Physics 360/Math 360 MWF 11:00 - 12:00 PM, Room: Dennis 231

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Assignment 3, Due Friday Jan 30

In this assignment, as in many of the future assignments, we will investigate a particular topic in depth, rather than solving several separate problems. This week, we focus on the Legendre polynomials, which have broad applicability in mathematical physics, especially in the modeling of spherically symmetric systems.

The text of the following problems is taken (with some small changes) from Boyce and DiPrima, Chapter 5, section 3.

The following problems deal with the Legendre equation:

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0 \tag{1}$$

As we showed in class, the point $x = 0$ is an ordinary point of (1), and the distance from the origin to the nearest zero of $P(x) = 1 - x^2$ is 1. Thus the radius of convergence of series solutions about $x = 0$ is at least 1. Also, note that we only need to consider $\alpha > -1$: if $\alpha \leq 1$, make the substitution $\alpha = (-1+\gamma)$, $\gamma \geq 0$. Plugging this in gives $(1-x^2)y'' - 2xy' + \gamma(\gamma+1)y = 0$.

In this problem set, we will examine the solutions of the Legendre equation, and we will spend some time showing how to write those solutions in the form that they are typically presented.

Problem 1 This problem follows the convention of choosing a fundamental set of solutions such that

$$\begin{aligned}
y_1(x) &= 1 + b_2(x-x_0)^2 + ... \\
y_2(x) &= (x-x_0) + c_2(x-x_0)^3 + ... \\
b_2 + c_2 &= a_2
\end{aligned}$$

(Note that these series have already included the fact that one will be even and one will be odd, a fact that you'll show below.)

We can, of course, use these to build any set of solutions that we'd like, and it's a particularly convenient convention. Show that two solutions of the Legendre equation for $|x| < 1$ are

$$\begin{aligned}
y_1(x) &= 1 - \frac{\alpha(\alpha+1)}{2!}x^2 + \frac{\alpha(\alpha-2)(\alpha+1)(\alpha+3)}{4!}x^4 \\
&\quad + \sum_{m=3}^{\infty}(-1)^m \frac{\alpha\cdots(\alpha-2m+2)(\alpha+1)\cdots(\alpha+2m-1)}{(2m)!}x^{2m}, \\
y_2(x) &= x - \frac{(\alpha-1)(\alpha+2)}{3!}x^3 + \frac{(\alpha-1)(\alpha-3)(\alpha+2)(\alpha+4)}{5!}x^5 \\
&\quad + \sum_{m=3}^{\infty} \frac{(\alpha-1)\cdots(\alpha-2m+1)(\alpha+2)\cdots(\alpha+2m)}{(2m+1)!}x^{2m+1}
\end{aligned}$$

For y_1, you must calculate the first several terms. You may do the proof by induction for the general term for extra credit. For y_2, calculating the first 2-3 terms is sufficient.

Problem 2 Show that, if α is zero or a positive even integer $2n$, the series solution y_1 reduces to a polynomial of degree $2n$ containing only even powers of x. Find the polynomials corresponding to $\alpha = 0, 2, 4$. Similarly, show that if α is a positive odd integer $2n+1$, the series solution y_2 reduces to a polynomial of degree $2n+1$ containing only odd powers of x. Find the polynomials corresponding to $\alpha = 1, 3, 5$.

Problem 3 The Legendre polynomial $P_n(x)$ is defined as the polynomial solution of the Legendre equation with $\alpha = n$ that also satisfies the condition $P_n(1) = 1$.

(a) Using the results of Problem 2, find the first five Legendre polynomials, $P_0(x), ..., P_5(x)$.

(b) Plot the graphs of $P_0(x), ..., P_5(x)$ in the range for which we've demonstrated convergence, $|x| \leq 1$. You may use whatever graphing package you'd like, including but not limited to Wolfram alpha (e.g. go to wolframalpha.com and type "plot 0.5*(3x^2-1) from -1 to 1" in the box), Matlab, Mathematica, Python+matplotlib, etc.

(c) Find the zeros of $P_0(x), ..., P_5(x)$.

Problem 4 This problem is only for extra credit; it is not required It can be shown (you do not need to do this) that the general formula for $P_n(x)$ is

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (2n-2k)!}{k!(n-k)!(n-2k)!} x^{n-2k},$$

where $\lfloor n/2 \rfloor$ (called the floor of $n/2$) denotes the greatest integer less than or equal to $n/2$. By observing the form of $P_n(x)$ for n even and n odd, show that $P_n(-1) = -1^n$

Problem 5 The Legendre polynomials play an important role in mathematical physics. For example, solving the potential equation (Laplace's equation) in spherical coordinates, we encounter the equation

$$\frac{d^2 F(\phi)}{d\phi^2} + \cot\phi \frac{dF(\phi)}{d\phi} + n(n+1)F(\phi) = 0, \qquad 0 < \phi < \pi$$

Show that the change of variables $x = \cos\phi$ leads to the Legendre equation with $\alpha = n$ for $y = f(x) = F(\cos^{-1}(x))$

Hint: you may need to use the fact that

$$\sin(\arccos(x)) = \sqrt{1-x^2}; \qquad \cot(\arccos(x)) = \frac{x}{\sqrt{1-x^2}}$$

Problem 6 Just mention the results. No need to verify. Show that for $n = 0, 1, 2, 3$, the corresponding Legendre polynomial is given by the famous Rodrigues formula:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

Rodrigues's formula is true for all positive integers n, so you may show the general case instead of the specific cases if you'd like. Full credit will be given for just verifying the first several terms.

Problem 7 Show that the Legendre equation can also be written as

$$[(1-x^2)y']' = -\alpha(\alpha+1)y$$

It then follows that

$$[(1-x^2)P_n^{'}(x)]' = -n(n+1)P_n(x) \tag{2}$$

and

$$[(1-x^2)P_m^{'}(x)]' = -m(m+1)P_m(x). \tag{3}$$

By multiplying (2) by $P_m(x)$ and (3) by $P_n(x)$, **integrating by parts**, and then subtracting one equation from the other, show that

$$\int_{-1}^{1} P_n(x)P_m(x)dx = 0 \qquad \text{if} \quad n \neq m \tag{4}$$

This property (4) of the Legendre polynomials is known as the orthogonality property. If $m = n$, it can be shown that the value of the integral in (4) is $2/(2n+1)$.

Problem 8 Given a polynomial f of degree n, it is possible to express f as a linear combination of $P_0, P_1, ..., P_n$:

$$f(x) = \sum_{k=0}^{n} a_k P_k(x) \tag{5}$$

Note that, since the $n+1$ polynomials $P_0, ..., P_n$ are linearly independent, and the degree of P_k is k, any polynomial of degree n can be expressed as (5).
Using the result of Problem 7, you can show that

$$a_k = \frac{2k+1}{2}\int_{-1}^{1} f(x)P_k(x)dx$$

but you don't have to! You're done!