Analysis Preliminary Exam Study Guide

NDSU

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Chapter 1

Measure Theory

1.1 σ-Algebras

Definition 1. Let X be a set. $\mathcal{A} \subset \mathcal{P}(X)$ is an algebra if

i. For $A \in \mathcal{A}, A^c \in \mathcal{A}$

ii. If $A, B \in \mathcal{A}, A \cup B \in \mathcal{A}$

We say that $\mathcal{A}$ is a σ-algebra if

iii. If A_j is a countable collection of sets in $\mathcal{A}$, then $\bigcup_{j\in\mathbb{N}} A_j \in \mathcal{A}$.

Exercise 1 (May 19)**.** Let $\mathcal{A}$ be an algebra of sets that is closed under countable increasing unions. Show that $\mathcal{A}$ is a σ-algebra.

Exercise 2 (Jan 18)**.** Let $\mathcal{S}$ be the collection of all subsets of $[0, 1)$ which can be written as a finite union of intervals of the form $[a, b) \subseteq [0, 1)$. Show that $\mathcal{S}$ is an algebra of sets, but is not a σ-algebra.

Definition 2. Since the intersection of σ-algebra is itself a σ-algebra, for a given collection of sets $\mathcal{E}$ we can define $M(\mathcal{E})$ as the intersections of all σ-algebras containing $\mathcal{E}$. We say $M(\mathcal{E})$ is the σ-algebra generated by $\mathcal{E}$. If $\mathcal{E}$ is all the open sets on X, then $M(\mathcal{E})$ is the Borel σ-algebra.

The Borel σ-algebra on $\mathbb{R}$ can be generated with several different sets. For example, one could use closed sets. Since the complement of every open set is closed (and vice versa) we generate the same σ-algebra.

Fact: If $\mathcal{E} \subseteq M(\mathcal{F})$, then $M(\mathcal{E}) \subseteq M(\mathcal{F})$

Exercise 3. Show that the closed rays $(-\infty, a]$ will generate the Borel $\sigma-$algebra.