4.1

$$v(t) = h(t) \otimes \sigma(t) = \int_{-\infty}^{t} h(t-\tau)\sigma(\tau)\,d\tau$$

and $h(t) = \begin{cases} HKte^{-Kt} & , t \geq 0 \\ 0 & , \text{otherwise} \end{cases}$

$$\dot{v}(t) = \frac{d}{dt}\int_{-\infty}^{t} h(t-\tau)\sigma(\tau)\,d\tau = \int_{-\infty}^{t} \frac{\partial}{\partial t} h(t-\tau)\sigma(\tau)\,d\tau + \underbrace{h(t-t)}_{=0}\sigma(t)\underbrace{\frac{dt}{dt}}_{=1}$$

Leibniz rule

$$\underbrace{- h(t-(-\infty))\,\sigma(-\infty)\,\frac{d(-\infty)}{dt}}_{=0}$$

$$= \int_{-\infty}^{t} \frac{\partial}{\partial t}\Big(h(t-\tau)\sigma(\tau)\Big)d\tau$$

product rule

$$= \int_{-\infty}^{t} \frac{\partial}{\partial t}\Big(h(t-\tau)\Big)\,\sigma(\tau)\,d\tau + \underbrace{\int_{-\infty}^{t} h(t-\tau)\frac{\partial}{\partial t}\sigma(\tau)\,d\tau}_{=0}$$

product rule

$$= \int_{-\infty}^{t} HK\,e^{-K(t-\tau)}\sigma(\tau)\,d\tau - \int_{-\infty}^{t} HK^2(t-\tau)\,e^{-K(t-\tau)}\sigma(\tau)\,d\tau$$

$$= HKe^{-Kt}\int_{-\infty}^{t} e^{K\tau}\sigma(\tau)\,d\tau \quad - K\int_{-\infty}^{t} h(t-\tau)\sigma(\tau)\,d\tau$$

$$\dot{v}(t) = \underbrace{\qquad ''\qquad}_{①} \quad \underbrace{-Kv(t)}_{②}$$

product rule

$$\boxed{\frac{d②}{dt} = -K\dot{v}(t)}$$

$$\frac{d①}{dt} = \frac{d}{dt}HKe^{-Kt}\cdot\int_{-\infty}^{t} e^{K\tau}\sigma(\tau)\,d\tau + HKe^{-Kt}\cdot\frac{d}{dt}\left(\int_{-\infty}^{t} e^{K\tau}\sigma(\tau)\,d\tau\right)$$

$$= -HK^2e^{-Kt}\int_{-\infty}^{t} e^{K\tau}\sigma(\tau)\,d\tau + HKe^{-Kt}\cdot\left(\underbrace{\int_{-\infty}^{t}\frac{\partial}{\partial t}e^{K\tau}\sigma(\tau)\,d\tau}_{=0} + e^{Kt}\sigma(t)\frac{dt}{dt}\right.$$

Leibniz rule for 2nd part

$$\left.\underbrace{- e^{-K\cdot\infty}\,\sigma(-\infty)\,\frac{d(-\infty)}{dt}}_{=0}\right)$$

$$= -HK^2e^{-Kt}\int_{-\infty}^{t} e^{K\tau}\sigma(\tau)\,d\tau + HK\underbrace{e^{-Kt}e^{Kt}}_{=1}\sigma(t)$$

$$= -K\cdot① + HK\sigma(t)$$

$$t) = H\kappa\sigma(t) - H\kappa^2 \int_{-\infty}^{t} e^{-\kappa(t-\tau)} \sigma(\tau)\, d\tau - \kappa\dot{v}(t)$$

$$= H\kappa\sigma(t) - \kappa \cdot ① - \kappa\dot{v}(t)$$

$$= H\kappa\sigma(t) - \kappa\dot{v}(t) - \kappa^2 v(t) - \kappa\dot{v}(t)$$

$$\boxed{① = \dot{v}(t) - ② = \dot{v}(t) + \kappa v(t)}$$

$$\Rightarrow \ddot{v}(t) = H\kappa\sigma(t) - 2\kappa\dot{v}(t) - \kappa^2 v(t)$$

4.2

a) $\ddot{x} = -f\dot{x} - k^2 x + u(t)$

$$\begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -k^2 & -f \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

b) $u(t) = a \cdot z(t)$ and $\ddot{z} = -f_z \dot{z} - k_z^2 z + u_z(t)$

$$\vec{x} = \begin{pmatrix} x \\ \dot{x} \\ z \\ \dot{z} \end{pmatrix} \qquad \dot{\vec{x}} = \begin{pmatrix} \dot{x} \\ \ddot{x} \\ \dot{z} \\ \ddot{z} \end{pmatrix}$$

$$\dot{\vec{x}} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ -k^2 & -f & a & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -k_z^2 & -f_z \end{pmatrix}}_{A} \vec{x} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ u_z \end{pmatrix}}_{\vec{u}}$$

the entries for A and $\vec{u}$ can be seen above

c) $\sigma(t) = a \cdot s(v_z(t)) + u(t)$

$$s(v) = \frac{1}{1 + e^{-rv}} - \frac{1}{2}$$

from equation (3) we know
$\ddot{v}(t) = Hk\,\sigma(t) - 2k\dot{v}(t) - k^2 v(t)$
$\Rightarrow$ we can replace $\sigma(t)$ by the term above :

$$\ddot{v}(t) = Hk(a \cdot s(v_z(t)) + u(t)) - 2k\dot{v}(t) - k^2 v(t)$$

to make $s(v)$ simpler we use Taylor expansion around 0

$$s(v) \approx s(0) + \frac{s'(0)}{1} \cdot v + \frac{s''(0)}{2} v^2 + \dots$$

$$\approx 0 + \frac{r e^{r0}}{(e^{r0}+1)^2} \cdot v - \frac{r^2(e^{r0}-1) \cdot e^{r \cdot 0}}{(e^{r0}+1)^3} \cdot v^2 + \dots$$

$$\approx 0 + \frac{r}{4} \cdot v - 0 \cdot v^2 + \dots$$

$$\approx \frac{rv}{4}$$

$$\Rightarrow \ddot{v}(t) = Hk\left(a \cdot \frac{r \cdot v_z(t)}{4} + u(t)\right) - 2k\dot{v}(t) - k^2 v(t)$$

we now define:

$$\vec{v} = \begin{pmatrix} v \\ \dot{v} \\ v_z \\ \dot{v}_z \end{pmatrix} \quad \text{and} \quad \dot{\vec{v}} = \begin{pmatrix} \dot{v} \\ \ddot{v} \\ \dot{v}_z \\ \ddot{v}_z \end{pmatrix}$$

since v_z is also defined by equ. (3):

$$\ddot{v}_z(t) = Hk\left(a \cdot \frac{r \cdot v_z(t)}{4} + u(t)\right) - 2k\dot{v}_z(t) - k^2 v_z(t)$$

$$= \left(\frac{Hkar}{4} - k^2\right) v_z(t) - 2k\dot{v}_z(t) + Hku(t)$$

from that we can derive the linear system

$$\dot{\vec{v}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -k^2 & -2k & \frac{Hkar}{4} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \left(\frac{Hkar}{4} - k^2\right) & -2k \end{pmatrix} \cdot \vec{v} + \begin{pmatrix} 0 \\ Hk \\ 0 \\ Hk \end{pmatrix} u(t)$$

$\Rightarrow$

$$f = f_z = 2k$$

$$a_{ERP} = \frac{Hkar}{4}$$

$$K = K$$

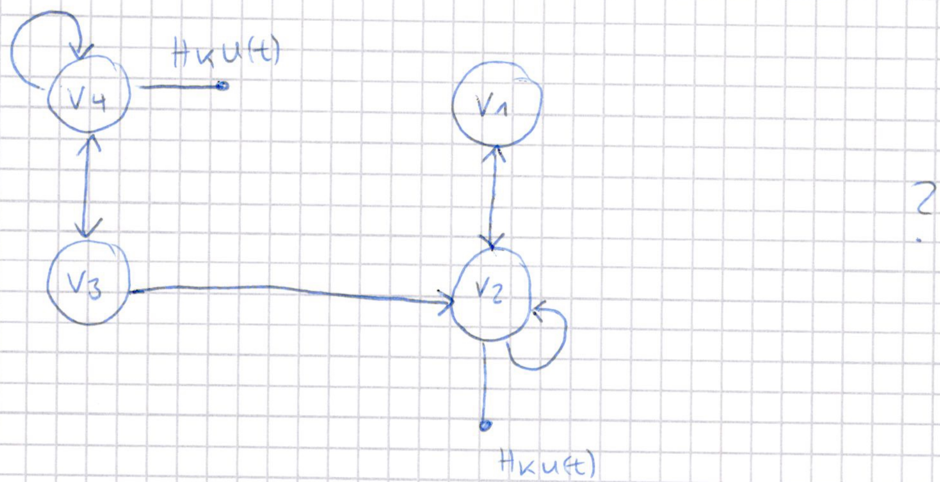
$$-K_z^2 = \frac{Hkar}{4} - K^2 \Rightarrow K_z = \sqrt{K^2 - \frac{Hkar}{4}}$$

$$\left(= a_{ERP} - K^2\right) \qquad = \sqrt{\frac{4K^2 - Hkar}{4}}$$

$$= \frac{1}{2}\sqrt{K^2\left(4 - \frac{Har}{K}\right)}$$

$$= \frac{K}{2}\sqrt{4 - Har/K}$$

d)



4.3

c) $\vec{y}(t) = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_3 \end{pmatrix}$ and $\vec{x} = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix}$

$$\vec{y}(t) = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{=L} \vec{x}(t)$$

L would be the identity matrix if $\vec{y}(t)$ and $\vec{x}(t)$ are defined as above. One can see that the activities of x_1 and x_3 are only dependent on states x_2, x_4 and not on the driving input.