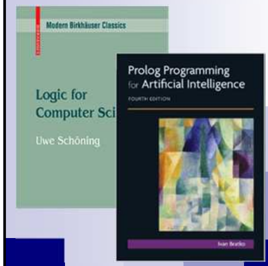

FLORIDA INTERNATIONAL UNIVERSITY

First Order Logic (Examples #1)

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Free and bounded occurrences

Find free variables in the following formulas:

1. $\forall x.(p(x) \rightarrow \exists y.\neg q(f(x), y, f(y)))$
2. $\forall x(\exists y.r(x, f(y)) \rightarrow r(x, y))$
3. $\forall z.(p(z) \rightarrow \exists y.(\exists x.q(x, y, z) \vee q(z, y, x)))$
4. $\forall z \exists u \exists y.(q(z, u, g(u, y)) \vee r(u, g(z, u)))$
5. $\forall z \exists x \exists y(q(z, u, g(u, y)) \vee r(u, g(z, u)))$

Solutions

1. *no free variables*
2. *y free*
3. *x free*
4. *no free variables*
5. *u free*

Interpretations

What is the meaning of the followinf FOL formulas?

1. $bought(Frank, dvd)$
   1. *"Frank bought a dvd."*
2. $\exists x.bought(Frank, x)$
   2. *"Frank bought something."*
3. $\forall x.(bought(Frank, x) \rightarrow bought(Susan, x))$
   3. *"Susan bought everything that Frank bought."*
5. $\forall x \exists y.bought(x, y)$
   5. *"Everyone bought something."*
6. $\exists x \forall y.bought(x, y)$
   6. *"Someone bought everything."*

Formalization

Define an appropriate language and formalize the following sentences using FOL formulas.

All Students are smart.

$$\forall x.(Student(x) \rightarrow Smart(x))$$

There exists a smart student.

$$\exists x.(Student(x) \wedge Smart(x))$$

Every student loves some student.

$$\forall x.(Student(x) \rightarrow \exists y.(Student(y) \wedge Loves(x, y)))$$

Every student loves some other student.

$$\forall x.(Student(x) \rightarrow \exists y.(Student(y) \wedge \neg(x = y) \wedge Loves(x, y)))$$

There is a student who is loved by every other student.

$$\exists x.(Student(x) \wedge \forall y.(Student(y) \wedge \neg(x = y) \rightarrow Loves(y, x)))$$

Formalization (cont...)

Bill is a student.

$$Student(Bill)$$

Bill takes either Analysis or Geometry (but not both).

$$Takes(Bill, Analysis) \leftrightarrow \neg Takes(Bill, Geometry)$$

Bill takes Analysis and Geometry.

$$Takes(Bill, Analysis) \wedge Takes(Bill, Geometry)$$

Bill has at least one sister.

$$\exists x. SisterOf(x, Bill)$$

Bill has no sister.

$$\neg \exists x. SisterOf(x, Bill)$$

Formalization (cont...)

Bill has at most one sister.

$$\forall x \forall y. (SisterOf(x, Bill) \wedge SisterOf(y, Bill) \rightarrow x = y)$$

Bill has (exactly) one sister.

$$\exists x. (SisterOf(x, Bill) \wedge \forall y. (SisterOf(y, Bill) \rightarrow x = y))$$

Every student who takes Analysis also takes Geometry.

$$\forall x. (Student(x) \wedge Takes(x, Analysis) \rightarrow Takes(x, Geometry))$$

Every student takes at least one course.

$$\forall x. (Student(x) \rightarrow \exists y. (Course(y) \wedge Takes(x, y)))$$

Formalization (cont…)

Define an appropriate language and formalize the following sentences in FOL:

Language	*Constants:* A, B, C, D, E, F;
	Predicates: $On^2, Above^2, Free^1, Red^1, Green^1$.

"A is above C, D is on E and above F."

$$Above(A, C) \wedge Above(D, F) \wedge On(D, E)$$

"A is green while C is not."

$$Green(A) \wedge \neg Green(C)$$

"Everything that is free has nothing on it."

$$\forall x.(Free(x) \rightarrow \neg \exists y.On(y, x))$$

"Everything that is not green and is above B, is red."

$$\forall x.(\neg Green(x) \wedge Above(x, B) \rightarrow Red(x))$$

Formalization (cont…)

Consider the following sentences:

1. *All actors and journalists invited to the party are late.*
2. *There is at least a person who is on time.*
3. *There is at least an invited person who is neither a journalist nor an actor.*

Solutions

1. $\forall x.((a(x) \vee j(x)) \wedge i(x) \rightarrow l(x))$
2. $\exists x.\neg l(x)$
3. $\exists x.(i(x) \wedge \neg a(x) \wedge \neg j(x))$

Formalization (cont…)

1. $\forall x.((a(x) \vee j(x)) \wedge i(x) \to l(x))$
2. $\exists x.\neg l(x)$
3. $\exists x.(i(x) \wedge \neg a(x) \wedge \neg j(x))$

Prove that 3 is not a logical consequence of 1 and 2.

It's sufficient to find an interpretation $\mathcal{I}$ for which the logical consequence does not hold:

	l(x)	*a(x)*	*j(x)*	*i(x)*
Bob	*F*	*T*	*F*	*F*
Tom	*T*	*T*	*F*	*T*
Mary	*T*	*F*	*T*	*T*

Formalization (cont…)

For each of the following properties, write a formula which is true in the graphs that satisfies the property:

0. "each node has at most one color"
1. "connected nodes don't have the same color"
2. "the graph contains only 2 yellow nodes"
3. "for each color there is at least a node with this color"

Language

- a binary predicate edge, where **edge(n,m)** means that node n is connected to node m
- a binary predicate color, where **color(n,x)** means that node n has color x
- the following constant: **yellow**

Formalization (cont…)

Language

- a binary predicate edge, where **edge(n,m)** means that node n is connected to node m
- a binary predicate color, where **color(n,x)** means that node n has color x
- the following constants: **yellow**

Axiom 0.

"each node has at most one color"

$$\forall n \forall x.(\mathsf{color}(n,x) \rightarrow \neg \exists y.(y \neq x \wedge \mathsf{color}(n,y)))$$

Formalization (cont…)

Language

- a binary predicate edge, where **edge(n,m)** means that node n is connected to node m
- a binary predicate color, where **color(n,x)** means that node n has color x
- the following constants: **yellow**

Axiom 1.

"connected nodes don't have the same color"

$$\forall n \forall m \forall x.(\mathsf{edge}(n,m) \wedge \mathsf{color}(n,x) \rightarrow \neg \mathsf{color}(m,x))$$

Formalization (cont…)

Language

- a binary predicate edge, where **edge(n,m)** means that node n is connected to node m
- a binary predicate color, where **color(n,x)** means that node n has color x
- the following constants: **yellow**

Axiom 2.

“the graph contains only 2 yellow nodes”

$$\exists n \exists n'.(\text{color}(n, \text{yellow}) \wedge \text{color}(n', \text{yellow}) \wedge n \neq n' \wedge$$
$$\forall m.(m \neq n \wedge m \neq n' \rightarrow \neg \text{color}(m, \text{yellow})))$$

Formalization (cont…)

Language

- a binary predicate edge, where **edge(n,m)** means that node n is connected to node m
- a binary predicate color, where **color(n,x)** means that node n has color x
- the following constants: **yellow**

Axiom 3.

“for each color there is at least a node with this color“

$$\forall x \exists n.\text{color}(n, x)$$

The Barber Paradox

https://www.youtube.com/watch?v=qQs2ZHV_WBk

Formal resolution in:
https://proofwiki.org/wiki/Barber_Paradox/Resolution_2