G2++: Short Rate, Forward Dynamics and Proof
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Chapter 1. G2++ with constant volatility

I. Short Rate Dynamics Definition:

$$r_t = x_t + y_t + \varphi_t |\, r_0, x_0, y_0$$

With:

$$dx_t = -a x_t dt + \sigma_1 dW_{1,t}$$
$$dy_t = -b x_t dt + \sigma_2 dW_{2,t}$$

Where:

$$dW_{1,t} \cdot dW_{2,t} = \rho dt, -1 \le \rho \le 1$$
$$r_0, a, b, \sigma_1, \sigma_2 > 0$$
$$x_0, y_0 = 0$$

1.1 Short rate dynamics moments:

$Let: F(x_t, t, T) := x_t e^{-a(T-t)}$

$$\xRightarrow{Ito's\ Lemma} dF = \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial x} dx + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(dx)^2 = a e^{-a(T-t)} x_t dt + e^{-a(T-t)} dx + \emptyset$$
$$= a e^{-a(T-t)} x_t dt - a e^{-a(T-t)} x_t dt + \sigma_1 e^{-a(T-t)} dW_{1,t} = \sigma_1 e^{-a(T-t)} dW_{1,t}$$
$$\therefore \int_s^t dF(x_u, u, t) = \sigma_1 \int_s^t e^{-a(t-u)}\, dW_{1,u}$$
$$\therefore x_u e^{-a(t-u)}|_s^t = \sigma_1 \int_s^t e^{-a(t-u)}\, dW_{1,u}$$
$$\therefore x_t - x_s e^{-a(t-s)} = \sigma_1 \int_s^t e^{-a(t-u)}\, dW_{1,u}$$
$$\therefore x_t = x_s e^{-a(t-s)} + \sigma_1 \int_s^t e^{-a(t-u)}\, dW_{1,u}$$

Identically, we can conclude:

$$y_t = y_s e^{-b(t-s)} + \sigma_2 \int_s^t e^{-b(t-u)}\, dW_{2,u}$$

Hence,

$$r_u|\mathcal{F}_t = x_t e^{-a(u-t)} + y_t e^{-b(u-t)} + \sigma_1 \int_t^u e^{-a(u-s)}\, dW_{1,s} + \sigma_2 \int_t^u e^{-b(u-s)}\, dW_{2,s} + \varphi_t \ \mathbf{(1)}$$

1.2 Short rate dynamics moments:

$E[r_t|\mathcal{F}_s] = x_s e^{-a(t-s)} + y_s e^{-b(t-s)} + \varphi_t$

$$Var[r_t|\mathcal{F}_s] = E\left[\left(\sigma_1 \int_s^t e^{-a(t-u)}\, dW_{1,u}\right)^2\right] + E\left[\left(\sigma_2 \int_s^t e^{-b(t-u)}\, dW_{2,u}\right)^2\right]$$
$$+ E\left[2\left(\sigma_1 \int_s^t e^{-a(t-u)}\, dW_{1,u}\right)\left(\sigma_2 \int_s^t e^{-b(t-u)}\, dW_{2,u}\right)\right]$$
$$= \sigma_1^2 \int_s^t e^{-2a(t-u)}\, du + \sigma_2^2 \int_s^t e^{-2b(t-u)}\, du + 2\rho\sigma_1\sigma_2 \int_s^t e^{-(a+b)(t-u)} du$$
$$= \frac{\sigma_1^2}{2a}\left[1 - e^{-2a(t-s)}\right] + \frac{\sigma_2^2}{2b}\left[1 - e^{-2b(t-s)}\right] + 2\rho\frac{\sigma_1\sigma_2}{a+b}\left[1 - e^{-(a+b)(t-s)}\right]$$

II. Pricing of Zero Coupon Bond

Define:

$$I(t,T)=\int_t^T [x_u+y_u|\mathcal{F}_t]du$$

$$=\int_t^T x_u e^{-a(u-t)}du+\int_t^T y_u e^{-b(u-t)}du+\sigma_1\int_t^T\int_t^u e^{-a(u-s)}dW_{1,s}\,du+\sigma_2\int_t^T\int_t^u e^{-b(u-s)}dW_{2,s}\,d$$

$$=x_t\int_t^T e^{-a(u-t)}du+y_t\int_t^T e^{-b(u-t)}du+\sigma_1\int_t^T\int_t^u e^{-a(u-s)}dW_{1,s}\,du$$
$$+\sigma_2\int_t^T\int_t^u e^{-b(u-s)}dW_{2,s}\,du$$
$$=x_t\int_t^T e^{-a(u-t)}du+y_t\int_t^T e^{-b(u-t)}du+\sigma_1\int_t^T e^{as}\int_s^T e^{-au}\,du dW_{1,s}$$
$$+\sigma_2\int_t^T e^{bs}\int_s^T e^{-bu}\,du dW_{2,s}$$
$$=x_t\int_t^T e^{-a(u-t)}du+y_t\int_t^T e^{-b(u-t)}du+\sigma_1\int_t^T\frac{1-e^{-a(T-s)}}{a}dW_{1,s}+\sigma_2\int_t^T\frac{1-e^{-b(T-s)}}{b}dW_{2,s}$$

What if volatility is not constant but a time dependent function?

Note, in case σ_1,σ_2 are positive deterministic (continuous) function $\sigma_1(t),\sigma_2(t)$, we have:

$$I(t,T)=\int_t^T [x_u+y_u|\mathcal{F}_t]du$$

$$=\int_t^T x_t e^{-a(u-t)}du+\int_t^T y_t e^{-b(u-t)}du+\int_t^T\int_t^u \sigma_1(s)e^{-a(u-s)}dW_{1,s}\,du+\int_t^T\int_t^u \sigma_2(s)e^{-b(u-s)}dW_{2,s}\,du$$

Apply Fubini's Theorem:

$$I(t,T)=\int_t^T x_t e^{-a(u-t)}du+\int_t^T y_t e^{-b(u-t)}du+\int_t^T \sigma_1(s)e^{as}\int_s^T e^{-au}\,du dW_{1,s}+\int_t^T \sigma_2(s)e^{bs}\int_s^T e^{-bu}\,du dW_{2,s}$$
$$=\int_t^T x_t e^{-a(u-t)}du+\int_t^T y_t e^{-b(u-t)}du+\int_t^T\frac{\sigma_1(s)}{a}\left[1-e^{-a(T-s)}\right]dW_{1,s}$$
$$+\int_t^T\frac{\sigma_2(s)}{b}\left[1-e^{-b(T-s)}\right]dW_{2,s}$$

Let's denote the mean as $\boldsymbol{M(t,T)}$, variance as $\boldsymbol{V(t,T)}$:

$$M(t,T)=E[I(t,T)|\mathcal{F}_t]=\int_t^T E\left[x_t e^{-a(u-t)}|\mathcal{F}_t\right]du+\int_t^T E\left[y_t e^{-b(u-t)}|\mathcal{F}_t\right]du$$
$$=\frac{1-e^{-a(T-t)}}{a}x_t+\frac{1-e^{-b(T-t)}}{b}y_t\ \textbf{(2)}$$

$$
\begin{aligned}
V(t,T) &= Var[I(t,T)|\mathcal{F}_t] \\
&= E\left[\left(\frac{\sigma_1}{a}\int_t^T [1-e^{-a(T-s)}]dW_{1,s}\right)^2\right] + E\left[\left(\frac{\sigma_2}{b}\int_t^T [1-e^{-b(T-s)}]dW_{2,s}\right)^2\right] \\
&+ E\left[2\left(\frac{\sigma_1}{a}\int_t^T [1-e^{-a(T-s)}]dW_{1,s}\right)\left(\frac{\sigma_2}{b}\int_t^T [1-e^{-b(T-s)}]dW_{2,s}\right)\right] \\
&= \frac{\sigma_1^2}{a^2}\left[T-t+\frac{2}{a}e^{-a(T-t)}-\frac{1}{2a}e^{-2a(T-t)}-\frac{3}{2a}\right] + \frac{\sigma_2^2}{b^2}\left[T-t+\frac{2}{b}e^{-b(T-t)}-\frac{1}{2b}e^{-2b(T-t)}-\frac{3}{2b}\right] \\
&+ 2\rho\frac{\sigma_1\sigma_2}{ab}\left[T-t+\frac{e^{-a(T-t)}-1}{a}+\frac{e^{-b(T-t)}-1}{b}-\frac{e^{-(a+b)(T-t)}-1}{a+b}\right] \ (\mathbf{3})
\end{aligned}
$$

Therefore, we can formulate solution to evaluate bond price via Ito's Lemma:

$$
P(t,T) = E\left\{e^{-\int_t^T r_u du}|\mathcal{F}_t\right\} = E\left\{e^{-\left\{\int_t^T \varphi_u ds + \int_t^T [x_u+y_u]du\right\}}|\mathcal{F}_t\right\} = \exp\left[-\int_t^T \varphi_u\, ds - M(t,T) + \frac{1}{2}V(t,T)\right] \ (\mathbf{4})
$$

2.1 Tracking current market condition: φ_t

$$
\begin{aligned}
&\because (\mathbf{4}) \xrightarrow{yields} P^M(0,T) = \exp\left[-\int_0^T \varphi_s ds - M(0,T) + \frac{1}{2}V(0,T)\right] = \exp\left[-\int_0^T \varphi_s ds - \emptyset + \frac{1}{2}V(0,T)\right] \\
&= \exp\left[-\int_0^T \varphi_s ds + \frac{1}{2}V(0,T)\right] \\
&\therefore \exp\left\{-\int_0^T \varphi_s ds\right\} = P^M(0,T)\exp\left[-\frac{1}{2}V(0,T)\right]
\end{aligned}
$$

Or equivalently in differential form:

$$
\begin{aligned}
\varphi_T &= -\frac{\partial lnP^M(0,T)}{\partial T} + \frac{1}{2}\times\frac{\partial V(0,T)}{\partial T} \\
&= f^M(0,T) + \frac{\sigma_1^2}{2a^2}(1-e^{-aT})^2 + \frac{\sigma_2^2}{2b^2}(1-e^{-bT})^2 + \rho\frac{\sigma_1\sigma_2}{ab}(1-e^{-aT})(1-e^{-bT}) \\
\therefore \exp\left\{-\int_t^T \varphi_s ds\right\} &= \exp\left\{-\left[\int_0^T \varphi_s ds - \int_0^s \varphi_s ds\right]\right\} = \frac{P^M(0,T)\exp\left[-\frac{1}{2}V(0,T)\right]}{P^M(0,t)\exp\left[-\frac{1}{2}V(0,t)\right]} \\
&= \frac{P^M(0,T)}{P^M(0,t)}\exp\left\{\frac{1}{2}[V(0,t)-V(0,T)]\right\} \\
\therefore P^M(t,T) &= \exp\left[-\int_t^T \varphi_s\, ds - M(t,T) + \frac{1}{2}V(t,T)\right] \\
&= \frac{P^M(0,T)}{P^M(0,t)}\exp\left\{\frac{1}{2}[V(0,t)-V(0,T)]\right\}\cdot\exp\left[-M(t,T)+\frac{1}{2}V(t,T)\right] \\
&= \frac{P^M(0,T)}{P^M(0,t)}\exp\left\{\frac{1}{2}[V(t,T)-V(0,T)+V(0,t)] - M(t,T)\right\} \ (\mathbf{5})
\end{aligned}
$$

Chapter 2. G2++: Heath-Jarrow-Morton(HJM) Framework

I. Instantaneous forward rate definition:

$$f(t,T) := -\frac{\partial lnP(t,T)}{\partial T}$$

Thus, we have:

$$(\mathbf{4}) \xrightarrow{yields} f(t,T) = -\frac{\partial\left[-\int_t^T \varphi_s ds - M(t,T) + \frac{1}{2}V(t,T)\right]}{\partial T} == \varphi_T - \frac{1}{2}\frac{\partial V(t,T)}{\partial T} + \frac{\partial M(t,T)}{\partial T}$$

$$(\mathbf{2}) \xrightarrow{yields} \frac{\partial M(t,T)}{\partial T} = \frac{\partial\left[\frac{1-e^{-a(T-t)}}{a}x_t + \frac{1-e^{-b(T-t)}}{b}y_t\right]}{\partial T} = e^{-a(T-t)}x_t + e^{-b(T-t)}y_t$$

$$(\mathbf{3}) \xrightarrow{yields} \frac{\partial V(t,T)}{\partial T}|_{\mathcal{F}_t}$$

$$= \partial\left\{\frac{\sigma_1^2}{a^2}\left[T - t + \frac{2}{a}e^{-a(T-t)} - \frac{1}{2a}e^{-2a(T-t)} - \frac{3}{2a}\right]\right\}\Big/\partial T$$

$$+ \partial\left\{\frac{\sigma_1^2}{b^2}\left[T - t + \frac{2}{b}e^{-b(T-t)} - \frac{1}{2b}e^{-2b(T-t)} - \frac{3}{2b}\right]\right\}\Big/\partial T$$

$$+ \partial\left\{2\rho\frac{\sigma_1\sigma_2}{ab}\left[T - t + \frac{e^{-a(T-t)}-1}{a} + \frac{e^{-b(T-t)}-1}{b} - \frac{e^{-(a+b)(T-t)}-1}{a+b}\right]\right\}\Big/\partial T$$

$$= \frac{\sigma_1^2}{a^2}\left[1 - 2e^{-a(T-t)} + e^{-2a(T-t)}\right] + \frac{\sigma_2^2}{b^2}\left[1 - 2e^{-b(T-t)} + e^{-2b(T-t)}\right]$$
$$+ 2\rho\frac{\sigma_1\sigma_2}{ab}\left[1 - e^{-a(T-t)} - e^{-b(T-t)} + e^{-(a+b)(T-t)}\right]$$
$$= \frac{\sigma_1^2}{a^2}\left[1 - e^{-a(T-t)}\right]^2 + \frac{\sigma_2^2}{b^2}\left[1 - e^{-b(T-t)}\right]^2 + 2\rho\frac{\sigma_1\sigma_2}{ab}\left[1 - e^{-a(T-t)}\right]\left[1 - e^{-b(T-t)}\right]$$

Therefore,

$$\therefore f(t,T) = \varphi_T - \frac{1}{2}\frac{\partial V(t,T)}{\partial T} + \frac{\partial M(t,T)}{\partial T}$$
$$= \varphi_T - \frac{\sigma_1^2}{2a^2}\left[1 - e^{-a(T-t)}\right]^2 - \frac{\sigma_2^2}{2b^2}\left[1 - e^{-b(T-t)}\right]^2 - \rho\frac{\sigma_1\sigma_2}{ab}\left[1 - e^{-a(T-t)}\right]\left[1 - e^{-b(T-t)}\right]$$
$$+ e^{-a(T-t)}x_t + e^{-b(T-t)}y_t$$

II. Instantaneous forward rate dynamics:

Apply Ito's Lemma, again:

$$\therefore df(t,T)|_T^{\mathcal{F}_T} = \frac{\partial f}{\partial T}dT + \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{1}{2}\left[\frac{\partial^2 f}{\partial x^2}(dx)^2 + 2\times\frac{\partial^2 f}{\partial x \partial y}dxdy + \frac{\partial^2 f}{\partial y^2}(dy)^2\right]$$

$$= \left\{\frac{\partial \varphi_T}{\partial T} - \frac{\sigma_1^2}{a}\left[1-e^{-a(T-t)}\right]e^{-a(T-t)} - \frac{\sigma_1^2}{b}\left[1-e^{-b(T-t)}\right]e^{-b(T-t)}\right.$$
$$\left. - \rho\frac{\sigma_1\sigma_2}{ab}\left\{ae^{-a(T-t)}\left[1-e^{-b(T-t)}\right] + be^{-b(T-t)}\left[1-e^{-a(T-t)}\right]\right\}\right\}\cdot dT + \sigma_1 e^{-a(T-t)}dW_{1,T}$$
$$+ \sigma_2 e^{-b(T-t)}dW_{2,T}$$

$$= \left\{\frac{\partial \varphi_T}{\partial T} - \frac{\sigma_1^2}{a}\left[1-e^{-a(T-t)}\right]e^{-a(T-t)} - \frac{\sigma_1^2}{b}\left[1-e^{-b(T-t)}\right]e^{-b(T-t)}\right.$$
$$\left. - \rho\frac{\sigma_1\sigma_2}{ab}\left[ae^{-a(T-t)} + be^{-b(T-t)} - (a+b)e^{-(a+b)(T-t)}\right]\right\}\cdot dT + \sigma_1 e^{-a(T-t)}dW_{1,T}$$
$$+ \sigma_2 e^{-b(T-t)}dW_{2,T}$$

Thus, we conclude that:

$$df(T,T) = d\varphi_T + \sigma_1 dW_{1,T} + \sigma_2 dW_{2,T}$$

If we denote $f_t = df(t,t)$, then we have:

$$df_t = d\varphi_t + \sigma_1 dW_{1,t} + \sigma_2 dW_{2,t}$$

Where:

$$dW_{1,t}\cdot dW_{2,t} = \rho dt$$
$$-1 \le \rho \le 1$$
$$a, b, \sigma_1, \sigma_2 > 0$$
$$f_0 = f(0,0) = \lim_{S\to T+} F(0,T,S)|_{T=0} = r_0 = \varphi_0 = -\frac{\partial lnP^M(0,T)}{\partial T}|_{T=0};$$

$$f_T = f_t + \Delta\varphi|_t^T + \sigma_1\Delta W_1|_t^T + \sigma_1\Delta W_2|_t^T$$

Chapter 3. HG2++: Aqueous Mercury Stable Polyatomic Cation

Okay, fine. It is really just Heston enhanced G2++. Congratulations, you are boring.

I. Short Rate Dynamics Definition:

$$r_t = x_t + y_t + \varphi_t | \, r_0, x_0, y_0$$

With:

$$\begin{bmatrix} dx_t \\ dy_t \\ dv_{1,t} \end{bmatrix} = \begin{bmatrix} -ax_t \\ -by_t \\ k_1(\theta_1 - v_{1,t}) \end{bmatrix} \cdot dt + \begin{bmatrix} \sigma_{1,t} & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \eta_1\sqrt{v_{1,t}} \end{bmatrix} \cdot \begin{bmatrix} dW_{1,t} \\ dW_{2,t} \\ dZ_{1,t} \end{bmatrix}$$

Where:

$$\begin{bmatrix} dW_{1,t} \\ dW_{2,t} \\ dZ_{1,t} \end{bmatrix} \cdot \begin{bmatrix} dW_{1,t} \\ dW_{2,t} \\ dZ_{1,t} \end{bmatrix}^T = \begin{bmatrix} dt & \rho_{12}dt & \rho_{13}dt \\ \cdot & dt & \rho_{23}dt \\ \cdot & \cdot & dt \end{bmatrix} (symetric); -1 \le \rho_{ij} \le 1$$

$$v_{1,t} := \sigma_{1,t}^2; \; v_{1,t=0} := \sigma_1^2$$

$$r_0, a, b, \sigma_1, \sigma_2, \eta_1, \eta_2 \ge 0$$

$$x_0, y_0 = 0; \; \varphi_0 = f_0 = r_0;$$

1.1 Short rate dynamics:

Similar to Chapter 1, we have:

$$\begin{cases} d\left[e^{-a(T-t)}x_t\right] = \sigma_{1,t}e^{-a(T-t)}dW_{1,t} \\ d\left[e^{-b(T-t)}y_t\right] = \sigma_2 e^{-b(T-t)}dW_{1,t} \end{cases}$$

Or in integral form:

$$\begin{cases} x_t = x_s e^{-a(t-s)} + \int_s^t \sigma_{1,u}e^{-a(t-u)}\,dW_{1,u} \\ y_t = y_s e^{-b(t-s)} + \sigma_2\int_s^t e^{-b(t-u)}\,dW_{2,u} \end{cases}$$

$$r_t = x_s e^{-a(t-s)} + y_s e^{-a(t-s)} + \int_s^t \sigma_{1,u}e^{-a(t-u)}\,dW_{1,u} + \sigma_2\int_s^t e^{-b(t-u)}\,dW_{2,u} + \varphi_t$$

1.2 Short rate moments:

$$E[r_t|\mathcal{F}_s] = x_s e^{-a(t-s)} + y_s e^{-b(t-s)} + \varphi_t$$

$$\begin{aligned} Var[r_t|\mathcal{F}_s] &= E\left[\left(\int_s^t \sigma_{1,u}e^{-a(t-u)}\,dW_{1,u}\right)^2\right] + E\left[\left(\int_s^t \sigma_{2,u}e^{-b(t-u)}\,dW_{2,u}\right)^2\right] \\ &+ E\left[2\left(\int_s^t \sigma_{1,u}e^{-a(t-u)}\,dW_{1,u}\right)\left(\int_s^t \sigma_{2,u}e^{-b(t-u)}\,dW_{2,u}\right)\right] \\ &= \int_s^t E\left[\sigma_{1,t}^2|\mathcal{F}_s\right]e^{-2a(t-u)}\,du + \sigma_2^2\int_s^t e^{-2b(t-u)}\,du + 2\rho_{12}\sigma_2\int_s^t E\left[\sigma_{1,t}|\mathcal{F}_s\right]e^{-(a+b)(t-u)}\,du \end{aligned}$$

1.3 Variance dynamics:

$$Let: F_1\left(v_{1,t},t,T\right) := e^{-k_1(T-t)}v_{1,t} \overset{equivalent}{\Longleftrightarrow} v_{1,t} := F_1\left(v_{1,t},t,T\right)e^{k_1(T-t)}$$

$$\overset{Ito's\ Lemma}{\Longrightarrow} dF = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}dx + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(dx)^2$$

$$= k_1 v_{1,t}e^{-k_1(T-t)}dt + e^{-k_1(T-t)}\left[k_1\theta_1 dt - k_1 v_{1,t}dt + \eta_1\sqrt{v_{1,t}}dZ_{1,t}\right] + \emptyset$$

$$= k_1\theta_1 e^{-k_1(T-t)}dt + \eta_1\sqrt{v_{1,t}}e^{-k_1(T-t)}dZ_{1,t} = k_1\theta_1 e^{-k_1(T-t)}dt + \eta_1\sigma_{1,t}e^{-k_1(T-t)}dZ_{1,t}$$

Thus we have:

$$d\left[e^{-k_1(T-t)}v_{1,t}\right] = k_1\theta_1 e^{-k_1(T-t)}dt + \eta_1\sqrt{v_{1,t}}e^{-k_1(T-t)}dZ_{1,t}$$

Or in integral form:

$$v_{1,t} = v_{1,s}e^{-k_1(t-s)} + \theta_1\left[1 - e^{-k_1(t-s)}\right] + \eta_1\int_s^t v_{1,u}^{\frac{1}{2}}\, e^{-k_1(t-u)}\, dZ_{1,u}$$

1.4 Variance moments:

$$E\left[v_{1,t}\middle|\mathcal{F}_s\right] = v_{1,s}e^{-k_1(t-s)} + \theta_1\left[1 - e^{-k_1(t-s)}\right] = \left(v_{1,s} - \theta_1\right)e^{-k_1(t-s)} + \theta_1$$

$$Var\left[v_{1,t}\middle|\mathcal{F}_s\right] = E\left[\eta_1^2\left(\int_s^t v_{1,u}^{\frac{1}{2}}\, e^{-\frac{k_1}{2}(t-u)}\, dW_{1,u}\right)^2\right] = E\left[\eta_1^2\left(\int_s^t F_{1,u}^{\frac{1}{2}}\, e^{-\frac{k_1}{2}(t-u)}\, dW_{1,u}\right)^2\right]$$

$$= \eta_1^2\int_s^t E\left[F_{1,t}\middle|\mathcal{F}_s\right] e^{-k_1(t-u)}\, du = \eta_1^2\int_s^t \left\{v_{1,s} + \theta_1\left[1 - e^{-k_1(t-u)}\right]\right\} e^{-k_1(t-u)}\, du$$

$$= \eta_1^2\left[\int_s^t v_{1,s}e^{-k_1(t-u)}\, du + \theta_1\int_s^t e^{-k_1(t-u)}\, du - \theta_1\int_s^t e^{-2k_1(t-u)}\, du\right]$$

$$= \frac{\eta_1^2}{k_1}v_{1,s}\left[1 - e^{-k_1(t-s)}\right] + \frac{\eta_1^2\theta_1}{2k_1}\left\{2\left[1 - e^{-k_1(t-s)}\right] - \left[1 - e^{-2k_1(t-s)}\right]\right\}$$

$$= \frac{\eta_1^2}{k_1}v_{1,s}\left[1 - e^{-k_1(t-s)}\right] + \frac{\eta_1^2\theta_1}{2k_1}\left[1 - e^{-k_1(t-s)}\right]^2$$

1.4 Volatility Dynamics:

$Let: s(x_t, t, T) := v_t^{\frac{1}{2}} e^{-\frac{k_1}{2}(T-t)}; then\ v_t^{\frac{1}{2}} = s_t e^{\frac{k_1}{2}(T-t)}; v_t^{-\frac{1}{2}} = s_t^{-1} e^{-\frac{k_1}{2}(T-t)}$

Then, we have:

$$\frac{\partial s_t}{\partial v_{1,t}} = \frac{1}{2} v_{1,t}^{-\frac{1}{2}} e^{-\frac{k_1}{2}(T-t)}; \frac{\partial s_t}{\partial t} = \frac{1}{2} k_1 v_{1,t}^{\frac{1}{2}} e^{-\frac{k_1}{2}(T-t)}; \frac{\partial^2 s_t}{\partial v_{1,t}^2} = -\frac{1}{4} v_{1,t}^{-\frac{3}{2}} e^{-\frac{k_1}{2}(T-t)}$$

Recall:

$$dv_{1,t} = k_1(\theta_1 - v_{1,t})dt + \eta_1 \sqrt{v_{1,t}} dZ_{1,t}$$

Thus:

$$\begin{aligned} ds_t &= \frac{1}{2} v_t^{-\frac{1}{2}} e^{-\frac{k_1}{2}(T-t)} k_1 \theta_1 dt - \frac{1}{2} k_1 v_t^{-\frac{1}{2}} v_t e^{-\frac{k_1}{2}(T-t)} dt + \frac{1}{2} v_t^{-\frac{1}{2}} e^{-\frac{k_1}{2}(T-t)} \eta_1 \sqrt{v_{1,t}} dZ_{1,t} + \frac{1}{2} k_1 v_t^{\frac{1}{2}} e^{-\frac{k_1}{2}(T-t)} dt - \frac{1}{2} \\ &\quad \times \frac{1}{4} v_t^{-\frac{3}{2}} e^{-\frac{k_1}{2}(T-t)} \eta_1^2 v_{1,t} dt \\ &= \frac{1}{2} v_t^{-\frac{1}{2}} e^{-\frac{k_1}{2}(T-t)} k_1 \theta_1 dt - \frac{1}{8} v_t^{-\frac{3}{2}} e^{-\frac{k_1}{2}(T-t)} \eta_1^2 v_{1,t} dt + \frac{1}{2} v_t^{-\frac{1}{2}} e^{-\frac{k_1}{2}(T-t)} \eta_1 \sqrt{v_{1,t}} dZ_{1,t} \\ &= \frac{1}{2} v_t^{-\frac{1}{2}} e^{-\frac{k_1}{2}(T-t)} k_1 \theta_1 dt - \frac{1}{8} v_t^{-\frac{1}{2}} e^{-\frac{k_1}{2}(T-t)} \eta_1^2 dt + \frac{1}{2} e^{-\frac{k_1}{2}(T-t)} \eta_1 dZ_{1,t} \\ &= \frac{4k_1\theta_1 - \eta_1^2}{8} v_t^{-\frac{1}{2}} e^{-\frac{k_1}{2}(T-t)} dt + \frac{\eta_1}{2} e^{-\frac{k_1}{2}(T-t)} dZ_{1,t} \\ &= \frac{4k_1\theta_1 - \eta_1^2}{8} s_t^{-1} e^{-k_1(T-t)} dt + \frac{\eta_1}{2} e^{-\frac{k_1}{2}(T-t)} dZ_{1,t} \end{aligned}$$

$$\therefore \begin{cases} ds_t = \dfrac{(4k_1\theta_1 - \eta_1^2)}{8} s_t^{-1} e^{-k_1(T-t)} dt + \dfrac{\eta_1}{2} e^{-\frac{k_1}{2}(T-t)} dZ_{1,t} \\ d\left[v_t^{\frac{1}{2}} e^{-\frac{k_1}{2}(T-t)}\right] = \dfrac{(4k_1\theta_1 - \eta_1^2)}{8} v_t^{-\frac{1}{2}} e^{-\frac{k_1}{2}(T-t)} dt + \dfrac{\eta_1}{2} e^{-\frac{k_1}{2}(T-t)} dZ_{1,t} \end{cases}$$

$$\therefore v_{1,T}^{\frac{1}{2}} = v_{1,t}^{\frac{1}{2}} e^{-\frac{k_1}{2}(T-t)} + \frac{4k_1\theta_1 - \eta_1^2}{8} \int_t^T v_{1,t}^{-\frac{1}{2}} e^{-\frac{k_1}{2}(T-t)} dt + \frac{\eta_1}{2} \int_t^T e^{-\frac{k_1}{2}(T-t)} dZ_{1,t}$$

$$\therefore Var\left[v_{1,T}^{\frac{1}{2}}\right] = Var[\sigma_{1,T} | \mathcal{F}_t] \approx \frac{\eta_1^2}{4} \int_t^T e^{-k_1(T-t)} dt = \frac{\eta_1^2}{4k_1}\left[1 - e^{-k_1(T-t)}\right]$$

Therefore:

$$\because Var[\sigma_{1,t} | \mathcal{F}_s] := E[v_{1,t} | \mathcal{F}_s] - \{E[\sigma_{1,t} | \mathcal{F}_s]\}^2$$

And recall:

$$E[v_{1,T} | \mathcal{F}_t] := E[\sigma_{1,T}^2 | \mathcal{F}_t] = v_{1,t} e^{-k_1(T-t)} + \theta_1\left[1 - e^{-k_1(T-t)}\right]$$

$$\begin{aligned} \therefore E[\sigma_{1,T} | \mathcal{F}_t] &= \sqrt{E[\sigma_{1,T}^2 | \mathcal{F}_t] - Var[\sigma_{1,T} | \mathcal{F}_t]} \approx \sqrt{v_{1,t} e^{-k_1(T-t)} + \theta_1[1 - e^{-k_1(T-t)}] - \frac{\eta_1^2}{4k_1}[1 - e^{-k_1(T-t)}]} \\ &= \sqrt{v_{1,t} e^{-k_1(T-t)} + \left(\theta_1 - \frac{\eta_1^2}{4k_1}\right)[1 - e^{-k_1(T-t)}]} = \sqrt{\left(v_{1,t} + \frac{\eta_1^2}{4k_1} - \theta_1\right) e^{-k_1(T-t)} + \theta_1 - \frac{\eta_1^2}{4k_1}} \end{aligned}$$

II. Pricing of Zero Coupon Bond

Define:

$$I(t,T) = \int_t^T [x_T + y_T | \mathcal{F}_t] dT$$

$$= \int_t^T x_t e^{-a(T-t)} dT + \int_t^T y_t e^{-b(T-t)} dT + \int_t^T \int_t^T \sigma_{1,u} e^{-a(T-u)} dW_{1,u}\, dT + \int_t^T \int_t^T \sigma_{2,u} e^{-b(T-u)} dW_{2,u}\, dT$$

Let's denote the mean as $\boldsymbol{M(t,T)}$, variance as $\boldsymbol{V(t,T)}$:

$$M(t,T) = E[I(t,T)|\mathcal{F}_t] = \frac{1-e^{-a(T-t)}}{a} x_t + \frac{1-e^{-b(T-t)}}{b} y_t$$

$$V(t,T) = Var[I(t,T)|\mathcal{F}_t] = E\left[\left(\int_t^T \int_t^T \sigma_{1,u} e^{-a(T-u)} dW_{1,u}\, dT + \int_t^T \int_t^T \sigma_{2,u} e^{-b(T-u)} dW_{2,u}\, dT\right)^2\right]$$

$$= E\left[\left(\int_t^T \int_t^T \sigma_{1,u} e^{-a(T-u)}\, dT dW_{1,u} + \int_t^T \int_t^T \sigma_{2,u} e^{-b(T-u)}\, dT dW_{2,u}\right)^2\right]$$

$$= E\left[\left(\int_t^T \sigma_{1,u} e^{au} \int_t^T e^{-aT}\, dT dW_{1,u} + \int_t^T \sigma_{2,u} e^{bu} \int_t^T e^{-bT}\, dT dW_{2,u}\right)^2\right]$$

$$= E\left[\left(\int_t^T \frac{\sigma_{1,u}}{a}\left[1-e^{-a(T-u)}\right] dW_{1,u} + \int_t^T \frac{\sigma_{2,u}}{b}\left[1-e^{-b(T-u)}\right] dW_{2,u}\right)^2\right]$$

$$= E\left[\int_t^T \frac{\sigma_{1,u}^2}{a^2}\left[1-e^{-a(T-u)}\right]^2 du + \int_t^T \frac{\sigma_{2,u}^2}{b^2}\left[1-e^{-b(T-u)}\right]^2 du\right]$$

$$+ E\left\{2\frac{\rho_{12}}{ab}\int_t^T \sigma_{1,u}\sigma_{2,u}\left[1-e^{-a(T-u)}\right]\left[1-e^{-b(T-u)}\right] du\right\}$$

$$\therefore V(t,T) = Var[I(t,T)|\mathcal{F}_t]$$

$$= \int_t^T \frac{E\left[\sigma_{1,u}^2|\mathcal{F}_t\right]}{a^2}\left[1-e^{-a(T-u)}\right]^2 du + \int_t^T \frac{E\left[\sigma_{2,u}^2|\mathcal{F}_t\right]}{b^2}\left[1-e^{-b(T-u)}\right]^2 du$$

$$+ 2\frac{\rho_{12}}{ab}\int_t^T E\left[\sigma_{1,u}\sigma_{2,u}|\mathcal{F}_t\right]\left[1-e^{-a(T-u)}\right]\left[1-e^{-b(T-u)}\right]du$$

$$= \int_t^T \frac{E\left[v_{1,u}|\mathcal{F}_t\right]}{a^2}\left[1-e^{-a(T-u)}\right]^2 du + \int_t^T \frac{E\left[v_{2,u}|\mathcal{F}_t\right]}{b^2}\left[1-e^{-b(T-u)}\right]^2 du$$

$$+ 2\frac{\rho_{12}}{ab}\int_t^T E\left[\sigma_{1,u}\sigma_{2,u}|\mathcal{F}_t\right]\left[1-e^{-a(T-u)}\right]\left[1-e^{-b(T-u)}\right]du$$

$$= \int_t^T \frac{E\left[v_{1,u}|\mathcal{F}_t\right]}{a^2}\left[1-e^{-a(T-u)}\right]^2 du + \frac{\sigma_2^2}{b^2}\int_t^T \left[1-e^{-b(T-u)}\right]^2 du$$

$$+ 2\frac{\rho_{12}\sigma_2}{ab}\int_t^T E\left[\sigma_{1,u}|\mathcal{F}_t\right]\left[1-e^{-a(T-u)}\right]\left[1-e^{-b(T-u)}\right]du$$

$$= \frac{1}{a^2}\int_t^T \left\{\left(v_{1,t}-\theta_1\right)e^{-k_1(T-t)}+\theta_1\right\}\left[1-e^{-a(T-u)}\right]^2 du$$

$$+ \frac{\sigma_2^2}{b^2}\left[T-t+\frac{2}{b}e^{-b(T-t)}-\frac{1}{2b}e^{-2b(T-t)}-\frac{3}{2b}\right]$$

$$+ 2\frac{\rho_{12}\sigma_2}{ab}\int_t^T \left\{\left[1-e^{-a(T-u)}-e^{-b(T-u)}+e^{-(a+b)(T-u)}\right]\right.$$

$$\left.\times\sqrt{\left(v_{1,t}+\frac{\eta_1^2}{4k_1}-\theta_1\right)e^{-k_1(T-t)}+\theta_1-\frac{\eta_1^2}{4k_1}}\right\}du\,\frac{\theta_1}{a^2}\left[T-t+\frac{2}{a}e^{-a(T-t)}-\frac{1}{2a}e^{-2a(T-t)}-\frac{3}{2a}\right]$$

$$+ \frac{\left(v_{1,t}-\theta_1\right)}{a^2}\left[\frac{1-e^{-k(T-t)}}{k}-2\frac{1-e^{-(a+k)(T-t)}}{a+k}+\frac{1-e^{-(2a+k)(T-t)}}{2a+k}\right]$$

$$+ \frac{\sigma_2^2}{b^2}\left[T-t+\frac{2}{b}e^{-b(T-t)}-\frac{1}{2b}e^{-2b(T-t)}-\frac{3}{2b}\right]$$

$$+ 2\frac{\rho_{12}\sigma_2}{ab}\int_t^T \left\{\left[1-e^{-a(T-u)}-e^{-b(T-u)}+e^{-(a+b)(T-u)}\right]\right.$$

$$\left.\times\sqrt{\left(v_{1,t}+\frac{\eta_1^2}{4k_1}-\theta_1\right)e^{-k_1(T-t)}+\theta_1-\frac{\eta_1^2}{4k_1}}\right\}du$$

2.1 Important Quantity:

$E[x_T|\mathcal{F}_t] = x_t e^{-a(T-t)}$

$E[y_T|\mathcal{F}_t] = y_t e^{-b(T-t)}$

$$
\begin{aligned}
Var[x_T|\mathcal{F}_t] &= \int_t^T E[\sigma_1^2|\mathcal{F}_t] e^{-2a(T-t)}\, dt \\
&= \int_t^T \frac{\eta_1^2}{k_1} v_{1,t} e^{-2a(T-t)} \left[1 - e^{-k_1(T-t)}\right] dt + \int_t^T \frac{\eta_1^2 \theta_1}{2k_1} e^{-2a(T-t)} \left[1 - e^{-k_1(T-t)}\right]^2 dt \\
&= \int_t^T \frac{\eta_1^2}{k_1} v_{1,t} \left[e^{-2a(T-t)} - e^{-(2a+k_1)\,(T-t)}\right] dt + \int_t^T \frac{\eta_1^2 \theta_1}{2k_1} e^{-2a(T-t)} \left[1 - e^{-k_1(T-t)}\right]^2 dt \\
&= \frac{\eta_1^2 v_{1,t}}{2ak_1} \left[1 - e^{-2a(T-t)}\right] - \frac{\eta_1^2 v_{1,t}}{(2a + k_1)k_1} \left[1 - e^{-(2a+k_1)\,(T-t)}\right] \\
&+ \int_t^T \frac{\eta_1^2 \theta_1}{2k_1} \left[e^{-2a(T-t)} - 2e^{-(2a+k_1)(T-t)} + e^{-2(a+k_1)(T-t)}\right] dt \\
&= \frac{\eta_1^2 v_{1,t}}{2ak_1} \left[1 - e^{-2a(T-t)}\right] - \frac{\eta_1^2 v_{1,t}}{(2a + k_1)k_1} \left[1 - e^{-(2a+k_1)\,(T-t)}\right] + \frac{\eta_1^2 \theta_1}{4ak_1} \left[1 - e^{-2a(T-t)}\right] \\
&- \frac{\eta_1^2 \theta_1}{(2a + k_1)k_1} \left[1 - e^{-(2a+k_1)(T-t)}\right] + \frac{\eta_1^2 \theta_1}{4(a + k_1)k_1} \left[1 - e^{-2(a+k_1)(T-t)}\right]
\end{aligned}
$$

$$Var[y_T|\mathcal{F}_t] = \sigma_2^2 \int_t^T e^{-2b(T-t)}\, dt = \frac{\sigma_2^2}{2b} \left[1 - e^{-2b(T-t)}\right]$$

$$M(t,T) = E[I(t,T)|\mathcal{F}_t] = \frac{1 - e^{-a(T-t)}}{a} x_t + \frac{1 - e^{-b(T-t)}}{b} y_t$$

$$V(t,T) = Var[I(t,T)|\mathcal{F}_t]$$

2.2 Zero Coupon Bond Formula:

Additionally, we have (Please refer to Chapter 1):

$$
\begin{aligned}
P^M(t,T) &= \exp\left[-\int_t^T \varphi_s\, ds - M(t,T) + \frac{1}{2} V(t,T)\right] \\
&= \frac{P^M(0,T)}{P^M(0,t)} \exp\left\{\frac{1}{2} [V(0,t) - V(0,T)]\right\} \cdot \exp\left[-M(t,T) + \frac{1}{2} V(t,T)\right] \\
&= \frac{P^M(0,T)}{P^M(0,t)} \exp\left\{\frac{1}{2} [V(t,T) - V(0,T) + V(0,t)] - M(t,T)\right\}
\end{aligned}
$$

Where:

$$M(t,T) = E[I(t,T)|\mathcal{F}_t] = \frac{1 - e^{-a(T-t)}}{a} x_t + \frac{1 - e^{-b(T-t)}}{b} y_t$$

With:

$$\begin{cases} x_t = x_s e^{-a(t-s)} + \int_s^t \sigma_{1,u} e^{-a(t-u)}\, dW_{1,u} \\ y_t = y_s e^{-b(t-s)} + \sigma_2 \int_s^t e^{-b(t-u)}\, dW_{2,u} \\ x_T|\mathcal{F}_t \sim N\left(x_t e^{-a(T-t)}, Var[x_T|\mathcal{F}_t]\right) \\ y_T|\mathcal{F}_t \sim N(y_t e^{-a(T-t)}, \frac{\sigma_2^2}{2b}\left[1-e^{-2b(t-u)}\right]) \end{cases}$$

And:

$$V(t,T) = Var[I(t,T)|\mathcal{F}_t]$$

With:

$$\begin{cases} v_{1,t} = v_{1,s} e^{-k_1(t-s)} + \theta_1\left[1-e^{-k_1(t-s)}\right] + \eta_1 \int_s^t v_{1,u}^{\frac{1}{2}}\, e^{-k_1(t-u)}\, dZ_{1,u} \\ v_{1,T}|\mathcal{F}_t \sim N\left(v_{1,t} e^{-k_1(T-t)} + \theta_1\left[1-e^{-k_1(T-t)}\right], \frac{\eta_1^2}{k_1} v_{1,t}\left[1-e^{-k_1(T-t)}\right] + \frac{\eta_1^2\theta_1}{2k_1}\left[1-e^{-k_1(T-t)}\right]^2\right) \end{cases}$$

Here, we recall again, by definition:

$$v_{1,t} := \sigma_{1,t}^2; v_{2,t} := \sigma_{2,t}^2$$

Chapter 4. CHICAGO 2++

Conditional Heteroskedastic Instantaneous Correlation Augmented Gaussian Opacity 2++

I. Short Rate Dynamics Formulation:

$$r_t = x_t + y_t + \varphi_t | \, r_0, x_0, y_0$$

With:

$$\begin{bmatrix} dx_t \\ dy_t \\ dv_{1,t} \end{bmatrix} = \begin{bmatrix} -ax_t \\ -by_t \\ 2k_1\theta_1\sigma_{1,t} - (2k_1 - \eta_1^2)v_t \end{bmatrix} \cdot dt + \begin{bmatrix} \sigma_{1,t} & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 2\eta_1 v_{1,t} \end{bmatrix} \cdot \begin{bmatrix} dW_{1,t} \\ dW_{2,t} \\ dW_{3,t} \end{bmatrix} \; (\mathbf{4.1.0})$$

And:

$$v_t := \sigma_{1,t}^2; \; \sigma_{1,t_0} := \sigma_1$$
$$\rho_{t_0} := \rho_0$$

Where

$$\begin{bmatrix} dW_{1,t} \\ dW_{2,t} \\ dW_{3,t} \end{bmatrix} \cdot \begin{bmatrix} dW_{1,t} \\ dW_{2,t} \\ dW_{3,t} \end{bmatrix}^T = \begin{bmatrix} dt & \rho_t dt & \rho_{13} dt \\ \cdot & dt & \rho_{23} dt \\ \cdot & \cdot & dt \end{bmatrix} (symetric); -1 \le \rho_{ij} \le 1$$

Also, correlation between (dx_t, dy_t) follows **van Emmerich** process:

$$d\rho_t = \lambda(\rho_\vartheta - \rho_t)dt + \xi\sqrt{1-\rho_t^2}dZ_t \; (\mathbf{4.1.1})$$

With:

$$dZ_t \cdot \begin{bmatrix} dW_{1,t} \\ dW_{2,t} \\ dW_{3,t} \end{bmatrix} = \begin{bmatrix} 0 \cdot dt \\ 0 \cdot dt \\ 0 \cdot dt \end{bmatrix}$$

And,

$$a, \sigma_1 \ge 0; \; b, \sigma_2 \ge 0$$
$$k_1, \theta_1, \eta_1 \ge 0$$
$$\lambda, \xi \ge 0$$
$$-1 \le \rho_0, \rho_\vartheta \le 1$$
$$x_0, y_0 = 0; \; \varphi_0 = f_0 = r_0;$$
$$\eta_1^2 \le 2k_1;$$

II. Dynamics Breakdown

2.1 Factorial correlation

$$\mathbf{4.1.1} \xrightarrow{\textit{yields}} \begin{cases} d\left[\rho_t e^{-\lambda(T-t)}\right] = \lambda\rho_\vartheta e^{-\lambda(T-t)}dt + \xi\sqrt{1-\rho_t^2}e^{-\lambda(T-t)}dZ_t \\ \rho_T = \rho_t e^{-\lambda(T-t)} + \rho_\vartheta\left[1 - e^{-\lambda(T-t)}\right] + \xi\int_t^T \sqrt{1-\rho_t^2}e^{-\lambda(T-t)}dZ_t \; (\mathbf{4.2.0}) \\ d[\rho_t^2] = [\xi^2 - (2\lambda + \xi^2)\rho_t^2]dt + 2\lambda\rho_\vartheta\rho_t dt + 2\xi\rho_t\sqrt{1-\rho_t^2}dZ_t \end{cases}$$

$$\therefore d\left[\rho_t^2 e^{-(2\lambda+\xi^2)(T-t)}\right]$$
$$= \xi^2 e^{-(2\lambda+\xi^2)(T-t)}dt + 2\lambda\rho_\vartheta\rho_t e^{-(2\lambda+\xi^2)(T-t)}dt + 2\xi\sqrt{\rho_t^2 - (\rho_t^2)^2}e^{-(2\lambda+\xi^2)(T-t)}dZ_t \; (\mathbf{4.2.1})$$

2.2 Variance and volatility:

$$\because dv_{1,t} = \left[2k_1\theta_1\sigma_{1,t} - (2k_1 - \eta_1^2)v_t\right]dt + 2\eta_1 v_{1,t} dW_{3,t}$$

$$\therefore \begin{cases} d\sigma_{1,t} = k_1\left(\theta_1 - \sigma_{1,t}\right)dt + \eta_1\sigma_{1,t}dW_{3,t} \\ d\left[\sigma_{1,t}e^{-k_1(T-t)}\right] = k_1\theta_1 e^{-k_1(T-t)}dt + \eta_1\sigma_{1,t}e^{-k_1(T-t)}dW_{3,t} \\ \sigma_{1,T} = \sigma_{1,t}e^{-k_1(T-t)} + \theta_1\left[1 - e^{-k_1(T-t)}\right] + \eta_1\int_t^T \sigma_{1,t}e^{-k_1(T-t)}dW_{3,t} \end{cases} \quad (\mathbf{4.2.2})$$

Define:

$$V_t := v_t e^{-(2k_1 - \eta^2)(T-t)}$$

With Ito's Lemma, we have:

$$d\left[v_t e^{-(2k_1-\eta_1^2)(T-t)}\right] = 2k_1\theta_1\sigma_{1,t}e^{-(2k_1-\eta_1^2)(T-t)}dt + 2\eta_1 v_t e^{-(2k_1-\eta_1^2)(T-t)}dW_{3,t} \; (\mathbf{4.2.3})$$

Thus,

$$\therefore v_T = v_t e^{-(2k-\eta_1^2)(T-t)} + 2k_1\theta_1\int_t^T \sigma_{1,t}e^{-(2k_1-\eta_1^2)(T-t)}dt + 2\eta_1\int_t^T v_t e^{-(2k_1-\eta_1^2)(T-t)}dW_{3,t} \; (\mathbf{4.2.4})$$

We can also easily see that:

(1) Normal Market Condition: $\eta_1^2 < 2k_1$
Variance of risk premium factor is Ornstein-Uhlenbeck GARCH-M(1,1) process, with standard deviation converge to $\theta_1$at the speed of $(2k_1 - \eta_1^2)$.

Intuition 1: low level of shocks to variance (η) means convergence to market equilibrium happens faster, while underlying volatility converges to $\theta_1$at the speed of k_1.

$$\begin{cases} dv_{1,t} = \left[2k_1\theta_1\sigma_{1,t} - (2k_1 - \eta_1^2)v_t\right]dt + 2\eta_1 v_{1,t}dW_{3,t} \\ d\sigma_{1,t} = k_1\left(\theta_1 - \sigma_{1,t}\right)dt + \eta_1\sigma_{1,t}dW_{3,t} \end{cases}$$

Intuition 2: In long run, as $T \to \infty$, we have:

$$\begin{cases} dv_t \to \eta_1^2 v_t dt + 2\eta_1 v_t dW_{3,t} \\ d[\ln(v_t)] \to -\eta_1^2 dt + 2\eta_1 dW_{3,t} \\ d\sigma_{1,t} = k_1\left(\theta_1 - \sigma_{1,t}\right)dt + \eta_1\sigma_{1,t}dW_{3,t} \end{cases}$$

Model allows variance structure exhibit upward hump(or not) in short term, while still converge to stable long term equilibrium, which is precisely what has been observed in normal market(or financial crisis).

(2) Special case: $\eta_1^2 = 2k_1$
This signifies the beginning of Martians invasion, volatility maintains initial level over time and does not converge towards long term equilibrium.

(3)Apocalypse: $\eta_1^2 > 2k_1$
Intuition: high volatility of volatility (η) pushes the market equilibrium upwards, making volatility increase over time.

Therefore, we can conclude that features of this variance model are desirable as they comply with market volatility observations both in normal condition and financial crisis.

2.3 Short rate & factors:

$Let: F(x_t, t, T) := x_t e^{-a(T-t)}$

$$\xrightarrow{Ito's\ Lemma} dF = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}dx + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(dx)^2 = ae^{-a(T-t)}x_t dt + e^{-a(T-t)}dx + \emptyset$$

$$= ae^{-a(T-t)}x_t dt - ae^{-a(T-t)}x_t dt + \sigma_1 e^{-a(T-t)}dW_{1,t} = \sigma_1 e^{-a(T-t)}dW_{1,t}$$

$$\therefore \int_s^t dF(x_u, u) = \sigma_1 \int_s^t e^{-a(t-u)}\, dW_{1,u}$$

$$\therefore x_u e^{-a(t-u)}|_s^t = \sigma_1 \int_s^t e^{-a(t-u)}\, dW_{1,u}$$

$$\therefore x_t - x_s e^{-a(t-s)} = \sigma_1 \int_s^t e^{-a(t-u)}\, dW_{1,u}$$

$$\therefore x_t = x_s e^{-a(t-s)} + \sigma_1 \int_s^t e^{-a(t-u)}\, dW_{1,u}$$

Identically, we can conclude:

$$y_t = y_s e^{-b(t-s)} + \sigma_2 \int_s^t e^{-b(t-u)}\, dW_{2,u}$$

Thus, we have:

$$r_t = x_s e^{-a(t-s)} + y_s e^{-a(t-s)} + \int_s^t \sigma_{1,u} e^{-a(t-u)}\, dW_{1,u} + \sigma_2 \int_s^t e^{-b(t-u)}\, dW_{2,u} + \varphi_t$$

With

$$\begin{cases} \begin{cases} d\left[x_t e^{-a(T-t)}\right] = \sigma_{1,t} e^{-a(T-t)} dW_{1,t} \\ x_T = x_t e^{-a(T-t)} + \int_t^T \sigma_{1,t} e^{-a(T-t)}\, dW_{1,t} \end{cases} \\ \begin{cases} d\left[y_t e^{-b(T-t)}\right] = \sigma_2 e^{-b(T-t)} dW_{1,t} \\ y_T = y_t e^{-b(T-t)} + \sigma_2 \int_t^T e^{-b(T-t)}\, dW_{2,t} \end{cases} \end{cases} \quad \mathbf{(4.2.5)}$$

III. Moments:

3.1 Factorial correlation

$E[\rho_T|\mathcal{F}_t] = \rho_t e^{-\lambda(T-t)} + \theta_c\left[1 - e^{-\lambda(T-t)}\right] = (\rho_t - \rho_\vartheta)e^{-\lambda(T-t)} + \rho_\vartheta$

$\mathbf{4.2.5} \xrightarrow{yields} \rho_T^2$

$$= \rho_t^2 e^{-(2\lambda+\xi^2)(T-t)} + \frac{\xi^2}{2\lambda+\xi^2}\left[1 - e^{-(2\lambda+\xi^2)(T-t)}\right] + 2\lambda\rho_\vartheta \int_t^T \rho_t e^{-(2\lambda+\xi^2)(T-t)}dt + 2\xi \int_t^T \sqrt{\rho_t^2 - (\rho_t^2)^2} e^{-(2\lambda+\xi^2)(T-t)}dZ_t$$

$$\therefore E[\rho_T^2|\mathcal{F}_t] = \rho_t^2 e^{-(2\lambda+\xi^2)(T-t)} + \frac{\xi^2}{2\lambda+\xi^2}\left[1 - e^{-(2\lambda+\xi^2)(T-t)}\right] + 2\lambda\rho_\vartheta \int_t^T E[\rho_T|\mathcal{F}_t] e^{-(2\lambda+\xi^2)(T-t)}dt$$

$$= \rho_t^2 e^{-(2\lambda+\xi^2)(T-t)} + \frac{\xi^2}{2\lambda+\xi^2}\left[1 - e^{-(2\lambda+\xi^2)(T-t)}\right] + 2\lambda\rho_\vartheta \int_t^T \left[(\rho_t - \rho_\vartheta)e^{-\lambda(T-t)} + \rho_\vartheta\right] e^{-(2\lambda+\xi^2)(T-t)}dt$$

$$\therefore E[\rho_T^2|\mathcal{F}_t] = \rho_t^2 e^{-(2\lambda+\xi^2)(T-t)} + \frac{\xi^2 + 2\lambda\rho_\vartheta^2}{2\lambda+\xi^2}\left[1 - e^{-(2\lambda+\xi^2)(T-t)}\right] + \frac{2\lambda\rho_\vartheta(\rho_t - \rho_\vartheta)}{3\lambda+\xi^2}\left[1 - e^{-(3\lambda+\xi^2)(T-t)}\right]$$

$$\therefore Var[\rho_T|\mathcal{F}_t] = E\left[\left(\xi \int_t^T \sqrt{1-\rho_t^2}\, e^{-\lambda(T-t)}dZ_t\right)^2\right] = \xi^2 \int_t^T E[(1-\rho_T^2)|\mathcal{F}_t] e^{-2\lambda(T-t)}dt$$

$$= \frac{\xi^2}{2\lambda}\left[1 - e^{-2\lambda(T-t)}\right] - \xi^2 \int_t^T E[\rho_T^2|\mathcal{F}_t] e^{-2\lambda(T-t)}dt$$

$$= \frac{\xi^2}{2\lambda}\left[1 - e^{-2\lambda(T-t)}\right] - \xi^2 \int_t^T \left\{\rho_t^2 e^{-(4\lambda+\xi^2)(T-t)} + \frac{\xi^2 + 2\lambda\rho_\vartheta^2}{2\lambda+\xi^2}\left[e^{-2\lambda(T-t)} - e^{-(4\lambda+\xi^2)(T-t)}\right] + \frac{2\lambda\rho_\vartheta(\rho_t - \rho_\vartheta)}{3\lambda+\xi^2}\left[e^{-2\lambda(T-t)} - e^{-(5\lambda+\xi^2)(T-t)}\right]\right\}dt$$

$$= \xi^2 \left\{\frac{1}{2\lambda}\left[1 - e^{-2\lambda(T-t)}\right] - \frac{\rho_t^2}{4\lambda+\xi^2}\left[1 - e^{-(4\lambda+\xi^2)(T-t)}\right] - \frac{\xi^2 + 2\lambda\rho_\vartheta^2}{(2\lambda+\xi^2)2\lambda}\left[1 - e^{-2\lambda(T-t)}\right] + \frac{\xi^2 + 2\lambda\rho_\vartheta^2}{(2\lambda+\xi^2)(4\lambda+\xi^2)}\left[1 - e^{-(4\lambda+\xi^2)(T-t)}\right] - \frac{2\lambda\rho_\vartheta(\rho_t - \rho_\vartheta)}{(3\lambda+\xi^2)2\lambda}\left[1 - e^{-2\lambda(T-t)}\right] + \frac{2\lambda\rho_\vartheta(\rho_t - \rho_\vartheta)}{(3\lambda+\xi^2)(5\lambda+\xi^2)}\left[1 - e^{-(5\lambda+\xi^2)(T-t)}\right]\right\}$$

$$\therefore Var[\rho_T|\mathcal{F}_t] = \xi^2 \cdot \left\{\left[1 - e^{-2\lambda(T-t)}\right]\left[\frac{1}{2\lambda} - \frac{\xi^2 + 2\lambda\rho_\vartheta^2}{(2\lambda+\xi^2)2\lambda} - \frac{2\lambda\rho_\vartheta(\rho_t - \rho_\vartheta)}{(3\lambda+\xi^2)2\lambda}\right] + \left[1 - e^{-(4\lambda+\xi^2)(T-t)}\right]\left[\frac{\xi^2 + 2\lambda\rho_\vartheta^2}{(2\lambda+\xi^2)(4\lambda+\xi^2)} - \frac{\rho_t^2}{4\lambda+\xi^2}\right] + \left[1 - e^{-(5\lambda+\xi^2)(T-t)}\right]\frac{2\lambda\rho_\vartheta(\rho_t - \rho_\vartheta)}{(3\lambda+\xi^2)(5\lambda+\xi^2)}\right\}$$

3.2 Variance and volatility

$$E\left[\sigma_{1,T}|\mathcal{F}_t\right] = \sigma_{1,t}e^{-k_1(T-t)} + \theta_1\left[1 - e^{-k_1(T-t)}\right] = \left(\sigma_{1,t} - \theta_1\right)e^{-k_1(T-t)} + \theta_1$$

$$\because E[v_T|\mathcal{F}_t] = v_t e^{-\left(2k_1-\eta_1^2\right)(T-t)} + 2k_1\theta_1\int_t^T E\left[\sigma_{1,T}|\mathcal{F}_t\right]e^{-\left(2k_1-\eta_1^2\right)(T-t)}dt$$

$$= v_t e^{-\left(2k_1-\eta_1^2\right)(T-t)} + 2k_1\theta_1\int_t^T \left[\left(\sigma_{1,t} - \theta_1\right)e^{-k_1(T-t)} + \theta_1\right]e^{-\left(2k_1-\eta_1^2\right)(T-t)}dt$$

$$\therefore E[v_T|\mathcal{F}_t] = v_t e^{-\left(2k-\eta_1^2\right)(T-t)} + 2k_1\theta_1\left(\sigma_{1,t} - \theta_1\right)\frac{1 - e^{-\left(3k_1-\eta_1^2\right)(T-t)}}{3k_1 - \eta_1^2} + 2k_1\theta_1^2\frac{1 - e^{-\left(2k_1-\eta_1^2\right)(T-t)}}{2k_1 - \eta_1^2}$$

$$\because Var\left[\sigma_{1,T}|\mathcal{F}_t\right] = \eta_1^2\int_t^T E\left[v_{1,T}|\mathcal{F}_t\right]e^{-2k_1(T-t)}dt$$

$$= \eta_1^2\int_t^T \left\{v_t e^{-\left(2k_1-\eta_1^2\right)(T-t)} + \frac{2k_1\theta_1\left(\sigma_{1,t} - \theta_1\right)}{3k_1 - \eta_1^2}\left[1 - e^{-\left(3k_1-\eta_1^2\right)(T-t)}\right]\right.$$
$$\left. + \frac{2k_1\theta_1^2}{2k_1 - \eta_1^2}\left[1 - e^{-\left(2k_1-\eta_1^2\right)(T-t)}\right]\right\}e^{-2k_1(T-t)}dt$$
$$= \eta_1^2\int_t^T \left\{v_t e^{-\left(4k_1-\eta_1^2\right)(T-t)} + \frac{2k_1\theta_1\left(\sigma_{1,t} - \theta_1\right)}{3k_1 - \eta_1^2}\left[e^{-2k_1(T-t)} - e^{-\left(5k_1-\eta_1^2\right)(T-t)}\right]\right.$$
$$\left. + \frac{2k_1\theta_1^2}{2k_1 - \eta_1^2}\left[e^{-2k_1(T-t)} - e^{-\left(4k_1-\eta_1^2\right)(T-t)}\right]\right\}dt$$
$$= \frac{\eta_1^2 v_t}{4k_1 - \eta_1^2}\left[1 - e^{-\left(4k_1-\eta_1^2\right)(T-t)}\right] + \frac{\eta_1^2 2k_1\theta_1\left(\sigma_{1,t} - \theta_1\right)}{\left(3k_1 - \eta_1^2\right)2k_1}\left[1 - e^{-2k_1(T-t)}\right]$$
$$- \frac{\eta_1^2 2k_1\theta_1\left(\sigma_{1,t} - \theta_1\right)}{\left(3k_1 - \eta_1^2\right)\left(5k_1 - \eta_1^2\right)}\left[1 - e^{-\left(5k_1-\eta_1^2\right)(T-t)}\right] + \frac{\eta_1^2 2k_1\theta_1^2}{\left(2k_1 - \eta_1^2\right)2k_1}\left[1 - e^{-2k_1(T-t)}\right]$$
$$- \frac{\eta_1^2 2k_1\theta_1^2}{\left(2k_1 - \eta_1^2\right)\left(4k_1 - \eta_1^2\right)}\left[1 - e^{-\left(4k_1-\eta_1^2\right)(T-t)}\right]$$

$$\therefore Var\left[\sigma_{1,T}|\mathcal{F}_t\right]$$
$$= \eta_1^2\left\{\left[1 - e^{-2k_1(T-t)}\right]\left[\frac{\theta_1\left(\sigma_{1,t} - \theta_1\right)}{\left(3k_1 - \eta_1^2\right)} + \frac{\theta_1^2}{\left(2k_1 - \eta_1^2\right)}\right]\right.$$
$$+ \left[1 - e^{-\left(4k_1-\eta_1^2\right)(T-t)}\right]\left[\frac{v_t}{4k_1 - \eta_1^2} - \frac{2k_1\theta_1^2}{\left(2k_1 - \eta_1^2\right)\left(4k_1 - \eta_1^2\right)}\right]$$
$$\left. - \left[1 - e^{-\left(5k_1-\eta_1^2\right)(T-t)}\right]\frac{2k_1\theta_1\left(\sigma_{1,t} - \theta_1\right)}{\left(3k_1 - \eta_1^2\right)\left(5k_1 - \eta_1^2\right)}\right\}$$

3.3 Short rate factorial

$E[x_T|\mathcal{F}_t] = x_t e^{-a(T-t)}$

$E[y_T|\mathcal{F}_t] = y_t e^{-b(T-t)}$

$$\because Var[x_T|\mathcal{F}_t] = \int_t^T E\left[\sigma_{1,T}^2|\mathcal{F}_t\right] e^{-2a(T-t)}\,dt := \int_t^T E[v_t|\mathcal{F}_t] e^{-2a(T-t)}\,dt$$

$$= \int_t^T \left\{ v_t e^{-(2k_1-\eta_1^2)(T-t)} + \frac{2k_1\theta_1(\sigma_{1,t}-\theta_1)}{3k_1-\eta_1^2}\left[1-e^{-(3k_1-\eta_1^2)(T-t)}\right] + \frac{2k_1\theta_1^2}{2k_1-\eta_1^2}\left[1-e^{-(2k_1-\eta_1^2)(T-t)}\right]\right\} e^{-2a(T-t)}\,dt$$

$$= \int_t^T \left\{ v_t e^{-(2k_1+2a-\eta_1^2)(T-t)} + \frac{2k_1\theta_1(\sigma_{1,t}-\theta_1)}{3k_1-\eta_1^2}\left[e^{-2a(T-t)}-e^{-(3k_1+2a-\eta_1^2)(T-t)}\right] + \frac{2k_1\theta_1^2}{2k_1-\eta_1^2}\left[e^{-2a(T-t)}-e^{-(2k_1+2a-\eta_1^2)(T-t)}\right]\right\} dt$$

$$= \frac{v_t}{2k_1+2a-\eta_1^2}\left[1-e^{-(2k_1+2a-\eta_1^2)(T-t)}\right] + \frac{2k_1\theta_1(\sigma_{1,t}-\theta_1)}{(3k_1-\eta_1^2)2a}\left[1-e^{-2a(T-t)}\right] - \frac{2k_1\theta_1(\sigma_{1,t}-\theta_1)}{(3k_1-\eta_1^2)(3k_1+2a-\eta_1^2)}\left[1-e^{-(3k_1+2a-\eta_1^2)(T-t)}\right] + \frac{2k_1\theta_1^2}{(2k_1-\eta_1^2)2a}\left[1-e^{-2a(T-t)}\right] - \frac{2k_1\theta_1^2}{(2k_1-\eta_1^2)(2k_1+2a-\eta_1^2)}\left[1-e^{-(2k_1+2a-\eta_1^2)(T-t)}\right]$$

$$\therefore Var[x_T|\mathcal{F}_t] = \left[1-e^{-2a(T-t)}\right]\left[\frac{k_1\theta_1(\sigma_{1,t}-\theta_1)}{(3k_1-\eta_1^2)a} + \frac{k_1\theta_1^2}{(2k_1-\eta_1^2)a}\right] + \left[1-e^{-(2k_1+2a-\eta_1^2)(T-t)}\right]\left[\frac{v_t}{2k_1+2a-\eta_1^2} - \frac{2k_1\theta_1^2}{(2k_1-\eta_1^2)(2k_1+2a-\eta_1^2)}\right] - \left[1-e^{-(3k_1+2a-\eta_1^2)(T-t)}\right]\frac{2k_1\theta_1(\sigma_{1,t}-\theta_1)}{(3k_1-\eta_1^2)(3k_1+2a-\eta_1^2)}$$

$$Var[y_T|\mathcal{F}_t] = \sigma_2^2\int_t^T e^{-2b(T-t)}\,dt = \frac{\sigma_2^2}{2b}\left[1-e^{-2b(T-t)}\right]$$

3.4 Short rate

$$E[r_t|\mathcal{F}_s] = x_s e^{-a(t-s)} + y_s e^{-b(t-s)} + \varphi_t$$

$$\begin{aligned}
Var[r_t|\mathcal{F}_s] &= E\left[\left(\int_s^t \sigma_{1,u} e^{-a(t-u)}\, dW_{1,u}\right)^2\right] + E\left[\left(\int_s^t \sigma_{2,u} e^{-b(t-u)}\, dW_{2,u}\right)^2\right] \\
&\quad + E\left[2\left(\int_s^t \sigma_{1,u} e^{-a(t-u)}\, dW_{1,u}\right)\left(\int_s^t \sigma_{2,u} e^{-b(t-u)}\, dW_{2,u}\right)\right] \\
&= \int_s^t E\left[\sigma_{1,t}^2|\mathcal{F}_s\right] e^{-2a(t-u)}\, du + \sigma_2^2 \int_s^t e^{-2b(t-u)}\, du + 2\sigma_2 \int_s^t E\left[\rho_t \sigma_{1,t}|\mathcal{F}_s\right] e^{-(a+b)(t-u)}\, du \\
&:= \int_s^t E[v_t|\mathcal{F}_s] e^{-2a(t-u)}\, du + \sigma_2^2 \int_s^t e^{-2b(t-u)}\, du + 2\sigma_2 \int_s^t E[\rho_t|\mathcal{F}_s] E\left[\sigma_{1,t}|\mathcal{F}_s\right] e^{-(a+b)(t-u)}\, du \\
&= \int_s^t \left\{ v_t e^{-\left(2k-\eta_1^2\right)(T-t)} + \frac{2k_1\theta_1\left(\sigma_{1,t}-\theta_1\right)}{3k_1-\eta_1^2}\left[1-e^{-\left(3k_1-\eta_1^2\right)(T-t)}\right] \right. \\
&\quad \left. + \frac{2k_1\theta_1^2}{2k_1-\eta_1^2}\left[1-e^{-\left(2k_1-\eta_1^2\right)(T-t)}\right] \right\} e^{-2a(t-u)}\, du + \sigma_2^2 \int_s^t e^{-2b(t-u)}\, du \\
&\quad + 2\sigma_2 \left\{\left[(\rho_t-\rho_\vartheta)e^{-\lambda(T-t)}+\rho_\vartheta\right]\left[\left(\sigma_{1,t}-\theta_1\right)e^{-k_1(T-t)}+\theta_1\right] e^{-(a+b)(t-u)}\right\} \\
&= \left[1-e^{-2a(t-u)}\right]\left[\frac{2k_1\theta_1\left(\sigma_{1,t}-\theta_1\right)}{(3k_1-\eta_1^2)2a} + \frac{2k_1\theta_1^2}{(2k_1-\eta_1^2)2a}\right] \\
&\quad + \left[1-e^{-\left(2k_1+2a-\eta_1^2\right)(T-t)}\right]\left[\frac{v_t}{2k_1+2a-\eta_1^2} - \frac{2k_1\theta_1^2}{(2k_1-\eta_1^2)(2k_1+2a-\eta_1^2)}\right] \\
&\quad - \left[1-e^{-\left(3k_1+2a-\eta_1^2\right)(T-t)}\right]\frac{2k_1\theta_1\left(\sigma_{1,t}-\theta_1\right)}{(3k_1-\eta_1^2)(3k_1+2a-\eta_1^2)} + \frac{\sigma_2^2}{2b}\left[1-e^{-2b(t-u)}\right] \\
&\quad + 2\left\{\frac{(\rho_t-\rho_\vartheta)\left(\sigma_{1,t}-\theta_1\right)}{a+b+\lambda+k_1}\left[1-e^{-(a+b+\lambda+k_1)(T-t)}\right] + \frac{\theta_1(\rho_t-\rho_\vartheta)}{a+b+\lambda}\left[1-e^{-(a+b+\lambda)(T-t)}\right] \right. \\
&\quad \left. + \frac{\rho_\vartheta\left(\sigma_{1,t}-\theta_1\right)}{a+b+k_1}\left[1-e^{-(a+b+k_1)(T-t)}\right] + \frac{\rho_\vartheta\theta_1}{a+b}\left[1-e^{-(a+b)(T-t)}\right]\right\}
\end{aligned}$$

IV. Pricing of Zero Coupon Bonds

4.1 Discount factor

Define:

$$I(t,T)=\int_t^T [x_T+y_T|\mathcal{F}_t]dT$$

$$=\int_t^T x_t e^{-a(T-t)}dT+\int_t^T y_t e^{-b(T-t)}dT+\int_t^T\int_t^T \sigma_{1,u}e^{-a(T-u)}dW_{1,u}\,dT+\int_t^T\int_t^T \sigma_{2,u}e^{-b(T-u)}dW_{2,u}\,dT$$

Let's denote the mean as $\boldsymbol{M(t,T)}$, variance as $\boldsymbol{V(t,T)}$:

$$M(t,T)=E[I(t,T)|\mathcal{F}_t]=\frac{1-e^{-a(T-t)}}{a}x_t+\frac{1-e^{-b(T-t)}}{b}y_t$$

$$V(t,T)=Var[I(t,T)|\mathcal{F}_t]=E\left[\left(\int_t^T\int_t^T \sigma_{1,u}e^{-a(T-u)}dW_{1,u}\,dT+\int_t^T\int_t^T \sigma_{2,u}e^{-b(T-u)}dW_{2,u}\,dT\right)^2\right]$$

$$=E\left[\left(\int_t^T\int_t^T \sigma_{1,u}e^{-a(T-u)}\,dTdW_{1,u}+\int_t^T\int_t^T \sigma_{2,u}e^{-b(T-u)}\,dTdW_{2,u}\right)^2\right]$$

$$=E\left[\left(\int_t^T \sigma_{1,u}e^{au}\int_t^T e^{-aT}\,dTdW_{1,u}+\int_t^T \sigma_{2,u}e^{bu}\int_t^T e^{-bT}\,dTdW_{2,u}\right)^2\right]$$

$$=E\left[\left(\int_t^T \frac{\sigma_{1,u}}{a}\left[1-e^{-a(T-u)}\right]dW_{1,u}+\int_t^T \frac{\sigma_{2,u}}{b}\left[1-e^{-b(T-u)}\right]dW_{2,u}\right)^2\right]$$

$$=E\left[\int_t^T \frac{\sigma_{1,u}^2}{a^2}\left[1-e^{-a(T-u)}\right]^2du+\int_t^T \frac{\sigma_{2,u}^2}{b^2}\left[1-e^{-b(T-u)}\right]^2du\right]$$

$$+E\left\{\frac{2}{ab}\int_t^T \rho_t\sigma_{1,u}\sigma_{2,u}\left[1-e^{-a(T-u)}\right]\left[1-e^{-b(T-u)}\right]du\right\}$$

Since no correlation between $(\rho_T,\sigma_{1,T})$, we have:

$$V(t,T)=\frac{1}{a^2}\int_t^T E\left[v_{1,T}|\mathcal{F}_t\right]\left[1-e^{-a(T-t)}\right]^2dt+\frac{\sigma_2^2}{b^2}\int_t^T\left[1-e^{-b(T-t)}\right]^2dt$$

$$+\frac{2\sigma_2}{ab}\int_t^T E[\rho_T|\mathcal{F}_t]E\left[\sigma_{1,T}|\mathcal{F}_t\right]\left[1-e^{-a(T-t)}\right]\left[1-e^{-b(T-t)}\right]dt$$

Define:

$$\mathrm{X}(t,T)=\frac{1}{a^2}\int_t^T E\left[v_{1,T}|\mathcal{F}_t\right]\left[1-e^{-a(T-t)}\right]^2dt$$

$$\Psi(t,T)=\frac{\sigma_2^2}{b^2}\int_t^T\left[1-e^{-a(T-t)}\right]^2dt$$

$$\Omega(t,T)=\frac{2\sigma_2}{ab}\int_t^T E[\rho_T|\mathcal{F}_t]E\left[\sigma_{1,T}|\mathcal{F}_t\right]\left[1-e^{-a(T-t)}\right]\left[1-e^{-b(T-t)}\right]dt$$

Naturally, we have:

$$V(t,T):=\mathrm{X}(t,T)+\Psi(t,T)+\Omega(t,T)$$

$$\therefore \mathrm{X}(t,T) = \frac{1}{a^2}\int_t^T \left\{ v_t e^{-(2k-\eta_1^2)(T-t)} + 2k_1\theta_1(\sigma_{1,t}-\theta_1)\frac{1-e^{-(3k_1-\eta_1^2)(T-t)}}{3k_1-\eta_1^2} + 2k_1\theta_1^2\frac{1-e^{-(2k_1-\eta_1^2)(T-t)}}{2k_1-\eta_1^2} \right\} \left[1 + e^{-2a(T-t)} - 2e^{-a(T-t)}\right] dt$$

$$= \frac{1}{a^2} \cdot \left\{ \int_t^T \left[v_t e^{-(2k-\eta_1^2)(T-t)} + 2k_1\theta_1(\sigma_{1,t}-\theta_1)\frac{1-e^{-(3k_1-\eta_1^2)(T-t)}}{3k_1-\eta_1^2} + 2k_1\theta_1^2\frac{1-e^{-(2k_1-\eta_1^2)(T-t)}}{2k_1-\eta_1^2} \right] dt \right.$$

$$+ \int_t^T \left[v_t e^{-(2k_1+2a-\eta_1^2)(T-t)} + \frac{2k_1\theta_1(\sigma_{1,t}-\theta_1)}{3k_1-\eta_1^2}\left[e^{-2a(T-t)} - e^{-(3k_1+2a-\eta_1^2)(T-t)}\right] + \frac{2k_1\theta_1^2}{2k_1-\eta_1^2}\left[e^{-2a(T-t)} - e^{-(2k_1+2a-\eta_1^2)(T-t)}\right] \right] dt$$

$$\left. - 2\int_t^T \left[v_t e^{-(2k_1+a-\eta_1^2)(T-t)} + \frac{2k_1\theta_1(\sigma_{1,t}-\theta_1)}{3k_1-\eta_1^2}\left[e^{-a(T-t)} - e^{-(3k_1+a-\eta_1^2)(T-t)}\right] + \frac{2k_1\theta_1^2}{2k_1-\eta_1^2}\left[e^{-a(T-t)} - e^{-(2k_1+a-\eta_1^2)(T-t)}\right] \right] dt \right\}$$

$$= \frac{1}{a^2}\left\{ v_t \frac{1-e^{-(2k_1-\eta_1^2)(T-t)}}{2k-\eta_1^2} + \frac{2k_1\theta_1(\sigma_{1,t}-\theta_1)}{3k_1-\eta_1^2}\left[T-t-\frac{1-e^{-(3k_1-\eta_1^2)(T-t)}}{3k_1-\eta_1^2}\right] \right.$$

$$+ \frac{2k_1\theta_1^2}{2k_1-\eta_1^2}\left[T-t-\frac{1-e^{-(2k_1-\eta_1^2)(T-t)}}{2k_1-\eta_1^2}\right] + v_t\frac{1-e^{-(2k_1+2a-\eta_1^2)(T-t)}}{2k_1+2a-\eta_1^2}$$

$$+ \frac{2k_1\theta_1(\sigma_{1,t}-\theta_1)}{3k_1-\eta_1^2}\left[\frac{1-e^{-2a(T-t)}}{2a} - \frac{1-e^{-(3k_1+2a-\eta_1^2)(T-t)}}{3k_1+2a-\eta_1^2}\right]$$

$$+ \frac{2k_1\theta_1^2}{2k_1-\eta_1^2}\left[\frac{1-e^{-2a(T-t)}}{2a} - \frac{1-e^{-(2k_1+2a-\eta_1^2)(T-t)}}{2k_1+2a-\eta_1^2}\right]$$

$$- 2\left\{ v_t\frac{1-e^{-(2k_1+a-\eta_1^2)(T-t)}}{2k_1+a-\eta_1^2} + \frac{2k_1\theta_1(\sigma_{1,t}-\theta_1)}{3k_1-\eta_1^2}\left[\frac{1-e^{-a(T-t)}}{a} - \frac{1-e^{-(3k_1+a-\eta_1^2)(T-t)}}{3k_1+a-\eta_1^2}\right] \right.$$

$$\left.\left. + \frac{2k_1\theta_1^2}{2k_1-\eta_1^2}\left[\frac{1-e^{-a(T-t)}}{a} - \frac{1-e^{-(2k_1+a-\eta_1^2)(T-t)}}{2k_1+a-\eta_1^2}\right] \right\} \right\}$$

$$\therefore \mathrm{X}(t,T) = \frac{1}{a^2}\left\{\left[\frac{2k_1\theta_1\left(\sigma_{1,t}-\theta_1\right)}{3k_1-\eta_1^2}+\frac{2k_1\theta_1^2}{2k_1-\eta_1^2}\right]\left[T-t+\frac{1-e^{-2a(T-t)}}{2a}-2\frac{1-e^{-a(T-t)}}{a}\right]\right.$$
$$+\left[v_t-\frac{2k_1\theta_1^2}{2k_1-\eta_1^2}\right]\left[\frac{1-e^{-\left(2k_1-\eta_1^2\right)(T-t)}}{2k-\eta_1^2}+\frac{1-e^{-\left(2k_1+2a-\eta_1^2\right)(T-t)}}{2k_1+2a-\eta_1^2}-2\frac{1-e^{-\left(2k_1+a-\eta_1^2\right)(T-t)}}{2k_1+a-\eta_1^2}\right]$$
$$\left.-\frac{2k_1\theta_1\left(\sigma_{1,t}-\theta_1\right)}{3k_1-\eta_1^2}\left[\frac{1-e^{-\left(3k_1-\eta_1^2\right)(T-t)}}{3k_1-\eta_1^2}+\frac{1-e^{-\left(3k_1+2a-\eta_1^2\right)(T-t)}}{3k_1+2a-\eta_1^2}-2\frac{1-e^{-\left(3k_1+a-\eta_1^2\right)(T-t)}}{3k_1+a-\eta_1^2}\right]\right\}$$

$$\therefore \Psi(t,T) = \frac{\sigma_2^2}{b^2}\left[T-t+\frac{2}{b}e^{-b(T-t)}-\frac{1}{2b}e^{-2b(T-t)}-\frac{3}{2b}\right]$$

$$
\begin{aligned}
\therefore \Omega(t,T) = \frac{2\sigma_2}{ab}\int_t^T & \left[(\rho_t-\rho_\vartheta)e^{-\lambda(T-t)}+\rho_\vartheta\right]\left[\left(\sigma_{1,t}-\theta_1\right)e^{-k_1(T-t)}+\theta_1\right]\left[1-e^{-a(T-t)}-e^{-b(T-t)}\right. \\
& \left.+e^{-(a+b)(T-t)}\right]dt \\
&= \frac{2\sigma_2}{ab}\int_t^T\left[(\rho_t-\rho_\vartheta)\left(\sigma_{1,t}-\theta_1\right)e^{-(\lambda+k_1)(T-t)}+\theta_1(\rho_t-\rho_\vartheta)e^{-\lambda(T-t)}+\rho_\vartheta\left(\sigma_{1,t}-\theta_1\right)e^{-k_1(T-t)}\right. \\
& \left.+\rho_\vartheta\theta_1\right]\left[1-e^{-a(T-t)}-e^{-b(T-t)}+e^{-(a+b)(T-t)}\right]dt \\
&= \frac{2\sigma_2}{ab}\left\{\int_t^T\left[(\rho_t-\rho_\vartheta)\left(\sigma_{1,t}-\theta_1\right)e^{-(\lambda+k_1)(T-t)}+\theta_1(\rho_t-\rho_\vartheta)e^{-\lambda(T-t)}+\rho_\vartheta\left(\sigma_{1,t}-\theta_1\right)e^{-k_1(T-t)}\right.\right. \\
& \left.+\rho_\vartheta\theta_1\right]dt \\
& -\int_t^T\left[(\rho_t-\rho_\vartheta)\left(\sigma_{1,t}-\theta_1\right)e^{-(a+\lambda+k_1)(T-t)}+\theta_1(\rho_t-\rho_\vartheta)e^{-(a+\lambda)(T-t)}\right. \\
& \left.+\rho_\vartheta\left(\sigma_{1,t}-\theta_1\right)e^{-(a+k_1)(T-t)}+\rho_\vartheta\theta_1e^{-a(T-t)}\right]dt \\
& -\int_t^T\left[(\rho_t-\rho_\vartheta)\left(\sigma_{1,t}-\theta_1\right)e^{-(b+\lambda+k_1)(T-t)}+\theta_1(\rho_t-\rho_\vartheta)e^{-(b+\lambda)(T-t)}\right. \\
& \left.+\rho_\vartheta\left(\sigma_{1,t}-\theta_1\right)e^{-(b+k_1)(T-t)}+\rho_\vartheta\theta_1e^{-b(T-t)}\right]dt \\
& +\int_t^T\left[(\rho_t-\rho_\vartheta)\left(\sigma_{1,t}-\theta_1\right)e^{-(a+b+\lambda+k_1)(T-t)}+\theta_1(\rho_t-\rho_\vartheta)e^{-(a+b+\lambda)(T-t)}\right. \\
& \left.\left.+\rho_\vartheta\left(\sigma_{1,t}-\theta_1\right)e^{-(a+b+k_1)(T-t)}+\rho_\vartheta\theta_1e^{-(a+b)(T-t)}\right]dt\right\} \\
&= \frac{2\sigma_2}{ab}\left\{(\rho_t-\rho_\vartheta)\left(\sigma_{1,t}-\theta_1\right)\frac{1-e^{-(\lambda+k_1)(T-t)}}{\lambda+k_1}+\theta_1(\rho_t-\rho_\vartheta)\frac{1-e^{-\lambda(T-t)}}{\lambda}\right. \\
& +\rho_\vartheta\left(\sigma_{1,t}-\theta_1\right)\frac{1-e^{-k_1(T-t)}}{k_1}+\rho_\vartheta\theta_1(T-t)-(\rho_t-\rho_\vartheta)\left(\sigma_{1,t}-\theta_1\right)\frac{1-e^{-(a+\lambda+k_1)(T-t)}}{a+\lambda+k_1} \\
& -\theta_1(\rho_t-\rho_\vartheta)\frac{1-e^{-(a+\lambda)(T-t)}}{a+\lambda}-\rho_\vartheta\left(\sigma_{1,t}-\theta_1\right)\frac{1-e^{-(a+k_1)(T-t)}}{a+k_1}-\rho_\vartheta\theta_1\frac{1-e^{-a(T-t)}}{a} \\
& -(\rho_t-\rho_\vartheta)\left(\sigma_{1,t}-\theta_1\right)\frac{1-e^{-(b+\lambda+k_1)(T-t)}}{b+\lambda+k_1}-\theta_1(\rho_t-\rho_\vartheta)\frac{1-e^{-(b+\lambda)(T-t)}}{b+\lambda} \\
& -\rho_\vartheta\left(\sigma_{1,t}-\theta_1\right)\frac{1-e^{-(b+k_1)(T-t)}}{b+k_1}-\rho_\vartheta\theta_1\frac{1-e^{-b(T-t)}}{b} \\
& +(\rho_t-\rho_\vartheta)\left(\sigma_{1,t}-\theta_1\right)\frac{1-e^{-(a+b+\lambda+k_1)(T-t)}}{a+b+\lambda+k_1}+\theta_1(\rho_t-\rho_\vartheta)\frac{1-e^{-(a+b+\lambda)(T-t)}}{a+b+\lambda} \\
& \left.+\rho_\vartheta\left(\sigma_{1,t}-\theta_1\right)\frac{1-e^{-(a+b+k_1)(T-t)}}{a+b+k_1}+\rho_\vartheta\theta_1\frac{1-e^{-(a+b)(T-t)}}{a+b}\right\}
\end{aligned}
$$

$$\therefore \Omega(t,T) = \frac{2\sigma_2}{ab}\left\{\rho_\vartheta\theta_1\left[T-t-\frac{1-e^{-a(T-t)}}{a}-\frac{1-e^{-b(T-t)}}{b}+\frac{1-e^{-(a+b)(T-t)}}{a+b}\right]\right.$$
$$+(\rho_t-\rho_\vartheta)\left(\sigma_{1,t}-\theta_1\right)\left[\frac{1-e^{-(\lambda+k_1)(T-t)}}{\lambda+k_1}-\frac{1-e^{-(a+\lambda+k_1)(T-t)}}{a+\lambda+k_1}-\frac{1-e^{-(b+\lambda+k_1)(T-t)}}{b+\lambda+k_1}\right.$$
$$\left.+\frac{1-e^{-(a+b+\lambda+k_1)(T-t)}}{a+b+\lambda+k_1}\right]$$
$$+\theta_1(\rho_t-\rho_\vartheta)\left[\frac{1-e^{-\lambda(T-t)}}{\lambda}-\frac{1-e^{-(a+\lambda)(T-t)}}{a+\lambda}-\frac{1-e^{-(b+\lambda)(T-t)}}{b+\lambda}+\frac{1-e^{-(a+b+\lambda)(T-t)}}{a+b+\lambda}\right]$$
$$\left.+\rho_\vartheta\left(\sigma_{1,t}-\theta_1\right)\left[\frac{1-e^{-k_1(T-t)}}{k_1}-\frac{1-e^{-(a+k_1)(T-t)}}{a+k_1}-\frac{1-e^{-(b+k_1)(T-t)}}{b+k_1}+\frac{1-e^{-(a+b+k_1)(T-t)}}{a+b+k_1}\right]\right\}$$

Finally, we arrive at:

$$V(t,T) := \mathrm{X}(t,T)+\Psi(t,T)+\Omega(t,T)$$

Where:

$$\mathrm{X}(t,T) = \frac{1}{a^2}\left\{\left[\frac{2k_1\theta_1\left(\sigma_{1,t}-\theta_1\right)}{3k_1-\eta_1^2}+\frac{2k_1\theta_1^2}{2k_1-\eta_1^2}\right]\left[T-t+\frac{2}{a}e^{-a(T-t)}-\frac{1}{2a}e^{-2a(T-t)}-\frac{3}{2a}\right]\right.$$
$$+\left[v_t-\frac{2k_1\theta_1^2}{2k_1-\eta_1^2}\right]\left[\frac{1-e^{-\left(2k_1-\eta_1^2\right)(T-t)}}{2k-\eta_1^2}+\frac{1-e^{-\left(2k_1+2a-\eta_1^2\right)(T-t)}}{2k_1+2a-\eta_1^2}-2\frac{1-e^{-\left(2k_1+a-\eta_1^2\right)(T-t)}}{2k_1+a-\eta_1^2}\right]$$
$$\left.-\frac{2k_1\theta_1\left(\sigma_{1,t}-\theta_1\right)}{3k_1-\eta_1^2}\left[\frac{1-e^{-\left(3k_1-\eta_1^2\right)(T-t)}}{3k_1-\eta_1^2}+\frac{1-e^{-\left(3k_1+2a-\eta_1^2\right)(T-t)}}{3k_1+2a-\eta_1^2}-2\frac{1-e^{-\left(3k_1+a-\eta_1^2\right)(T-t)}}{3k_1+a-\eta_1^2}\right]\right\}$$

$$\Psi(t,T) = \frac{\sigma_2^2}{b^2}\left[T-t+\frac{2}{b}e^{-b(T-t)}-\frac{1}{2b}e^{-2b(T-t)}-\frac{3}{2b}\right]$$

$$\Omega(t,T) = \frac{2\sigma_2}{ab}\left\{\rho_\vartheta\theta_1\left[T-t-\frac{1-e^{-a(T-t)}}{a}-\frac{1-e^{-b(T-t)}}{b}+\frac{1-e^{-(a+b)(T-t)}}{a+b}\right]\right.$$
$$+(\rho_t-\rho_\vartheta)\left(\sigma_{1,t}-\theta_1\right)\left[\frac{1-e^{-(\lambda+k_1)(T-t)}}{\lambda+k_1}-\frac{1-e^{-(a+\lambda+k_1)(T-t)}}{a+\lambda+k_1}-\frac{1-e^{-(b+\lambda+k_1)(T-t)}}{b+\lambda+k_1}\right.$$
$$\left.+\frac{1-e^{-(a+b+\lambda+k_1)(T-t)}}{a+b+\lambda+k_1}\right]$$
$$+\theta_1(\rho_t-\rho_\vartheta)\left[\frac{1-e^{-\lambda(T-t)}}{\lambda}-\frac{1-e^{-(a+\lambda)(T-t)}}{a+\lambda}-\frac{1-e^{-(b+\lambda)(T-t)}}{b+\lambda}+\frac{1-e^{-(a+b+\lambda)(T-t)}}{a+b+\lambda}\right]$$
$$\left.+\rho_\vartheta\left(\sigma_{1,t}-\theta_1\right)\left[\frac{1-e^{-k_1(T-t)}}{k_1}-\frac{1-e^{-(a+k_1)(T-t)}}{a+k_1}-\frac{1-e^{-(b+k_1)(T-t)}}{b+k_1}+\frac{1-e^{-(a+b+k_1)(T-t)}}{a+b+k_1}\right]\right\}$$

4.2 Zero Coupon Bond Formula:

Additionally, we have (Please refer to Chapter 1 for derivation):

$$
\begin{aligned}
P^M(t,T) &= \exp\left[-\int_t^T \varphi_s\,ds - M(t,T) + \frac{1}{2}V(t,T)\right] \\
&= \frac{P^M(0,T)}{P^M(0,t)}\exp\left\{\frac{1}{2}[V(0,t) - V(0,T)]\right\}\cdot\exp\left[-M(t,T) + \frac{1}{2}V(t,T)\right] \\
&= \frac{P^M(0,T)}{P^M(0,t)}\exp\left\{\frac{1}{2}[V(t,T) - V(0,T) + V(0,t)] - M(t,T)\right\}
\end{aligned}
$$

Where:

$$M(t,T) = E[I(t,T)|\mathcal{F}_t] = \frac{1-e^{-a(T-t)}}{a}x_t + \frac{1-e^{-b(T-t)}}{b}y_t$$

With:

$$
\begin{cases}
x_T = x_t e^{-a(T-t)} + \int_t^T \sigma_{1,t} e^{-a(T-t)} dW_{1,t} \\
y_T = y_t e^{-b(T-t)} + \int_t^T \sigma_{2,t} e^{-b(T-t)} dW_{2,t} \\
x_T|\mathcal{F}_t \sim N\left(x_t e^{-a(T-t)}, Var[x_T|\mathcal{F}_t]\right) \\
y_T|\mathcal{F}_t \sim N(y_t e^{-a(T-t)}, \frac{\sigma_2^2}{2b}\left[1 - e^{-2b(t-u)}\right])
\end{cases}
$$

And:

$$V(t,T) = Var[I(t,T)|\mathcal{F}_t] = \mathrm{X}(t,T) + \Psi(t,T) + \Omega(t,T)$$

Where:

$$
\begin{cases}
\sigma_{1,T} = \sigma_{1,t} e^{-k_1(T-t)} + \theta_1\left[1 - e^{-k_1(T-t)}\right] + \eta_1 \int_t^T \sigma_{1,t} e^{-k_1(T-t)} dW_{3,t} \\
v_T = v_t e^{-(2k_1-\eta_1^2)(T-t)} + \frac{2k_1}{2k_1-\eta_1^2}\theta_1\sigma_{1,t}\left[1 - e^{-(2k_1-\eta_1^2)(T-t)}\right] + 2\eta_1 \int_t^T v_t e^{-(2k_1-\eta_1^2)(T-t)} dW_{3,t} \\
\rho_T = \rho_t e^{-\lambda(T-t)} + \rho_\vartheta\left[1 - e^{-\lambda(T-t)}\right] + \xi \int_t^T \sqrt{1-\rho_t^2}\, e^{-\lambda(T-t)} dZ_t
\end{cases}
$$

Or in differential form:

$$
\begin{cases}
d\left[\sigma_{1,t} e^{-k_1(T-t)}\right] = k_1\theta_1 e^{-k_1(T-t)} dt + \eta_1\sigma_{1,t} e^{-k_1(T-t)} dW_{3,t} \\
d\left[v_t e^{-(2k_1-\eta_1^2)(T-t)}\right] = 2k_1\theta_1\sigma_{1,t} e^{-(2k_1-\eta_1^2)(T-t)} dt + 2\eta_1 v_t e^{-(2k_1-\eta_1^2)(T-t)} dW_{3,t} \\
d\left[\rho_t e^{-\lambda(T-t)}\right] = \lambda\rho_\vartheta e^{-\lambda(T-t)} dt + \xi\sqrt{1-\rho_t^2}\, e^{-\lambda(T-t)} dZ_t
\end{cases}
$$

And finally, the correlation matrix:

$$
C_t = \begin{bmatrix} 1 & \rho_t & \rho_{13} \\ \cdot & 1 & \rho_{23} \\ \cdot & \cdot & 1 \end{bmatrix};\ L = \begin{bmatrix} 1 & 0 & 0 \\ \rho_t & \sqrt{1-\rho_t^2} & 0 \\ \rho_{13} & \frac{\rho_{23}-\rho_{13}\rho_t}{\sqrt{1-\rho_t^2}} & \sqrt{1-\rho_{13}^2 - \left(\frac{\rho_{23}-\rho_{13}\rho_t}{\sqrt{1-\rho_t^2}}\right)^2} \end{bmatrix};\ C_t = L_t \cdot L_t^T
$$

Chapter 5. CHICAGO 2++ Simulation Algorithmic Setup

I. Random Sample
For each path j, do:
1. Draw sample $\tilde{z}^j_{1,t}, \tilde{z}^j_{2,t}, \tilde{z}^j_{3,t}, \tilde{z}^j_{4,t} \sim N(0,1), i.i.d$
2. Correlate sample:

$$Z^j_{1,t} = \tilde{z}^j_{1,t} \text{ :: x}$$

$$Z^j_{2,t} = \rho_{t_i} \cdot \tilde{z}^j_{1,t} + \sqrt{1-\rho^2_{t_i}} \cdot \tilde{z}^j_{2,t} \text{:: y}$$

$$Z^j_{3,t} = \rho_{13} \cdot \tilde{z}^j_{1,t} + \frac{\rho_{23}-\rho_{13}\rho_t}{\sqrt{1-\rho^2_{t_i}}} \cdot \tilde{z}^j_{2,t} + \sqrt{1-\rho^2_{13} - \left(\frac{\rho_{23}-\rho_{13}\rho_{t_i}}{\sqrt{1-\rho^2_{t_i}}}\right)^2} \cdot \tilde{z}^j_{3,t} \text{ :: v}$$

$$Z^j_{4,t} = \tilde{z}^j_{4,t} \text{:: rho}$$

II. Dynamic Evolution
2.1 Ito-Taylor
(1) Correlation

$$\begin{aligned}\rho_t = {} & (\rho_0 - \rho_\vartheta)e^{-\lambda(t-t_0)} + \rho_\vartheta \\ & + \left\{\left[1-e^{-2\lambda(t-t_0)}\right]\left[\frac{1}{2\lambda} - \frac{\xi^2+2\lambda\rho^2_\vartheta}{(2\lambda+\xi^2)2\lambda} - \frac{\rho_\vartheta(\rho_0-\rho_\vartheta)}{(3\lambda+\xi^2)}\right]\right. \\ & + \left[1-e^{-(4\lambda+\xi^2)(t-t_0)}\right]\left[\frac{\xi^2+2\lambda\rho^2_\vartheta}{(2\lambda+\xi^2)(4\lambda+\xi^2)} - \frac{\rho^2_0}{4\lambda+\xi^2}\right] \\ & \left. + \left[1-e^{-(5\lambda+\xi^2)(t-t_0)}\right]\frac{2\lambda\rho_\vartheta(\rho_0-\rho_\vartheta)}{(3\lambda+\xi^2)(5\lambda+\xi^2)}\right\}^{1/2} \cdot \xi \cdot \sqrt{t-t_0} \cdot Z^j_{4,t}\end{aligned}$$

(2) Volatility:

$$\begin{aligned}\sigma_{1,t} = {} & (\sigma_1 - \theta_1)e^{-k_1(t-t_0)} + \theta_1 \\ & + \left\{\left[1-e^{-2k_1(t-t_0)}\right]\left[\frac{\theta_1(\sigma_1-\theta_1)}{(3k_1-\eta^2_1)} + \frac{\theta^2_1}{(2k_1-\eta^2_1)}\right]\right. \\ & + \left[1-e^{-(4k_1-\eta^2_1)(t-t_0)}\right]\left[\frac{\sigma^2_1}{4k_1-\eta^2_1} - \frac{2k_1\theta^2_1}{(2k_1-\eta^2_1)(4k_1-\eta^2_1)}\right] \\ & \left. - \left[1-e^{-(5k_1-\eta^2_1)(t-t_0)}\right]\frac{2k_1\theta_1(\sigma_1-\theta_1)}{(3k_1-\eta^2_1)(5k_1-\eta^2_1)}\right\}^{1/2} \cdot \eta_1 \cdot \sqrt{t-t_0} \cdot Z^{\boldsymbol{j}}_{3,t}\end{aligned}$$

(3) Factors:

$$\begin{aligned}x_t = {} & x_0 e^{-a(t-t_0)} \\ & + \left\{\left[1-e^{-2a(t-t_0)}\right]\left[\frac{k_1\theta_1(\sigma_1-\theta_1)}{(3k_1-\eta^2_1)a} + \frac{k_1\theta^2_1}{(2k_1-\eta^2_1)a}\right]\right. \\ & + \left[1-e^{-(2k_1+2a-\eta^2_1)(t-t_0)}\right]\left[\frac{\sigma^2_1}{2k_1+2a-\eta^2_1} - \frac{2k_1\theta^2_1}{(2k_1-\eta^2_1)(2k_1+2a-\eta^2_1)}\right] \\ & \left. - \left[1-e^{-(3k_1+2a-\eta^2_1)(t-t_0)}\right]\frac{2k_1\theta_1(\sigma_1-\theta_1)}{(3k_1-\eta^2_1)(3k_1+2a-\eta^2_1)}\right\}^{1/2} \cdot \sqrt{t-t_0} \cdot Z^j_{1,t}\end{aligned}$$

$$y_t = y_0 e^{-b(t-t_0)} + \sqrt{\frac{\sigma_2^2}{2b}\left[1 - e^{-2b(t-t_0)}\right]} \cdot \sqrt{t - t_0} \cdot Z_{2,t}^j$$

2.2 Euler-Maruyama

For each term i and path j:

(1) Correlation

$$\rho_{t_i} = \rho_{t_{i-1}} + \lambda\left(\rho_\vartheta - \rho_{t_{i-1}}\right)\Delta t_{i-1} + \xi\sqrt{1 - \rho_{t_{i-1}}^2} \cdot \sqrt{\Delta t_{i-1}} \cdot Z_{4,t_i}^j$$

(2) Volatility:

$$\sigma_{1,t_i} = \sigma_{1,t_{i-1}} + k_1\left(\theta_1 - \sigma_{1,t_{i-1}}\right)\Delta t_{i-1} + \eta_1\sigma_{1,t_{i-1}} \cdot \sqrt{\Delta t_{i-1}} \cdot Z_{3,t}^{\boldsymbol{j}}$$

(3) Factors:

$$x_{t_i} = x_{t_{i-1}} - a x_{t_{i-1}}\Delta t_{i-1} + \sigma_{1,t_{i-1}}\sqrt{\Delta t_{i-1}} \cdot Z_{1,t}^j$$
$$y_{t_i} = y_{t_{i-1}} - b y_{t_{i-1}}\Delta t_{i-1} + \sigma_2\sqrt{\Delta t_{i-1}} \cdot Z_{2,t}^j$$

Where:

$$\rho_{t_0} = \rho_0; \sigma_{1,t_0} = \sigma_1; x_0 = 0; y_0 = 0;\ \Delta t_{i-1} = t_i - t_{i-1}; i = 1, \dots, n$$

2.2 Predictor-Corrector

2.2.1 Generalization

If we have:

$$dx_t = f(x_t)dt + \sum_{j=1}^{m} g_j(x_t)dW_j$$

Then, we have:

Predictor:

$$\hat{x}_{t_i} = x_{t_{i-1}} + f\left(x_{t_{i-1}}\right)\Delta t_{i-1} + \sum_{j=1}^{m} g_j\left(x_{t_{i-1}}\right)\Delta W_j$$

Corrector:

$$x_{t_i} = x_{t_{i-1}} + \left[\alpha\hat{f}\left(x_{t_{i-1}}, \beta\right) + (1-\alpha)\hat{f}\left(\hat{x}_{t_i}, \beta\right)\right]\Delta t_{i-1} + \sum_{j=1}^{m}\left[\beta g_j\left(x_{t_{i-1}}\right) + (1-\beta)g_j\left(\hat{x}_{t_i}\right)\right]\Delta W_j$$

Where:

$$\hat{f}(x_t, \beta) = f(x_t) - \beta\sum_{j=1}^{m}\sum_{k=1}^{m} g_{kj}(x_t) \cdot \frac{\partial g_j(x_t)}{\partial x_k(t)}$$

2.2.2 Predictor-Corrector for CHICAGO 2++

For each term t_i and path j:

(1) Correlation

Predictor:

$$\hat{\rho}_{t_i} = \rho_{t_{i-1}} + \lambda(\rho_\vartheta - \rho_{t_{i-1}})\Delta t_{i-1} + \xi\sqrt{1-\rho_{t_{i-1}}^2} \cdot \sqrt{\Delta t_{i-1}} \cdot Z_{4,t_i}^j$$

Corrector:

$$\rho_{t_i} = \rho_{t_{i-1}} + \left[\alpha_\rho \hat{f}_\rho(\rho_{t_{i-1}}, \beta_\rho) + (1-\alpha_\rho)\hat{f}_\rho(\hat{\rho}_{t_i}, \beta_\rho)\right]\Delta t_{i-1} + \xi\left[\beta_\rho\sqrt{1-\rho_{t_{i-1}}^2} + (1-\beta_\rho)\sqrt{1-\hat{\rho}_{t_i}^2}\right] \cdot \sqrt{\Delta t_{i-1}} \cdot Z_{4,t_i}^j$$

Where:

$$\hat{f}_\rho(\rho_t, \beta_\rho) = \lambda(\rho_\vartheta - \rho_t) - \beta_\rho\xi\sqrt{1-\rho_t^2} \cdot \left(-\frac{\xi\rho_t}{\sqrt{1-\rho_t^2}}\right) = \lambda(\rho_\vartheta - \rho_t) + \beta_\rho\xi^2\rho_t$$

(2) Volatility:

Predictor:

$$\hat{\sigma}_{1,t_i} = \sigma_{1,t_{i-1}} + k_1(\theta_1 - \sigma_{1,t_{i-1}})\Delta t_{i-1} + \eta_1\sigma_{1,t_{i-1}} \cdot \sqrt{\Delta t_{i-1}} \cdot Z_{3,t_i}^{\boldsymbol{j}}$$

Corrector:

$$\sigma_{1,t_i} = \sigma_{1,t_{i-1}} + \left[\alpha_v \hat{f}_v(\sigma_{t_{i-1}}, \beta_v) + (1-\alpha_v)\hat{f}_v(\hat{\sigma}_{t_i}, \beta_v)\right]\Delta t_{i-1} + \eta_1\left[\beta_v\sigma_{1,t_{i-1}} + (1-\beta_v)\hat{\sigma}_{1,t_i}\right] \cdot \sqrt{\Delta t_{i-1}} \cdot Z_{3,t_i}^j$$

Where:

$$\hat{f}_v(\sigma_t, \beta_v) = k_1(\theta_1 - \sigma_{1,t}) - \beta_v\eta_1^2\sigma_{1,t}$$

(3) Factors:

Predictor:

$$\hat{x}_{t_i} = x_{t_{i-1}} - a x_{t_{i-1}}\Delta t_{i-1} + \sigma_{1,t_{i-1}}\sqrt{\Delta t_{i-1}} \cdot Z_{1,t_i}^j$$

Corrector:

$$x_{t_i} = x_{t_{i-1}} - a \cdot \left[\alpha_x x_{t_{i-1}} + (1-\alpha_x)\hat{x}_{t_i}\right]\Delta t_{i-1} + \sigma_{1,t_{i-1}}\sqrt{\Delta t_{i-1}} \cdot Z_{1,t_i}^j$$

Predictor:

$$\hat{y}_{t_i} = y_{t_{i-1}} - b y_{t_{i-1}}\Delta t_{i-1} + \sigma_2\sqrt{\Delta t_{i-1}} \cdot Z_{2,t_i}^j$$

Corrector:

$$y_{t_i} = y_{t_{i-1}} - b \cdot \left[\alpha_y y_{t_{i-1}} + (1-\alpha_y)\hat{y}_{t_i}\right]\Delta t_{i-1} + \sigma_2\sqrt{\Delta t_{i-1}} \cdot Z_{2,t_i}^j$$

III. Zero Coupon Bond Price

$$P^j(t,T) = \exp\left[-\int_t^T \varphi_s\,ds - M(t,T) + \frac{1}{2}V(t,T)\right]$$
$$= \frac{P^M(0,T)}{P^M(0,t)}\exp\left\{\frac{1}{2}\left[V(t,T) - V(0,T) + V(0,t)\right] - M(t,T)\right\}$$

Where:

$$M(t,T) = \frac{1-e^{-a(T-t)}}{a}x_t + \frac{1-e^{-b(T-t)}}{b}y_t$$

$$\begin{aligned}
V(t,T) = \frac{1}{a^2}\Bigg\{&\left[\frac{2k_1\theta_1(\sigma_{1,t}-\theta_1)}{3k_1-\eta_1^2} + \frac{2k_1\theta_1^2}{2k_1-\eta_1^2}\right]\left[T-t+\frac{2}{a}e^{-a(T-t)} - \frac{1}{2a}e^{-2a(T-t)} - \frac{3}{2a}\right] \\
&+ \left[v_t - \frac{2k_1\theta_1^2}{2k_1-\eta_1^2}\right]\left[\frac{1-e^{-(2k_1-\eta_1^2)(T-t)}}{2k-\eta_1^2} + \frac{1-e^{-(2k_1+2a-\eta_1^2)(T-t)}}{2k_1+2a-\eta_1^2}\right. \\
&\left. - 2\,\frac{1-e^{-(2k_1+a-\eta_1^2)(T-t)}}{2k_1+a-\eta_1^2}\right] \\
&- \frac{2k_1\theta_1(\sigma_{1,t}-\theta_1)}{3k_1-\eta_1^2}\left[\frac{1-e^{-(3k_1-\eta_1^2)(T-t)}}{3k_1-\eta_1^2} + \frac{1-e^{-(3k_1+2a-\eta_1^2)(T-t)}}{3k_1+2a-\eta_1^2}\right. \\
&\left. - 2\,\frac{1-e^{-(3k_1+a-\eta_1^2)(T-t)}}{3k_1+a-\eta_1^2}\right]\Bigg\} + \frac{\sigma_2^2}{b^2}\left[T-t+\frac{2}{b}e^{-b(T-t)} - \frac{1}{2b}e^{-2b(T-t)} - \frac{3}{2b}\right] \\
&+ \frac{2\sigma_2}{ab}\Bigg\{\rho_\vartheta\theta_1\left[T-t-\frac{1-e^{-a(T-t)}}{a} - \frac{1-e^{-b(T-t)}}{b} + \frac{1-e^{-(a+b)(T-t)}}{a+b}\right] \\
&+ (\rho_t-\rho_\vartheta)(\sigma_{1,t}-\theta_1)\left[\frac{1-e^{-(\lambda+k_1)(T-t)}}{\lambda+k_1} - \frac{1-e^{-(a+\lambda+k_1)(T-t)}}{a+\lambda+k_1} - \frac{1-e^{-(b+\lambda+k_1)(T-t)}}{b+\lambda+k_1}\right. \\
&\left. + \frac{1-e^{-(a+b+\lambda+k_1)(T-t)}}{a+b+\lambda+k_1}\right] \\
&+ \theta_1(\rho_t-\rho_\vartheta)\left[\frac{1-e^{-\lambda(T-t)}}{\lambda} - \frac{1-e^{-(a+\lambda)(T-t)}}{a+\lambda} - \frac{1-e^{-(b+\lambda)(T-t)}}{b+\lambda} + \frac{1-e^{-(a+b+\lambda)(T-t)}}{a+b+\lambda}\right] \\
&+ \rho_\vartheta(\sigma_{1,t}-\theta_1)\left[\frac{1-e^{-k_1(T-t)}}{k_1} - \frac{1-e^{-(a+k_1)(T-t)}}{a+k_1} - \frac{1-e^{-(b+k_1)(T-t)}}{b+k_1}\right. \\
&\left. + \frac{1-e^{-(a+b+k_1)(T-t)}}{a+b+k_1}\right]\Bigg\}
\end{aligned}$$