exists y(\neg\forall x P(x) \lor \forall x Q(x, y))$	implication
$\equiv \forall y\neg(\neg\forall x P(x) \lor \forall x Q(x, y))$	$\neg\exists x A \equiv \forall x\neg A$ (de Morgan's)
$\equiv \forall y(\forall x P(x) \land \neg\forall x Q(x, y))$	de Morgan's, double neg.
$\equiv \forall y(\forall x P(x) \land \exists x\neg Q(x, y))$	$\neg\forall x A \equiv \exists x\neg A$ (de Morgan's)
$\equiv \forall y(\forall x P(x) \land \exists z\neg Q(z, y))$	rectification
$\equiv \forall y\forall x(P(x) \land \exists z\neg Q(z, y))$	$\forall x A\land B \equiv \forall x(A\land B)$ (x not free in B)
$\equiv \forall y\forall x\exists z(P(x) \land \neg Q(z, y))$	$A\land\exists z B \equiv \exists z(A\land B)$ (z not free in A)

Find the prenex normal form of

$$\forall x(\exists y R(x,y) \wedge \forall y \neg S(x,y) \rightarrow \neg(\exists y R(x,y) \wedge P))$$

Solution:

- According to Step 1, we must eliminate $\rightarrow$, which yields

$$\forall x(\neg(\exists y R(x,y) \wedge \forall y \neg S(x,y)) \vee \neg(\exists y R(x,y) \wedge P))$$

- We move all negations inwards, which yields:

$$\forall x(\forall y \neg R(x,y) \vee \exists y S(x,y) \vee \forall y \neg R(x,y) \vee \neg P).$$

- Next, all variables are standardized apart:

$$\forall x(\forall y_1 \neg R(x,y_1) \vee \exists y_2 S(x,y_2) \vee \forall y_3 \neg R(x,y_3) \vee \neg P)$$

- We can now move all quantifiers in front, which yields

$$\forall x \forall y_1 \exists y_2 \forall y_3 (\neg R(x,y_1) \vee S(x,y_2) \vee \neg R(x,y_3) \vee \neg P).$$

Transformation to prenex normal forms: example

$A = \exists z(\exists x Q(x,z) \vee \exists x P(x)) \rightarrow \neg(\neg\exists x P(x) \wedge \forall x \exists z Q(z,x))$.

1. Eliminating $\rightarrow$:

$A \equiv \neg\exists z(\exists x Q(x,z) \vee \exists x P(x)) \vee \neg(\neg\exists x P(x) \wedge \forall x \exists z Q(z,x))$

2. Importing the negation:

$A \equiv \forall z(\neg\exists x Q(x,z) \wedge \neg\exists x P(x)) \vee (\neg\neg\exists x P(x) \vee \neg\forall x \exists z Q(z,x))$

$\equiv \forall z(\forall x \neg Q(x,z) \wedge \forall x \neg P(x)) \vee (\exists x P(x) \vee \exists x \forall z \neg Q(z,x))$.

3. Using the equivalences (a) and (b):

$A \equiv \forall z \forall x(\neg Q(x,z) \wedge \neg P(x)) \vee \exists x(P(x) \vee \forall z \neg Q(z,x))$.

4. Renaming:

$A \equiv \forall z \forall x(\neg Q(x,z) \wedge \neg P(x)) \vee \exists y(P(y) \vee \forall w \neg Q(w,y))$.

5. Using the equivalences (c)-(f) to pull the quantifiers in front:

$A \equiv \forall z \forall x \exists y \forall w((\neg Q(x,z) \wedge \neg P(x)) \vee P(y) \vee \neg Q(w,y))$.

6. The resulting formula is in a prenex DNF.

For a prenex CNF we have to distribute the $\vee$ over $\wedge$:

$A \equiv \forall z \forall x \exists y \forall w((\neg Q(x,z) \vee P(y) \vee \neg Q(w,y)) \wedge (\neg P(x) \vee P(y) \vee \neg Q(w,y)))$.

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From:

http://www2.imm.dtu.dk/courses/02286/Slides/FirstOrderLogicPrenexFormsSkolemizationClausalFormTrans.pdf

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$$
\begin{aligned}
\exists x(\forall y Friend(x,y)\ \&\ \neg(\exists y Foe(y,x))) &\iff \exists x(\forall y\, Friend(x,y)\ \&\ (\forall y\, \neg Foe(y,x))) && (1)\\
&\iff \exists x(\forall y\, Friend(x,y)\ \&\ \forall y\, \neg Foe(y,x)) && (2)\\
&\iff \exists x \forall y\, (Friend(x,y)\ \&\ \neg Foe(y,x)) && (3)
\end{aligned}
$$

From:

https://math.stackexchange.com/questions/1486954/transform-a-formula-into-prenex-normal-form