$$\frac{1}{h_\ell^u}\sum\left(J^{ug}_{\ell+1/2,m,n}-J^{ug}_{\ell-1/2,m,n}\right)+\Sigma^g_{t_{\ell,m,n}}\phi^g_{\ell mn}$$

$$=\sum_{h=1}^{G}\Sigma^{h\to g}_{s_{\ell mn}}\phi^h_{\ell mn}+\frac{1}{k}\sum_{h=1}^{G}\nu\Sigma^{h\to g}_{f\,\ell mn}\phi^h_{\ell mn}$$

Balance own cell

Inside a cell

$$\nabla\cdot\vec{J}^g+\Sigma^g_t(\vec{r})\phi^g(\vec{r})=\sum_{h=1}^{G}\Sigma^{h\to g}_s(\vec{r})\phi^h(\vec{r})+\frac{1}{k}\sum_{h=1}^{G}\nu\Sigma^{h\to g}_f(\vec{r})\phi^h(\vec{r})$$

Diffusion Approx.

$$\vec{J}^g=-D^g(\vec{r})\nabla\phi^g(\vec{r})$$

3-D

$$-D^g_{\ell mn}\frac{\partial^2\phi^g}{\partial x^2}-D^g_{\ell mn}\frac{\partial^2\phi^g}{\partial y^2}-D^g_{\ell mn}\frac{\partial^2\phi^g}{\partial z^2}+\Sigma^g_{t}(\vec{r})\phi^g(\vec{r})=$$

$$\sum_{h=1}^{G}\Sigma^{h\to g}_{s}(\vec{r})\phi^h(\vec{r})+\frac{1}{k}\sum_{h=1}^{G}\nu\Sigma^{h\to g}_f(\vec{r})\phi^h(\vec{r})$$

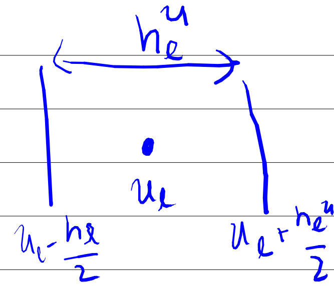
Integrate over transverse directions (v+w) → switch notation

$$-D^g_{\ell mn}\frac{\partial^2}{\partial u^2}\int_{v_m-h^v_m/2}^{v_m+h^v_m/2}dv\int_{w_n-h^w_n/2}^{w_n+h^w_n/2}dw\,\phi(u,v,w)+-D^g_{\ell mn}\frac{\partial^2}{\partial v^2}\int dv\int dw\,\phi(u,v,w)$$

$$+-D^g_{\ell mn}\frac{\partial^2}{\partial w^2}\int_v dv\int_w dw\,\phi(u,v,w)+\Sigma^g_{t_{\ell mn}}\iint\phi(u,v,w)=\sum\Sigma^{h\to g}_{s_g}\iint\phi^h+\frac{1}{k}\sum\nu\Sigma^{h\to g}_{f_g}\iint\phi$$

Define

Coordinates inside a cell: $\frac{u-u_\ell}{h_\ell^u} \equiv u'$



Transverse Integrated Flux: $$\int_{v_m-\frac{h_m^v}{2}}^{v_m+\frac{h_m^v}{2}} dv \int_{w_n-\frac{h_n^w}{2}}^{w_n+\frac{h_n^w}{2}} dw\, \phi^g(u',v',w') \frac{1}{h_m^v h_n^w} = \bar{\phi}_{\ell mn}^{ug}(u')$$

Transverse Integrated Leakage (performed one of the ints)

$$L_{\ell mn}^{ug}(u') = \frac{D_{\ell mn}^g}{h_m^v h_n^w}\left[\int_{w_n-h_n^w/2}^{w_n+h_n^w/2} dw \frac{\partial}{\partial v}\phi_{\ell mn}^g \Big|_{v_m-h_m^v/2}^{v_m+h_m^v/2} + \int_{v_m-h_m^v/2}^{v_m+h_m^v/2} dv \frac{\partial}{\partial w}\phi_{\ell mn}^g \Big|_{w_n-h_n^w/2}^{w_n+h_n^w/2}\right]$$

divide thru equ. from transverse flux

Equation Becomes (now eqs are decoupled)

$$-D_{\ell mn}^g \frac{d^2}{du'^2}\bar{\phi}_{\ell mn}^{ug}(u') + \Sigma_{t\ell mn}^g \bar{\phi}_{\ell mn}^{ug}(u')$$
$$= \sum_{h=1}^{G} \Sigma_{s\ell mn}^{h\to g} \bar{\phi}_{\ell mn}^{ug}(u') + \frac{1}{k}\sum_{h=1}^{G} \nu\Sigma_{f\ell mn}^{h\to g} \bar{\phi}_{\ell mn}^{ug}(u') + L_{\ell mn}^{ug}(u')$$

Expand Transverse Integrated Flux

$$\bar{\phi}^{ug}_{lmn}(u') = \bar{\phi}^{g}_{lmn}\left[1 + \sum_{t=1}^{5} a^{ug}_{t\,lmn} P_t(u')\right]$$

Quadratic Transverse Leakage Approximation

$$L^{ug}_{lmn}\left(\frac{u-u_l}{h^u_l}\right) = L^{ug}_{0_{lmn}} + L^{ug}_{1_{lmn}} \frac{u-u_l}{h^u_l} + L^{ug}_{2_{lmn}}\left(\frac{u-u_l}{h^u_l}\right)^2$$

Define average transverse leakage in cell:

$$\bar{L}^{ug}_{lmn} = \frac{1}{h^u_l}\int_{u_l-\frac{h^u_l}{2}}^{u_l+\frac{h^u_l}{2}} L^{ug}_{lmn}\left(\frac{u-u_l}{h^u_l}\right) du$$

Integrate polynomial over adjacent cells:

$$\bar{L}^{ug}_{lmn} = \frac{1}{h^u_l}\int_{u_l-\frac{h^u_l}{2}}^{u_l+h^u_l/2}\left(L^{ug}_{0_{lmn}} + L^{ug}_{1_{lmn}}\left(\frac{u-u_l}{h^u_l}\right) + L^{ug}_{2_{lmn}}\left(\frac{u-u_l}{h^u_l}\right)^2\right) du$$

$$\bar{L}^{ug}_{l-1,m,n} = \frac{1}{h^u_{l-1}}\int_{u_l-\frac{h^u_l}{2}-h^u_{l-1}}^{u_l-\frac{h^u_l}{2}}\left(L^{ug}_{0_{lmn}} + L^{ug}_{1_{lmn}}\left(\frac{u-u_l}{h^u_l}\right) + L^{ug}_{2_{lmn}}\left(\frac{u-u_l}{h^u_l}\right)^2\right) du$$

$$\bar{L}^{ug}_{l+1,m,n} = \frac{1}{h^u_{l+1}}\int_{u_l+\frac{h^u_l}{2}}^{u_l+\frac{h^u_l}{2}+h^u_{l+1}}\left(L^{ug}_{0_{lmn}} + L^{ug}_{1_{lmn}}\left(\frac{u-u_l}{h_l}\right) + L^{ug}_{2_{lmn}}\left(\frac{u-u_l}{h^u_l}\right)^2\right) du$$

3 previous equations are solved simultaneously
This was done with Mathematica
The results are below (file saved in bode folder)

$$L^{ug}_{0_{lmn}} = \overline{L}^{ug}_{lmn} - \frac{1}{12} L^{ug}_{2_{lmn}}$$

$$L^{ug}_{1_{lmn}} = \frac{h^u_l}{h^u_{l-1}+h^u_l+h^u_{l+1}} \left[\left(\overline{L}^{ug}_{l+1,m,n} - \overline{L}^{ug}_{lmn} \right) \frac{2h^u_{l-1}+h^u_l}{h^u_l+h^u_{l+1}} - \left(\overline{L}^{ug}_{l-1,m} - \overline{L}^{ug}_{lmn} \right) \frac{h^u_l+2h^u_{l+1}}{h^u_{l-1}+h^u_l} \right]$$

$$L^{ug}_{2_{lmn}} = \frac{3\left(h^u_l\right)^2}{h^u_{l-1}+h^u_l+h^u_{l+1}} \left[\left(\overline{L}^{ug}_{l+1,m,n} - \overline{L}^{ug}_{lmn} \right) \frac{1}{h^u_l+h^u_{l+1}} + \left(\overline{L}^{ug}_{l-1,m,n} - \overline{L}^{ug}_{lmn} \right) \frac{1}{h^u_{l-1}+h^u_l} \right]$$

The average transverse leakage can be backed out from nodal balance:

$$\overline{L}^{ug}_{lmn} = \frac{\overline{J}^{vg}_{l,m-1/2,n} - \overline{J}^{vg}_{l,m+1/2,n}}{h^v_m} + \frac{\overline{J}^{wg}_{lmn-1/2} - \overline{J}^{wg}_{lmn+1/2}}{h^w_n}$$

Polynomials for 5^{th} order NEM:

$P_0(\xi)=1 \quad P_1(\xi)=\xi \quad P_2(\xi)=\xi^2-\frac{1}{12}$

$P_3(\xi)=\xi^3-\xi/4 \quad P_4(\xi)=\xi^4-3\xi^2/10+1/80$

$P_5(\xi)=\xi^5-\xi^3/3+\xi/48$

Note 10 eqs. per group are needed for a two node problem - 5 expansion coeffs per node per group

Constraint #2 - Longitudinal Leakage

$$Q^{ug}_{lmn}=\frac{\bar{J}^{ug}_{l+1/2,mn}-\bar{J}^{ug}_{l-1/2,mn}}{h^u_l}$$

Net current

$$\bar{J}^{ug}_{lmn}\left(\frac{u-u_l}{h^u_l}\right)=-D^g_{lm}\frac{d\bar{\phi}^{ug}_{lm}}{du'}\frac{du'}{du}$$

→ we know expansion from before

→ need chain rule here for coordinate system change ⇒ $\frac{d\phi}{du}$ ⇒

$$\bar{J}^{ug}_{lmn}\left(\frac{u-u_l}{h^u_l}\right)=\frac{-D^g_{lmn}}{h^u_l}\bar{\phi}^g_{lmn}\sum_{t=1}^{5}a^{ug}_{t\,lmn}P'_t\left(\frac{u-u_l}{h_l}\right)$$

→ from chain

$P'_0(\xi)=0 \quad P'_1(\xi)=1 \quad P'_2(\xi)=2\xi \quad P'_3(\xi)=2\xi^2-1/4$

$P'_5(\xi)=5\xi^4-\xi^2+1/48 \quad P'_4(\xi)=4\xi^3-3/5\,\xi$

Right Face: $u = \frac{u_e + h^u_e/2 - u_e}{h^u_e} = \frac{1}{2}$

$P_0'(1/2)=0 \quad P_1'(1/2)=1 \quad P_2'(\frac{1}{2})=1 \quad P_3'(\frac{1}{2})=\frac{1}{2}$

$P_4'(1/2)=1/5 \quad P_5'(1/2)=\frac{1}{12}$

Left Face: $u = \frac{u_e - h^u_e/2 - u_e}{h^u_e} = -\frac{1}{2}$

$P_0'(-1/2)=0 \quad P_1'(-\frac{1}{2})=1 \quad P_2'(-\frac{1}{2})=-1 \quad P_3'(-\frac{1}{2})=\frac{1}{2}$

$P_4'(-1/2)=-\frac{1}{5} \quad P_5(-1/2)=\frac{1}{12}$

Thus we can express longitudinal current as:

$$\overline{J}^{ug}_{\ell\pm 1/2,m,n} = -\frac{D^g_{\ell mn}}{h^u_e}\overline{\phi}^g_{\ell mn}\left[a^{ug}_{1\ell mn} \pm a^{ug}_{2\ell mn} + \frac{1}{2}a^{ug}_{3\ell mn} \pm \frac{1}{5}a^{ug}_{4\ell mn} + \frac{1}{12}a^{ug}_{5\ell mn}\right]$$

Substituting these values into the net longitudinal current over a cell in u-direction: (only - terms remain)

$$Q^{ug}_{\ell mn} = \frac{\overline{J}^{ug}_{\ell+1/2,mn} - \overline{J}^{ug}_{\ell-1/2,mn}}{h^u_e} = -2\frac{D^g_{\ell mn}}{h^u_e}\overline{\phi}^g_{\ell mn}\left[a^{ug}_{2\ell mn} + \frac{1}{5}a^{ug}_{4\ell mn}\right]$$

$$\boxed{Q^{ug}_{\ell mn} = -2\frac{D^g_{\ell mn}}{h^u_e}\overline{\phi}^g_{\ell mn}\left[a^{ug}_{2\ell mn} + \frac{1}{5}a^{ug}_{4\ell mn}\right]}$$

Constraint 1

Smith said others use node average fluxes here.

Constraint 3 Net Current Continuum

$$-\frac{D^g_{\ell mn}}{h^u_\ell}\,\bar{\bar{\phi}}^g_{\ell mn}\left[a^{ug}_{1_{\ell mn}}+a^{ug}_{2_{\ell mn}}+\frac{1}{2}a^{ug}_{3_{\ell mn}}+\frac{1}{5}a^{ug}_{4_{\ell mn}}+\frac{1}{12}a^{ug}_{5_{\ell mn}}\right]$$

$$=-\frac{D^g_{\ell+1,m,n}}{h^u_{\ell+1}}\,\bar{\bar{\phi}}^g_{\ell+1,m,n}\left[a^{ug}_{1_{\ell+1,m,n}}-a^{ug}_{2_{\ell+1,m,n}}+\frac{1}{2}a^{ug}_{3_{\ell+1,m,n}}-\frac{1}{5}a^{ug}_{4_{\ell+1,m,n}}+\frac{1}{12}a^{ug}_{5_{\ell+1,m,n}}\right]$$

Constraint 4 $\bar{\phi}$ continuity

$P_0(\pm\frac{1}{2})=1 \quad P_1(\pm\frac{1}{2})=\pm\frac{1}{2} \quad P_2(\pm\frac{1}{2})=1/6$

Note all polynomials > order 2 $=0$ at $\pm 1/2$

$$\frac{1}{f^{ug+}_{\ell mn}}\,\bar{\bar{\phi}}^g_{\ell mn}\left[1+\frac{1}{2}a^{ug}_{1_{\ell mn}}+\frac{1}{6}a^{ug}_{2_{\ell mn}}\right]=$$

$$\frac{1}{f^{ug-}_{\ell+1,m,n}}\,\bar{\bar{\phi}}^g_{\ell+1,m,n}\left[1-\frac{1}{2}a^{ug}_{1_{\ell+1,m,n}}+\frac{1}{6}a^{ug}_{2_{\ell+1,m,n}}\right]$$

Weighted Residual Constraints

Start w/ Transverse Integrated

$$-D^g_{lmn}\frac{d^2}{du^2}\bar{\phi}^{ug}_{lmn}\left(\frac{u-u_l}{h_l^u}\right)+\Sigma^g_{t_{lmn}}\bar{\phi}^{ug}_{lmn}\left(\frac{u-u_l}{h_l^u}\right)$$

$$=\sum_h \Sigma^{h\to g}_{s_{lmn}}\bar{\phi}^{uh}\left(\frac{u-u_l}{h_l^u}\right)+\frac{1}{k}\sum_h \nu\Sigma^{h\to g}_{f_{lmn}}\bar{\phi}^{uh}\left(\frac{u-u_l}{h_l^u}\right)+L^{ug}_{lmn}\left(\frac{u-u_l}{h_l^u}\right)$$

Pull summation out + write in general:

$$\sum_h\left(-D^h_{lmn}\frac{d^2}{du^2}\delta_{gh}+\Sigma^h_{t_{lmn}}\delta_{gh}-\Sigma^{h\to g}_{s_{lmn}}-\frac{1}{k}\nu\Sigma^{h\to g}_{f_{lmn}}\right)\bar{\phi}^{uh}_{lmn}\left(\frac{u-u_l}{h_l^u}\right)$$

$$=L^{ug}_{lmn}\left(\frac{u-u_l}{h_l^u}\right)$$

Define: $\tilde{\Sigma}^{h\to g}_{lmn}=\Sigma^h_{t_{lmn}}\delta_{gh}-\Sigma^{h\to g}_{s_{lmn}}-\frac{1}{k}\nu\Sigma^{h\to g}_{f_{lmn}}$

Rewrite Equation:

$$\sum_h\left(-D^h_{lmn}\frac{d^2}{du^2}\delta_{gh}+\tilde{\Sigma}^{h\to g}_{lmn}\right)\bar{\phi}^{uh}_{lmn}\left(\frac{u-u_l}{h_l^u}\right)=L^{ug}_{lmn}\left(\frac{u-u_l}{h_l^u}\right)$$

Write Flux out:

$$\bar{\phi}^{ug}(\xi)=\bar{\phi}^g_{lmn}\left[1+a^{ug}_{1_{lmn}}\xi+a^{ug}_{2_{lmn}}\left(\xi^2-\tfrac{1}{12}\right)+a^{ug}_{3_{lmn}}\left(\xi^3-\tfrac{\xi}{4}\right)+a^{ug}_{4_{lmn}}\left(\xi^4-\tfrac{3}{10}\xi^2+\tfrac{1}{80}\right)+a^{ug}_{5_{lmn}}\left(\xi^5-\tfrac{1}{3}\xi^3+\tfrac{1}{48}\xi\right)\right]$$

Derivative:

$$\frac{d\bar{\phi}^{ug}}{d\xi}=\bar{\phi}^g_{lmn}\left[a^{ug}_{1_{lmn}}+a^{ug}_{2_{lmn}}(2\xi)+a^{ug}_{3_{lmn}}\left(3\xi^2-\tfrac{1}{4}\right)+a^{ug}_{4_{lmn}}\left(4\xi^3-\tfrac{3}{5}\xi\right)+a^{ug}_{5_{lmn}}\left(5\xi^4-\xi^2+\tfrac{1}{48}\right)\right]$$

2nd Derivative

$$\frac{d^2}{d\xi^2}\bar{\phi}^{ug}=\bar{\phi}^g_{lmn}\left[2a^{ug}_{2_{lmn}}+6a^{ug}_{3_{lmn}}\xi+a^{ug}_{4_{lmn}}\left(12\xi^2-\tfrac{3}{5}\right)+a^{ug}_{5_{lmn}}\left(20\xi^3-2\xi\right)\right]$$

Substitute flux expansion + flux derivative:

$$\sum_h \Big(-\frac{D^h_{\ell m}}{h^{u}_{\ell}} \Big(2a^{uh}_{2_{\ell m}} + 6\, a^{uh}_{3_{\ell m}} \xi + a^{uh}_{4_{\ell m}} (12\xi^2 - 3/5) + a^{uh}_{5_{\ell m}} (20\,\xi^3 - 2\,\xi) \Big) \delta_{gh}$$

$$+ \tau^{h\to g}_{\ell m} \Big(1 + a^{ug}_{1_{\ell m}} \xi + a^{uh}_{2_{\ell m}} (\xi^2 - 1/12) + a^{uh}_{3_{\ell m}} (\xi^3 - \xi/4) + a^{uh}_{4_{\ell m}} (\xi^4 - 3/10\, \xi^2 + 1/80)$$

$$+ a^{uh}_{5_{\ell m}} (\xi^5 - 1/3\, \xi^3 + 1/48\, \xi) \Big) \Big) \bar{\bar{\Phi}}^{uh}_{\ell m} = L^{ug}_{0_{\ell m}} + L^{ug}_{1_{\ell m}} \xi + L^{ug}_{2_{\ell m}} \xi^2$$

Multiply by ξ + integrate over $-1/2$ to $1/2$: 5 + 6

$$\sum_h \Big[\tau^{h\to g}_{\ell m} a^{uh}_{1_{\ell m}} - \Big(\tfrac{1}{10} \tau^{h\to g}_{\ell m} + \delta_{gh}\, 6 \frac{D^h_{\ell m}}{h^{u2}_{\ell}} \Big) a^{uh}_{3_{\ell m}} - \Big(\frac{1}{420} \tau^{h\to g}_{\ell m} + \delta_{gh} \frac{D^h_{\ell m}}{h^{u2}_{\ell}} \Big) a^{uh}_{5_{\ell m}} \Big] \bar{\bar{\Phi}}^{h}_{\ell} = L^{ug}_{1_{\ell m}}$$

Multiply by $(\xi^2 - 1/12)$ + integrate over $-1/2$ to $1/2$: 7 + 8

$$\sum_h \Big[\tau^{h\to g}_{\ell m} a^{uh}_{2_{\ell m}} - \Big(\tfrac{3}{35} \tau^{h\to g}_{\ell m} + \delta_{gh}\, 12 \frac{D^h_{\ell m}}{h^{u2}_{\ell}} \Big) a^{uh}_{4_{\ell m}} \Big] \bar{\bar{\Phi}}^{h}_{\ell m} = L^{ug}_{2_{\ell m}}$$

Multiply by $(\xi^3 - \xi/4)$ + integrate over $-1/2$ to $1/2$: 9 + 10

$$\sum_h \Big[\tau^{h\to g}_{\ell m} a^{uh}_{1_{\ell m}} - \Big(\tfrac{1}{7} \tau^{h\to g}_{\ell m} + \delta_{gh}\, 6 \frac{D^h_{\ell m}}{h^{u2}_{\ell}} \Big) a^{uh}_{3_{\ell m}} - \delta_{gh} \frac{1}{7} \frac{D^h_{\ell m}}{h^{u2}_{\ell}} a^{h}_{5_{\ell m}} \Big] \bar{\bar{\Phi}}^{h}_{\ell} = L^{ug}_{1_{\ell m}}$$

$$\tau^{h\to g}_{\ell m} = \Sigma^h_{t_{\ell m}} \delta_{gh} - \Sigma^{h\to g}_{s_{\ell m}} - \frac{1}{k} \nu \Sigma^{h\to g}_{f_{\ell m}}$$