Instructor: Michael Lerner, CST 213, Phone: 765-983-1784

Name: _____

Assignment 3, Due as specified

For Wednesday Jan 31

As you know, there's an exam coming up covering complex numbers. I won't cover logs on the exam, but I will cover things like complex powers, geometric representations, and trig functions. So, the following (short) problems are due on Wednesday:

§2.5 51

§2.7 9

§2.9 1, 5, 12

§2.10 16, 17, 20

For Friday Feb 2

Using a table like the one in Boas §12.2, solve problem 12.1.1

Also For Friday Feb 2

In this assignment, as in many of the future assignments, we will investigate a particular topic in depth, rather than solving several separate problems. This week, we focus on the Legendre polynomials, which have broad applicability in mathematical physics, especially in the modeling of spherically symmetric systems.

The text of the following problems is taken (with some small changes) from Boyce and DiPrima, Chapter 5, section 3.

The following problems deal with the Legendre equation:

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0 \tag{1}$$

Following the convention of choosing a fundamental set of solutions such that

$$y_1(x) = 1 + b_2(x - x_0)^2 + ...$$
$$y_2(x) = (x - x_0) + c_2(x - x_0)^3 + ...$$
$$b_2 + c_2 = a_2$$

(Note that these series have already included the fact that one will be even and one will be odd, a fact that you'll show below.)

Two solutions of the Legendre equation for $|x| < 1$ are

$$y_1(x) = 1 - \frac{\alpha(\alpha+1)}{2!}x^2 + \frac{\alpha(\alpha-2)(\alpha+1)(\alpha+3)}{4!}x^4$$
$$+ \sum_{m=3}^{\infty}(-1)^m \frac{\alpha \cdots (\alpha-2m+2)(\alpha+1) \cdots (\alpha+2m-1)}{(2m)!}x^{2m},$$
$$y_2(x) = x - \frac{(\alpha-1)(\alpha+2)}{3!}x^3 + \frac{(\alpha-1)(\alpha-3)(\alpha+2)(\alpha+4)}{5!}x^5$$
$$+ \sum_{m=3}^{\infty} \frac{(\alpha-1) \cdots (\alpha-2m+1)(\alpha+2) \cdots (\alpha+2m)}{(2m+1)!}x^{2m+1}$$

Problem 1 Write out the first 4 terms for y_1 and y_2.

Problem 2 Show that, if α is zero or a positive even integer $2n$, the series solution y_1 reduces to a polynomial of degree $2n$ containing only even powers of x. Find the polynomials corresponding to $\alpha = 0, 2, 4$. Similarly, show that if α is a positive odd integer $2n+1$, the series solution y_2 reduces to a polynomial of degree $2n+1$ containing only odd powers of x. Find the polynomials corresponding to $\alpha = 1, 3, 5$.

Problem 3 The Legendre polynomial $P_n(x)$ is defined as the polynomial solution of the Legendre equation with $\alpha = n$ that also satisfies the condition $P_n(1) = 1$.

(a) Using the results of Problem 2, find the first five Legendre polynomials, $P_0(x), ..., P_5(x)$.

(b) Plot the graphs of $P_0(x), ..., P_5(x)$ in the range for which we've demonstrated convergence, $|x| \leq 1$. You can check your answers with Wolfram Alpha (e.g. go to wolframalpha.com and type "plot 0.5*(3x^2-1) from -1 to 1" in the box), but you need to print out solutions here using Python + matplotlib. For this, please refer to the in-class notebooks and ask your instructor/classmates for help early and often!

(c) Find the zeros of $P_0(x), ..., P_5(x)$.

For Wednesday Feb 6

Problem 4 The Legendre polynomials play an important role in mathematical physics. For example, solving the potential equation (Laplace's equation) in spherical coordinates, we encounter the equation

$$\frac{d^2 F(\phi)}{d\phi^2} + \cot\phi \frac{dF(\phi)}{d\phi} + n(n+1)F(\phi) = 0, \qquad 0 < \phi < \pi$$

Show that the change of variables $x = \cos\phi$ leads to the Legendre equation with $\alpha = n$ for $y = f(x) = F(\cos^{-1}(x))$

Hint: you may need to use the fact that

$$\sin(\arccos(x)) = \sqrt{1-x^2}; \qquad \cot(\arccos(x)) = \frac{x}{\sqrt{1-x^2}}$$

Problem 5 Show that the Legendre equation can also be written as

$$[(1 - x^2)y']' = -\alpha(\alpha + 1)y$$

It then follows that

$$[(1 - x^2)P_n'(x)]' = -n(n + 1)P_n(x) \tag{2}$$

and

$$[(1 - x^2)P_m'(x)]' = -m(m + 1)P_m(x). \tag{3}$$

By multiplying (2) by $P_m(x)$ and (3) by $P_n(x)$, **integrating by parts**, and then subtracting one equation from the other, show that

$$\int_{-1}^{1} P_n(x)P_m(x)dx = 0 \qquad \text{if} \quad n \neq m \tag{4}$$

This property (4) of the Legendre polynomials is known as the orthogonality property. If $m = n$, it can be shown that the value of the integral in (4) is $2/(2n + 1)$.

Given a polynomial f of degree n, it is possible to express f as a linear combination of $P_0, P_1, ..., P_n$:

$$f(x) = \sum_{k=0}^{n} a_k P_k(x) \tag{5}$$

Note that, since the $n + 1$ polynomials $P_0, ..., P_n$ are linearly independent, and the degree of P_k is k, any polynomial of degree n can be expressed as (5).
Using the result of Problem 7, you can show that

$$a_k = \frac{2k + 1}{2} \int_{-1}^{1} f(x)P_k(x)dx$$

but you don't have to! You're done!