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Propositional Logic (III)

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Horn Formulas

- An important special case of **CNF formulas** which often occurs in practical applications are the **Horn formulas** (named after the logician *Alfred Horn**)

Definition (**Horn formula**)

- A formula **F** in **CNF** is a **Horn formula** if every disjunction set in **F** contains at most one positive literal.
- **Examples:**

$$F = (A \vee \neg B) \wedge (\neg C \vee \neg A \vee D) \wedge (\neg A \vee \neg B) \wedge D \wedge \neg E$$

$$G = (A \vee \neg B) \wedge (C \vee \neg A \vee D)$$

F is a **Horn formula** and G is not.

Alfred Horn (1918-2001) was an American mathematician notable for his work in lattice theory and universal algebra.

Horn Formulas

- **Horn formulas** can be (equivalently) rewritten in a more intuitive way, namely as **implications**.
- Remember that:

$$F \rightarrow G \equiv \neg F \vee G$$

$$F \equiv F \vee \text{False} \equiv \text{False} \vee F \equiv \neg(\text{True}) \rightarrow F \equiv 1 \rightarrow F$$

$$\neg F \equiv \neg F \vee \text{False} \equiv F \rightarrow \text{False} \equiv F \rightarrow 0$$

- **Example:**

Formula F:

$$F = (A \vee \neg B) \wedge (\neg C \vee \neg A \vee D) \wedge (\neg A \vee \neg B) \wedge D \wedge \neg E$$

Can be rewritten as:

$$F \equiv (B \rightarrow A) \wedge (C \wedge A \rightarrow D) \wedge (A \wedge B \rightarrow 0) \wedge (1 \rightarrow D) \wedge (E \rightarrow 0)$$

- Here, **0** stands for an arbitrary **unsatisfiable** formula and **1** for an arbitrary **tautology**.

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Horn Formulas

Example: Formula F in **CNF:**

$$F = (A \vee \neg B) \wedge (\neg C \vee \neg A \vee D) \wedge (\neg A \vee \neg B) \wedge D \wedge \neg E$$

Can be rewritten as:

$$F \equiv (B \rightarrow A) \wedge (C \wedge A \rightarrow D) \wedge (A \wedge B \rightarrow 0) \wedge (1 \rightarrow D) \wedge (E \rightarrow 0)$$

- The **general rules** are:
  - Write the **negative literals** to the left of the implication sign. Ex: $\neg C \vee \neg A \vee D \equiv (C \wedge A) \rightarrow D$
  - Write the **positive literal** (if any) at the right of the implication sign. Ex: $A \vee \neg B \equiv B \rightarrow A$
  - Write **1** (or **True**) to the left of the implication sign if there is no negative single literal. Ex: $D \equiv 1 \rightarrow D$
  - Write **0** (or **False**) at the right of the implication if there is no positive single literal. Ex: $\neg E \equiv E \rightarrow 0$

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Horn Formulas

- Remember that one on the main goals of the **propositional logic** is:
  - The search for efficient algorithms which decide **satisfiability** (or **validity**) of formulas.
  - We know that:
    - it is enough to have a test for **unsatisfiability** because a formula is **valid** if and only if its negation is **unsatisfiable**.
    - Using **truth-tables**, it is always possible to find out whether a formula is **satisfiable** or **unsatisfiable**. But we have observed already that an algorithm based on constructing the full truth-table of a formula necessarily runs in **exponential time**.

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Satisfiability of Horn Formulas

- **Good News!**
  - For **Horn formulas** there exists an **efficient test** for **satisfiability** which works as follows:
- **Input:** a **Horn formula F**

```
{
    mark all occurrences of True in F
    while there is a conjunction A1 ∧ A2 ∧…∧ An → B in F such
            that all Ai are already marked and B is not,
      mark B
    end while
    if False is marked           % some original Ak → 0 is marked
        return "unsatisfiable"
    else
        return "satisfiable"
    endif
}
```

- **Complexity**: O**(nm)**, n - # of atomic formulas in F and m – length of F (number of implications).

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Satisfiability of Horn Formulas

```
{
    mark all occurrences of True in F
    while there is a conjunction A1 ∧ A2 ∧…∧ An → B in F such
            that all Ai are already marked and B is not,
       mark B
    end while
    if False is marked            % Ak → 0 is marked
        return "unsatisfiable"
    else
        return "satisfiable"
    endif
}
```

- **Example #1:**

  $F \equiv (B \rightarrow A) \wedge (C \wedge A \rightarrow D) \wedge (A \wedge B \rightarrow 0) \wedge (1 \rightarrow D) \wedge (E \rightarrow 0)$

  - **Returning:** F is satisfiable
  - One possible **model** for F:
    - $\mathcal{A}$(A)=$\mathcal{A}$(B)=$\mathcal{A}$(C)=$\mathcal{A}$(E)= 0 and $\mathcal{A}$(D)= 1

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Satisfiability of Horn Formulas

```
{
    mark all occurrences of True in F
    while there is a conjunction A1 ∧ A2 ∧…∧ An → B in F such
            that all Ai are already marked and B is not,
       mark B
    end while
    if False is marked            % Ak → 0 is marked
        return "unsatisfiable"
    else
        return "satisfiable"
    endif
}
```

- **Example #2:**
- F_2= $(1\rightarrow B)\wedge(1\rightarrow D)\wedge(F\rightarrow 0)\wedge(A\wedge B\wedge D\rightarrow E)\wedge(E\rightarrow D)\wedge(1\rightarrow C)\wedge(C\rightarrow A)$
  - **Returning:** F_2 is satisfiable
  - One possible **model** for F_2:
    - $\mathcal{A}$(F)= 0
    - $\mathcal{A}$(A)=$\mathcal{A}$(B)=$\mathcal{A}$(C)=$\mathcal{A}$(D)=$\mathcal{A}$(E)= 1

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Satisfiability of Horn Formulas

```
{
    mark all occurrences of True in F
    while there is a conjunction A1 ∧ A2 ∧...∧ An → B in F such
            that all Ai are already marked and B is not,
      mark B
    end while
    if False is marked            % Ak → 0 is marked
        return "unsatisfiable"
    else
        return "satisfiable"
    endif
}
```

- **Example #3:**

  $F_3 = (A \wedge B \wedge D \to A) \wedge (B \wedge C \to A) \wedge (A \wedge D \to E)$

  No occurrences of the form $(1 \to A)$ in F_3

  Nothing is marked

  Returning: F_3 is satisfiable

  One possible **model** for F_3:

  - $\mathcal{A}(A)=\mathcal{A}(B)=\mathcal{A}(C)=\mathcal{A}(D)=\mathcal{A}(E)= 0$

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Satisfiability of Horn Formulas

```
{
    mark all occurrences of True in F
    while there is a conjunction A1 ∧ A2 ∧...∧ An → B in F such
            that all Ai are already marked and B is not,
      mark B
    end while
    if False is marked            % Ak → 0 is marked
        return "unsatisfiable"
    else
        return "satisfiable"
    endif
}
```

- **Example #4:**

  $F_4 = (1 \to B) \wedge (1 \to D) \wedge (E \to 0) \wedge (A \wedge B \wedge D \to E) \wedge (E \to D) \wedge (1 \to C) \wedge (C \to A)$

  False is marked!

  Returning: F_4 is unsatisfiable

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Satisfiability of Horn Formulas

Complementary material:

- **Theory**

https://www21.in.tum.de/teaching/logik/SS17/Slides/horn-prop.pdf

- **Theory & Exercises**

http://sites.cs.queensu.ca/courses/cisc204/Record/Week09/Horn%20Clauses%20and%20Satisfiability_review.pdf