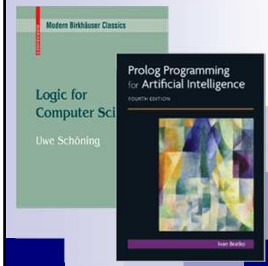

FLORIDA INTERNATIONAL UNIVERSITY

Propositional Logic (I)

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Logic…

- Logic is the science of formal principles of **reasoning** or correct **inference**.
- Historically, logic originated with the ancient Greek philosopher Aristotle.
- **Example:**
  - All men are mortal
  - All Greeks are men.
    - Hence (implication) all Greeks are mortal.
- We don't need to understand what are "men", "mortal", "Greeks" to identify the validity of this inference.
- We can use symbols:
  - All **A** are **B**
  - All **C** are **A**. Hence all **C** are **B**

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Logic…

- **Axioms of Aristotelean Logic*:**
  - **Identity**. Everything is what it **is** and acts accordingly.
    - Symbolically: **A is A**.
  - **Contradictions do not exist**.
    - Symbolically: **A and non-A cannot both be the case**.
  - **Either-or**. Everything must either be or not be.
    - Symbolically: **Either A or non-A**.

*Aristotle's Logical Works: The ***Organon*** (384–322 B.C.)

Why study Logic in Computer Science?

- **Logic** has a **deep impact** on **Computer Science**.
- Some examples:
  - **Propositional logic** – <u>the foundation of computers and circuitry</u>
  - **Databases** – **query** languages (Ex. **SQL**)
  - **Programming languages** (e.g. **PROLOG**)
  - **Design, Validation** and **Verification**
  - **Artificial Intelligence** (e.g. inference systems)
  - …

Propositional Logic. Foundations

- **Propositional logic** explores simple grammatical connections, like **and**, **or** and **not**, between the simplest "**atomic sentences**".
- **Examples:**
  - A = "Paris is the capital of France"
  - B = "Mice chase elephants"
- Such atomic components can be either **true** or **false**. In our understanding of the world, A is true but B is false.
- **More Examples:**
  - *Seinfeld* was the best TV comedy of all time. (Not a Statement – opinion)
  - Did you watch *The Godfather?* (Not a statement – a question)
  - Mice chase elephants. (statement – false)
  - I am lying. (If he *is* lying, then he is telling the truth, and vice versa - **liar's paradox**)

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Propositional Logic. Foundations

- The subject of **propositional logic** is to declare formally how such "**truth values**" of the **atomic components** extend to a truth value of a **more complex structure**.
- We are interested in how the notion of a truth value extends from simple objects to more complex objects.
- Our interest is concentrated on the **truth value of the sentence**.

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Propositional Logic. Foundations

- Note that in **propositional logic** we ignore what the underlying meaning of an **atomic sentence** is.
- For example. If
  - A: "Charlie is getting sick"
  - B: "Charlie is consulting a doctor"

  then there is a big difference in "language" whether we say "**A and B**" or "**B and A**".
- All **atomic sentences** (now called **atomic formulas**) are enumerated as A_1, A_2, A_3,... ignoring the possible "meanings" of such formulas.

Propositional Logic. Syntax

- **<u>Definition</u>** (**syntax of propositional logic**)
  - An **atomic formula** has the form A_i where i = 1, 2, 3, ... defined by the following inductive process:
  - **Formulas** are
    - All atomic formulas are formulas.
    - For every formula **F**, ¬**F** is a formula.
    - For all formulas **F** and **G**, also (**F** ∨ **G**) and (**F** ∧ **G**) are formulas.
- A formula of the form:
  - ¬**F** is called **negation** of **F**.
  - **(F ∨ G)** is called **disjunction** of **F** and **G**.
  - **(F ∧ G)** is the **conjunction** of **F** and **G**.
- Any formula F which occurs in another formula G is called a **subformula** of G.

Propositional Logic. Syntax

- **Examples:**

  F = $\neg((A_5 \land A_6) \lor \neg A_3)$ is a **formula**,

  and all **subformulas** of F are:

  $\neg((A_5 \land A_6) \lor \neg A_3)$,

  $((A_5 \land A_6) \lor \neg A_3)$,

  $(A_5 \land A_6)$,

  A_5,

  A_6,

  $\neg A_3$,

  A_3

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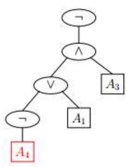
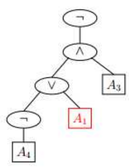
Propositional Logic. Syntax

- **Examples:**
  - **F =** $\neg((\neg A_4 \lor A_1) \land A_3)$ is a **formula. Subformulas** of **F** are the **subtrees** of the **syntax tree.**

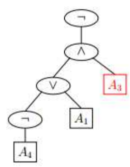
A_4



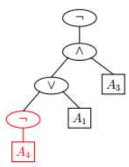
A_1



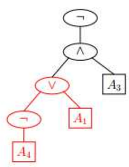
A_3



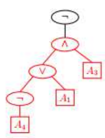
$\neg A_4$



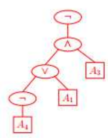
$(\neg A_4 \lor A_1)$



$((\neg A_4 \lor A_1) \land A_3)$



$\neg((\neg A_4 \lor A_1) \land A_3)$



Propositional Logic. Semantics

- **Definition** (**semantic of propositional logic**):
  - The elements of the set {0, 1} are called **truth values**.
  - An **assignment** is a function $\mathcal{A}$: D → {0, 1}, where D is any subset of the atomic formulas.
  - Given an assignment $\mathcal{A}$, we extend it to a function $\mathcal{A}'$: E → {0, 1}, where D ⊆ E is the set of formulas that can be built up using only the atomic formulas from D.

1. For every atomic formula $A_i \in \mathbf{D}$, $\mathcal{A}'(A_i) = \mathcal{A}(A_i)$.
2. $\mathcal{A}'((F \wedge G)) = \begin{cases} 1, & \text{if } \mathcal{A}'(F) = 1 \text{ and } \mathcal{A}'(G) = 1 \\ 0, & \text{otherwise} \end{cases}$
3. $\mathcal{A}'((F \vee G)) = \begin{cases} 1, & \text{if } \mathcal{A}'(F) = 1 \text{ or } \mathcal{A}'(G) = 1 \\ 0, & \text{otherwise} \end{cases}$
4. $\mathcal{A}'(\neg F) = \begin{cases} 1, & \text{if } \mathcal{A}'(F) = 0 \\ 0, & \text{otherwise} \end{cases}$

Propositional Logic. Semantics

Example: Let $\mathcal{A}(A) = 1$, $\mathcal{A}(B) = 1$ and $\mathcal{A}(C) = 0$. Then we obtain:

$$\begin{aligned}
\mathcal{A}(\neg((A \wedge B) \vee C)) &= \begin{cases} 1, & \text{if } \mathcal{A}(((A \wedge B) \vee C)) = 0 \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} 0, & \text{if } \mathcal{A}(((A \wedge B) \vee C)) = 1 \\ 1, & \text{otherwise} \end{cases} \\
&= \begin{cases} 0, & \text{if } \mathcal{A}((A \wedge B)) = 1 \text{ or } \mathcal{A}(C) = 1 \\ 1, & \text{otherwise} \end{cases} \\
&= \begin{cases} 0, & \text{if } \mathcal{A}((A \wedge B)) = 1 \quad (\text{ because } \mathcal{A}(C) = 0\ ) \\ 1, & \text{otherwise} \end{cases} \\
&= \begin{cases} 0, & \text{if } \mathcal{A}(A) = 1 \text{ and } \mathcal{A}(B) = 1 \\ 1, & \text{otherwise} \end{cases} \\
&= 0
\end{aligned}$$

Propositional Logic. Semantics

- **Truth tables** define the semantics (=meaning) of the operators
- By convention: **0 = (F)alse, 1 = (T)rue**

CONJUNCTION

$\mathcal{A}(F)$	$\mathcal{A}(G)$	$\mathcal{A}((F \wedge G))$
0	0	0
0	1	0
1	0	0
1	1	1

Propositional Logic. Semantics

- **Truth tables** define the semantics (=meaning) of the operators
- By convention: **0 = (F)alse, 1 = (T)rue**

DISJUNCTION

$\mathcal{A}(F)$	$\mathcal{A}(G)$	$\mathcal{A}((F \vee G))$
0	0	0
0	1	1
1	0	1
1	1	1

Propositional Logic. Semantics

- **Truth tables** define the semantics (=meaning) of the operators
- By convention: **0 = (F)alse, 1 = (T)rue**

NEGATION

$\mathcal{A}(F)$	$\mathcal{A}(\neg F)$
0	1
1	0

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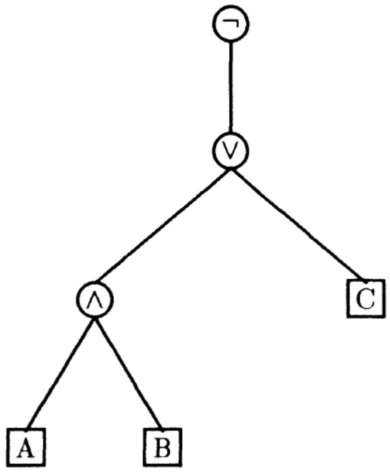
Propositional Logic. Semantics

- Using these tables, we can determine the truth value of a formula F
- For example for **F =** $\neg$**((A** $\wedge$ **B)** $\vee$ **C)**, we have the **syntax tree**:



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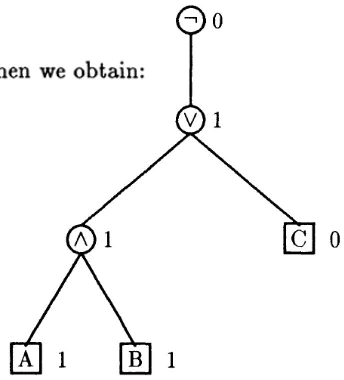
Propositional Logic. Semantics

- The **truth value** of F is obtained:
  - by marking all **leaves of this tree** with the truth values given by the assignment $\mathcal{A}$,
  - by determining the values of the inner nodes according to the above tables.
- The mark at the root gives the truth value of F under the given assignment $\mathcal{A}$

Let $\mathcal{A}(A) = 1$, $\mathcal{A}(B) = 1$ and $\mathcal{A}(C) = 0$. Then we obtain:



Propositional Logic. Semantics

- From the definition of $\mathcal{A}$ (F) it can be seen that:
  - the symbol "$\wedge$" is intended to model the spoken word "**and**",
  - the symbol "$\vee$" is intended to model the spoken word "**or**",
  - the symbol "$\neg$" models "**not**"

Propositional Logic. Semantics

- For propositions F and G, the **implication** $F \rightarrow G$ is false when F is true and G is false, and is true otherwise. F is called the **premise** or **hypothesis**, and G is called the **conclusion**.
- Ways to read $F \rightarrow G$: If F then G; F is sufficient for G; F implies G; and G follows from F

$\mathcal{A}(F)$	$\mathcal{A}(G)$	$\mathcal{A}((F \rightarrow G))$
0	0	1
0	1	1
1	0	0
1	1	1

- **Exercise:** Verify the table's values for "$\rightarrow$" using the fact that:

$(F_1 \rightarrow F_2)$ in place of $(\neg F_1 \vee F_2)$ (**implication law**)

$(F_1 \rightarrow F_2)$ in place of $(\neg F_1 \rightarrow \neg F_2)$ (inverse statement)

$(F_1 \rightarrow F_2)$ in place of $(\neg F_2 \rightarrow \neg F_1)$ (contrapositive statement)

Propositional Logic. Semantics

Implication Example

- **If** you earn more than 89% of the possible points in the COT-3541 course, **then** you will get an A in COT-3541.
- Rewritten in logical notation:
  - $F \rightarrow G$, where
  - F: You earn more than 89% of the possible points in COT-3541.
  - G: You will get an A in COT-3541.

$\mathcal{A}(F)$	$\mathcal{A}(G)$	$\mathcal{A}((F \rightarrow G))$
0	0	1
0	1	1
1	0	0
1	1	1

Interpretation of $F \rightarrow G$:

Think of **$F \rightarrow G$** being true as long as I do what I promised:

- If **F** is **false**, you should not expect an A, so I have honored the commitment (**$F \rightarrow G$ is true**).
- If **F** is **true**, **$F \rightarrow G$** is true only if **G** is also true, i.e. if I give you an A.

Propositional Logic. Semantics

- The **biconditional** of statements F and G, denoted F $\leftrightarrow$ G, is read "**F if and only if G**" (or "F is necessary and sufficient for G"), and is true if F and G have the same truth values, and false otherwise.
- Note: If P $\leftrightarrow$ Q is true, then P $\rightarrow$ Q and Q $\rightarrow$ P are true. Conversely if both P $\rightarrow$ Q and Q $\rightarrow$ P are true, then P $\leftrightarrow$ Q is true.

$\mathcal{A}(F)$	$\mathcal{A}(G)$	$\mathcal{A}((F \leftrightarrow G))$
0	0	1
0	1	0
1	0	0
1	1	1

Exercise: Verify that we can write:

$(F_1 \leftrightarrow F_2)$ in place of $((F_1 \wedge F_2) \vee (\neg F_1 \wedge \neg F_2))$

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Propositional Logic.

- Using the following symbolic representations
  - **A: I am innocent.**
  - **B: I have an alibi.**

  express the following in words.

1. **A** $\wedge$ **B**
   - Answer: "I am innocent and I have an alibi."
2. **A** $\rightarrow$ **B**
   - Answer: "If I am innocent, then I have an alibi."
3. $\neg$**B** $\rightarrow$ $\neg$**A**
   - Answer: "If I do not have an alibi, then I am not innocent."
4. **B** $\vee$ $\neg$**A**
   - Answer: "I have an alibi or I am not innocent."

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Propositional Logic.

- Using the following symbolic representations
  - **A: I am innocent.**
  - **B: I have an alibi.**
  - **C: I go to jail.**

  express the following in words.

  1. $(A \vee B) \rightarrow \neg C$
  2. $(A \wedge \neg B) \rightarrow C$
  3. $(\neg A \wedge B) \vee C$
  4. $(A \wedge C) \rightarrow \neg B$

Propositional Logic.

- **A: I am innocent.**
- **B: I have an alibi.**
- **C: I go to jail.**

express the following in words.

1. $\mathbf{(A \vee B) \rightarrow \neg C}$

   If I am innocent or have an alibi, then I do not go to jail.

Propositional Logic.

- Using the following symbolic representations
  - **A: I am innocent.**
  - **B: I have an alibi.**
  - **C: I go to jail.**

  express the following in words.

  2. $(\mathbf{A} \wedge \neg \mathbf{B}) \rightarrow \mathbf{C}$

  If I am innocent and do not have an alibi, then I go to jail.

Propositional Logic.

- Using the following symbolic representations
  - **A: I am innocent.**
  - **B: I have an alibi.**
  - **C: I go to jail.**

  express the following in words.

  3. $(\neg \mathbf{A} \wedge \mathbf{B}) \vee \mathbf{C}$

  I am not innocent and I have an alibi or I go to jail.

Propositional Logic.

- Using the following symbolic representations
  - **A: I am innocent.**
  - **B: I have an alibi.**
  - **C: I go to jail.**

  express the following in words.

  4. **$(A \wedge C) \rightarrow \neg B$**

     If I am innocent and go to jail, then I do not have an alibi.

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Propositional Logic.

- Translate the sentence to symbolic form.
  - **If you drink and drive, you are fined or you go to jail.**
- Atomic sentences:
  - A: You drink.
  - B: You drive.
  - C: You are fined.
  - D: You are jailed.

Answer: $(A \wedge B) \rightarrow (C \vee D)$.

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Propositional Logic.

- Translate the sentence to symbolic form.
  - **No integer number is greater than 3 and less than 4.**
- Atomic sentences:
  - A: An integer number.
  - B: A number greater than 3.
  - C: A number less than 4.

Answer: $\neg A \rightarrow (B \wedge C)$.

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Propositional Logic. Semantics

- Construct a **truth tables** for:
  - $F_1 = \neg(A \wedge B)$ and $F_2 = \neg A \vee \neg B$
  - $F_3 = \neg(A \vee B)$ and $F_4 = \neg A \wedge \neg B$
- **DeMorgan's Laws**

$$\neg(\mathbf{A} \wedge \mathbf{B}) \equiv \neg\mathbf{A} \vee \neg\mathbf{B}$$

$$\neg(\mathbf{A} \vee \mathbf{B}) \equiv \neg\mathbf{A} \wedge \neg\mathbf{B}$$

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Propositional Logic. Semantics

- **General Negations:**

$$\neg(\neg A) \equiv A$$
$$\neg(A \wedge B) \equiv \neg A \vee \neg B$$
$$\neg(A \vee B) \equiv \neg A \wedge \neg B$$
$$\neg(A \rightarrow B) \equiv A \wedge \neg B$$

Propositional Logic. Application

- **Combinatorial Logic Circuit:**

http://www.cs.duke.edu/courses/summer13/compsci230/restricted/lectures/L03.pdf