**Physics 360/Math 360   MF 12:00 - 12:50 PM, W 2:30-3:20, Room: CST 314**

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**Assignment 3, Due as specified**

**For Friday Jan 27**

Using a table like the one in Boas §12.2, solve problems 12.1.1 and 12.1.4

**For Wednesday Feb 1**

In this assignment, as in many of the future assignments, we will investigate a particular topic in depth, rather than solving several separate problems. This week, we focus on the Legendre polynomials, which have broad applicability in mathematical physics, especially in the modeling of spherically symmetric systems.

The text of the following problems is taken (with some small changes) from Boyce and DiPrima, Chapter 5, section 3.

The following problems deal with the Legendre equation:

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0 \tag{1}$$

Following the convention of choosing a fundamental set of solutions such that

$$\begin{aligned} y_1(x) &= 1 + b_2(x-x_0)^2 + ... \\ y_2(x) &= (x-x_0) + c_2(x-x_0)^3 + ... \\ b_2 + c_2 &= a_2 \end{aligned}$$

(Note that these series have already included the fact that one will be even and one will be odd, a fact that you'll show below.)

Two solutions of the Legendre equation for $|x| < 1$ are

$$\begin{aligned} y_1(x) &= 1 - \frac{\alpha(\alpha+1)}{2!}x^2 + \frac{\alpha(\alpha-2)(\alpha+1)(\alpha+3)}{4!}x^4 \\ &\quad + \sum_{m=3}^{\infty}(-1)^m \frac{\alpha\cdots(\alpha-2m+2)(\alpha+1)\cdots(\alpha+2m-1)}{(2m)!}x^{2m}, \\ y_2(x) &= x - \frac{(\alpha-1)(\alpha+2)}{3!}x^3 + \frac{(\alpha-1)(\alpha-3)(\alpha+2)(\alpha+4)}{5!}x^5 \\ &\quad + \sum_{m=3}^{\infty} \frac{(\alpha-1)\cdots(\alpha-2m+1)(\alpha+2)\cdots(\alpha+2m)}{(2m+1)!}x^{2m+1} \end{aligned}$$

**Problem 1** Write out the first 4 terms for $y_1$ and $y_2$.

**Problem 2** Show that, if $\alpha$ is zero or a positive even integer $2n$, the series solution $y_1$ reduces to a polynomial of degree $2n$ containing only even powers of $x$. Find the polynomials corresponding to

$\alpha = 0, 2, 4$. Similarly, show that if $\alpha$ is a positive odd integer $2n+1$, the series solution $y_2$ reduces to a polynomial of degree $2n+1$ containing only odd powers of $x$. Find the polynomials corresponding to $\alpha = 1, 3, 5$.

**Problem 3** The Legendre polynomial $P_n(x)$ is defined as the polynomial solution of the Legendre equation with $\alpha = n$ that also satisfies the condition $P_n(1) = 1$.

**(a)** Using the results of Problem 2, find the first five Legendre polynomials, $P_0(x), ..., P_5(x)$.

**(b)** Plot the graphs of $P_0(x), ..., P_5(x)$ in the range for which we've demonstrated convergence, $|x| \leq 1$. You may use whatever graphing package you'd like, including but not limited to Wolfram alpha (e.g. go to wolframalpha.com and type "plot 0.5*(3x^2-1) from -1 to 1" in the box), Matlab, Mathematica, Python+matplotlib, etc.

**(c)** Find the zeros of $P_0(x), ..., P_5(x)$.

**Problem 4** The Legendre polynomials play an important role in mathematical physics. For example, solving the potential equation (Laplace's equation) in spherical coordinates, we encounter the equation

$$\frac{d^2F(\phi)}{d\phi^2} + \cot\phi \frac{dF(\phi)}{d\phi} + n(n+1)F(\phi) = 0, \qquad 0 < \phi < \pi$$

Show that the change of variables $x = \cos\phi$ leads to the Legendre equation with $\alpha = n$ for $y = f(x) = F(\cos^{-1}(x))$

Hint: you may need to use the fact that

$$\sin(\arccos(x)) = \sqrt{1-x^2}; \qquad \cot(\arccos(x)) = \frac{x}{\sqrt{1-x^2}}$$

**For Friday Feb 3**

**Problem 5** Show that the Legendre equation can also be written as

$$[(1-x^2)y']' = -\alpha(\alpha+1)y$$

It then follows that

$$[(1-x^2)P_n'(x)]' = -n(n+1)P_n(x) \tag{2}$$

and

$$[(1-x^2)P_m'(x)]' = -m(m+1)P_m(x). \tag{3}$$

By multiplying (2) by $P_m(x)$ and (3) by $P_n(x)$, **integrating by parts**, and then subtracting one equation from the other, show that

$$\int_{-1}^{1} P_n(x)P_m(x)dx = 0 \qquad \text{if} \quad n \neq m \tag{4}$$

This property (4) of the Legendre polynomials is known as the orthogonality property. If $m = n$, it can be shown that the value of the integral in (4) is $2/(2n+1)$.

Given a polynomial $f$ of degree $n$, it is possible to express $f$ as a linear combination of $P_0, P_1, ..., P_n$:

$$f(x) = \sum_{k=0}^{n} a_k P_k(x) \tag{5}$$

Note that, since the $n+1$ polynomials $P_0, ..., P_n$ are linearly independent, and the degree of $P_k$ is $k$, any polynomial of degree $n$ can be expressed as (5).
Using the result of Problem 7, you can show that

$$a_k = \frac{2k+1}{2} \int_{-1}^{1} f(x) P_k(x) dx$$

**but you don't have to! You're done!**