Solutions to the Algebra Preliminary Exam Study Guide

James (Jimmy) Thorne

May 17, 2019

Exercise 1. *An element is called idempotent if $x^2 = x$. Show that if each element a in the ring R is idempotent, then R is a commutative ring. (Note: R is called a boolean ring)*

Proof. Let $x, y \in R$.

$$\begin{aligned} (x+y)^2 &= (x+y)(x+y) && (1)\\ &= x^2 + xy + yx + y^2 && (2)\\ &= x + xy + yx + y && (3) \end{aligned}$$

We also have the $(x+y)^2 = x+y\,(4)$ by the idempotent property, so we combine (3) and (4) to get

$$x + y = x + xy + yx + y \implies 0 = xy + yx \implies -xy = yx$$

By squaring both sides of this last expression we get

$$xy = yx$$

□