The method of Frobenius summary; Bessel intro

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1 A couple of definitions

Given a second-order differential equation

$$P(x)y'' + Q(x)y' + R(x)y = 0, \tag{1}$$

we'd like to be able to write down a general solution. Sadly, the world is not kind enough to let us do so. The method of Frobenius lets us write down solutions for most of the points that we find interesting. First off, we can write it as

$$y'' + p(x)y' + p(x)y = 0 \tag{2}$$

where

$$p(x) \equiv Q(x)/P(x) \qquad q(x) \equiv R(X)\P(x). \tag{3}$$

When doesn't this work? Well, clearly it doesn't work when $P(x)$ is 0. We call those **singular points**. $p(x)$ or $q(x)$ diverge there.[1] We call points where $p(x)$ and $q(x)$ do not diverge (i.e. $P(x) \neq 0$) **ordinary points**. Near ordinary points, we can find solutions with the power series methods we've used so far for, e.g., Airy's equation. However, when modeling physical systems, we find that we often care about singular points. They show up at physical boundaries, for instance. So, the reasoning goes, what if things didn't blow up "too quickly"? Could we write down a solution then?

Consider a singular point x_0. This means that $p(x)$ diverges as $x \to x_0$. If, however, $(x-x_0)p(x)$ and $(x-x_0)^2q(x)$ remain finite as $x \to x_0$, we call x_0 a **regular singular point**.[2]. The Method of Frobenius lets us write down solutions near such points.

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[1]You have to be careful here. We're only in trouble if $p(x)$ or $q(x)$ diverge. So, if someone hands you $xy'' + xy' + x^2y = 9$, you don't call 0 a singular point: although $P(x) = 0$, $p(x) = 1$ and $q(x) = x$.

[2]See the previous footnote. And irregular singular points are significantly more troublesome.

2 The main theorem

(Thm 5.6.1 from Boyce & DiPrima) Let $x = 0$ be a regular singular point of the differential equation

$$\begin{aligned} P(x)y'' + Q(x)y' + R(x)y &= 0 \\ x^2y'' + x[xp(x)]y' + [x^2q(x)]y &= 0 \end{aligned} \tag{4}$$

Then $xp(x)$ and $x^2q(x)$ are analytic at $x = 0$ with convergent power series expansions

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n, \qquad x^2q(x) = \sum_{n=0}^{\infty} q_n x^n \tag{5}$$

for $|x| < \rho$ where $\rho > 0$ is the minimum of the radii of convergence of the power series for $xp(x)$ and $x^2q(x)$. Let r_1 and r_2 be the roots of the indicial equation

$$F(r) = r(r-1) + p_0 r + q_0 = 0 \tag{6}$$

with $r_1 \geq r_2$ if r_1 and r_2 are real. Then in either the interval $-\rho < x < 0$ or the interval $0 < x < \rho$, there exists a solution of the form

$$y_1(x) = |x|^{r_1} \left[1 + \sum_{n=1}^{\infty} a_n(r_1) x^n \right], \tag{7}$$

where the $a_n(r_1)$ are given by the recurrence relation:

$$F(r+n)a_n + \sum_{k=0}^{n-1} a_k[(r+k)p_{n-k} + q_{n-k}] = 0, \qquad n \geq 1 \tag{8}$$

with $a_0 = 1$ and $r = r_1$. If $r_1 - r_2$ is neither zero nor a positive integer, then in either the interval $-\rho < x < 0$ or the interval $0 < x < \rho$, there exists a second solution of the form

$$y_2(x) = |x|^{r_2} \left[1 + \sum_{n=1}^{\infty} a_n(r_2) x^n \right]. \tag{9}$$

The $a_n(r_2)$ are also determined by the recurrence relation (8) with $a_0 = 1$ and $r = r_2$. The power series in (7) and (9) converge at least for $|x| < \rho$.

If $r_1 = r_2$, then the second solution is

$$y_2(x) = y_1(x) \ln |x| + |x|^{r_1} \left[1 + \sum_{n=1}^{\infty} b_n(r_1) x^n \right] \tag{10}$$

If $r_1 - r_2 = N \in \mathbb{N}$, then

$$y_2(x) = a y_1(x) \ln |x| + |x|^{r_2} \left[1 + \sum_{n=1}^{\infty} c_n(r_2) x^n \right]. \tag{11}$$

The coefficients $a_n(r_1), b_n(r_1), c_n(r_2)$ and the constant a can be determined by substituting the form of the series solutions for y into (4). The constant a may turn out to be zero, in which case there is no logarithmic term in the solution to (11). Each of the series in (10) and (11) converges at least for $|x| < \rho$ and defines a function that is analytic in some neighborhood of $x = 0$. In all three cases, the two solutions $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions for the given differential equation.

3 Bessel Equations

Bessel equations are perhaps the most widely used of the "special" functions (Bessel, Airy, Legendre, etc.) in mathematical physics. They are most commonly seen in problems with circular or cylindrical symmetry, including vibrations of a uniformly stretched circular membrane (e.g. a drum head), quantum mechanical problems with circular symmetry (e.g. this week's homework assignment), heat conduction in cylindrical objects, diffusion problems on a lattice, etc.

They are the solutions to Bessel's equation:

$$x^2y'' + xy' + (x^2 - v^2)y \tag{12}$$

where v is a constant, also called the **order** of the Bessel equation.

Given the form of (12), you will not be surprised to find that we plan on using the results of the preceding section in its analysis. We will cover three cases here (order 0, 1/2 and 1). We will see more applications in our triumphant return to series solutions later in the class (i.e. solving PDEs by separation of variables).

Problem 1 Is $x = 0$ is a regular singular point of (12)?

Problem 2 What is the indicial equation? What are r_1, r_2?

3.1 Order Zero

"order zero", as mentioned above, means $v = 0$:

$$L[y] = x^2y'' + xy' + x^2y \tag{13}$$

So, we will have $r_1 = r_2 = 0$. As per the theorem, we know that we'll have one solution of the form (7):

$$y(x) = \phi(x, r) = a_0x^r + x^r\sum_{n=1}^{\infty} a_nx^n.$$

Problem 3 After reminding yourself that

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^{r+n} \\ y'(x) &= \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1} \\ y''(x) &= \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n-2}, \end{aligned} \tag{14}$$

3a plug (14) into (13). You should find that you'll need to strip off the first two terms.

3b what is the recurrence relation? Don't forget to simplify the denominator.

You've seen these sorts of recurrence relations before. We'll ratchet our way up to a general form, so we start with the first term. Recall that a_0 is not zero. Plugging in $r = 0$, we see that the coefficient of x^r is zero, as it should be from the indicial equation $F(r) = 0$. We are not surprised to find that we must set a_1 to zero for the coefficient of $x^r + 1$ to be zero. So, our recurrence relation turns into

$$a_n(0) = -\frac{a_{n-2}}{n^2}$$

All of the odd terms are zero, and we write

$$a_{2m}(0) = -\frac{a_{2m-2}(0)}{(2m)^2}$$

Problem 4 Write out $a_2(0), a_4(0), a_6(0)$.

Looking at this, we see that we can split out $2^2, 2^4, 2^6$ from the denominators, writing the general term as

$$a_{2m}(0) = \frac{(-1)^m a_0}{2^{2m}(m!)^2}, m = 1, 2, 3, ...$$

and so (in what we've previously decided is a unique and lucky situation), we can write out the first solution as

$$\begin{aligned} y_1(x) &= a_0 \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2} \right] \\ &= a_0 \left[J_0(x) \right] \end{aligned} \tag{15}$$

where $J_0(x)$ is known as the **Bessel function of the first kind of order zero**. The results of Problem 1, combined with the theorem, tell us that J_0 is analytic at $x = 0$ and that the series (15) converges for all x.

Now we want to find $y_2(x)$. [Hint! This will give us Y_0, a Bessel function of the *second* kind of order zero!]. In the general case, we will need to plug back in and solve for the b_n terms. In this case, we are lucky enough that we can evaluate the derivative and just solve directly for ${a_n}'(0)$.