# Hands-on tutorial on ADVI

## Example #1 :

Given some $p(x)$, choose a mean-field variational approximation $q_\varphi(x)$, and find :

$$\varphi^* = \underset{\varphi}{\arg\min}\ \mathbb{KL}[q_\varphi(x) \,\|\, p(x)] := \mathcal{L}(\varphi)$$

where $q_\varphi(x) = \prod_{i=1}^{d} \mathcal{N}(x_i \mid \mu_i, \sigma_i^2)$

$$\mathcal{L}(\varphi) := \mathbb{E}_{x \sim q_\varphi(x)} \left[ \log q_\varphi(x) - \log p(x) \right] = \int \log \frac{q_\varphi(x)}{p(x)} \cdot q_\varphi(x)\, dx$$

## Example #2 : Bayesian Logistic regression

**Model:** $y = \sigma(w^T x)$, $\underbrace{y_i \sim \mathrm{Ber}(\sigma(w^T x_i))}_{\text{likelihood}}$, $\underbrace{\begin{bmatrix} w_1 \\ \vdots \\ w_d \end{bmatrix} \sim \mathcal{N}\left( \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \right)}_{\text{prior}}$

**Goal:** $$p(w \mid D) = \frac{p(D \mid w)\, p(w)}{p(D)}$$

$p(D) \hookrightarrow \int p(D \mid w)\, p(w)\, dw$ $\begin{cases} \text{marginal likelihood,} \\ \text{model evidence} \end{cases}$

**VI:** Approximate $p(w \mid D) \approx q_\varphi(w \mid D) \approx \prod_{i=1}^{d} \mathcal{N}(w_i \mid \mu_i, \sigma_i^2)$

**Recall:** $\varphi^* = \underset{\varphi}{\arg\min}\ \underline{\mathbb{KL}[q_\varphi(w \mid D) \,\|\, p(w \mid D)]} := \mathcal{L}(\varphi)$

$$\boxed{\mathcal{L}(\varphi) := -H[q_\varphi(w \mid D)] - \mathbb{E}_{w \sim q_\varphi(w \mid D)} \left[ \log p(D \mid w) + \log p(w) \right]}$$ Evidence Lower Bound (ELBO

Let's consider :

$$\log p(D) = \log \int p(D \mid w)\, p(w)\, dw \overset{\text{I.S.}}{=} \log \int p(D \mid w)\, p(w) \frac{q_\varphi(w \mid D)}{q_\varphi(w \mid D)}\, dw$$

$$\underline{\underline{q_\phi(w|D)}} \quad \circledast$$

Jensen's inequality :

If $f$ is a convex function and $X$ is some random variable, then :

$$f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)] \implies f\left(\int x\, p(x)\, dx\right) \leq \int f(x)\, p(x)$$

$$\circledast \overset{\text{Jensen's iq}}{\geq} \int \log \frac{p(D|w)\,p(w)}{q_\phi(w|D)} \cdot q_\phi(w|D)\, dw$$

$$= \underline{\underline{\mathbb{E}_{w \sim q_\phi(w|D)}}} \left[ \log p(D|w) + \log p(w) - \underline{\log q_\phi(w|D)} \right]$$

$$\implies -\log p(D) \leq -H[q_\phi(w|D)] - \mathbb{E}_{w \sim q_\phi(w|D)} \left[ \log p(D|w) + \log p( \right.$$

<u>Evidence lower bound (ELBO)</u>

i.e. $-\log p(D) \leq \underline{\mathbb{KL}\left[q_\phi(w|D) \,||\, p(w|D)\right]} := \mathcal{L}(\phi)$

$$\phi^* = \underset{\phi}{\arg\min} -\log p(D) = \underset{\phi}{\arg\max}\; p(D)$$

$$\boxed{\mathcal{L}(\phi) := \mathbb{E}_{w \sim q_\phi(w|D)} \left[ \log q_\phi(w|D) - (\log p(D|w) + \log p(w)) \right]}$$

$$-H[q_\phi(w|D)] = \mathbb{E}_{w \sim q_\phi(w|D)} [\log q_\phi(w|D)]$$