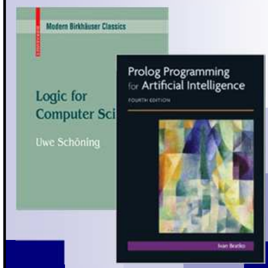

First Order Logic (II)

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Equivalences in FOL

- The concept of (semantic) **equivalence** can be translated into **FOL** in the obvious way:
  - two formulas F and G of **FOL** are equivalent (symbolically $F \equiv G$) if for all structures $\mathcal{A}$ which are suitable for both F and G,

$$\mathcal{A}(F) = \mathcal{A}(G)$$

- Also we observe that all equivalences which have been proved for formulas in propositional logic still hold in predicate logic, e.g. deMorgan's law:

$$\neg(F \wedge G) \equiv (\neg F \vee \neg G)$$

- For the purpose of manipulating formulas of FOL, to convert them to certain normal forms etc., we need equivalences which also **include quantifiers**.

Equivalences in FOL

- **Theorem:**
  - Let F and G be arbitrary formulas:

$$
\begin{aligned}
&1.\ \neg\forall x F \equiv \exists x \neg F \\
&\quad\ \neg\exists x F \equiv \forall x \neg F
\end{aligned}
$$

2. If x does not occur free in G, then

$$
\begin{aligned}
(\forall x F \wedge G) &\equiv \forall x(F \wedge G) \\
(\forall x F \vee G) &\equiv \forall x(F \vee G) \\
(\exists x F \wedge G) &\equiv \exists x(F \wedge G) \\
(\exists x F \vee G) &\equiv \exists x(F \vee G)
\end{aligned}
$$

$$
\begin{aligned}
&3.\ (\forall x F \wedge \forall x G) \equiv \forall x(F \wedge G) \\
&\quad\ (\exists x F \vee \exists x G) \equiv \exists x(F \vee G)
\end{aligned}
$$

$$
\begin{aligned}
&4.\ \forall x \forall y F \equiv \forall y \forall x F \\
&\quad\ \exists x \exists y F \equiv \exists y \exists x F
\end{aligned}
$$

Note: x does not occur free in G $\Leftrightarrow$ G does not contain x

Equivalences in FOL

- **Important notes**
  - $\forall x \exists y F(x,y) \not\equiv \exists y \forall x F(x,y)$. Why?

    Example: Assume that domain, U = $\mathbb{Z}$ and
    - F(x,y) = True if x <y, otherwise False
    - For every integer x there is an integer y such that x < y" is **true**, but
    - There is an integer y such that for every integer x, x < y is **false**
  - **We cannot conclude** that

    $\forall x F(x) \vee \forall x G(x) \equiv \forall x(F(x) \vee G(x))$
    - **Example:** Assume that domain, U = $\mathbb{N}$ ={0,1,2,…} and
      - F(x) = True, if (x = 0); otherwise False
      - G(x) = True, if (0 < x); otherwise False
      - Then
        - $\forall x(x = 0) \vee \forall x(0 < x)$ and $\forall x((x = 0) \vee (0 < x))$ are **not equivalent**, for $\mathbb{N}$
        - the first is **false** in $\mathbb{N}$, whereas
        - the second is **true** in $\mathbb{N}$

    $\forall x F(x) \vee \forall x G(x) \not\equiv \forall x(F(x) \vee G(x))$

Prenex form

Definition (prenex form):

- A formula is in **prenex** form it has the form

  $$\mathbf{Q_1x_1Q_2x_2 \ldots Q_nx_nF},$$

  where $\mathbf{Q_i} \in \{\exists,\forall\}$, n > 0, and the $\mathbf{x_i}$ are variables. Further, **F** does not contain a quantifier.

  A list of quantifiers $\mathbf{Q_1x_1Q_2x_2 \ldots Q_nx_n}$ is called **prefix** and **F** is called the **matrix** of a formula. Here **F** is represented in **CNF**.

Theorem:

- For every formula F there exists an **equivalent formula** G in **prenex form**.

Prenex form. Reduction

Algorithm (to transform a formula into ≡ **CN prenex Form**):

- **Step 1:**
  - Transform a formula into ≡ **CNF** (see algorithm from **COT3541_2.PDF** slides).
- **Step 2:**
  - Positioning of "¬" just before the predicates:
    - $\neg((\forall x)P(x)) \equiv (\exists x)(\neg P(x))$ (1)
    - $\neg((\exists x)P(x)) \equiv (\forall x)(\neg P(x))$ (2)
- **Step 3:**
  - Movement of the quantifiers with changing variables, if necessary:
    (by R we will denote a predicate without a variable x)
    - $(Qx)P(x) \vee R \equiv Qx(P(x) \vee R)$ (3)
    - $(Qx)P(x) \wedge R \equiv Qx(P(x) \wedge R)$ (4)
    - $(\forall x)P(x) \wedge (\forall x)R(x) \equiv \forall x(P(x) \wedge R(x))$ (5)
    - $(\exists x)P(x) \vee (\exists x)R(x) \equiv \exists x(P(x) \vee R(x))$ (6)
    - $(Q_1x)P(x) \wedge (Q_2x)R(x) \equiv (Q_1x)(Q_2 z)(P(x) \wedge R(z))$ (7)
    - $(Q_3x)P(x) \vee (Q_4x)R(x) \equiv (Q_3x)(Q_4z)(P(x) \vee R(z))$ (8)

Prenex form. Reduction

Example: Convert the following formula into ≡ **CN prenex form**:

$$(\forall x)(\forall y)((\exists z)(P(x,z)\wedge P(y,z))\rightarrow(\exists z)Q(x,y,z))$$

Attention with the (..)

$$(\forall x)(\forall y)((\exists z)(P(x,z)\wedge P(y,z))\rightarrow(\exists z)Q(x,y,z))$$

Implication rule

$$\equiv(\forall x)(\forall y)(\neg((\exists z)(P(x,z)\wedge P(y,z)))\vee(\exists z)Q(x,y,z))$$

deMorgan Law

$$\equiv(\forall x)(\forall y)((\forall z)(\neg P(x,z)\vee\neg P(y,z))\vee(\exists z)Q(x,y,z))$$

Changing variables

$$\equiv(\forall x)(\forall y)(\forall z)(\exists w)(\neg P(x,z)\vee\neg P(y,z)\vee Q(x,y,w))$$

Prenex form. Reduction

Exercise: Convert the following formula into ≡ **CN prenex form**:

$$F = (\forall x\exists yP(x,g(y,f(x)))\vee\neg Q(z))\vee\neg\forall xR(x,y)$$

Skolem normal form.

Definition (Skolem Normal Form):

- A formula is in **Skolem Normal Form** it has the **prenex normal form:**

$$\mathbf{Q_1x_1Q_2x_2\ldots Q_nx_nF},$$

where

$\mathbf{Q_i} \in \{\forall\}$, $1 \le i \le n$, and
the **matrix** of the formula doesn't contain free variables.

- **Examples (SNF):**
  - $\forall x \forall y (P(x) \vee \neg Q(a,y))$
  - $\forall y \forall z (\neg Q(g(y,f(z))) \vee \neg P(b))$

- The process of eliminating existential quantifiers and replacing the corresponding variable by a constant or a function is called **skolemisation**

Thoralf Albert Skolem (May/1887 – March/1963) was a Norwegian mathematician who worked in mathematical logic and set theory.

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Skolemisation

- Let a formula be already in a **prenex normal form:**

$$\mathbf{Q_1x_1Q_2x_2\ldots Q_nx_nF},$$

- where **F** is in a **CNF**.
  - Let $\mathbf{Q_r}$ be an existential quantifier in the prefix $\mathbf{Q_1x_1Q_2x_2\ldots Q_nx_n}$, $1 \le r \le n$.
- **Case #1:** If no universal quantifier appears before $\mathbf{Q_r}$, we do the following:
  - choose a new constant c different from other constants occurring in F;
  - replace all x_r appearing in F by c;
  - delete (Q_rx_r) from the prefix.
- **Example:** $(\exists \mathbf{x})(\forall y)(\forall z)(P(x,y) \wedge Q(x,z))$

$$\Downarrow \qquad \Downarrow$$

$$(\forall y)(\forall z)(P(\mathbf{a},y) \wedge Q(\mathbf{a},z))$$

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Skolemisation (cont…)

- Let a formula be already in a **prenex normal form**

$$\mathbf{Q_1x_1Q_2x_2 \ldots Q_nx_nF},$$

- where F is in a **conjunctive normal form**.
  - Let $\mathbf{Q_r}$ be an existential quantifier in the prefix $\mathbf{Q_1x_1Q_2x_2 \ldots Q_nx_n}$, $1 \le r \le n$.
- **Case #2:** If $Q_{s_1},\ldots,Q_{s_m}$ are all the universal quantifiers appearing before Q_r, $1 \le s_1 \le s_2 \le \ldots \le s_m \le r$, we do the following:
  - choose a new m-place function symbol f different from other function symbols occurring in F;
  - replace all y_r appearing in F by $f(x_{s_1},x_{s_2},\ldots,x_{s_m})$;
  - delete (Q_rx_r) from the prefix.
- **Example #1:** $(\forall x)(\exists y)(\exists z)((\neg P(x,y) \vee Q(x,z))$

$$\Downarrow$$

$$(\forall x)((\neg P(x,f(x)) \vee Q(x,g(x)))$$

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Skolemisation (cont…)

- **Example #2:**
  - Obtain a **Skolem normal form** of the formula

    $(\exists x)(\forall y)(\forall z)(\exists u)(\forall v)(\exists w)P(x,y,z,u,v,w)$
  - $(\exists x)$ is preceded by no universal quantifiers, $(\exists u)$ is preceded by $(\forall y)$ and $(\forall z)$, and $(\exists w)$ by $(\forall y)$, $(\forall z)$ and $(\forall v)$.
  - Therefore, we replace the existential variable x by a constant a, u by a two-place function f(y,z), and w by a three-place function g(y,z,v).
  - The **Skolem normal form** of the formula is

    $(\forall y)(\forall z)(\forall v)P(a,y,z,f(y,z),v,g(y,z,v))$

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Skolemisation (cont…)

- **Example #3:**
  - Obtain a **Skolem normal form** of the formula
    $(\forall x)(\exists y)(\exists z)((\neg P(x,y) \wedge Q(x,z)) \vee R(x,y,z))$.
  - First, the matrix is transformed into a **CNF**:
    $(\forall x)(\exists y)(\exists z)((\neg P(x,y) \vee R(x,y,z)) \wedge (Q(x,z) \vee R(x,y,z)))$
  - Then, since $(\exists y)$ and $(\exists z)$ are both preceded by $(\forall x)$, the existential variables y and z are replaced, respectively by one-place functions f(x) and g(x).
  - The **Skolem normal form** of the formula is

$(\forall x)((\neg P(x,f(x)) \vee R(x,f(x),g(x))) \wedge (Q(x,g(x)) \vee R(x,f(x),g(x))))$

Skolem form.

Exercise:

- Obtain a **Skolem normal form** of the formula:

$$F = \neg\exists x \forall y (P(x,y) \wedge \exists x (P(x,x) \rightarrow Q(z))).$$

Solution:

$$\begin{array}{lll} F & = & \neg\exists x \forall y (P(x,y) \wedge \exists x (P(x,x) \rightarrow Q(z))) \\ & \equiv & \forall x \exists y (\neg P(x,y) \vee \forall x (P(x,x) \wedge \neg Q(z))) \\ & \equiv & \forall x \exists y (\neg P(x,y) \vee \forall u (P(u,u) \wedge \neg Q(z))) \\ & \equiv & \forall x \exists y \forall u (\neg P(x,y) \vee (P(u,u) \wedge \neg Q(z))) \\ & \equiv_s & \exists z \forall x \exists y \forall u (\neg P(x,y) \vee (P(u,u) \wedge \neg Q(z))) \\ & \equiv_s & \forall x \forall u (\neg P(x,f(x)) \vee (P(u,u) \wedge \neg Q(a))). \end{array}$$

CNF:

$$F \equiv_s \forall x \forall u ((\neg P(x,f(x)) \vee P(u,u)) \wedge (\neg P(x,f(x)) \vee \neg Q(a)))$$

Skolem form is not unique!

- **Example:**
  - Consider the following formula:

    $F = (\forall x)P(x) \wedge (\exists y)Q(y)$

    $F_1 = (\forall x)(\exists y)(P(x) \wedge Q(y))$

    $F_2 = (\exists y)(\forall x)(P(x) \wedge Q(y))$

    F_1 and F_2 are **prenex normal forms**

    $F_1 \overset{\text{Skolem}}{\Rightarrow} S_1 = (\forall x)(P(x) \wedge Q(f(x)))$

    $F_2 \overset{\text{Skolem}}{\Rightarrow} S_2 = (\forall x)(P(x) \wedge Q(a))$

  We want to find **Skolem forms** which are as simple as possible then the strategy is to **move ∃ to the left as much is possible**

Skolem form. Logical equivalence

- Given a formula F and S that represents a **Skolem normal form of F**.

  We can write that F ≡ S?

- **Nope!**
  - Counter-example:
    - $F = (\exists x)P(x)$ and $S = P(a)$
    - Let $\mathcal{A} = (U_{\mathcal{A}}, I_{\mathcal{A}})$, $U_{\mathcal{A}} = \{1,2\}$ and $a^{\mathcal{A}} = 1$ and

      $\{P^{\mathcal{A}}(1) = \text{False}, P^{\mathcal{A}}(2) = \text{True}\}$
    - then $\mathcal{A} \models$ **F but** $\mathcal{A} \not\models$ **S** therefore in general **S ≢ F**

Skolem form. Preserving Inconsistency

Theorem: (inconsistency preservation of **Skolem form**)

Let F be a formula and S be a **set of clauses in Skolem normal form** that represents the formula F. **Then F is inconsistent/contradiction if and only if S is inconsistent/contradiction.**

(see the Proof in the text book for the course)

- Then we have:
  - **S ≡ F iif F is inconsistent**
  - **S ≢ F if F isn't inconsistent**
    - (we can find and interpretation I, such that I ⊨ S and I ⊭ F or reverse)

- **Prenex Normal Form** – preserves equivalence
- **Skolemization** – preserves satisfiability

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Skolem form. Preserving Inconsistency

Example (home work):

Suppose S = (∃x)(∀y)P(x,y) is a **prenex normal form** of F.

Prove that F is valid iff S′ = (∃x)P(x,f(x)) is valid.

Solution:

- F is valid ↔ ¬F is inconsistent ↔ ¬S is inconsistent
- ¬S ≡ ¬ (∃x)(∀y)P(x,y) ≡ (∀x)(∃y)¬P(x,y) = F′
- F′ is inconsistent ↔ F'_{Skol} is inconsistent, where F'_{Skol} is the Skolem normal form of the formula F′
- F'_{Skol} ≡ (∀x)¬P(x,f(x)) ≡ ¬((∃x)P(x,f(x))) ≡ ¬S′ ■
- ¬S′ is is inconsistent ↔ S′ is valid

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