NOTES ON EPIDEMIC MODELS

These notes serve as a basis for a graduation project on novel epidemiological models. UCR.

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CONTENTS

1. ADAPTIVE BEHAVIOR MODEL

1.1. **Model review.** We describe a new modeling technique presented in [2], which gives an adaptive approach to the construction of an epidemiological system. The classical approach is taking the system:

$$\frac{dS}{dt} = -C(\cdot)\beta SI/N$$
$$\frac{dI}{dt} = C(\cdot)\beta SI/N - \nu I$$
$$\frac{dZ}{dt} = \nu I.$$

Where:

- A population of N individuals is divided in three compartments: $N = S + I + R$. Here S are the susceptible individuals, I are the infected and R are the recovered individuals.
- β represents the likelihood that contact with an infected individual yields infection.
- ν is the rate of recovery.
- $C(\cdot)$ is the rate that susceptible contact infected, which means that $C(\cdot)\beta$ is the rate that susceptible individuals become infected.

In the classical setting, either $C(\cdot) = c$ (contacts are constant) or $C(\cdot) = cN$ (contacts are proportional to N). In the adaptive setting, the idea is that $C(\cdot)$ depends on the incentives different individuals have to vary their number of contacts. The costs and benefits of individual contact vary across health status.

The proposal is then to divide the individuals by health type. Let $Y = \{s, i, z\}$. For $h \in Y$ denote C^h the expected number of contacts made by an individual of type h. For $m, n \in Y$ we define

$$C^{mn}(\cdot) = C^m C^n N/(SC^s + IC^i + ZC^z).$$

The rate of contact between individuals of types m and n. In here C^m **is a choice** made by individuals of type m. In the classical model $C(\cdot) = C^{si}$.

People engage in contacts because there is a certain utility to gain from them. The adaptive approach models the utility for an individual of type $h \in Y$ depending on the current time, therefore we have an utility

$$u_t^h = u_t^h(C_t^h),$$

where t is the current time, and C_t^h is the expected number of contacts of an individual of type h made at time t. u_h^t is **the utility of making contacts for an individual of type** h **at time** t.

The utility function should be concave and should have a single peak with respect to the number of contacts, but should decrease with infection.

An example of an utility function provided in [2] is

$$u_t^h = (b^h C_t^h - (C_t^h)^2)^\gamma - a^h,$$

where γ, b^h, a^h are fixed parameters, with $b^s = b^z \geqslant b^i \geqslant 0$, $a^z = a^s = 0$, $\gamma > 0$, $a^i > 0$. **Intiutively, this means that during the infection period, the utility has a term that pauses it's increment. Note that for each state utility has a peak with respect to C_t^h.**

Recovered (and Immune) and Infected individuals: If the individual doesn't think that a change in their contacts will affect their health status, then for a given time t, the best thing would be to choose C_t^h such that u_t^h is maximized. This happens with individuals of types i and z. The optimal choice is $C_h^{t*} = 0.5 b^h$.

Susceptible individuals: The number of contacts a susceptible individual engages in might affect their health status, so the optimal choice of contacts C_t^{s*} is subjected to planning towards the future. Here is where the adaptive decision comes in, as a factor of future utility.

Let P_t^i the probability that an s-type individual becomes infected at time t. This depends on the current state of things and the selection of C_t^s, as

$$P_t^i = 1 - e^{-\beta I_t C_t^s C_t^{i*} / (S_t C_t^{s*} + I_t C_t^{i*} + Z_t C_t^{z*})}, \tag{1.1}$$

where C_t^{s*} is the optimal choice of other susceptible individuals the present susceptible individual might encounter.

To find the optimal C_t^{s*} we maximize the value function

$$V_t(s) = \max_{C^s \in X} \left\{ u_t^s(C_t^s) + \delta \left[(1 - P_t^i) V_{t+1}(s) + P_t^i V_{t+1}(i) \right] \right\}, \tag{1.2}$$

where X is the range of possible contacts, δ is a discount factor, $V_{t+1}(s)$ is the present value of expected utility if the individual remains susceptible and $V_{t+1}(i)$ the present value of expected utility if the individual becomes infected.

Solving equation (1.2) gives the first order condition:

$$\frac{\partial u_t}{\partial C_t^s} = \delta(V_{t+1}(s) - V_{t+1}(i)) \left(\frac{\partial P_t^i}{\delta C_t^s}\right). \tag{1.3}$$

The idea on how to select the optimal C_t^{s*} at time t depends on the current state of things and the previsions the individual does for the future. The adaptive approach proposes a continuous update of the selection made across time.

For each $t < \tau - 1$, the individual will solve (1.2). For that they will need to solve (1.3). This requires knowledge of $V_{t+1}(i)$, which it's modeled like:

$$V_{t+1}(i) = u_t^z(C_t^{z*}) \left[\left(\frac{1-\delta^{\tau+1}}{1-\delta}\right) - \left(\frac{1-(\delta(1-P^z))^{\tau+1}}{1-\delta(1-P^z)}\right)\right], \tag{1.4}$$

where τ is the **planning period** and $P^z = 1 - e^{-\nu}$ is the probability of recovery.

1.2. **Utility calculations for implementation.** Here we provide some notes on how to implement the adaptive model. First we solve some more explicitly some of the equations presented in the paper.

- In equation (1.1) we have that

$$P_t^i = 1 - \exp\left(-\frac{\beta I_t C_t^s C_t^{i*}}{\phi(t)}\right) = 1 - \exp\left(-\frac{0.5 \cdot \beta b^i \cdot I_t C_t^s}{\phi(t)}\right),$$

  where $\phi(t) = S_t C_t^{s*} + I_t C_t^{i*} + Z_t C_t^{z*} = S_t C_t^{s*} + 0.5 b^i \cdot I_t + 0.5 b^z \cdot Z_t$. Therefore

$$\frac{\partial P_t^i}{\partial C_t^s} = \beta I_t C_t^{i*} \cdot \exp\left(-\frac{\beta I_t C_t^s C_t^{i*}}{\phi(t)}\right) = 0.5 \beta b^i I_t \cdot \exp\left(-\frac{0.5 \cdot \beta b^i \cdot I_t C_t^s}{\phi(t)}\right)$$

- Given that $u_t^s = (b^s C_t^s - (C_t^s)^2)^\gamma - a^s$, then

$$\frac{\partial u_t^s}{C_t^s} = \gamma(b^s C_t^s - (C_t^s)^2)^{\gamma-1} + b^s - 2C_t^s.$$

- By definition $V_{t+1}(i) = u_t^z(z_t, C_t^{z*})\xi(\delta, \tau, P^z)$, where

$$\xi(\delta, \tau, P^z) = \left(\frac{1-\delta^{\tau+1}}{1-\delta}\right) - \left(\frac{1-(\delta(1-P^z))^{\tau+1}}{1-\delta(1-P^z)}\right)$$

  using the form of u_t^z and the value of C_t^{z*} this equals

$$V_{t+1}(i) = [(0.25 \cdot (b^z)^2)^\gamma - a^z] \cdot \xi(\delta, \tau, P^z).$$

  This is for $t \in [t_0, t_0 + \tau - 2]$ for $t = t_0 + \tau - 1$ we have

$$V_{t+1}(i) = V_{t_0+\tau} = u^i_{t_0+\tau} = \left(0.25 \cdot (b^i)^2\right)^\gamma - a^i$$

and for $t = t_0 + \tau$ we have $V_{t+1}(i) = V_{\tau+1}(i) = 0$. So in conclusion

$$V_{t+1}(i) = \begin{cases} [(0.25 \cdot (b^z)^2)^\gamma - a^z] \cdot \xi(\delta, \tau, P^z) & \text{if } t \in [t_0, t_0 + \tau - 2] \\ (0.25 \cdot (b^i)^2)^\gamma - a^i & \text{if } t = t_0 + \tau - 1 \\ 0 & \text{if } t = t_0 + \tau \end{cases} \tag{1.5}$$

- Using the above relations, then equation (1.3) can be written as

$$\gamma(b^s C^s_t - (C^s_t)^2)^{\gamma-1} + b^s - 2c^s_t = \delta(V_{t+1}(s) - V_{t+1}(i)) \cdot 0.5\beta b^i I_t \cdot \exp\left(-\frac{0.5 \cdot \beta b^i \cdot I_t C^s_t}{\phi(t)}\right),$$

thus we can clear $V_{t+1}(s)$ as

$$V_{t+1}(s) = \frac{\gamma(b^s C^s_t - (C^s_t)^2)^{\gamma-1} + b^s - 2C^s_t}{0.5 \cdot \delta\beta b^i \cdot I_t \cdot \exp\left(-\frac{0.5 \cdot \beta b^i \cdot I_t C^s_t}{\phi(t)}\right)} + V_{t+1}(i).$$

The idea now is to use **backward induction** over the planning period $[t, t + \tau]$.

a) At time $t = t_0 + \tau$ we have that $V_{t+1}(i) = 0$ so we need to maximize

$$V_{t+1}(s) = \frac{\gamma(b^s C^s_t - (C^s_t)^2)^{\gamma-1} + b^s - 2C^s_t}{0.5 \cdot \delta\beta b^i \cdot I_t \cdot \exp\left(-\frac{0.5 \cdot \beta b^i \cdot I_t C^s_t}{\phi(t)}\right)}$$

Once we find that max value, we store it as $V_{(t_0+\tau)+1}(s)$

b) Continue from $t = t_0 + \tau$ until $t = t_0$ using the values of $V_{t+1}(s)$ and $V_{t+1}(i)$. We use the Bellman equation here.

c) Get value of $V_{t_0}(s)$ and the argmax will be $C^{s*}_{t_0}$.

1.3. **Implementation draft.** A first draft of this implementation can be found here:

github/JimmyCalvoMonge

1.4. **Graphs and comparison.** We plot and compare the solutions of three models:

- The **ex-ante** model where C is constant and equals $0.5b^s$. This is the classical approach.
- The **ex-post** model, where $C^s = 0.5b^s$ is constant, but we use the formula

$$C = C(\cdot) = \frac{C^s C^i N}{SC^s + IC^i + ZC^z}.$$

Here $C^s \neq C^z \neq C^i$, so $C(\cdot)$ is **not** constant.

- The **adaptive** model, where C^s is determined with the adaptive approach.

The parameters used for this computation are:

$$\beta = 0.0925, \nu = 0.1823, b^i = 6.67, b^z = 10, b^s = b^z,$$
$$a^i = 1.826, a^z = a^s = 0, \gamma = 0.25, \tau = 12, \delta = 0.99986.$$

These are used in initial examples by [2]. Figures follow, with explanations.

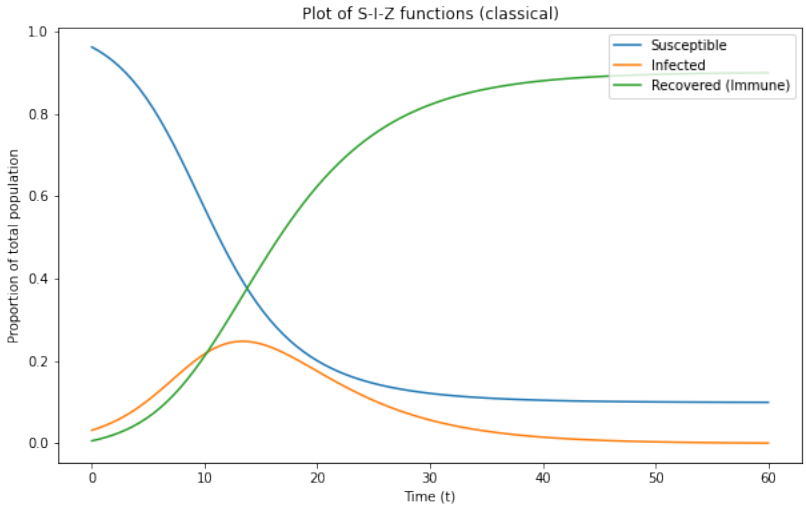

FIGURE 1. Solution through time of the classical system.

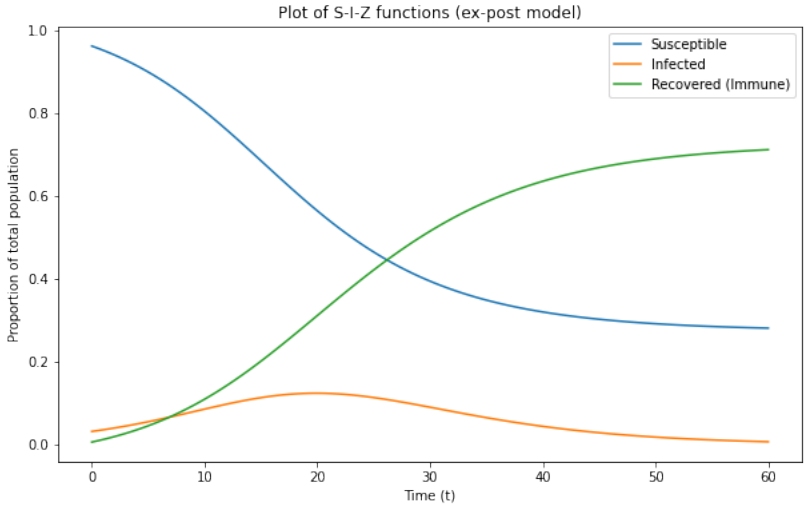

FIGURE 2. Solution through time of the ex-ante system.

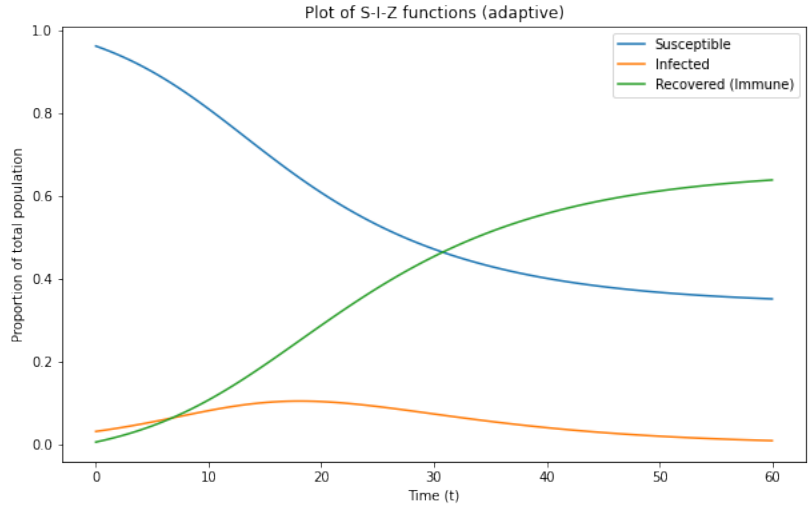

FIGURE 3. Solution through time of the ex-post (**Full adaptive**) system.

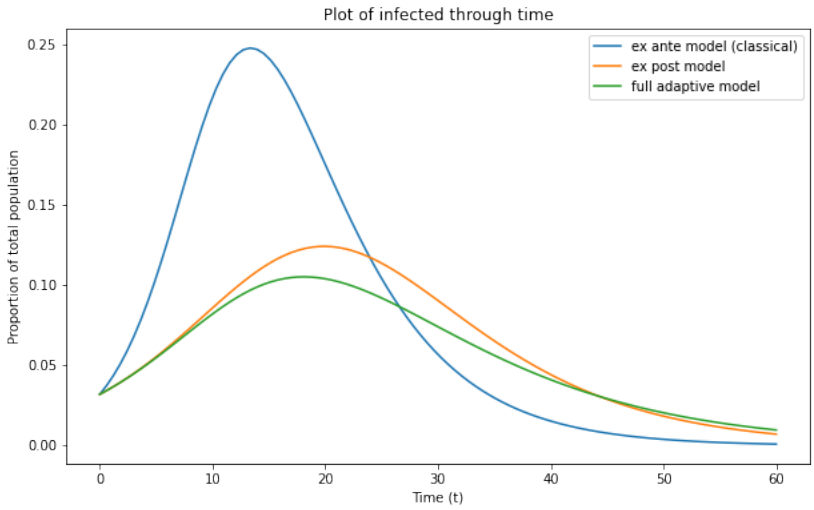

FIGURE 4. Comparison of disease prevalence between the three models

Some remarks:

- These parameters are taken from a flu-like pathogen studied in [3] where an R_0 of 1.8 was estimated using statistical data. This is called an **apparent** R_0.
- Using these parameters is easy to see that the R_0 from the classical model is ≈ 2.5.

- Using the adaptive approach, the R_0 is given by

$$R_0^i = \lim_{(S,I,Z)\to(N,0,0)} \frac{C(\cdot)\beta}{\nu} = \frac{C^i\beta}{\nu} \approx 1.67$$

- This means the classical approach **overestimates the value of R_0 in this case, and the prediction made by the ex-post model approach is more accurate.** (we assume the adaptive model is the reality).

2. Possible research steps

Some utilities for possible work, simulations and computations. Thoughts. Some calculations I did to be able to program the simulations.

2.1. **Contacts' model introducing relapse.** Let's study the adaptive model from a theoretical perspective. For now the adaptive computation of the $C^h(t)$ functions will be left aside, and we center on the form of $C(\cdot)$.

We generalize the model proposed in [4], using the approach in [2]. The model is now:

$$\frac{dS}{dt} = -C\beta\frac{SI}{N} + \mu N - \mu S. \tag{2.1}$$
$$\frac{dI}{dt} = C\beta\frac{SI}{N} + \phi\frac{ZI}{N} - (\gamma+\mu)I. \tag{2.2}$$
$$\frac{dZ}{dt} = \gamma I - \phi\frac{IZ}{N} - \mu Z. \tag{2.3}$$

Where

$$C = C(t,S,I,Z) = \frac{C^sC^iN}{SC^s + IC^i + ZC^z},$$

for functions $C^h = C^h(t,S,I,Z)$. Where C^h at time t is the average number of contacts (as proportion of the population) that individuals in compartment h engage in. We scale everything by N so that we get the system:

$$\frac{ds}{dt} = -C\beta si + \mu - \mu s. \tag{2.4}$$
$$\frac{di}{dt} = C\beta si + \phi zi - (\gamma+\mu)i. \tag{2.5}$$
$$\frac{dz}{dt} = \gamma i - \phi zi - \mu z. \tag{2.6}$$

Where now we have:

$$C = C(t, S, I, Z) = \frac{C^s C^i}{sC^s + iC^i + zC^z}.$$

We assume that $s + i + z = 1$. This is clearly a very general model. Here

$$R_0 = \frac{\beta}{\gamma + \mu} \lim_{(s,i,z)\to(1,0,0)} C^i.$$

If C^i is constant then we have $R_0 = \frac{\beta C^i}{\gamma+\mu}$.

- In article [2] this is called R_0^i. This depends on how infected individuals alter behavior in response to disease.
- Note that the article [4] is a special case of this model, where we put $C^s = C^i = \kappa$ and $C^z = \kappa(1 + \nu)$, because in that case we will have

$$C = \frac{\kappa \cdot \kappa}{s\kappa + i\kappa + (1 + \nu)\kappa z} = \frac{\kappa}{s + i + (1 + \nu)z} = \frac{\kappa}{1 - z + (1 + \nu)z} = \frac{\kappa}{1 + \nu z}.$$

  In this case susceptible and infected individuals have the same average contacts, and this average contact is higher for individuals with relapse. In general, the average contacts of each compartment depend on time, and also they depend on the size of all compartments at each time t, and they ought to be adaptive.
- The same proof from [4] applies here to show that the disease free equilibrium $(1, 0, 0)$ is stable if and only if $R_0 < 1$.

2.2. **Preliminary calculations for equilibra.** Calculations show that the endemic equilibria correspond to points of the form

$$P = \left(1 - i - \frac{\gamma i}{\phi i + \mu}, i, \frac{\gamma i}{\phi i + \mu}\right),$$

where i must satisfy a cubic equation $f(i) = x_3 i^3 + x_2 i^2 + x_1 i + x_0 = 0$, where the coefficients are:

$$
\begin{aligned}
x_3 &= R_\phi^2 R_0 + R_\mu R_\phi^2 \left(\frac{C^i}{C^s} - 1 \right) \\
x_2 &= R_\phi \left[R_0(1 - R_\phi) + R_\mu(R_0 + R_\phi) + R_\mu(1 - R_\mu) \left(\frac{C^z}{C^s} - 1 \right) + R_\mu(1 + R_\mu) \left(\frac{C^i}{C^s} - 1 \right) \right] \\
x_1 &= R_\mu \left[R_0(1 - R_\phi) + R_\phi(1 - R_0) + (1 - R_\mu) \left(\frac{C^z}{C^s} - 1 \right) + R_\mu R_\phi + R_\mu \left(\frac{C^i}{C^s} - 1 \right) \right] \\
x_0 &= R_\mu^2 (1 - R_0).
\end{aligned}
$$

Remark 2.1. When we are in the situation of [4], $C^s = C^i = \kappa$ and $C^z = \kappa(1 + \nu)$, thus $\frac{C^z}{C^s} - 1 = \nu$. We obtain the same coefficients as in that reference, because the red terms are zero and the orange terms are equal to ν.

2.3. **Simulations with simple functions as the C^s, C^i and C^z.**

TODO

2.4. **What next?** TODO

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