Damiano Steger

TM- Exercise 2

2.1

(a) The function $f(\underline{x}) = \sqrt{x_1^2 + x_2^2}$ is not invertible since it is not injective: it's a projection from a 2D space to 1D space.

(b) $y = \sqrt{x_1^2 + x_2^2} + \varepsilon \Rightarrow \varepsilon = \sqrt{x_1^2 + x_2^2} - y$

$$\rightarrow p(y|x) = \frac{2}{\sqrt{2\pi}} \cdot \exp\left(- \frac{\left(\sqrt{x_1^2 + x_2^2} - y\right)^2}{2 \cdot \frac{1}{4}} \right)$$

(c) $\underline{y = 5}$, $p(x) = 1$

$$\rightarrow \underset{\underline{x}}{\operatorname{argmax}}\ p(x|y) = \underset{\underline{x}}{\operatorname{argmax}} \frac{p(y|x) \cdot p(x)}{p(y)} = \underset{\underline{x}}{\operatorname{argmax}}\ p(y|x)\, p(x)$$

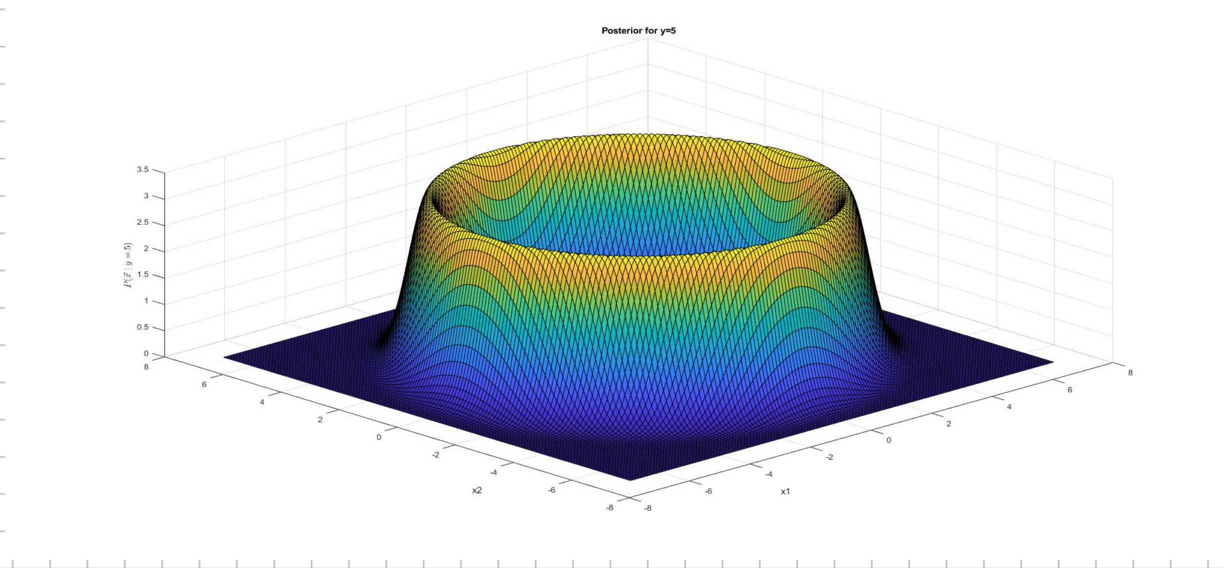
since does not depend on $p(y)$

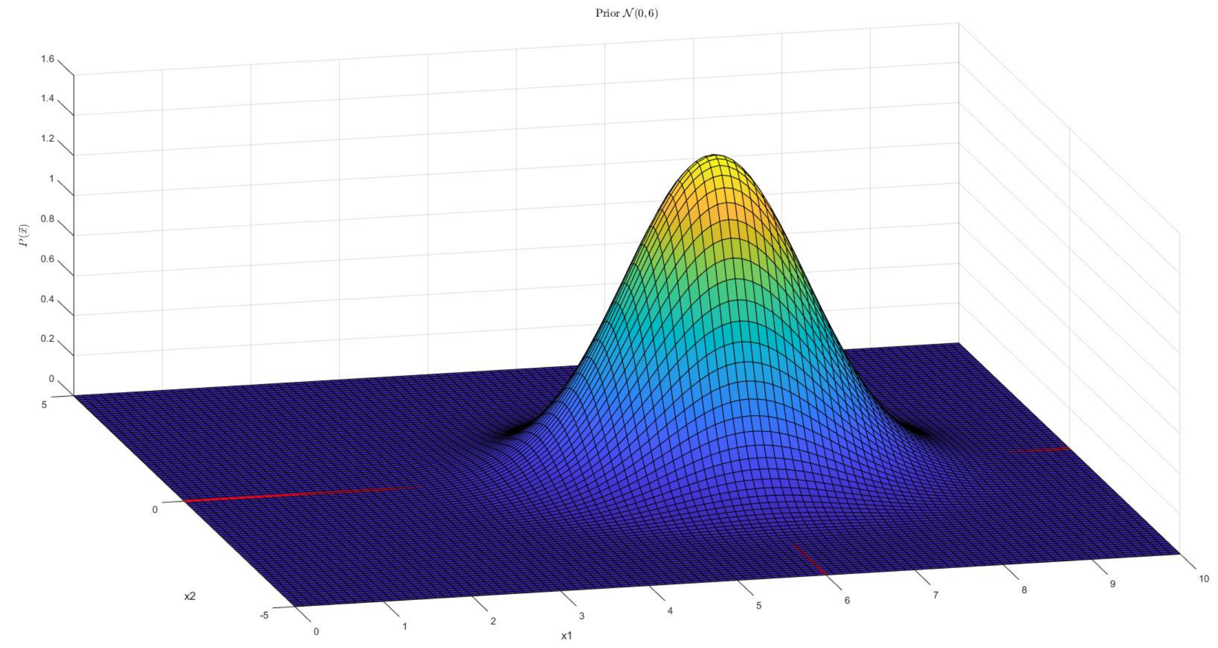
$$\rightarrow \underset{\underline{x}}{\operatorname{argmax}}\ p(y|x) \cdot 1 = \underset{x}{\operatorname{argmax}} \frac{2}{\sqrt{2\pi}} \cdot \exp\left(- \frac{\left(\sqrt{x_1^2 + x_2^2} - 5\right)^2}{1/2} \right)$$

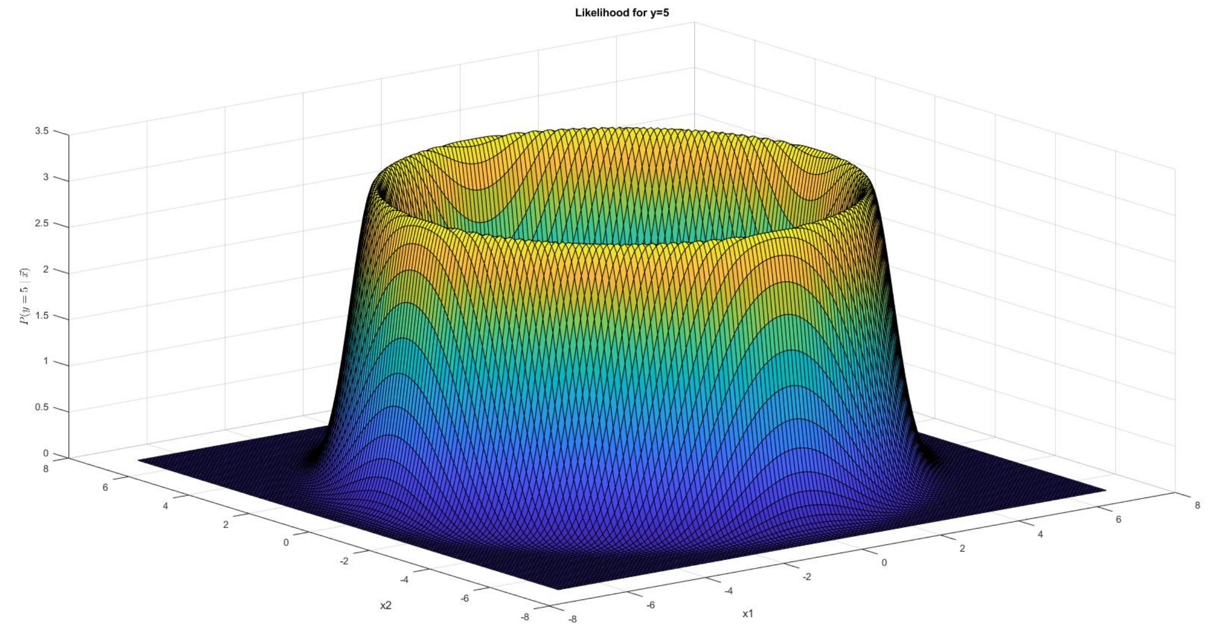
$$\rightarrow \max \text{ for } \sqrt{x_1^2 + x_2^2} = 5 \rightarrow \boxed{x_1^2 + x_2^2 = 25}$$

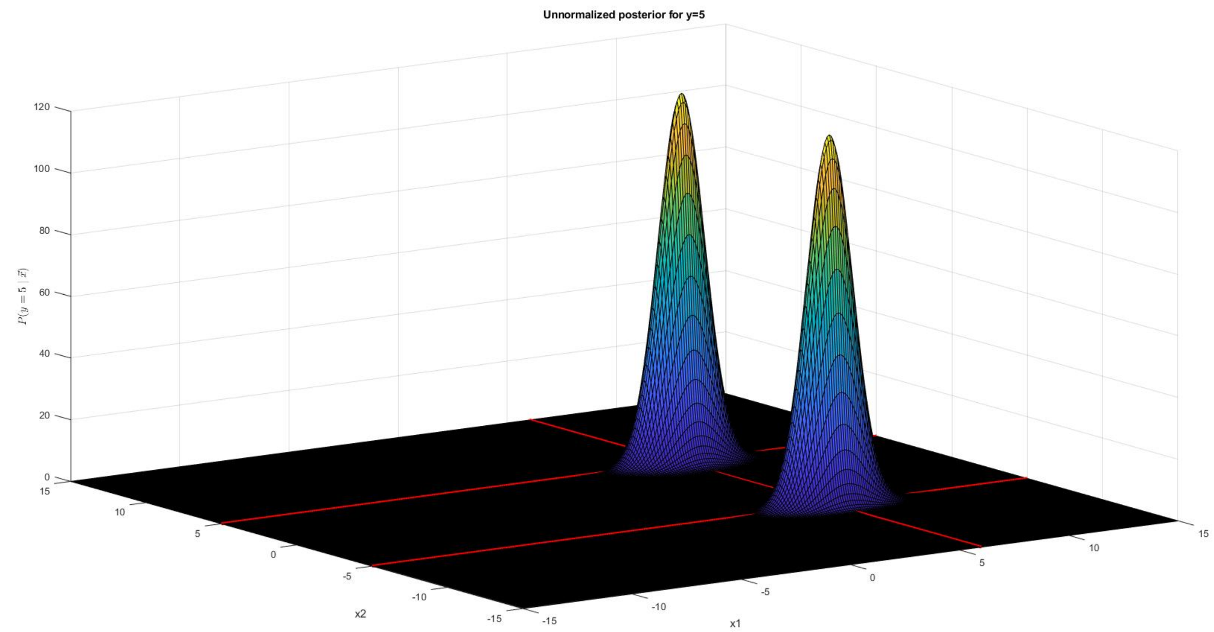
max value for all pairs of (x_1, x_2) sitting on the circle centered in 0 with radius 5.

(d)




Prior $\mathcal{N}(0,6)$
$P(\vec{x})$
x1
x2


Likelihood for y=5
$P(y=5 \mid \vec{x})$
x1
x2


Unnormalized posterior for y=5
$P(y=5 \mid \vec{x})$
x1
x2

$\underline{\mu} = [6 \;\; 0]^T$

(f) $p(\underline{x} \mid y=5) \propto p(y=5 \mid \underline{x})\, p(\underline{x}) = \frac{2}{\sqrt{2\pi}} \exp\left(-2\left(\sqrt{x_1^2+x_2^2}-5\right)^2\right) \cdot$

$\cdot \frac{1}{2\pi} \cdot \exp\left(-\underbrace{[\underline{x}-\underline{\mu}]^T \Sigma^{-1} [x-\underline{\mu}]}_{\circledast}\right)$ $\quad \circledast = (x_1-6)^2 + x_2^2$

$= \frac{2}{(2\pi)^{3/2}} \exp\left(-\left(2\left(\sqrt{x_1^2+x_2^2}-5\right)^2 + (x_1-6)^2 + x_2^2\right)\right)$

$\underset{\underline{x}}{\arg\max}\; p(\underline{x} \mid y=5) = \underset{\underline{x}}{\arg\min} \left[2\left(\sqrt{x_1^2+x_2^2}-5\right)^2 + (x_1-6)^2 + x_2^2\right] =$

$= \underset{x_2=0,\; x_1}{\arg\min} \left[\underbrace{2(x_1-5)^2 + (x_1-6)^2}_{(1)}\right]$

$\rightarrow \frac{d}{dx_1}(1) \overset{!}{=} 0 \rightarrow 2(2x_1-10) + 2x_1 - 12 = 0$

$\rightarrow 4x_1 - 20 + 2x_1 - 12 = 0 \rightarrow 6x_1 = 32 \rightarrow x_1 \cong 5{,}33$

$\rightarrow \text{MAP} = (\hat{x}_1, \hat{x}_2) \cong (5.33, 0)$

(g)