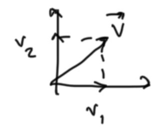
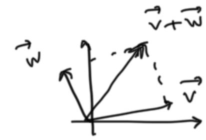
Primer on Linear Algebra

Vectors : tuple of elements

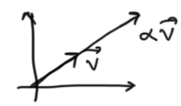
$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = [v_1, v_2]^T \quad , \quad \vec{v} \in \mathbb{R}^2$$



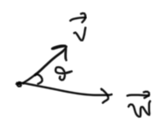
- vector addition : $\vec{v} + \vec{w} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}$



- multiplication by a scalar : $\alpha \in \mathbb{R}$, $\alpha \vec{v} = \begin{bmatrix} \alpha v_1 \\ \alpha v_2 \end{bmatrix}$



- elementwise multiplication : $\vec{v} \odot \vec{w} = \begin{bmatrix} v_1 w_1 \\ v_2 w_2 \end{bmatrix}$
- linear combination : $c_1 \vec{v} + c_2 \vec{w} = c_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + c_2 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} c_1 v_1 + c_2 w_1 \\ c_1 v_2 + c_2 w_2 \end{bmatrix}$
- dot/inner product : $\vec{v}^T \vec{w} = \underset{1\times \underline{2}}{[v_1 \; v_2]} \underset{\underline{2}\times 1}{\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}} = v_1 w_1 + v_2 w_2 = \sum_{i=1}^{2} v_i w_i$
- Norms :
  - Euclidean / L_2 / Length norm : $\|\vec{v}\|_2 = \sqrt{\vec{v}^T v} = \sqrt{v_1^2 + v_2^2}$
  - L_∞ / maximum norm : $\|\vec{v}\|_\infty = \max_i |v_i|$
  - L_1 norm : $\|\vec{v}\|_1 = \sum_i |v_i|$
- Unit vector : $\|\vec{u}\|_2 = 1$, $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|_2}$
- Cosine similarity : $\cos\theta = \dfrac{\vec{v}^T \vec{w}}{\|\vec{v}\|_2 \|\vec{w}\|_2}$



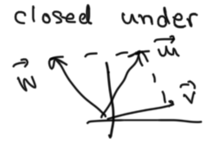
$\Longrightarrow$ $\vec{v}$ and $\vec{w}$ are orthogonal $(\vec{v} \perp \vec{w})$ iff $\begin{cases} \cos\theta = 0 \\ \vec{v}^T \vec{w} = 0 \end{cases}$

- Schwarz inequality : $|\vec{v}^T \vec{w}| \leq \|\vec{v}\|_2 \|\vec{w}\|_2$

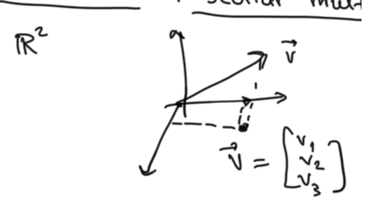
⊛ Vector space / Linear Space : closed under <u>addition</u> + <u>scalar mult</u>

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$, $\vec{v}, \vec{w} \in \mathbb{R}^2$

$\vec{v} \in \mathcal{V}$



closed under addition



- Linear (in)dependence : Assume a finite collection of vectors :

$$\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k \in V$$

Then define the linear combination:

$$\vec{v} = c_1\vec{x}_1 + c_2\vec{x}_2 + \ldots + c_k\vec{x}_k = \sum_{i=1}^{k} c_i\vec{x}_i$$

Definition:

$\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$ are linearly dependent if there exist a non-trivial linear combination for which $\sum_{i=1}^{k} c_i\vec{x}_i = 0$ with at least one $c_i \neq 0$.

If only the trivial solution exist, i.e. $c_1 = c_2 = \ldots = c_k = 0$ then $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$ are linearly independent.

In practice:

- $\vec{x}_1, \ldots, \vec{x}_k$ are linearly dependent $\begin{cases} \text{- if at least one of them is zero} \\ \text{- if two vectors identical} \\ \text{- if at least one of the } \vec{x}_i \text{ can be written as a linear combination of all the other vectors, i.e.} \end{cases}$

$$\vec{x}_i = \sum_{i \neq j} c_j x_j$$

- if $\vec{x}_1, \ldots, \vec{x}_k$ vectors are linearly independent, then they form a basis in V

$\Longrightarrow$ that any $\vec{v} \in V$ can be written as a linear combination of $\vec{x}_1, \ldots, \vec{x}_k$, i.e. $\vec{v} = \sum_{i=1}^{k} c_i\vec{x}_i$

Matrices: collections of vectors, linear systems of equations, linear ma[ps]

$$\underset{m\times n}{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \quad a_{ij} \in \mathbb{R}, \; i = 1, \ldots, m, \; j = 1, \ldots, n$$

columns → ; rows ↓

$$A \in \mathbb{R}^{m\times n} \longrightarrow \vec{a} \in \mathbb{R}^{m\cdot n}$$

neighboring dimensions should match!

- Multiplication: $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times k}$, $\underset{m\times k}{C} = \underset{m\times \underline{n}}{A}\underset{\underline{n}\times k}{B} \in \mathbb{R}^{m\times k}$

$$c_{ij} = \sum_{l=1}^{n} a_{il} b_{lj}, \quad i = 1, \ldots, m, \; j = 1, \ldots, k$$

—

i.e. taking the dot-product of the i-th row of A and j-th column of B.

⊛ Not a commutative operation, i.e. $AB \neq BA$

Not an element-wise operation, i.e. $c_{ij} \neq a_{ij} b_{ij}$

- Addition: $A, B \in \mathbb{R}^{m \times n}$, $C = A + B \in \mathbb{R}^{m \times n}$, $c_{ij} = a_{ij} + b_{ij}$

- Identity matrix: $I_n = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} \in \mathbb{R}^{n \times n}$, $AI = IA = A$

- Associativity: $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$, $C \in \mathbb{R}^{p \times q}$, then

$$\underbrace{\overset{m \times p}{(\underset{m \times n \; n \times p}{AB})} \overset{p \times q}{C}}_{m \times q} = \underbrace{\underset{m \times n}{A} \overset{n \times q}{(\underset{n \times p \; p \times q}{BC})}}_{m \times q}$$

- Distributivity: $A, B \in \mathbb{R}^{m \times n}$, $C, D \in \mathbb{R}^{n \times p}$

$$\begin{cases} (A+B)C = AC + BC \\ A(C+D) = AC + AD \end{cases}$$

- Inverse: $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times m}$, if $\overset{m \times m}{AB} = I_m$

then $B := A^{-1}$ is the inverse of A.

Remark: Not all matrices are invertible!

- Transpose: $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times m}$ if $b_{ij} = a_{ji}$ then B is the traspose of A, i.e. $B = A^T$

⊛ if A symmetric then $A = A^T$.

- Matrix rank: dimension of the vector space generated by its columns (or equivalently by its rows).

i.e. $\text{rank}(A) :=$ # linearly independent column vectors

- Nullspace: is formed by the remaining linearly dependent column vect