Monash University
Faculty of Information Technology

Lecture 2
Propositional Logic

Slides by David Albrecht (2011), modified by Graham Farr (2013).

FIT2014 Theory of Computation

Overview

- Propositions
- Connectives
- Tautologies
- Arguments
- Representing Knowledge

Definition

A **proposition** is

- A statement

which is

- Either true or false.

Examples

- **2 + 2 = 4**
  - is a proposition which is **true**.
- **The moon is made of cheese**
  - is a proposition which is **false**.
- **It will rain tomorrow**
  - is a proposition.
- **This statement is false**
  - is **not** a proposition.
- **Vote for Mickey Mouse**
  - is **not** a proposition.

Negation

P : **I have three children.**
¬P : I do **not** have three children.

Other Notation: ~P , $\overline{P}$

Truth table:

P	¬P
F	**T**
T	**F**

Connectives

- And ∧ (&)
- Or ∨
- Implies ⇒ (→)
- Equivalence ⇔ (↔)

Conjunction

P : **This subject is interesting.**
Q : **I am tired.**
P ∧ Q :

- This subject is interesting **and** I am tired.
- This subject is interesting **although** I am tired.
- This subject is interesting **but** I am tired.

P	Q	P ∧ Q
F	F	**F**
F	T	**F**
T	F	**F**
T	T	**T**

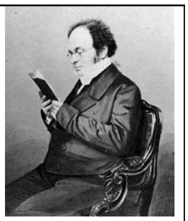
Disjunction

P : **Sue is a football player.**
Q: **Bob is lazy.**
P ∨ Q: Sue is a football player **or** Bob is lazy.
Note: Disjunction is inclusive.

P	Q	P∨Q
F	F	**F**
F	T	**T**
T	F	**T**
T	T	**T**

De Morgan's Laws

¬(P ∨ Q) = ¬P ∧ ¬Q
¬(P ∧ Q) = ¬P ∨ ¬Q



Augustus De Morgan (1806-1871)
http://www-history.mcs.st-andrews.ac.uk/Biographies/De_Morgan.html

Can prove using truth tables.
E.g.:

P	Q	P∨Q	¬(P∨Q)	¬P	¬Q	¬P∧¬Q
F	F	**T**	**T**	T	T	**T**
F	T	**T**	**F**	T	F	**F**
T	F	**T**	**F**	F	T	**F**
T	T	**T**	**F**	F	F	**F**

Conditional

P: **It is Tuesday.**
Q: **We are in Belgium.**
P ⇒ Q:

- **If** it is Tuesday **then** we are in Belgium.
- It's being Tuesday **implies** we are in Belgium.
- It's Tuesday **only if** we are in Belgium.
- It's being Tuesday is **sufficient** for us to be in Belgium.

Conditional Table

P	Q	P ⇒ Q
F	F	**T**
F	T	**T**
T	F	**F**
T	T	**T**

Grace Hopper (1906-1992)
http://www.cs.uoregon.edu/groups/wics/projects.php

P: **Grace is a COBOL expert.**
Q: **Grace can program.**

P ⇒ Q:
If Grace is a COBOL expert **then** she can program.

Biconditional

P : **The world will blow up.**
Q : **KAOS rules the world.**
P ⇔ Q :

- The world will blow up **if and only** if KAOS rules the world.
- KAOS ruling the world is a **necessary and sufficient** condition that the world will blow up.

Biconditional Table

P	Q	P $\Leftrightarrow$ Q
F	F	**T**
F	T	**F**
T	F	**F**
T	T	**T**

P: **It will rain tomorrow.**

Q: **I will wear a raincoat tomorrow.**

P $\Leftrightarrow$ Q:

I will wear a raincoat tomorrow **if and only if** it rains.

Interpretation

The truth table for a statement provides a record for all possible interpretations of that statement.

Example:

S: **If the price is less than $30 and I have at least $50, then I will buy that CD.**

- P: The price of the CD is less than $30.
- L: I have at least $50.
- B: I will buy that CD.
- S: **(P ∧ L) ⇒ B**

P	L	B	S
F	F	F	**T**
T	F	F	**T**
F	T	F	**T**
T	T	F	**F**
F	F	T	**T**
T	F	T	**T**
F	T	T	**T**
T	T	T	**T**

Definitions

A **tautology** is a statement whose interpretation is always true.

We say two statements P and Q are **logically equivalent** if they always have the same interpretation.

- **Their truth tables are the same.**
- **i.e.,** P $\Leftrightarrow$ Q is a ....

Examples of logical equivalence

P $\Rightarrow$ Q is logically equivalent to ¬P ∨ Q

P $\Leftrightarrow$ Q is logically equivalent to (¬P ∨ Q) ∧ (P ∨¬Q)

These can be proved using truth tables.

Fitness Example

S: **If I exercise then I will get fit, and I do exercise, so I will get fit.**

E: **I do exercise**.

F: **I will get fit**.

S: **(E ⇒ F) ∧ E ⇒ F**

Definition

An **argument** consists of:

- A set of propositions, $\mathbf{P_1, \ldots, P_n}$, called the **premises.**
- Another proposition, **C**, called the **conclusion.**

An argument is called **valid** if the statement

$$\mathbf{P_1 \wedge \ldots \wedge P_n \Rightarrow C}$$

is a tautology.

Mary's Exam Example

Today Mary has a Law exam or a Computer Science exam or both. She doesn't have a Law exam. Therefore she must have a Computer Science exam.

L: **Mary has a Law exam today**.

C: **Mary has a Computer Science exam today**.

Premises: $\mathbf{L \vee C, \neg L}$

Conclusion: **C**

Wumpus World

[based on a video game by Gregory Yob, c1972]

- The idea of the game is to find the gold.
- The cave has rooms that lie in a grid.
- Dangers
  - **The Wumpus** (You can smell a Wumpus in the next room or the same room)
  - **Pits** (You can feel a breeze in the next room)
  - **Bats** (You can hear the bats in the next room)
- You can use to kill a Wumpus with an arrow.

Example

			P
W	G	P	
			B
S		P	

- W represents the Wumpus
- P represents a Pit
- B represents Bats
- S is the Starting Position
- G represents where the gold is.

A Game

- No stench or breeze in square 1,1
  - **Therefore Wumpus is not in 1,2 or 2,1**
- Suppose you move to square 2,1 and detect a breeze and no stench there.
  - **Therefore there is a Pit in either 2,2 or 3,1**
- So you go back and up to square 1,2 and detect a stench and no breeze.
  - **Therefore the Wumpus is in square 1,3.**

Notation

- $W_{1,1}$
  - **The Wumpus is in square 1,1.**
- $S_{1,2}$
  - **A stench was detected in square 1,2.**
- $B_{2,1}$
  - **A breeze was detected in square 2,1.**

Etc.

Knowledge

- P_1: $\neg S_{1,1} \wedge \neg B_{1,1}$
- P_2: $\neg S_{2,1} \wedge B_{2,1}$
- P_3: $S_{1,2} \wedge \neg B_{1,2}$
- P_4: $\neg S_{1,1} \Rightarrow \neg W_{1,1} \wedge \neg W_{1,2} \wedge \neg W_{2,1}$
- P_5: $\neg S_{2,1} \Rightarrow \neg W_{1,1} \wedge \neg W_{2,1} \wedge \neg W_{2,2} \wedge \neg W_{3,1}$
- P_6: $\neg S_{1,2} \Rightarrow \neg W_{1,1} \wedge \neg W_{1,2} \wedge \neg W_{2,2} \wedge \neg W_{1,3}$
- P_7: $S_{1,2} \Rightarrow W_{1,3} \vee W_{1,2} \vee W_{2,2} \vee W_{1,1}$

Wumpus Argument

- Want to obtain the conclusion: $W_{1,3}$
- Need to show:

  $P_1 \wedge P_2 \wedge P_3 \wedge P_4 \wedge P_5 \wedge P_6 \wedge P_7 \Rightarrow W_{1,3}$

  is a tautology.
- Truth Table has 2^{12} = **4096** rows.

Problems

- Need to introduce a lot of propositions to represent any useful knowledge.
- Using truth tables to show validity requires:
  - **Exponential Space**
  - **Exponential Time**

History

- George Boole
  1815-1864

http://www-history.mcs.st-andrews.ac.uk/Biographies/Boole.html

Revision

- Propositions
- Connectives
- Tautologies & Logical Equivalence
- Arguments

Reading

- Sipser, pp. 14-15, 21-25.