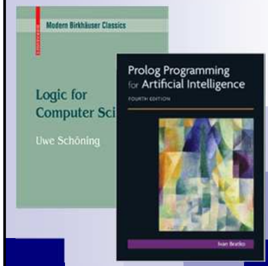


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# First Order Logic (I)

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## First Order Logic (FOL). Foundations

- **FOL** can be understood as an extension of **Propositional Logic**.
- For example, in **Propositional Logic** it was not possible to express that certain "objects" stand in certain relations, or that a property holds for all such objects, or that some object with a certain property exists.
- Here is a well known example from **calculus**:
  - For all $\varepsilon > 0$ there exists some $n_o$, such that for all $n > n_o$, abs(f(n) – a) $< \varepsilon$
  - The main concepts here are the verbal constructs ***for all*** and ***exists***, as well as the use of **functions** (*abs, mod*) and **relations** (>, =, <).

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## First Order Logic (FOL). Foundations

- **Propositional logic** assumes the world contains **facts (propositions)**
- **FOL** (like natural language) assumes the world contains:
  - **Objects:** people, houses, numbers, colors, …
  - **Relations:** has color, brother of, bigger than,…
  - **Facts:** father of, best friend, one more than, …
- **Facts** have a truth value **true** or **false**



## Syntax of First Order Logic

- A **variable** is of the form $x_i$ where i = 1, 2, 3, ....
- A **predicate** symbol the form $P_i^k$ and a **function** symbol of the form $f_i^k$ where i = 1, 2, 3 , …, and k = 0, 1,2, ....
  - Here, i is the **distinguishability** index and
  - k is called the **arity**. In the case of arity 0, we drop the parentheses, and just write $P_i^0$ or $f_i^0$
  - A **function symbol** of **arity 0** will also be called a **constant**.

## Syntax of First Order Logic (cont…)

- **Terms** (using inductive process):

  1. Each **variable** is a term.

  2. If **f** is a **function** symbol with arity k, and if $t_1, \ldots, t_k$ are terms, then $\mathbf{f(t_1, \ldots, t_k)}$ is a term.

- **Formulas** (of **FOL**) are defined inductively as follows:

  1. If **P** is a **predicate** symbol with arity k, and if $t_1, \ldots, t_k$ are **terms**, then $\mathbf{P(t_1, \ldots, t_k)}$ is a formula.

  2. For each formula **F**, $\neg$**F** is a formula.

  3. For all formulas **F** and **G**, (**F** $\wedge$ **G**) and (**F** $\vee$ **G**) are formulas.

  4. If x is a variable and **F** is a formula, then $\exists$**xF** and $\forall$**xF** are formulas.



## Syntax of First Order Logic (cont…)

- **Important points!**
  - If **F** is a formula, and **F** occurs as part of the formula **G**, then **F** is called a **subformula** of **G**.
  - All occurrences of a variable in a formula are discriminated into **bound** and **free** occurrences.
  - An occurrence of the variable x in the formula **F** is **bound** if x occurs within a subformula of **F** of the form $\exists$**xF** or $\forall$**xF**.
  - A variable in the formula is **free** if at least one occurrence of the variable is free.
  - A formula without occurrence of a **free variable** is called **closed**.
  - The symbols $\exists$ and $\forall$ are called **quantifiers**:
    - $\exists$ is the **existential** quantifier, and
    - $\forall$ is the **universal** quantifier.

## Syntax of First Order Logic (cont…)

**Example:** $F = (\exists x_1 P_5^2(x_1, f_2^1(x_2)) \vee \neg\forall x_2 P_4^2(x_2, f_7^2(f_4^0, f_5^1(x_3))))$ is a formula. All the subformulas of $F$ are:

$$F$$
$$\exists x_1 P_5^2(x_1, f_2^1(x_2))$$
$$P_5^2(x_1, f_2^1(x_2))$$
$$\neg\forall x_2 P_4^2(x_2, f_7^2(f_4^0, f_5^1(x_3)))$$
$$\forall x_2 P_4^2(x_2, f_7^2(f_4^0, f_5^1(x_3)))$$
$$P_4^2(x_2, f_7^2(f_4^0, f_5^1(x_3)))$$



## Syntax of First Order Logic (cont…)

**Example:** $F = (\exists x_1 P_5^2(x_1, f_2^1(x_2)) \vee \neg\forall x_2 P_4^2(x_2, f_7^2(f_4^0, f_5^1(x_3))))$

All the terms that occur in $F$ are:

$$x_1$$
$$x_2$$
$$f_2^1(x_2)$$
$$f_7^2(f_4^0, f_5^1(x_3))$$
$$f_4^0$$
$$f_5^1(x_3)$$
$$x_3$$

## Syntax of First Order Logic (cont…)

An occurrence of the variable x in the formula **F** is **bound** if x occurs within a subformula of **F** of the form ∃**xF** or ∀**xF**.

A variable in the formula is **free** if at least one occurrence of the variable is free.

A formula without occurrence of a **free variable** is called **closed**.

**Example:** $F = (\exists x_1 P_5^2(x_1, f_2^1(x_2)) \vee \neg\forall x_2 P_4^2(x_2, f_7^2(f_4^0, f_5^1(x_3))))$

- All occurrences of $x_1$ in F are bound.
- The first occurrence of $x_2$ is free, all others are bound, then $x_2$ is free.
- $x_3$ occurs free in F.
- The formula F is not closed.
- The term $f_4^0$ is an example for a constant.

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## Syntax of First Order Logic

- **Notation:**
  - If **FOL** we allow the same simplifying notations for formulas as in propositional logic.
  - Additionally, we allow the following abbreviations.
    - u, v, w, x, y, z - always stand for **variables**
    - a, b, c - always stand for **constants**
    - f, g, h - stand for **function symbols** where the arity can always be inferred from the context
    - P, Q, R - stand for **predicate symbols** where the arity can always be inferred from the context

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## Semantic of First Order Logic

- A **structure** is a pair $\mathcal{A} = (U_\mathcal{A}, I_\mathcal{A})$ where $U_\mathcal{A}$ is an arbitrary, non-empty set and is called the ground set or universe. Further, $I_\mathcal{A}$ is a mapping that maps
  - each k-ary **predicate** symbol P to a k-ary predicate on $U_\mathcal{A}$ (if $I_\mathcal{A}$ is defined on P).
  - each k-ary **function** symbol f to a k-ary function on $U_\mathcal{A}$ (if $I_\mathcal{A}$ is defined on f).
  - each **variable** x to an element of $U_\mathcal{A}$ (if $I_\mathcal{A}$ is defined on x).
- In other words, the **domain of $I_\mathcal{A}$** is a subset of
  $\{P_i^k, f_i^k, x_i$ for $i = 1, 2, \ldots,$ and $k = 0, 1,2,\ldots\}$, and
- The **range of** $I_\mathcal{A}$ is a subset of all predicates, functions, and single elements of $U_\mathcal{A}$.
- In the following, we abbreviate the notation and write $P^\mathcal{A}$ instead of $I_\mathcal{A}(P)$, $f^\mathcal{A}$ instead of $I_\mathcal{A}(f)$, and $x^\mathcal{A}$ instead $I_\mathcal{A}(x)$.

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## Semantic of First Order Logic (cont...)

Let $F$ be a formula and $\mathcal{A} = (U_\mathcal{A}, I_\mathcal{A})$ be a structure. $\mathcal{A}$ is called *suitable* for $F$ if $I_\mathcal{A}$ is defined for all predicate symbols, function symbols, and for all variables that occur free in $F$.

**Example:** $F = \forall x P(x, f(x)) \wedge Q(g(a,z))$ is a formula. Here, $P$ is a binary and $Q$ a unary predicate, $f$ is unary, $g$ a binary, and $a$ a 0-ary function symbol. The variable $z$ is free in $F$. An example for a structure $\mathcal{A} = (U_\mathcal{A}, I_\mathcal{A})$ which is suitable for $F$ is the following.

$$\begin{aligned}
U_\mathcal{A} &= \{0,1,2,3,\ldots\} = \mathbb{N}, \\
I_\mathcal{A}(P) &= P^\mathcal{A} = \{(m,n) \mid m,n \in U_\mathcal{A} \text{ and } m < n\}, \\
I_\mathcal{A}(Q) &= Q^\mathcal{A} := \{n \in U_\mathcal{A} \mid n \text{ is prime }\} \\
I_\mathcal{A}(f) &= f^\mathcal{A} = \text{ the successor function on } U_\mathcal{A}, \\
&\quad \text{hence } f^\mathcal{A}(n) = n+1, \\
I_\mathcal{A}(g) &= g^\mathcal{A} = \text{ the addition function on } U_\mathcal{A}, \\
&\quad \text{hence } g^\mathcal{A}(m,n) = m+n, \\
I_\mathcal{A}(a) &= a^\mathcal{A} = 2, \\
I_\mathcal{A}(z) &= z^\mathcal{A} = 3.
\end{aligned}$$

In this structure $F$ is obviously "true" (we will define this notion in a moment), because every natural number is smaller than its successor, and the sum of 2 and 3 is a prime number.

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## Semantic of First Order Logic (cont…)

- Now we can define the **(truth-)value** of the formula F, denoted $\mathcal{A}$(F), under the structure $\mathcal{A}$ by an inductive definition.

1. If $F$ has the form $F = P(t_1, \ldots, t_k)$ where $t_1, \ldots, t_k$ are terms and $P$ is a predicate symbol of arity $k$, then

$$\mathcal{A}(F) = \begin{cases} 1, & \text{if } (\mathcal{A}(t_1), \ldots, \mathcal{A}(t_k)) \in P^{\mathcal{A}} \\ 0, & \text{otherwise} \end{cases}$$

2. If $F$ has the form $F = \neg G$, then

$$\mathcal{A}(F) = \begin{cases} 1, & \text{if } \mathcal{A}(G) = 0 \\ 0, & \text{otherwise} \end{cases}$$

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## Semantic of First Order Logic (cont…)

3. If $F$ has the form $F = (G \wedge H)$, then

$$\mathcal{A}(F) = \begin{cases} 1, & \text{if } \mathcal{A}(G) = 1 \text{ and } \mathcal{A}(H) = 1 \\ 0, & \text{otherwise} \end{cases}$$

4. If $F$ has the form $F = (G \vee H)$, then

$$\mathcal{A}(F) = \begin{cases} 1, & \text{if } \mathcal{A}(G) = 1 \text{ or } \mathcal{A}(H) = 1 \\ 0, & \text{otherwise} \end{cases}$$

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## Semantic of First Order Logic (cont…)

5. If $F$ has the form $F = \forall x G$, then

$$\mathcal{A}(F) = \begin{cases} 1, & \text{if for all } u \in U_{\mathcal{A}},\ \mathcal{A}_{[x/u]}(G) = 1 \\ 0, & \text{otherwise} \end{cases}$$

Here, $\mathcal{A}_{[x/u]}$ is the structure $\mathcal{A}'$, which is identical to $\mathcal{A}$ with the exception of the definition of $x^{\mathcal{A}'}$ : No matter whether $I_{\mathcal{A}}$ is defined on $x$ or not, we let $x^{\mathcal{A}'} = u$.

6. If $F$ has the form $F = \exists x G$, then

$$\mathcal{A}(F) = \begin{cases} 1, & \text{if there exists some } u \in U_{\mathcal{A}} \text{ such that } \mathcal{A}_{[x/u]}(G) = 1 \\ 0, & \text{otherwise} \end{cases}$$

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## Semantic of First Order Logic (cont…)

- If for a formula F and a **suitable structure** $\mathcal{A}$ we have $\mathcal{A}$(F) = 1, then we denote this by , $\mathcal{A} \models$ F (we say, F is true in $\mathcal{A}$, or $\mathcal{A}$ is a model for F).
- If every suitable structure for F is a model for F, then we denote this by $\models$F (F is **valid** or **tautology**), otherwise $\not\models$ F.
- If there is at least one model for the formula F then F is called **satisfiable**, and otherwise **unsatisfiable** (or **contradictory**).

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## Propositional Logic vs FOL

- Analogously to **propositional logic**, it can be shown

  **F is valid if and only if ¬F is unsatisfiable.**

- Now is easy to see that **FOL** can be understood as an extension of **Propositional Logic** in the following sense:
  - If all predicate symbols are required to have arity 0 (then there is no use for variables, quantifiers, and terms), essentially we get the formulas in propositional logic where the predicates $P_i^0$ play the role of the atomic formulas $A_i$ in propositional logic.

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## Propositional Logic vs FOL

- It even suffices not to use variables (and therefore also no quantifiers) such that FOL "degenerates" to propositional logic.
- **Example:**

$$F = (Q(a) \vee \neg R(f(b), c)) \wedge P(a, b)$$

be a formula without variables (but with predicate symbols of arity greater than 0). By identifying different atomic formulas in $F$ with different atomic formulas $A_i$ of propositional logic, such as

$$\begin{array}{rcl} Q(a) & \longleftrightarrow & A_1 \\ R(f(b), c) & \longleftrightarrow & A_2 \\ P(a, b) & \longleftrightarrow & A_3 \end{array}$$

we get

$$F' = (A_1 \vee \neg A_2) \wedge A_3 \ .$$

Obviously, a formula obtained like $F'$ from $F$ is satisfiable (or valid) if and only if $F$ is satisfiable (or valid).

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## Propositional Logic vs FOL (cont...)

- It's easy to see that a formula without occurrences of a **quantifier** can be transformed into an equivalent formula in **CNF** or **DNF** where only the tools from **propositional logic** are needed.
- Although **FOL** is expressionally more "powerful" than **propositional logic** (i.e. more statements in colloquial language can be expressed formally), it is not powerful enough to express every conceivable statement (e.g. in mathematics).
- We can obtain an even stronger power if we allow also **quantifications** that range over **predicate** or **function symbols**, like

$$F = \forall P \exists f \forall x P(f(x))$$

- This is a matter of the so-called **second order predicate logic**.



## Exercise:

- The **structure**

$$U_{\mathcal{A}} = \mathbb{N},\ P^{\mathcal{A}} = \{(m,n) \mid m,n \in \mathbb{N}, m < n\}$$

is a **model** for the formula

$$F = \exists x \exists y \exists z (P(x,y) \wedge P(z,y) \wedge P(x,z) \wedge \neg P(z,x))\ ?$$

**Answer: Yes. Let x=3, y=5, and z=4**