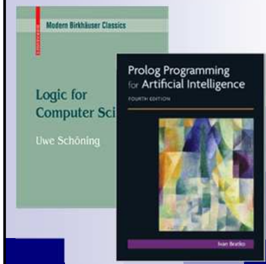

FLORIDA INTERNATIONAL UNIVERSITY

First Order Logic (V)

Dr. Antonio L. Bajuelos

School of Computing & Information Sciences

Resolution principle for FOL.

- Using the **unification principle**, we are now in a situation to formulate the **resolution principle** for **FOL**.

<u>DEFINITION (resolution in FOL):</u>

Let C_1, C_2 and R be clauses (in predicate logic). Then R is called a *resolvent* of C_1, C_2 if the following holds.

1. There exist certain substitutions s_1 and s_2 which are variable renamings so that $C_1 s_1$ and $C_2 s_2$ do not contain the same variable.

2. There is a set of literals $L_1, \ldots, L_m \in C_1 s_1$ $(m \geq 1)$ and $L'_1, \ldots, L'_n \in C_2 s_2$ $(n \geq 1)$, such that $\mathbf{L} = \{\overline{L_1}, \overline{L_2}, \ldots, \overline{L_m}, L'_1, L'_2, \ldots, L'_n\}$ is unifiable. Let sub be a most general unifier for $\mathbf{L}$.

3. R has the form

$$R = ((C_1 s_1 - \{L_1, \ldots, L_m\}) \cup (C_2 s_2 - \{L'_1, \ldots, L'_n\}))sub .$$

2

Resolution principle for FOL.

- Example of the **resolvent**:

$$\{\underline{P(f(x))}, \neg Q(z), \underline{P(z)}\} \qquad\qquad \{\underline{\neg P(x)}, R(g(x), a)\}$$

$$\{\neg Q(f(x)), R(g(f(x)), a)\}$$

Substitutions???

3

Resolution principle for FOL.

Resolution Theorem (of FOL)

- Let F be a closed formula in **Skolem** form with its matrix F*. Then, F is **unsatisfiable** iff □ ∈ Res*(F*).

(see proof in the Textbook, Chapter 2, section 2.5)

4

Resolution principle for FOL.

Example:

- Consider the **two clauses**:

  C1 = {P(f(x),g(y)),Q(x,y)}

  C2 = {¬P(f(f(a)),g(z)),Q(f(a),g(z))}

- These **two clauses** contain the following literals:

  L1 = {P(f(x),g(y))}

  L2 = {¬P(f(f(a)),g(z))}

- It is easy to check that the literals in L1 and L2 are **unifiable** with **MGU** = {[x/f(a)], [y/z]}
- so the clauses C1 and C2 are clashing, and their **resolvent** is:

  Res(C1,C2) = {Q(f(a),z),Q(f(a),g(z))}

5

Resolution principle for FOL.

<u>Example #1:</u>

- In this example, we will prove that the set of clauses (1)-(4) is **unsatisfiable**.
  1. ¬p(x,y) ∨ p(y,x)
  2. ¬p(x,y) ∨ ¬p(y,z) ∨ p(x,z)
  3. p(x,f(x))
  4. ¬p(x,x)
- **Resolution…**

3′. p(x′,f(x′))	3 Rename x to x′
5. p(f(x),x)	1,3′ [x′/x], [y/f(x)]
3″. p(x″, f(x″))	3 Rename x to x″
6. ¬p(f(x), z) ∨ p(x, z)	3″,2 [y/f (x)], [x″/x]
5‴. p(f(x‴), x‴)	5 Rename x to x‴
7. p(x,x)	6,5 [z/x], [x‴/x]
4″″. ¬p(x″″, x″″)	4 Rename x to x″″
8. □	7,4″″ [x″″/x]

6

Resolution principle for FOL.

Example #2:

- We will show, using the general resolution procedure, that the following set of clauses is **unsatisfiable**:
  1. $\neg p(x) \vee q(x) \vee r(x,f(x))$
  2. $\neg p(x) \vee q(x) \vee s(f(x))$
  3. $t(a)$
  4. $p(a)$
  5. $\neg r(a,y) \vee t(y)$
  6. $\neg t(x) \vee \neg q(x)$
  7. $\neg t(x) \vee \neg s(x)$
- **Resolution…**

8. $\neg q(a)$	3,6 [x/a]
9. $\neg p(a) \vee s(f(a))$	2,8 [x/a]
10. $\neg p(a) \vee r(a,f(a))$	1,8 [x/a]
11. $s(f(a))$	4,9
12. $r(a,f(a))$	4,10
13. $t(f(a))$	5,12 [y/f(a)]
14. $\neg t(f(a))$	11,7 [x/f(a)]
15. ❑	13,14

7

Resolution principle for FOL.

Example #3:

- Use resolution to prove that a **relation R $\subseteq$ AxA is reflexive if it is transitive and symmetric.**

Solution:

- Translate in terms of logic.
- Define R(x,y) to be true if x is related to y.
- Since A is the **domain** we have

  $\forall x \exists y R(x,y)$.
- The relation is **transitive**, that is,

  $\forall x \forall y \forall z((R(x,y) \wedge R(y,z)) \rightarrow R(x,z))$.
- The relation is **symmetric**, that is

  $\forall x \forall y(R(x,y) \rightarrow R(y,x))$.
- We want to conclude from these premises that the relation is **reflexive**, that is,

  $\forall x R(x,x)$.
- To <u>prove the theorem</u> by **refutation**, <u>negate the conclusion</u>

  $\exists x \neg R(x,x)$.

8

Resolution principle for FOL.
Example #3 (cont…):

Original clauses are:

The **domain** we have

$\forall x \exists y R(x,y)$

Transitive:

$\forall x \forall y \forall z((R(x,y) \wedge R(y,z)) \rightarrow R(x,z))$

Symmetric:

$\forall x \forall y(R(x,y) \rightarrow R(y,x))$

Reflexive:

$\forall x R(x,x)$

To prove the theorem by refutation, negate the conclusion

$\exists x \neg R(x,x)$.

After Skolemization:

The **domain** we have

$R(x,f(x))$

Transitive:

$\neg R(x,y) \vee \neg R(y,z) \vee R(x,z))$

Symmetric:

$\neg R(x,y) \vee R(y,x)$

Reflexive:

To prove the theorem by refutation, negate the conclusion

$\neg R(a,a)$.

Resolution principle for FOL.
Example #3 (cont…):

- **Clauses:**
  1. $\neg R(a,a)$ // Conclusion negation
  2. $R(x,f(x))$
  3. $\neg R(x,y) \vee \neg R(y,z) \vee R(x,z)$
  4. $\neg R(x,y) \vee R(y,x)$

- **Resolution:**
  5. $\neg R(a,y) \vee \neg R(y,a) \vee R(a,a)$ 3, with [x/a] [z/a]
  6. $\neg R(a,y) \vee \neg R(y,a)$ resolve 1 and 5
  7. $R(a,f(a))$ 2 with [x/a]
  8. $\neg R(a,f(a)) \vee \neg R(f(a),a)$ 6 with [y/f(a)]
  9. $\neg R(f(a),a)$ resolve 7 and 8
  10. $\neg R(a,f(a)) \vee R(f(a),a)$ 4 with [x/a] [y/a]
  11. $\neg R(a,f(a))$ resolve 9 and 10
  12. $R(a,f(a))$ 2 with [x/a]
  13. $\square$ 11, 12 **empty clause – original claim is valid**

Resolution principle for FOL.

Example #4:

Prove that everybody has a grandparent, provided everybody has a parent.

- Let **P(x,y)** represent **x is a parent of y**. The **premise** can now be stated as

  $\forall x \exists y P(y,x)$.

- From this we must be able to **conclude** that there exists a parent of a parent, which can be expressed as

  $\forall x \exists y \exists z (P(z,y) \wedge P(y,x))$.

- We must thus prove that

  $\forall x \exists y P(y,x) \vDash \forall x \exists y \exists z (P(z,y) \wedge P(y,x))$

Resolution principle for FOL.

Example #4 (cont…):

Prove that everybody has a grandparent is a logical consequence that everybody has a parent.

P(x,y): x is a parent of y

$\forall x \exists y P(y,x) \vDash \forall x \exists y \exists z (P(z,y) \wedge P(y,x))$

- We add the **negation of the conclusion** to the set of premises, which yields:

  $\forall x \exists y P(y,x) \wedge \exists x \forall y \forall z (\neg P(z,y) \vee \neg P(y,x))$

- Eliminate the existential quantifiers (**skolemization** process):

  $\forall x \exists y P(y,x) \wedge \exists w \forall v \forall z (\neg P(z,v) \vee \neg P(v,w))$

  $\forall x \exists y \exists w \forall v \forall z (P(y,x) \wedge (\neg P(z,v) \vee \neg P(v,w)))$

  $\forall x \forall v \forall z (P(f(x),x) \wedge (\neg P(z,v) \vee \neg P(v,g(x))))$

- After dropping the universal quantifiers, this yields:

  $S = \{\{P(f(x),x)\}, \{\neg P(z,v), \neg P(v,g(x))\}\}$

Resolution principle for FOL.

Example #4 (cont…):

Prove that everybody has a grandparent, provided everybody has a parent.

$$\forall x \exists y P(y,x) \models \forall x \exists y \exists z (P(z,y) \wedge P(y,x))$$

$$\Updownarrow$$

$\{\{P(f(x),x)\}, \{\neg P(z,v), \neg P(v,g(x))\}\}$ is **unsatisfiable**

- **Resolution…**

1. $P(f(x),x)$	Given
2. $\neg P(z,v) \vee \neg P(v,g(x))$	Given
3. $P(f(g(a)),g(a))$	1 with $[x/g(a)]$
4. $\neg P(z,f(g(a))) \vee \neg P(f(g(a)),g(a))$	2 with $[x/a]$ and $[v/f(g(a))]$
5. $\neg P(z,f(g(a)))$	Resolve 3 and 4
6. $P(f(f(g(a))),f(g(a)))$	1 with $[x/f(g(a))]$
7. $\neg P(f(f(g(a))),f(g(a)))$	5 with $[z/f(f(g(a)))]$

8. $\square$ **empty clause** (contradiction) from 6 and 7

This means that the original argument is valid.