FIT2014 - Theory Of Computation

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1 Lecture 1

- Natural Languages, English, Chinese, French, Auslan, etc.
- Programming Languages
- Mathematics
- State Diagrams
- Music
- Feynman Diagrams

Components of language include an alphabet, which are the basic elements that a language is made up of. This is a set of letters of characters. The rules (or grammar) tell you what words belong to the language. They are also the syntax. The meaning of the language is the semantics.

A word is a finite string of characters belonging to the language, while a language is a set of words. For english, the alphabet is a, b, c ... z A B ... Z. It also includes English, punctuation marks and blank space. While the words is in a standard dictionary aardvark, zygote, woot. An empty word is

For Java, the alphabet is unicode characters, while the words are programs.

If we have a language that is made of a and b, the words are a b ab abbb bab.... A lot of these words can be expressed by a^2.

Even-Even - All the strings that contain an even number of a's and an even number of b's. IE, aa, bb, aaaa,aa,bb,

Palindromes - All the strings which are the same if they are spelt backwards. IE, a, b, aa, bb, aaa,aba, bbb

Double word - All the strings are formed by two copies of a string joined together. IE, aa, bb, aaaa, abab, baba.

Symbols

- $\subset$ Subset
- $\subseteq$ Subset Equals
- $\in$ Contains
- ϵ Empty Word
- ϕ Empty Language

Definition: Specifies the precise meaning of certain things Theorem: A mathematical statement that has been proved to be true. Has some close, but less significant relatives in Proposition and Lemma. Proof: A step-by-step argument that establishes, logically and with certainty, that something is true.

Existential statements: Statements that require one suitable example for a proof (There exists a palindrome in English). Universal Statement: Statements where you need to cover every possible case. One way is to go through every possibility in turn and check each one. However the number of things to check may be huge, or infinite. So usually we want to reason in a way can apple to many different possibilities at once.

2 Lecture 2 - Propositional Logic

A proposition is a statement which is either true or false. IE, $2 + 2 = 4$, or The moon is made of cheese. This statement is false, or Vote for Mickey Mouse, are not propositions.

2.1 Connectives

2.1.1 Negation

Symbol: $\neg$

P: I have three children.

$\neg P$: I do not have three children

P	¬ P
F	T
T	F

2.1.2 Conjuction (and)

Symbol: & or $\wedge$

P: This subject is interesting. Q: I am tired. P^Q : This subject is interesting and I'm tired. This subject is interesting although I am tried. This subject is interesting but I am tired.

P	Q	P & Q
F	F	F
F	T	F
T	F	F
T	T	T

2.1.3 Disjunction (Or)

Symbol: $\vee$

P: Sue is a football player Q: Bob is lazy. P $\vee$ Q: Sure is a football player or Bob is lazy.

P	Q	P & Q
F	F	F
F	T	T
T	F	T
T	T	T

2.1.4 Conditional

Symbol: $\Rightarrow$

P: It is tuesday. Q: We are in Belgium $P \Rightarrow Q$ If it is Tuesday then we are in Belgium.
It's being Tuesday implies we are in Belgium.
It's Tuesday only if we are in Belgium.
It's being Tuesday is sufficient for us to be in Belgium.

P	Q	$P \Rightarrow Q$
F	F	T
F	T	T
T	F	F
T	T	T

2.1.5 Biconditional

Symbol: $\Leftrightarrow$

P: It will rain tomorrow.
Q: I will wear a raincoat tomorrow.
$P \Leftrightarrow Q$: I will wear a raincoat tomorrow if and only if it rains.

P	Q	$P \Rightarrow Q$
F	F	T
F	T	F
T	F	F
T	T	T

2.2 De Morgan's Laws

$\neg(P \vee Q) = \neg P \wedge \neg Q$
$\neg(P \wedge Q) = \neg P \vee \neg Q$

This can be proven using truth tables.

P	Q	$P \vee Q$	$\neg(P \vee Q)$
F	F	F	T
F	T	T	F
T	F	T	F
T	T	T	F

$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$
T	T	T
T	F	F
F	T	F
F	F	F

2.3 Definitions

- Tautology - A statement whose interpretation is always true.

2.4 Logically Equivalent

Logically Equivalent, two statements that have the same truth table. For example:
$P \Rightarrow Q$ is logically equal to $\neg P \vee Q$.
$P \Leftrightarrow Q$ is logically equal to $(\neg P \vee Q) \wedge (P \vee \neg Q)$.

2.5 Solving Real Problems

Definition

An **argument** consists of:

- A set of propositions, $\mathbf{P_1, \ldots, P_n}$, called the **premises.**
- Another proposition, **C**, called the **conclusion.**

An argument is called **valid** if the statement

$$\mathbf{P_1 \wedge \ldots \wedge P_n \Rightarrow C}$$

is a tautology.

Mary's Exam Example

Today Mary has a Law exam or a Computer Science exam or both. She doesn't have a Law exam. Therefore she must have a Computer Science exam.

L: **Mary has a Law exam today**.

C: **Mary has a Computer Science exam today**.

Premises: $\mathbf{L \vee C, \neg L}$

Conclusion: **C**

Wumpus World

[based on a video game by Gregory Yob, c1972]

- The idea of the game is to find the gold.
- The cave has rooms that lie in a grid.
- Dangers
  - **The Wumpus** (You can smell a Wumpus in the next room or the same room)
  - **Pits** (You can feel a breeze in the next room)
  - **Bats** (You can hear the bats in the next room)
- You can use to kill a Wumpus with an arrow.

Example

			P
W	G	P	
			B
S		P	

- W represents the Wumpus
- P represents a Pit
- B represents Bats
- S is the Starting Position
- G represents where the gold is.

A Game

- No stench or breeze in square 1,1
  - **Therefore Wumpus is not in 1,2 or 2,1**
- Suppose you move to square 2,1 and detect a breeze and no stench there.
  - **Therefore there is a Pit in either 2,2 or 3,1**
- So you go back and up to square 1,2 and detect a stench and no breeze.
  - **Therefore the Wumpus is in square 1,3.**

Notation

- $W_{1,1}$
  - **The Wumpus is in square 1,1.**
- $S_{1,2}$
  - **A stench was detected in square 1,2.**
- $B_{2,1}$
  - **A breeze was detected in square 2,1.**

Etc.

Knowledge

- P_1: $\neg S_{1,1} \wedge \neg B_{1,1}$
- P_2: $\neg S_{2,1} \wedge B_{2,1}$
- P_3: $S_{1,2} \wedge \neg B_{1,2}$
- P_4: $\neg S_{1,1} \Rightarrow \neg W_{1,1} \wedge \neg W_{1,2} \wedge \neg W_{2,1}$
- P_5: $\neg S_{2,1} \Rightarrow \neg W_{1,1} \wedge \neg W_{2,1} \wedge \neg W_{2,2} \wedge \neg W_{3,1}$
- P_6: $\neg S_{1,2} \Rightarrow \neg W_{1,1} \wedge \neg W_{1,2} \wedge \neg W_{2,2} \wedge \neg W_{1,3}$
- P_7: $S_{1,2} \Rightarrow W_{1,3} \vee W_{1,2} \vee W_{2,2} \vee W_{1,1}$

Wumpus Argument

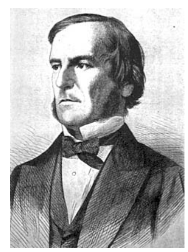
- Want to obtain the conclusion: $W_{1,3}$
- Need to show:
  $\mathbf{P_1 \wedge P_2 \wedge P_3 \wedge P_4 \wedge P_5 \wedge P_6 \wedge P_7 \Rightarrow W_{1,3}}$
  is a tautology.
- Truth Table has $\mathbf{2^{12} = 4096}$ rows.

Problems

- Need to introduce a lot of propositions to represent any useful knowledge.
- Using truth tables to show validity requires:
  - **Exponential Space**
  - **Exponential Time**

History

- George Boole
  1815-1864



http://www-history.mcs.st-andrews.ac.uk/Biographies/Boole.html

Revision

- Propositions
- Connectives
- Tautologies & Logical Equivalence
- Arguments

Reading

- Sipser, pp. 14-15, 21-25.

3 Lecture 3 - Predicate Logic, Prolog

In predicate logic, there are 3 types of objects, Constant symbols, Function symbols and Individual Variables. Constant symbols are names which refer to exactly one object (like socrates, medusa, 1 or 2). Function symbols relate some objects to exactly one object (motherOf, kingOf, plus, times). They can have complex names. Individual variables are variables which can relate to any object (X, Y).

A term is a logical expression which refers to an object.

The equality symbol (=) is used to state that two objects are the same (IE rebecca = rebecca, fatherOf(john) = henry.

An example of predicate logic is:

- All men are mortal.
- Socrates is a man.
- Thereforce Socrates is mortal.

The objects in this example are socrates and set of people, while the properties are man and mortal.

3.1 Sentences

Atomic sentences: A predicate symbol followed by a list of terms in brackets. EG taller(motherOf(claire), mary). Complex sentences: Atomic sentences joined together by logical connectives man(socrates) $\Rightarrow$ mortal(socrates).

3.2 Quantification

Universal quantification means that the statement applies to all objects. $\forall$ = For All

An example is All Dogs Are Happy:
$\forall x(dog(x) \Rightarrow happy(x))$

No dog is happy/All dogs are unhappy.
$\forall x(dog(x) \Rightarrow \neg happy(x))$

For statements that apply only about some object, use $\exists$

Some dogs are happy.
$\exists X(dog(x) \wedge happy(x))$

Some dogs are not happy.
$\exists X(dog(x) \wedge \neg happy(x))$

These quantifiers are related.
$\neg\forall x$ is equal to $\exists y\neg$

"Not all dogs are happy" is the same as "There exists an unhappy dog".

3.3 More examples

There is an app which is loved by every student. This means that there is an app X and if Y is a student, then Y loves it.
$\exists X(app(X) \wedge \forall Y(student(Y) \Rightarrow loves(Y, X)))$

Every student loves some app. For every student Y there exists an app X that she loves.
$\forall Y(student(Y) \Rightarrow \exists X(app(X) \wedge loves(Y, X)))$

4 Lecture 4 - Proofs

There is no systemic method for finding proofs for theorems. Discovering proofs is an art as well as a science. There are proofs of many kind, the main being Proof by Construction, Proof by cases, Proof by contradiction, Proof by Induction. Some proofs are a mix of these.

4.1 Proof by Construction

Proof by construction is often known as proof by example. It can be used where the theorem asserts the existence of some object with a specific property. In this case you can just give the example and show it has the property.

4.2 Proof by Cases

This is also known as proof by exhaustion, or if there are a lot of cases, brute force. First you identify a number of different cases which cover all possibilities, then prove the theorem for each of these cases.

4.3 Proof by Contradiction

Start by assuming the negation of the statement you want to prove. Deduce from this a contradiction. Therefore the statement must be true.

5 Lecture 5 - Regular Expressions

The language with no words is ϕ Empty Language.
The language consisting only of the empty words is ϵ Empty Word
The language { w } consists only of the single word w.

Alternatives are indicated by $\cup$, so $1 \cup 2 \cup 3 \cup 4$ is {1,2,3,4}.

Groupings are indicated by ( ), so the expression $(ab \cup ba)(e \cup g)$ maps to $\{abe, bag, bas, bag\}$
Finite languages consist of a finite number of words, like $\{ababa, abb., abbaba\}$.
This maps to the regular expression $abaaba \cup abbbba \cup abbaba$,
which maps to $ab(aa \cup bb \cup ba)ba$,
which maps down to $ab(a \cup b)aha \cup abb(b \cup a)ba$.
The asterisk (*) is called a Kleene Star, which means zero or more times.
$a*$ represents $\{\epsilon, a, aa, aaa, aaaa, ...\}$
$(ba)*$ represents $\{\epsilon, ba, baba, bababa, babababa, ...\}$

6 Lecture 6

7 Lecture 7

8 Lecture 8