G2++: Gaussian Mean Reversion with 2 Degrees of Freedom

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I. Instantaneous Interest Rate Dynamics

1.1 Definition

$$r_t = x_t + y_t + \varphi_t | x_{0,} y_{0,} \varphi_0$$

Where:

$$dx_t = -a\, x_t\, dt + \sigma_1\, dW_{1,t}$$
$$dy_t = -b\, x_t\, dt + \sigma_2\, dW_{2,t}$$

With:

$$dW_{1,t} \cdot dW_{2,t} = \rho dt\,;$$
$$-1 \le \rho \le 1\,;$$
$$x_{0,}\, y_0 = 0\,;$$
$$r_{0,}\, a, b, \sigma_{1,}\, \sigma_2 \ge 0\,;$$

1.2 Moments Analysis

Let:

$$F(x_s, s, u) \stackrel{\text{def}}{=} x_s e^{-a(u-s)}, u \ge s$$

Thus, per Ito’s Lemma:

$$\begin{aligned} dF(x_u, s, u) &= \frac{\partial F}{\partial s} ds + \frac{\partial F}{\partial x_s} dx_s + \frac{1}{2} \frac{\partial^2 F}{\partial x_s^2} (dx_s)^2 \\ &= a e^{-a(u-s)} x_s\, ds + e^{-a(u-s)} dx_s + \emptyset \\ &= \sigma_1 e^{-a(u-s)} dW_{1,s} \end{aligned}$$

Or, equivalently, in integral form:

$$\int_t^u dF(x_s, s, u) = \sigma_1 \int_t^u e^{-a(u-s)} dW_{1,s}$$

Thus, we can conclude:

$$x_u = x_t e^{-a(u-t)} + \sigma_1 \int_t^u e^{-a(u-s)} dW_{1,s}$$
$$y_u = y_t e^{-b(u-t)} + \sigma_2 \int_t^u e^{-b(u-s)} dW_{2,s}$$

Therefore:

$$\begin{aligned} E[r_u | F_t] &= x_t e^{-a(u-t)} + y_t e^{-b(u-t)} + \varphi_t \\ Var[r_u | F_t] &= \sigma_1^2 \int_t^u e^{-2a(u-s)} ds + \sigma_2^2 \int_t^u e^{-2b(u-s)} ds + 2\, \sigma_1\, \sigma_2 \int_t^u e^{-(a+b)(u-s)} dW_{1,s} \cdot dW_{2,s} \\ &= \frac{\sigma_1^2}{2a} [1 - e^{-2a(u-t)}] + \frac{\sigma_2^2}{2b} [1 - e^{-2b(u-t)}] + 2\rho \frac{\sigma_1 \sigma_2}{a+b} [1 - e^{-(a+b)(u-t)}] \end{aligned}$$

II. Pricing of Zero Coupon Bond

1.1 Forward Bond Price

Define:

$$I(t,T)=\int_t^T (x_u+y_u|F_t)\,du$$

$$\begin{aligned}
\therefore I(t,T) &= \int_t^T (x_u+y_u|F_t)du \\
&= x_t\int_t^T e^{-a(u-t)}du+y_t\int_t^T e^{-b(u-t)}du+\sigma_1\int_t^T\int_t^u e^{-a(u-s)}dW_{1,s}du+\sigma_2\int_t^T\int_t^u e^{-b(u-s)}dW_{2,s}du \\
&= x_t\int_t^T e^{-a(u-t)}du+y_t\int_t^T e^{-b(u-t)}du+\sigma_1\int_t^T\int_s^T e^{-a(u-s)}du\,dW_{1,s}+\sigma_2\int_t^T\int_s^T e^{-b(u-s)}du\,dW_{2,s} \\
&= \frac{1-e^{-a(T-t)}}{a}x_t+\frac{1-e^{-b(T-t)}}{b}y_t+\frac{\sigma_1}{a}\int_t^T[1-e^{-a(T-s)}]dW_{1,s}+\frac{\sigma_2}{b}\int_t^T[1-e^{-b(T-s)}]dW_{2,s}
\end{aligned}$$

Let's denote:

$$\begin{aligned}
M(t,T) &\stackrel{\text{def}}{=} E[I(t,T)] \\
V(t,T) &\stackrel{\text{def}}{=} Var[I(t,T)]
\end{aligned}$$

Then, we should conclude:

$$\begin{aligned}
M(t,T) &= \frac{1-e^{-a(T-t)}}{a}x_t+\frac{1-e^{-b(T-t)}}{b}y_t \\
V(t,T) &\stackrel{\text{def}}{=} E[I(t,T)^2]-E[I(t,T)]^2 \\
&= E[I(t,T)^2]-M(t,T)^2 \\
&= \frac{\sigma_1^2}{a^2}\int_t^T[1-e^{-a(T-s)}]^2ds+\frac{\sigma_2^2}{b^2}\int_t^T[1-e^{-b(T-s)}]^2ds+2\rho\frac{\sigma_1\sigma_2}{ab}\int_t^T[1-e^{-a(T-s)}][1-e^{-b(T-s)}]ds \\
&= \frac{\sigma_1^2}{a^2}[T-t+\frac{2}{a}e^{-a(T-t)}-\frac{1}{2a}e^{-2a(T-t)}-\frac{3}{2a}] \\
&+ \frac{\sigma_2^2}{b^2}[T-t+\frac{2}{b}e^{-b(T-t)}-\frac{1}{2b}e^{-2b(T-t)}-\frac{3}{2b}] \\
&+ 2\rho\frac{\sigma_1\sigma_2}{ab}[T-t+\frac{e^{-a(T-t)}-1}{a}+\frac{e^{-b(T-t)}-1}{b}-\frac{e^{-(a+b)(T-t)}-1}{a+b}]
\end{aligned}$$

Therefore, forward bond price can be evaluated as:

$$\begin{aligned}
P(t,T) &= E(e^{-\int_t^T r_u du}|F_t) \\
&= E(e^{-\int_t^T x_u+y_u+\varphi_u du}|F_t) \\
&= \exp[-\int_t^T \varphi_u du-M(t,T)+\frac{1}{2}V(t,T)]
\end{aligned}$$

1.2 Tracking current yield curve φ_t

Since:

$$\begin{aligned} P^M(0,T) &= \exp[-\int_0^T \varphi_u\,du - M(0,T) + \frac{1}{2}V(0,T)] \\ &= \exp[-\int_0^T \varphi_u\,du - \cancel{\emptyset} + \frac{1}{2}V(0,T)] \\ &= \exp[-\int_0^T \varphi_u\,du + \frac{1}{2}V(0,T)] \end{aligned}$$

Therefore:

$$\exp(-\int_0^T \varphi_u\,du) = P^M(0,T)\exp[-\frac{1}{2}V(0,T)]$$

Or, equivalently, in differential form:

$$\begin{aligned} \varphi_T &= -\frac{\partial \ln P^M(0,T)}{\partial T} + \frac{1}{2}\times\frac{\partial V(0,T)}{\partial T} \\ &= f^M(0,T) + \frac{\sigma_1^2}{2a^2}(1-e^{-aT})^2 + \frac{\sigma_2^2}{2b^2}(1-e^{-bT})^2 + \rho\frac{\sigma_1\sigma_2}{ab}(1-e^{-aT})(1-e^{-bT}) \end{aligned}$$

1.3 Pricing Formula for Zero Coupon Bond:

Furthermore, we obtain:

$$\begin{aligned} \exp(-\int_t^T \varphi_u\,du) &= \frac{\exp(-\int_0^T \varphi_u\,du)}{\exp(-\int_0^t \varphi_u\,du)} \\ &= \frac{P^M(0,T)\exp[-\frac{1}{2}V(0,T)]}{P^M(0,t)\exp[-\frac{1}{2}V(0,t)]} \\ &= \frac{P^M(0,T)}{P^M(0,t)}\exp[\frac{V(0,t)-V(0,T)}{2}] \end{aligned}$$

Finally, we arrive at:

$$P(t,T) = \frac{P^M(0,T)}{P^M(0,t)}\exp[\frac{V(t,T)+V(0,t)-V(0,T)}{2} - M(t,T)]$$