NOTES ON EPIDEMIC MODELS

These notes serve as a basis for a graduation project on novel epidemiological models. UCR.

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Contents

1. ADAPTIVE BEHAVIOR MODEL

1.1. **Model review.** We describe a new modeling technique presented in [2], which gives an adaptive approach to the construction of an epidemiological system. The classical approach is taking the system:

$$\begin{aligned}
\frac{dS}{dt} &= -C(\cdot)\beta SI/N \\
\frac{dI}{dt} &= C(\cdot)\beta SI/N - \nu I \\
\frac{dZ}{dt} &= \nu I.
\end{aligned}$$

Where:

- A population of N individuals is divided in three compartments: $N = S + I + R$. Here S are the susceptible individuals, I are the infected and R are the recovered individuals.
- β represents the likelihood that contact with an infected individual yields infection.
- ν is the rate of recovery.
- $C(\cdot)$ is the rate that susceptible contact infected, which means that $C(\cdot)\beta$ is the rate that susceptible individuals become infected.

In the classical setting, either $C(\cdot) = c$ (contacts are constant) or $C(\cdot) = cN$ (contacts are proportional to N). In the adaptive setting, the idea is that $C(\cdot)$ depends on the incentives different individuals have to vary their number of contacts. The costs and benefits of individual contact vary across health status.

The proposal is then to divide the individuals by health type. Let $Y = \{s, i, z\}$. For $h \in Y$ denote C^h the expected number of contacts made by an individual of type h. For $m, n \in Y$ we define

$$C^{mn}(\cdot) = C^m C^n N/(SC^s + IC^i + ZC^z).$$

The rate of contact between individuals of types m and n. In here C^m **is a choice** made by individuals of type m. In the classical model $C(\cdot) = C^{si}$.

People engage in contacts because there is a certain utility to gain from them. The adaptive approach models the utility for an individual of type $h \in Y$ depending on the current time, therefore we have an utility

$$u_t^h = u_t^h(C_t^h),$$

where t is the current time, and C_t^h is the expected number of contacts of an individual of type h made at time t. u_h^t is **the utility of making contacts for an individual of type h at time t**. The utility function should be concave and should have a single peak with respect to the number of contacts, but should decrease with infection.

An example of an utility function provided in [2] is

$$u_t^h = (b^h C_t^h - (C_t^h)^2)^\gamma - a^h,$$

where γ, b^h, a^h are fixed parameters, with $b^s = b^z \geq b^i \geq 0$, $a^z = a^s = 0$, $\gamma > 0$, $a^i > 0$. **Intiutively, this means that during the infection period, the utility has a term that pauses it's increment. Note that for each state utility has a peak with respect to C_t^h.**

Recovered (and Immune) and Infected individuals: If the individual doesn't think that a change in their contacts will affect their health status, then for a given time t, the best thing would be to choose C_t^h such that u_t^h is maximized. This happens with individuals of types i and z. The optimal choice is $C_h^{t*} = 0.5b^h$.

Susceptible individuals: The number of contacts a susceptible individual engages in might affect their health status, so the optimal choice of contacts C_t^{s*} is subjected to planning towards the future. Here is where the adaptive decision comes in, as a factor of future utility.

Let P_t^i the probability that an s-type individual becomes infected at time t. This depends on the current state of things and the selection of C_t^s, as

$$P_t^i = 1 - e^{-\beta I_t C_t^s C_t^{i*}/(S_t C_t^{s*} + I_t C_t^{i*} + Z_t C_t^{z*})}, \tag{1.1}$$

where C_t^{s*} is the optimal choice of other susceptible individuals the present susceptible individual might encounter.

To find the optimal C_t^{s*} we maximize the value function

$$V_t(s) = \max_{C^s \in X} \left\{ u_t^s(C_t^s) + \delta \left[(1 - P_t^i) V_{t+1}(s) + P_t^i V_{t+1}(i) \right] \right\}, \tag{1.2}$$

where X is the range of possible contacts, δ is a discount factor, $V_{t+1}(s)$ is the present value of expected utility if the individual remains susceptible and $V_{t+1}(i)$ the present value of expected utility if the individual becomes infected.

Solving equation (1.2) gives the first order condition:

$$\frac{\partial u_t}{\partial C_t^s} = \delta(V_{t+1}(s) - V_{t+1}(i)) \left(\frac{\partial P_t^i}{\delta C_t^s} \right). \tag{1.3}$$

The idea on how to select the optimal C_t^{s*} at time t depends on the current state of things and the previsions the individual does for the future. The adaptive approach proposes a continuous update of the selection made across time.

For each $t < \tau - 1$, the individual will solve (1.2). For that they will need to solve (1.3). This requires knowledge of $V_{t+1}(i)$, which it's modeled like:

$$V_{t+1}(i) = u_t^z(C_t^{z*}) \left[\left(\frac{1-\delta^{\tau+1}}{1-\delta} \right) - \left(\frac{1-(\delta(1-P^z))^{\tau+1}}{1-\delta(1-P^z)} \right) \right], \tag{1.4}$$

where τ is the **planning period** and $P^z = 1 - e^{-\nu}$ is the probability of recovery.

1.2. **Utility calculations for implementation.** Here we provide some notes on how to implement the adaptive model. First we solve some more explicitly some of the equations presented in the paper.

- In equation (1.1) we have that

$$P_t^i = 1 - \exp\left(-\frac{\beta I_t C_t^s C_t^{i*}}{\phi(t)} \right) = 1 - \exp\left(-\frac{0.5 \cdot \beta b^i \cdot I_t C_t^s}{\phi(t)} \right),$$

where $\phi(t) = S_t C_t^{s*} + I_t C_t^{i*} + Z_t C_t^{z*} = S_t C_t^{s*} + 0.5 b^i \cdot I_t + 0.5 b^z \cdot Z_t$. Therefore

$$\frac{\partial P_t^i}{\partial C_t^s} = \frac{\beta I_t C_t^{i*}}{\phi(t)} \cdot \exp\left(-\frac{\beta I_t C_t^s C_t^{i*}}{\phi(t)} \right) = \frac{0.5 \beta b^i I_t}{\phi(t)} \cdot \exp\left(-\frac{0.5 \cdot \beta b^i \cdot I_t C_t^s}{\phi(t)} \right)$$

- Given that $u_t^s = (b^s C_t^s - (C_t^s)^2)^\gamma - a^s$, then

$$\frac{\partial u_t^s}{C_t^s} = \gamma(b^s C_t^s - (C_t^s)^2)^{\gamma-1} \cdot (b^s - 2C_t^s).$$

- By definition $V_{t+1}(i) = u_t^z(z_t, C_t^{z*})\xi(\delta, \tau, P^z)$, where
$$\xi(\delta, \tau, P^z) = \left(\frac{1-\delta^{\tau+1}}{1-\delta}\right) - \left(\frac{1-(\delta(1-P^z))^{\tau+1}}{1-\delta(1-P^z)}\right)$$
using the form of u_t^z and the value of C_t^{z*} this equals
$$V_{t+1}(i) = [(0.25 \cdot (b^z)^2)^\gamma - a^z] \cdot \xi(\delta, \tau, P^z).$$
This is for $t \in [t_0, t_0 + \tau - 2]$ for $t = t_0 + \tau - 1$ we have
$$V_{t+1}(i) = V_{t_0+\tau} = u_{t_0+\tau}^i = \left(0.25 \cdot (b^i)^2\right)^\gamma - a^i$$
and for $t = t_0 + \tau$ we have $V_{t+1}(i) = V_{\tau+1}(i) = 0$. So in conclusion
$$V_{t+1}(i) = \begin{cases} [(0.25 \cdot (b^z)^2)^\gamma - a^z] \cdot \xi(\delta, \tau, P^z) & \text{if } t \in [t_0, t_0 + \tau - 2] \\ \left(0.25 \cdot (b^i)^2\right)^\gamma - a^i & \text{if } t = t_0 + \tau - 1 \\ 0 & \text{if } t = t_0 + \tau \end{cases} \tag{1.5}$$
- Using the above relations, then equation (1.3) can be written as
$$\gamma(b^s C_t^s - (C_t^s)^2)^{\gamma-1} \cdot (b^s - 2C_t^s) = \delta(V_{t+1}(s) - V_{t+1}(i)) \cdot \frac{0.5\beta b^i I_t}{\phi(t)} \cdot \exp\left(-\frac{0.5 \cdot \beta b^i \cdot I_t C_t^s}{\phi(t)}\right),$$
thus we can clear $V_{t+1}(s)$ as
$$V_{t+1}(s) = \frac{\gamma(b^s C_t^s - (C_t^s)^2)^{\gamma-1} \cdot (b^s - 2C_t^s)}{\frac{0.5 \cdot \delta\beta b^i \cdot I_t}{\phi(t)} \cdot \exp\left(-\frac{0.5 \cdot \beta b^i \cdot I_t C_t^s}{\phi(t)}\right)} + V_{t+1}(i).$$
The idea now is to use **backward induction** over the planning period $[t_0, t_0 + \tau]$.
  a) At time $t = t_0 + \tau$ we have that $V_{t+1}(i) = 0$ so we need to maximize
$$V_{t+1}(s) = \frac{\gamma(b^s C_t^s - (C_t^s)^2)^{\gamma-1} \cdot (b^s - 2C_t^s)}{\frac{0.5 \cdot \delta\beta b^i \cdot I_t}{\phi(t)} \cdot \exp\left(-\frac{0.5 \cdot \beta b^i \cdot I_t C_t^s}{\phi(t)}\right)}$$
  Once we find that max value, we store it as $V_{(t_0+\tau)+1}(s)$
  b) Continue from $t = t_0 + \tau$ until $t = t_0$ using the values of $V_{t+1}(s)$ and $V_{t+1}(i)$. We use the Bellman equation here.
  c) Get value of $V_{t_0}(s)$ and the argmax will be $C_{t_0}^{s*}$.

1.3. **Implementation draft.** A first draft of this implementation can be found here:

github/JimmyCalvoMonge/adaptive

1.4. **Graphs and comparison.** We plot and compare the solutions of three models:

- The **ex-ante** model where C is constant and equals $0.5b^s$. This is the classical approach.
- The **ex-post** model, where $C^s = 0.5b^s$ is constant, but we use the formula

$$C = C(\cdot) = \frac{C^s C^i N}{SC^s + IC^i + ZC^z}.$$

  Here $C^s \neq C^z \neq C^i$, so $C(\cdot)$ is **not** constant.
- The **adaptive** model, where C^s is determined with the adaptive approach.

The parameters used for this computation are:

$$\beta = 0.0925, \nu = 0.1823, b^i = 6.67, b^z = 10, b^s = b^z,$$
$$a^i = 1.826, a^z = a^s = 0, \gamma = 0.25, \tau = 12, \delta = 0.99986.$$

These are used in initial examples by [2]. Figures follow, with explanations.

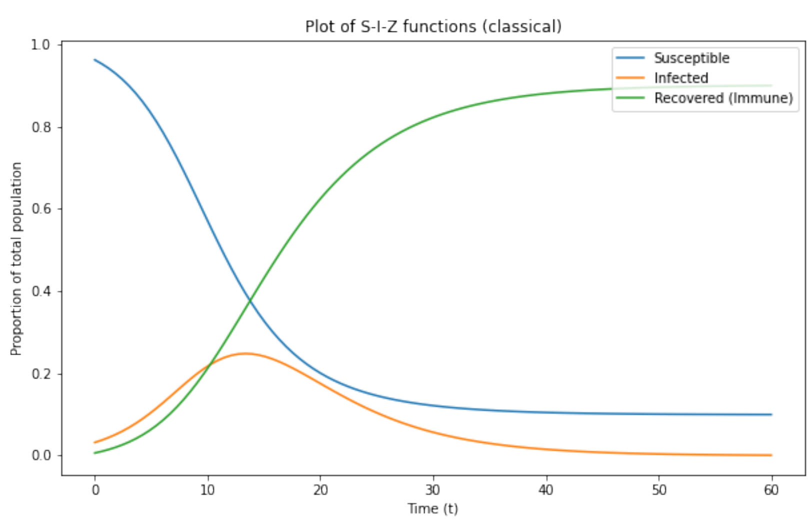


FIGURE 1. Solution through time of the classical system.

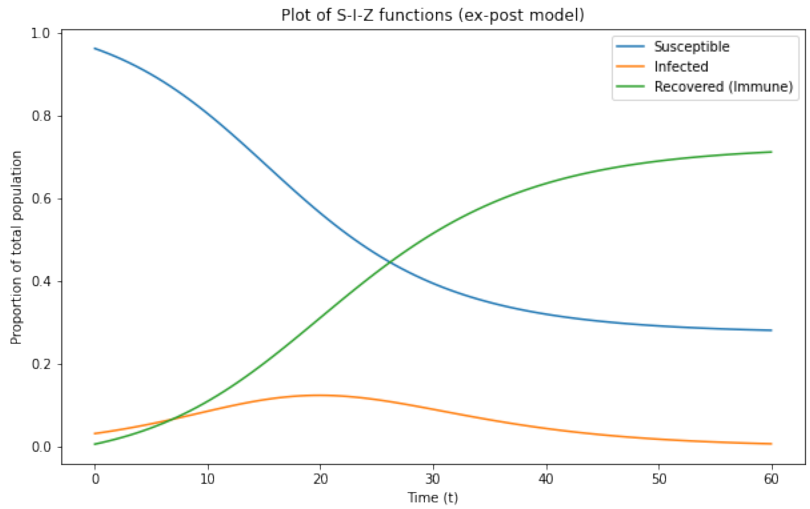


FIGURE 2. Solution through time of the ex-ante system.

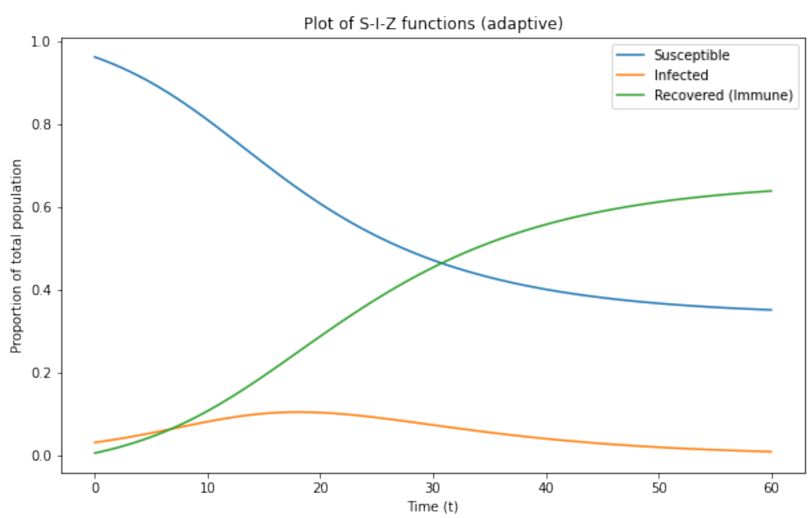


FIGURE 3. Solution through time of the ex-post (**Full adaptive**) system.

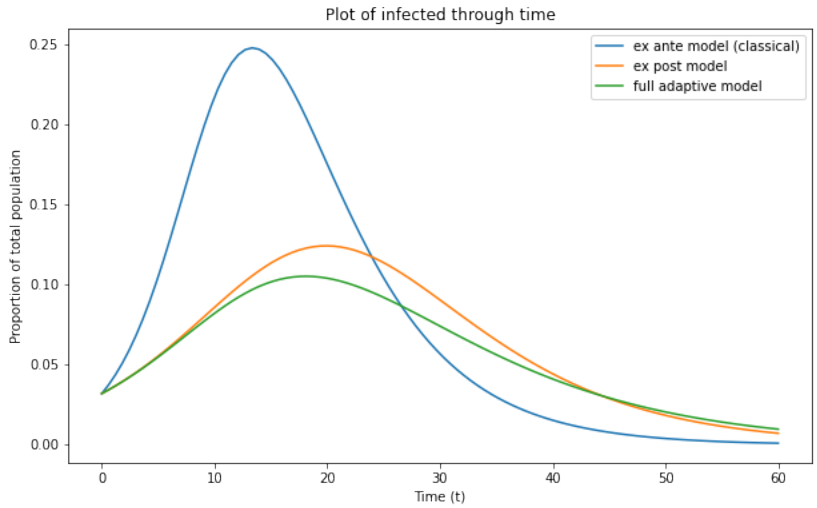


FIGURE 4. Comparison of disease prevalence between the three models

Some remarks:

- These parameters are taken from a flu-like pathogen studied in [3] where an R_0 of 1.8 was estimated using statistical data. This is called an **apparent** R_0.
- Using these parameters is easy to see that the R_0 from the classical model is ≈ 2.5.
- Using the adaptive approach, the R_0 is given by

$$R_0^i = \lim_{(S,I,Z)\to(N,0,0)} \frac{C(\cdot)\beta}{\nu} = \frac{C^i\beta}{\nu} \approx 1.67$$

- This means the classical approach **overestimates the value of R_0 in this case, and the prediction made by the ex-post model approach is more accurate.** (we assume the adaptive model is the reality).

2. POSSIBLE RESEARCH STEPS

Some utilities for possible work, simulations and computations. Thoughts. Some calculations I did to be able to program the simulations.

2.1. **Contacts' model introducing relapse.** Let's study the adaptive model from a theoretical perspective. For now the adaptive computation of the $C^h(t)$ functions will be left aside, and we center on the form of $C(\cdot)$.

We generalize the model proposed in [4], using the approach in [2]. The model is now:

$$\frac{dS}{dt} = -C\beta\frac{SI}{N} + \mu N - \mu S. \tag{2.1}$$
$$\frac{dI}{dt} = C\beta\frac{SI}{N} + \phi\frac{ZI}{N} - (\gamma+\mu)I. \tag{2.2}$$
$$\frac{dZ}{dt} = \gamma I - \phi\frac{IZ}{N} - \mu Z. \tag{2.3}$$

Where

$$C = C(t,S,I,Z) = \frac{C^sC^iN}{SC^s + IC^i + ZC^z},$$

for functions $C^h = C^h(t,S,I,Z)$. Where C^h at time t is the average number of contacts that individuals in compartment h engage in. We scale everything by N so that we get the system:

$$\frac{ds}{dt} = -C\beta si + \mu - \mu s. \tag{2.4}$$
$$\frac{di}{dt} = C\beta si + \phi zi - (\gamma+\mu)i. \tag{2.5}$$
$$\frac{dz}{dt} = \gamma i - \phi zi - \mu z. \tag{2.6}$$

Where now we have:

$$C = C(t,S,I,Z) = \frac{C^sC^i}{sC^s + iC^i + zC^z}.$$

We assume that $s+i+z=1$. This is clearly a very general model. Here

$$R_0 = \frac{\beta}{\gamma + \mu} \lim_{(s,i,z)\to(1,0,0)} C^i.$$

If C^i is constant then we have $R_0 = \frac{\beta C^i}{\gamma+\mu}$.

- In article [2] this is called R_0^i. This depends on how infected individuals alter behavior in response to disease.
- Note that the article [4] is a special case of this model, where we put $C^s = C^i = \kappa$ and $C^z = \kappa(1+\nu)$, because in that case we will have

$$C = \frac{\kappa \cdot \kappa}{s\kappa + i\kappa + (1+\nu)\kappa z} = \frac{\kappa}{s + i + (1+\nu)z} = \frac{\kappa}{1 - z + (1+\nu)z} = \frac{\kappa}{1+\nu z}.$$

  In this case susceptible and infected individuals have the same average contacts, and this average contact is higher for individuals with relapse.
- The same proof from [4] applies here to show that the disease free equilibrium $(1,0,0)$ is stable if and only if $R_0 < 1$.

2.2. **Preliminary calculations for equilibra.** Calculations show that the endemic equilibria correspond to points of the form

$$P = \left(1 - i - \frac{\gamma i}{\phi i + \mu}, i, \frac{\gamma i}{\phi i + \mu}\right),$$

where i must satisfy a cubic equation $f(i) = x_3 i^3 + x_2 i^2 + x_1 i + x_0 = 0$, where the coefficients are:

$$\begin{aligned}
x_3 &= R_\phi^2 R_0 - R_\mu R_\phi^2 (1-\kappa) \\
x_2 &= R_\phi \Big[R_0(1-R_\phi) + R_\mu(R_0 + R_\phi) - R_\mu(1-R_\mu)(1-\theta) - R_\mu(1+R_\mu)(1-\kappa)\Big] \\
x_1 &= R_\mu \Big[R_0(1-R_\phi) + R_\phi(1-R_0) - (1-R_\mu)(1-\theta) + R_\mu R_\phi - R_\mu(1-\kappa)\Big] \\
x_0 &= R_\mu^2(1-R_0),
\end{aligned} \tag{2.7}$$

where $\kappa = \frac{C^i}{C^s}$ and $\theta = \frac{C^z}{C^s}$ and

$$R_\mu = \frac{\mu}{\mu+\gamma}, \quad R_\phi = \frac{\phi}{\mu+\gamma}. \tag{2.8}$$

Remark 2.1. When we are in the situation of [4], $C^s = C^i$ and $C^z = C^i(1+\nu)$, thus $\frac{C^z}{C^s} - 1 = \nu$. We obtain the same coefficients as in that article.

Remark 2.2. See the appendix for the calculation details.

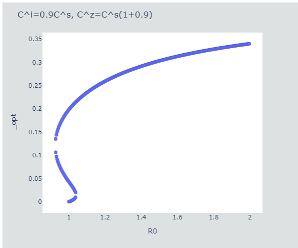
2.3. **Simulations with constant functions as the C^s, C^i and C^z.** In the next plots we show what happens with the bifurcation plots obtained for this model for different configurations of the C^s, C^i and C^z functions. Note that the polynomial equations only depend on the terms $\kappa := \frac{C^i}{C^s}$ and $\theta := \frac{C^z}{C^s}$.

In our simulations we take C^s to vary between 0 and 1 and create several scenarios for C^i and C^z, which will depend on C^s. In all simulations C^i is lower than C^s (contacts for infected individuals decrease), however we divide the case in which $C^z > C^s$ and $C^z < C^s$, as we will see they behave differently. See next page.

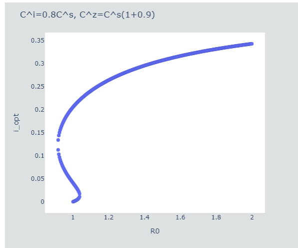
These are the model parameters for this simulations, taken from an example in [4].

$$\mu = 0.00015, \quad \gamma = 0.0027, \quad \beta = 0.009, \phi = 0.0044.$$

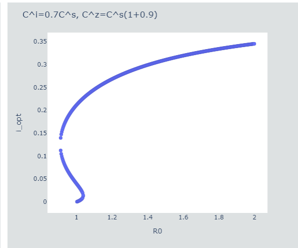
2.3.1. *Case where $C^z > C^s$ (higher contact rate in recovered individuals with relapse).* In this scenario, for individuals that are recovered, average contacts increase.



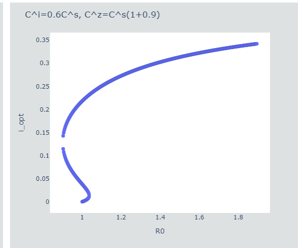
(A) $C^i = 0.9C^s$



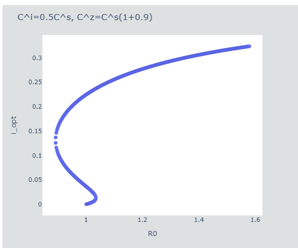
(B) $C^i = 0.8C^s$



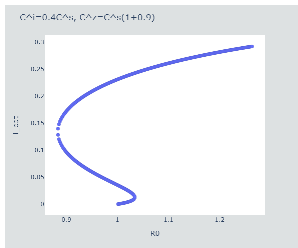
(C) $C^i = 0.7C^s$



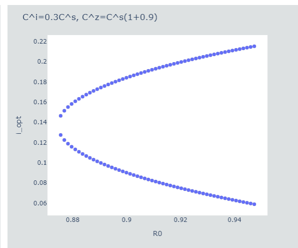
(D) $C^i = 0.6C^s$



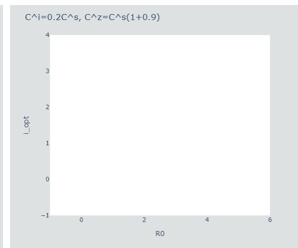
(E) $C^i = 0.5C^s$



(F) $C^i = 0.4C^s$



(G) $C^i = 0.3C^s$



(H) $C^i = 0.2C^s$

FIGURE 5. In all these cases $C^z = C^s(1+\nu)$, where $\nu = 0.9$ and we display cases of $C^i = \kappa C^s$. A cubic bifurcation plot can be found for sufficiently high values of κ.

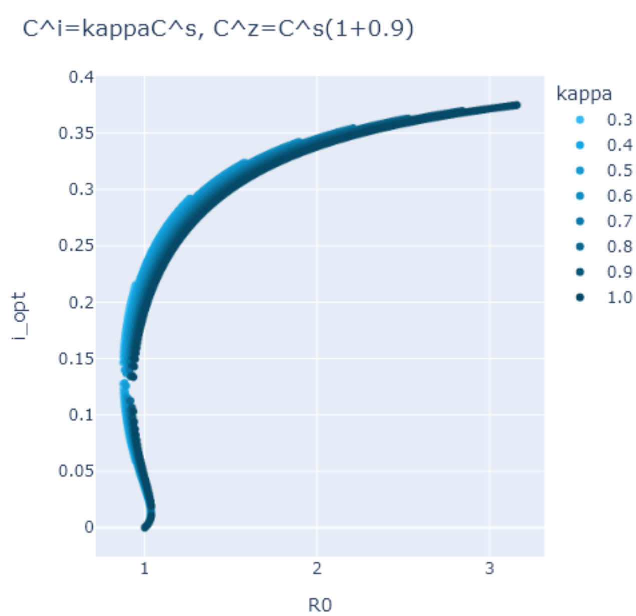
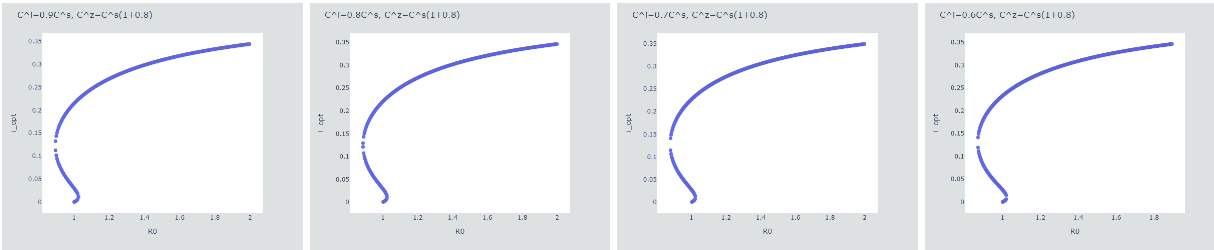
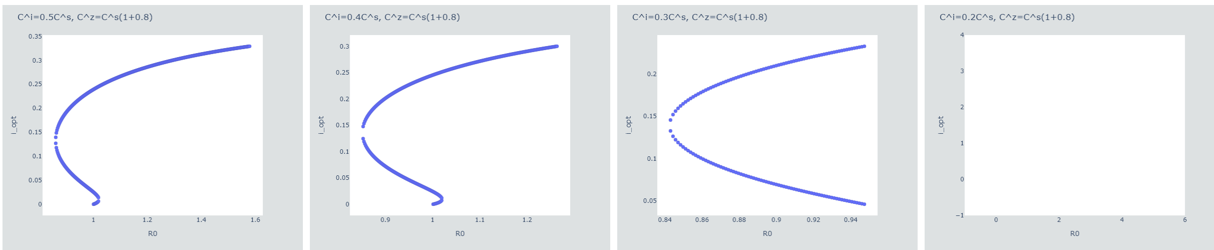


FIGURE 6. Comparison of all curves for $C^z = C^s(1+0.9)$. Effect of decreasing κ for $C^i = \kappa C^s$.



(A) $C^i = 0.9C^s$ (B) $C^i = 0.8C^s$ (C) $C^i = 0.7C^s$ (D) $C^i = 0.6C^s$



(E) $C^i = 0.5C^s$ (F) $C^i = 0.4C^s$ (G) $C^i = 0.3C^s$ (H) $C^i = 0.2C^s$

FIGURE 7. In all these cases $C^z = C^s(1+\nu)$, where $\nu = 0.8$ and we display cases of $C^i = \kappa C^s$. A cubic bifurcation plot can be found for sufficiently high values of κ.

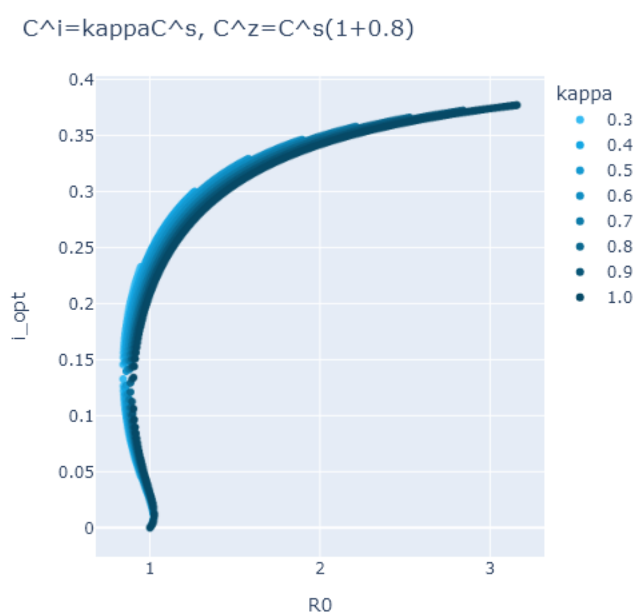
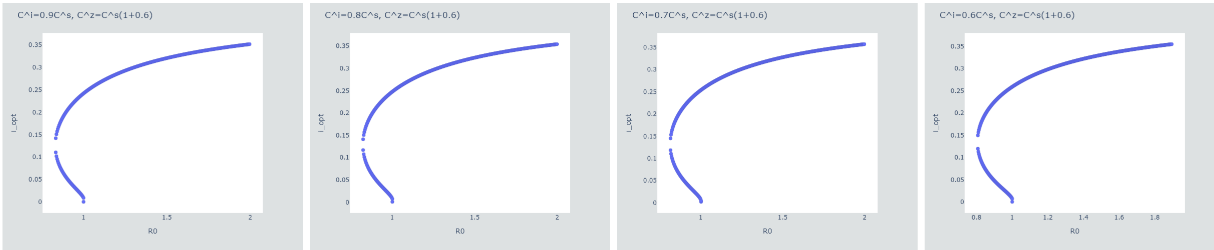
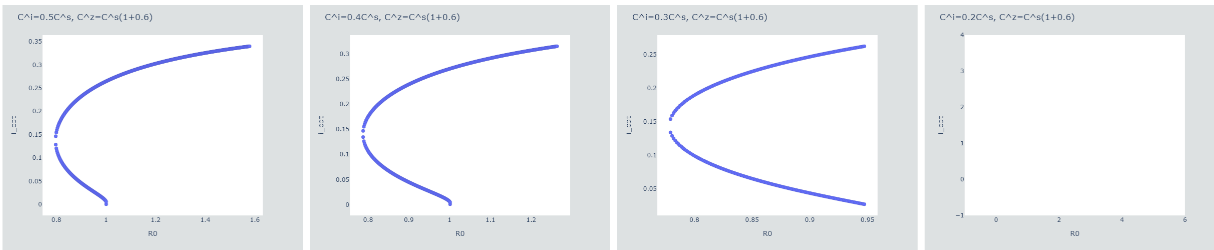


FIGURE 8. Comparison of all curves for $C^z = C^s(1+0.8)$. Effect of decreasing κ for $C^i = \kappa C^s$.



(A) $C^i = 0.9C^s$ (B) $C^i = 0.8C^s$ (C) $C^i = 0.7C^s$ (D) $C^i = 0.6C^s$



(E) $C^i = 0.5C^s$ (F) $C^i = 0.4C^s$ (G) $C^i = 0.3C^s$ (H) $C^i = 0.2C^s$

FIGURE 9. In all these cases $C^z = C^s(1+\nu)$, where $\nu = 0.6$ and we display cases of $C^i = \kappa C^s$. A cubic bifurcation plot can be found for sufficiently high values of κ.

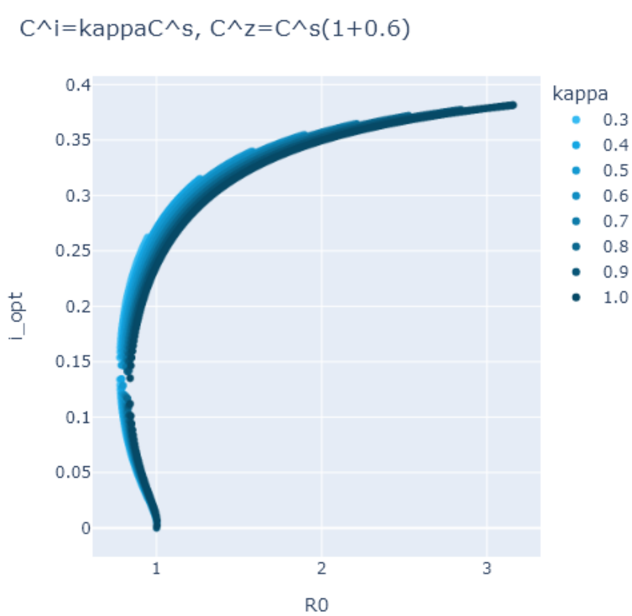
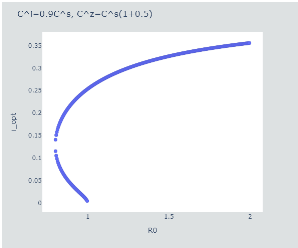
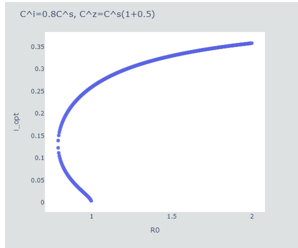


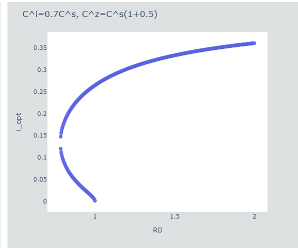
FIGURE 10. Comparison of all curves for $C^z = C^s(1 + 0.6)$. Effect of decreasing κ for $C^i = \kappa C^s$.



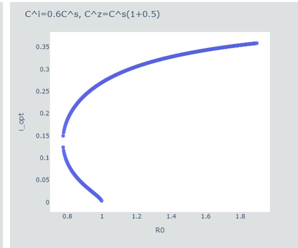
(A) $C^i = 0.9C^s$



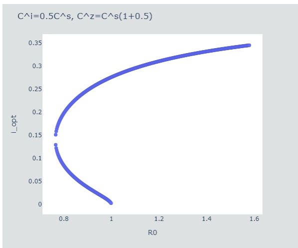
(B) $C^i = 0.8C^s$



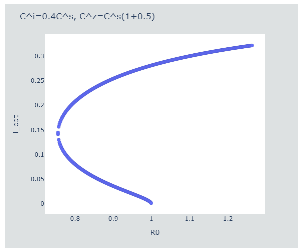
(C) $C^i = 0.7C^s$



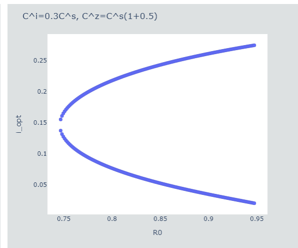
(D) $C^i = 0.6C^s$



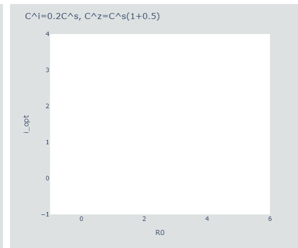
(E) $C^i = 0.5C^s$



(F) $C^i = 0.4C^s$



(G) $C^i = 0.3C^s$



(H) $C^i = 0.2C^s$

FIGURE 11. In all these cases $C^z = C^s(1+\nu)$, where $\nu = 0.5$ and we display cases of $C^i = \kappa C^s$. A cubic bifurcation plot can be found for sufficiently high values of κ. Here the cubic bifurcation plot is more sutil.

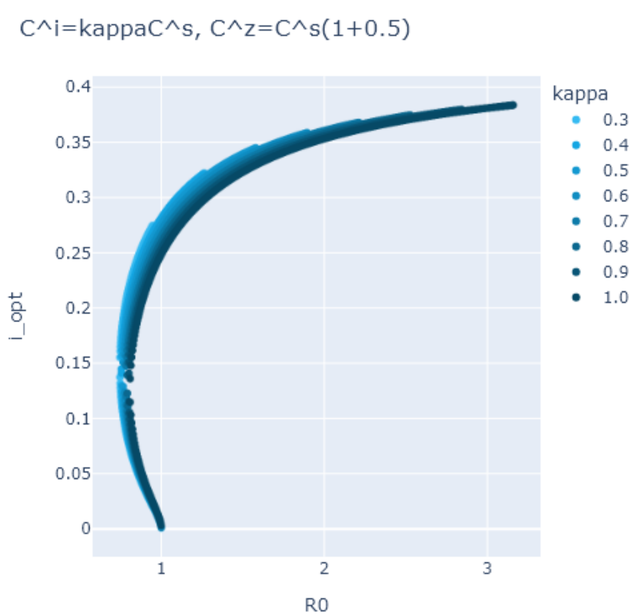
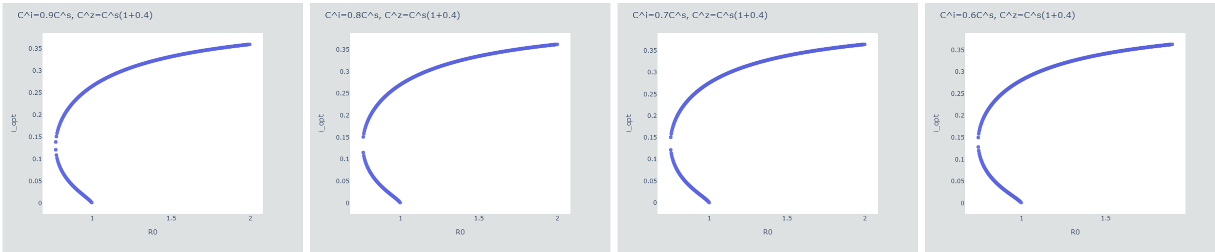
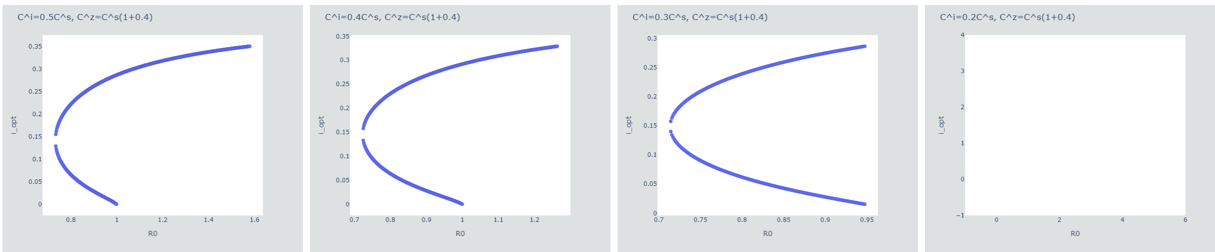


FIGURE 12. Comparison of all curves for $C^z = C^s(1 + 0.5)$. Effect of decreasing κ for $C^i = \kappa C^s$.



(A) $C^i = 0.9C^s$ (B) $C^i = 0.8C^s$ (C) $C^i = 0.7C^s$ (D) $C^i = 0.6C^s$



(E) $C^i = 0.5C^s$ (F) $C^i = 0.4C^s$ (G) $C^i = 0.3C^s$ (H) $C^i = 0.2C^s$

FIGURE 13. In all these cases $C^z = C^s(1+\nu)$, where $\nu = 0.4$ and we display cases of $C^i = \kappa C^s$. A cubic bifurcation plot can be found. Some graphs have a very mild cubic part.

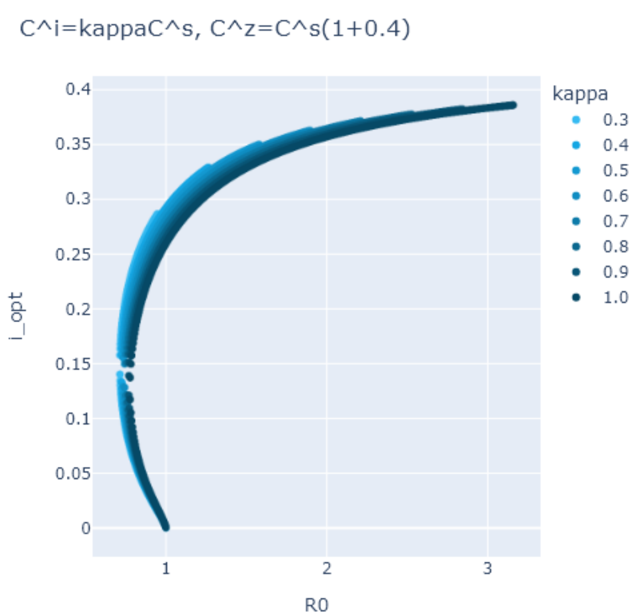
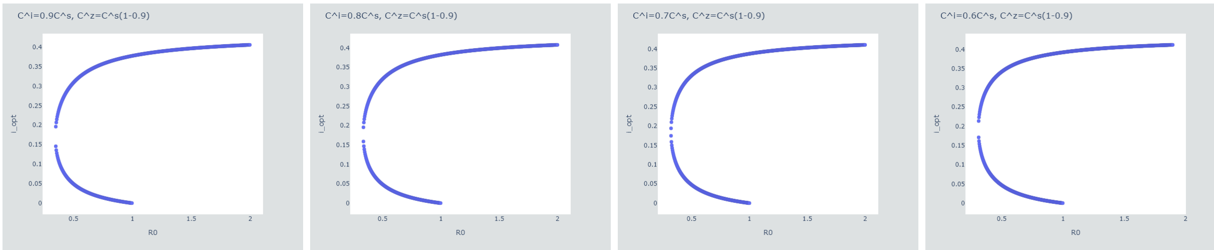
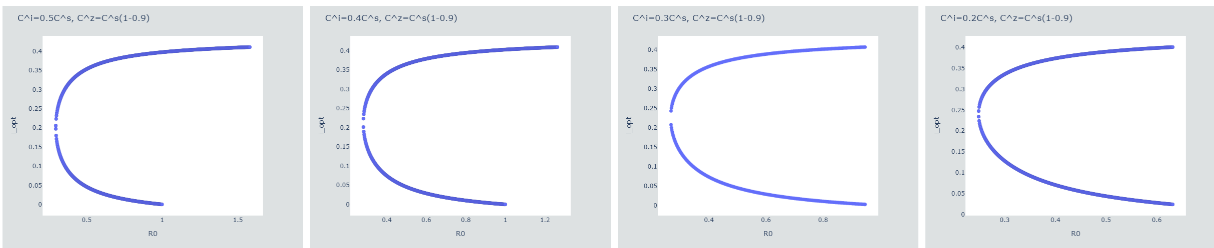


FIGURE 14. Comparison of all curves for $C^z = C^s(1+0.4)$. Effect of decreasing κ for $C^i = \kappa C^s$.

2.3.2. *Case where $C^z < C^s$ (lower contact rate in recovered individuals with relapse).* In this scenario, for individuals that are recovered, average contacts decrease.



(A) $C^i = 0.9C^s$ (B) $C^i = 0.8C^s$ (C) $C^i = 0.7C^s$ (D) $C^i = 0.6C^s$



(E) $C^i = 0.5C^s$ (F) $C^i = 0.4C^s$ (G) $C^i = 0.3C^s$ (H) $C^i = 0.2C^s$

FIGURE 15. In all these cases $C^z = C^s(1 - \nu)$, where $\nu = 0.9$ and we display cases of $C^i = \kappa C^s$. A cubic bifurcation plot cannot be found.

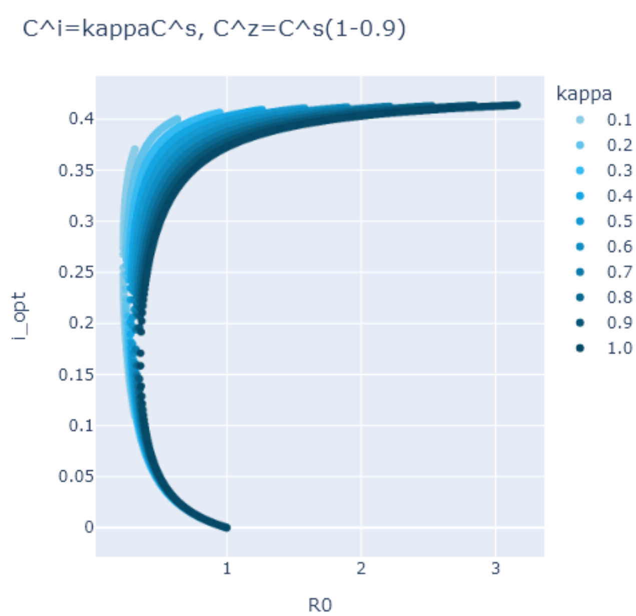
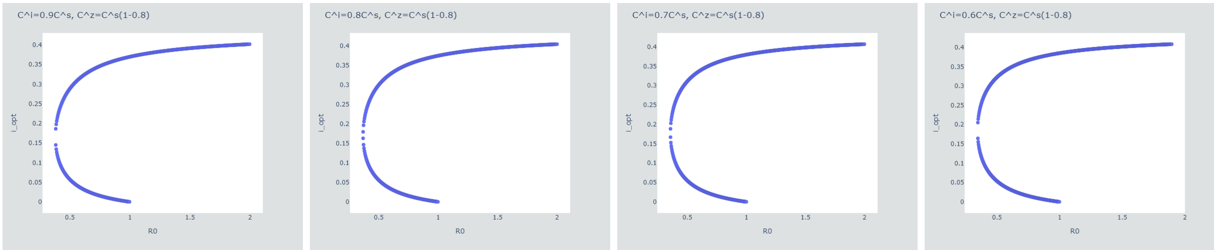
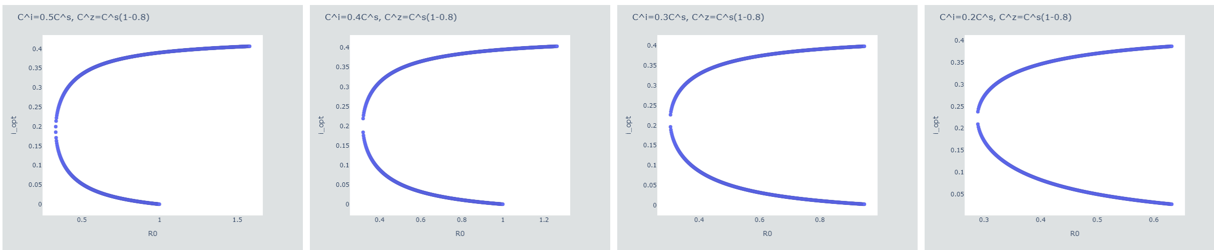


FIGURE 16. Comparison of all curves for $C^z = C^s(1 - 0.9)$. Effect of decreasing κ for $C^i = \kappa C^s$.



(A) $C^i = 0.9C^s$ (B) $C^i = 0.8C^s$ (C) $C^i = 0.7C^s$ (D) $C^i = 0.6C^s$



(E) $C^i = 0.5C^s$ (F) $C^i = 0.4C^s$ (G) $C^i = 0.3C^s$ (H) $C^i = 0.2C^s$

FIGURE 17. In all these cases $C^z = C^s(1-\nu)$, where $\nu = 0.8$ and we display cases of $C^i = \kappa C^s$. Similar behavior as above.

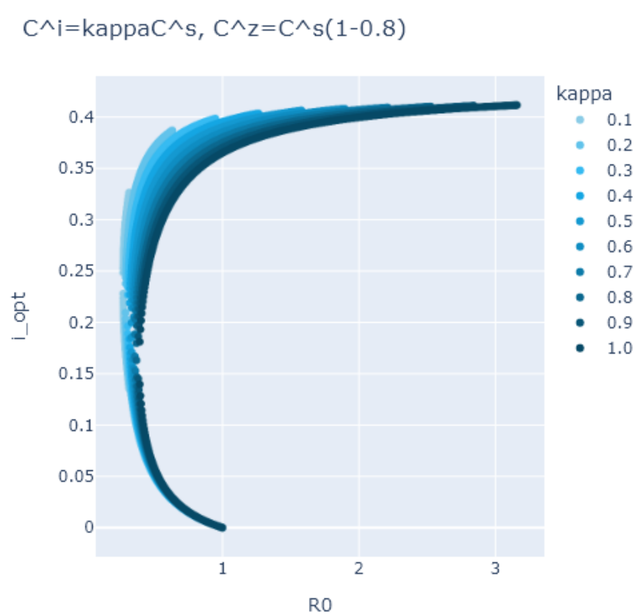
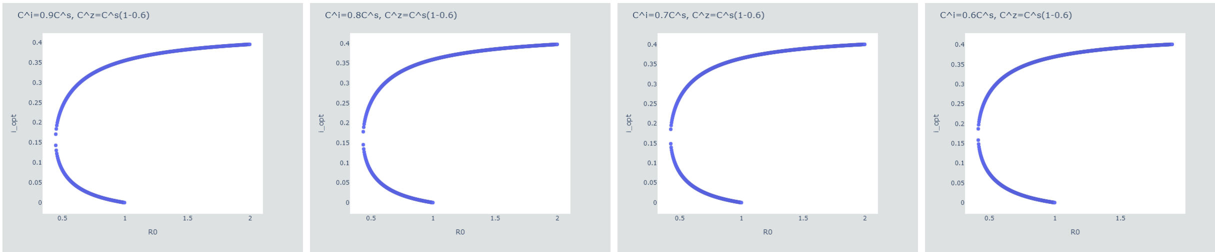
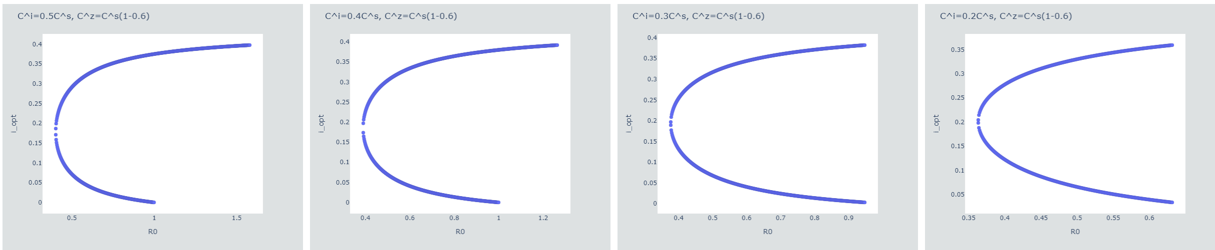


FIGURE 18. Comparison of all curves for $C^z = C^s(1-0.8)$. Effect of decreasing κ for $C^i = \kappa C^s$.



(A) $C^i = 0.9C^s$ (B) $C^i = 0.8C^s$ (C) $C^i = 0.7C^s$ (D) $C^i = 0.6C^s$



(E) $C^i = 0.5C^s$ (F) $C^i = 0.4C^s$ (G) $C^i = 0.3C^s$ (H) $C^i = 0.2C^s$

FIGURE 19. In all these cases $C^z = C^s(1-\nu)$, where $\nu = 0.6$ and we display cases of $C^i = \kappa C^s$. Similar behavior as above.

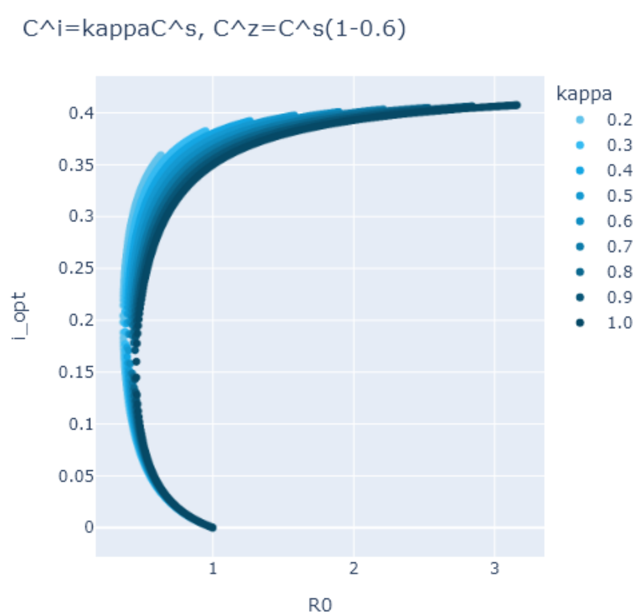
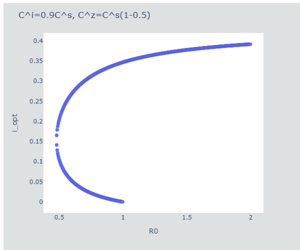
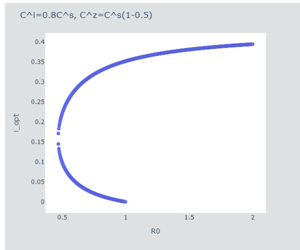


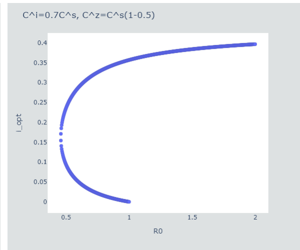
FIGURE 20. Comparison of all curves for $C^z = C^s(1-0.6)$. Effect of decreasing κ for $C^i = \kappa C^s$.



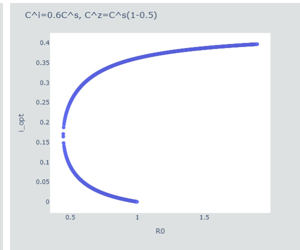
(A) $C^i = 0.9C^s$



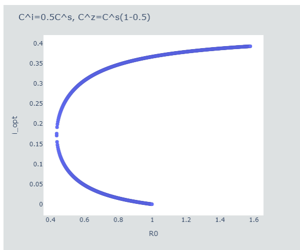
(B) $C^i = 0.8C^s$



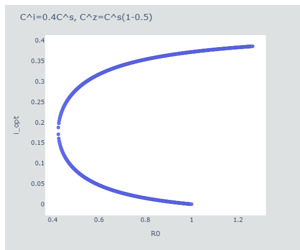
(C) $C^i = 0.7C^s$



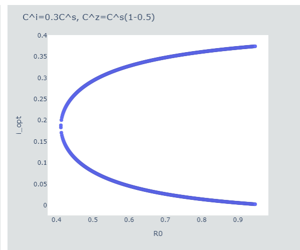
(D) $C^i = 0.6C^s$



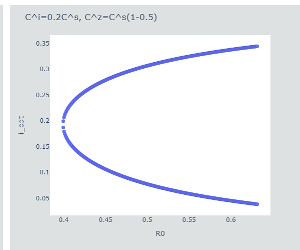
(E) $C^i = 0.5C^s$



(F) $C^i = 0.4C^s$



(G) $C^i = 0.3C^s$



(H) $C^i = 0.2C^s$

FIGURE 21. In all these cases $C^z = C^s(1-\nu)$, where $\nu = 0.5$ and we display cases of $C^i = \kappa C^s$. Similar results. C^z is larger in comparison to C^s now.

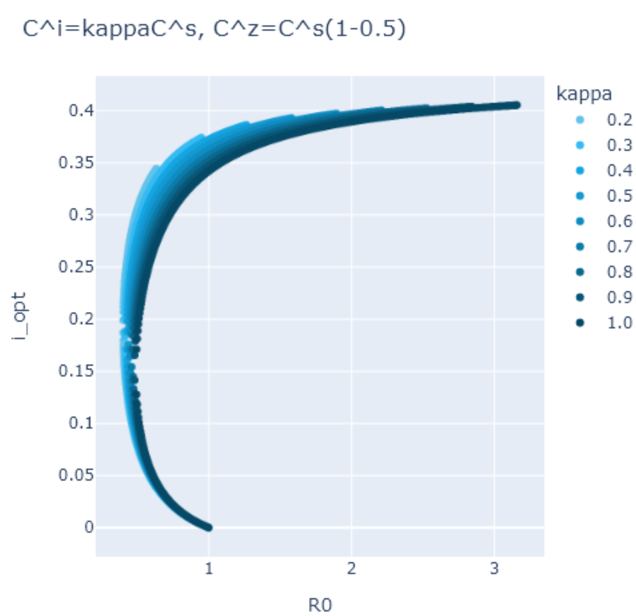
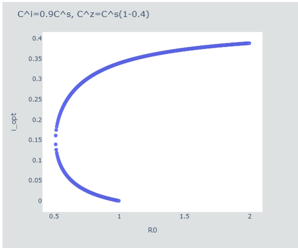
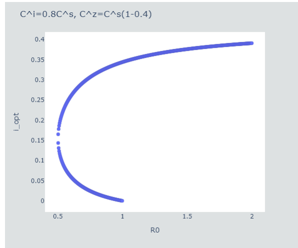


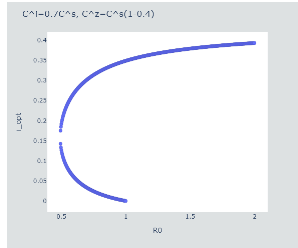
FIGURE 22. Comparison of all curves for $C^z = C^s(1-0.5)$. Effect of decreasing κ for $C^i = \kappa C^s$.



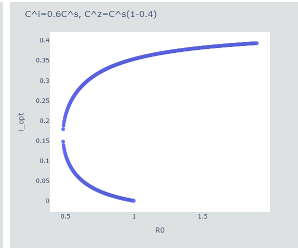
(A) $C^i = 0.9C^s$



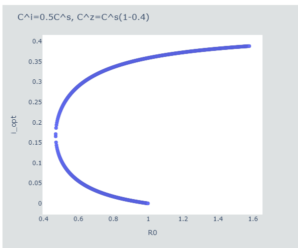
(B) $C^i = 0.8C^s$



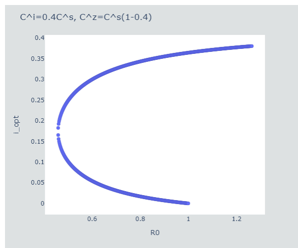
(C) $C^i = 0.7C^s$



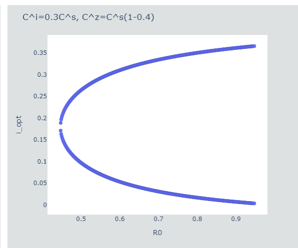
(D) $C^i = 0.6C^s$



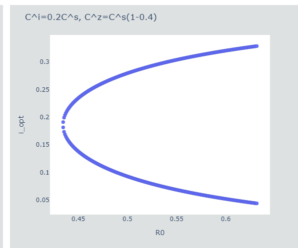
(E) $C^i = 0.5C^s$



(F) $C^i = 0.4C^s$



(G) $C^i = 0.3C^s$



(H) $C^i = 0.2C^s$

FIGURE 23. In all these cases $C^z = C^s(1-\nu)$, where $\nu = 0.4$ and we display cases of $C^i = \kappa C^s$. Similar results. C^z is larger in comparison to C^s now.

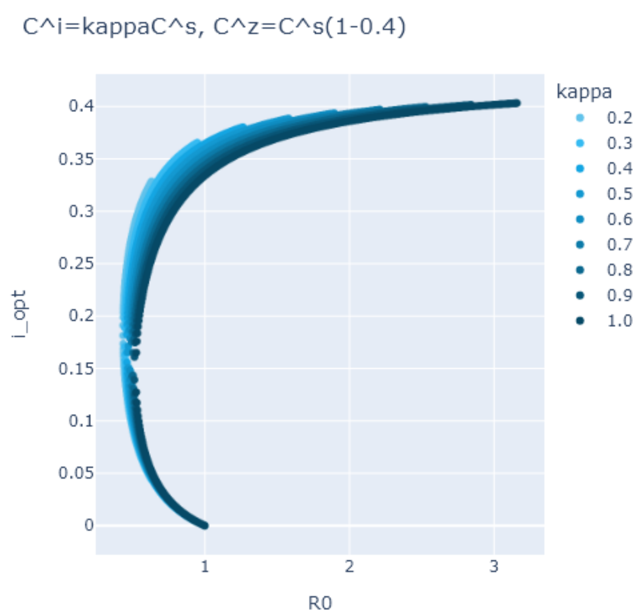
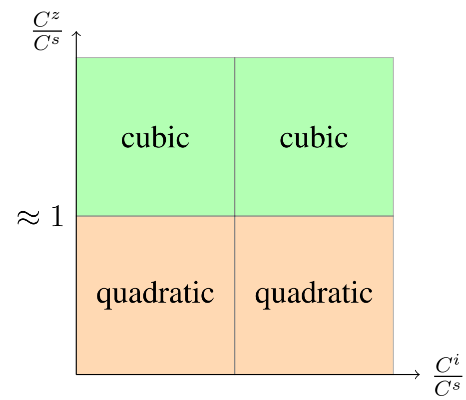


FIGURE 24. Comparison of all curves for $C^z = C^s(1-0.4)$. Effect of decreasing κ for $C^i = \kappa C^s$.

2.3.3. *Remarks.*

- When $C^z > C^s$ (this is: $\theta = \frac{C^z}{C^s} > 1$), lets say $C^z = C^s(1+\nu)$, a cubic behavior can be found.
- When $C^z < C^s$ (this is: $\frac{C^z}{C^s} < 1$), lets say $C^z = C^s(1-\nu)$, a cubic behavior is never found and we obtain regular backwards quadratic bifurcation plots. The next figure explains the results of these simulations:



2.4. **Theoretical Results.** Based on our simulations, we propose the following result.

Theorem 2.3. *Let μ, γ, ϕ real positive numbers, and define R_μ and R_ϕ as in (2.8).*

Suppose that

$$R_\phi > \frac{1 + 2R_\mu^2}{R_\mu^2 - R_\mu + 1}, \tag{2.9}$$

then there exists $0 < \theta_1 < \theta_2$ such that for every $\theta \in [\theta_1, \theta_2]$ and every $\kappa \in [0,1]$, there's an $R_0 > 0$ such that the polynomial equation:

$$f(X) = x_0 + x_1 X + x_2 X^2 + x_3 X^3 = 0,$$

where $x_0, \cdots, x_3$ are defined by (2.7), has three distinct real roots in the $[0,1]$ interval. And, in fact, for each (κ, θ) combination, this will happen for all R_0 in a neighborhood of the form $[1, 1+\epsilon_\theta]$, where the ϵ_θ only depends on θ, and is independent of κ.

In other words, there is a range of the fraction $\theta = \frac{C^z}{C^i}$ which will yield a cubic bifurcation plot, under our condition (2.9), independently of the value of $\kappa = \frac{C^i}{C^s}$. To prove this we make use of the general theory of Sturm sequences. Which we recall now.

Definition 2.4. Let $p_0(x) \in \mathbb{R}[x]$ a polynomial with real coefficients of degree $n \geq 0$. A sequence $S = p_0(x), p_1(x), p_2(x), \cdots, p_m(x)$ of polynomials in $\mathbb{R}[x]$ is called a **Sturm sequence** for $p_0(x)$ if it satisfies the following properties:

a) The last term $p_m(x)$ of the sequence is either always positive or always negative on the real line.
b) No two consecutive $p_i(x)$ are simultaneously zero for x a real number.
c) Suppose that α is a real root of $p_i(x)$, for some i with $0 < i < m$. Then $p_{i-1}(\alpha)$ and $p_{i+1}(\alpha)$ have opposite signs (note that neither is zero by (b) above).
d) At any real root α of $p_0(x)$, the graph of $p(x)$ "crosses" the x-axis at α, or, in other words, the values of $p(x)$ close to α and on either side of α it are of opposite sign.

Remark 2.5. This is taken from these notes: Sturm's method for the number of real roots of a real polynomial.

- When the polynomial $p_0(x)$ has no repeated roots, then the sequence
$$p_0(x), p_0'(x), p_0^{(2)}(x), p_0^{(3)}(x), \cdots, p_0^{(n)}(x),$$
of higher derivatives of $p_0(x)$ is a Sturm sequence for $p_0(x)$.
- The **canonical Sturm sequence** is obtained with $p_1(x) = p_0'(x)$ and $p_{i+1}(x)$ being the negative of the residue of dividing $p_i(x)$ by $p_{i-1}(x)$ for $i \geq 1$.

Proposition 2.6 (Sturm's theorem)**.** Let $p(x)$ be a non-zero polynomial with real coefficients. The number of distinct real roots (counted without multiplicity) of $p(x)$ in an interval (a, b] of the real line is given by $V_S(a) - V_S(b)$, where $V_S(c)$ is the number of consecutive sign changes of the values of the polynomials of a Sturm sequence S at $x = c$.

With all of this we can start the proof of theorem (2.3).

Proof. Let's assume that the polynomial $f(X)$ has no repeated roots. In this case, as discussed in remark (2.5), the sequence of higher derivatives of $f(X)$ forms a Sturm sequence for $f(X)$. This sequence is then

$$[x_0 + x_1X + x_2X^2 + x_3X^3, x_1 + 2x_2X + 3x_3X^2, 2x_2 + 6x_3X, 6x_3].$$

Its values at $x = 0$ are $(x_0, x_1, 2x_2, 6x_3)$ and at $x = 1$ are $(x_0 + x_1 + x_2 + x_3, x_1 + 2x_2 + 3x_3, 2x_2 + 6x_3, 6x_3)$. We will try to find R_0's for which the signs of these sequences are $-, +, -, +$ at $x = 0$ and $+, +, +, +$ at $x = 1$. This would prove the existence of three different roots in the $[0, 1)$ interval, according to Sturm's Theorem, because $V_S(0) = 3$ and $V_S(1) = 0$.

Thus, we determine a region in which the following inequalities occur:

$$\begin{aligned}
&x_0 < 0, \quad x_1 > 0, \quad x_2 < 0, \quad x_3 > 0 \\
&x_0 + x_1 + x_2 + x_3 > 0, \quad x_1 + 2x_2 + 3x_3 > 0, \quad 2x_2 + 6x_3 > 0.
\end{aligned}$$

Note that if $R_0 > 1$ then $x_0 < 0$ and if $R_0 > R_\mu$, then for all $\kappa \in [0, 1]$ we have

$$x_3 = R_\phi^2[R_0 - R_\mu(1 - \kappa)] > R_\phi^2[R_0 - R_\mu] > 0.$$

Because $R_\mu < 1$ then $R_0 > 1$ is sufficient to ensure $x_0 < 0$ and $x_3 > 0$. Now, for all the other inequalities it's enough to have just these:

$$x_2 < 0, \quad x_0 + x_1 > 0, \quad x_2 + x_3 > 0. \tag{2.10}$$

Indeed, the second inequality would imply $x_1 > -x_0 > 0$ (because $x_0 < 0$), the last two inequalities combined would imply $x_0 + x_1 + x_2 + x_3 > 0$ and also we would have

$$\begin{aligned}
&x_1 + 2x_2 + 3x_3 > x_1 + 2x_2 + 2x_3 = x_1 + 2(x_2 + x_3) > 0, \text{ and} \\
&2x_2 + 6x_3 > 2x_2 + 2x_3 = 2(x_2 + x_3) > 0.
\end{aligned}$$

We now move to a coordinate plane with R_0 in the x-axis and θ in the y-axis. The coefficients $x_0, \cdots, x_3$ can be seen as linear equations in this plane, given by:

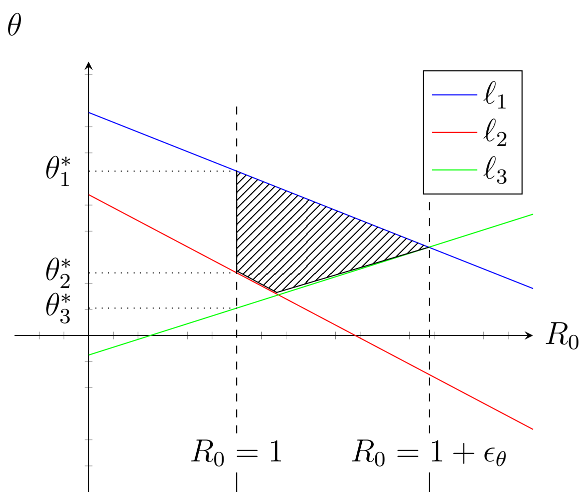
$$\begin{aligned}
x_3 &= AR_0 - B(1 - \kappa) \\
x_2 &= C + DR_0 - E(1 - \kappa) - F(1 - \theta) \\
x_1 &= G + HR_0 - I(1 - \kappa) - J(1 - \theta) \\
x_0 &= K(1 - R_0),
\end{aligned}$$

where the constants $A, B, \cdots, K$ depend only on R_ϕ and R_μ and are given by

$$
\begin{aligned}
&A := R_\phi^2, \quad B := R_\mu R_\phi^2, \quad C := R_\phi R_\mu^2, \quad D := R_\phi(1-R_\phi) + R_\phi R_\mu,\\
&E := R_\phi R_\mu(1+R_\mu), \quad F := R_\phi R_\mu(1-R_\mu), \quad G := R_\phi R_\mu(1+R_\mu)\\
&H := R_\mu(1-R_\phi) - R_\mu R_\phi, \quad I := R_\mu^2, \quad J := R_\mu(1-R_\mu) \quad K := R_\mu^2.
\end{aligned}
$$

Note that all constants are positive, except perhaps D and H (we don't know the sign of $1 - R_\phi$).

Geometrically, the inequalities in (2.10) refer to an area that is below the line $\ell_1 := \{x_2 = 0\}$ and above the lines $\ell_2 := \{x_2 + x_3 = 0\}$ and $\ell_3 := \{x_0 + x_1 = 0\}$ in the (R_0, θ) plane. Consider the intersections of these three lines with the $R_0 = 1$ vertical line. If we prove that the θ coordinate of intersection of the ℓ_1 with this vertical axis is bigger than the θ coordinates of intersections of the other two (ℓ_2 and ℓ_3), then there would be an interval to the right of $R_0 = 1$ which is below the line ℓ_1 and above both ℓ_1 and ℓ_2. See the next figure for a visual intuition.



Actually, this is a specific example and we don't know the signs of the slopes of lines ℓ_1, ℓ_2 and ℓ_3. However, it's their intersections at $R_0 = 1$ which will define the existence of a region below ℓ_1 and above both ℓ_2 and ℓ_3 in a neighborhood of $R_0 = 1$.

The intersection of line ℓ_2 at $R_0 = 1$ gives us

$$\theta_1^* = 1 - \frac{C+D}{F} + \frac{E}{F}(1-\kappa),$$

and the intersection of line ℓ_2 at $R_0 = 1$ gives us

$$\theta_2^* = 1 - \frac{C+D}{F} + \frac{E}{F}(1-\kappa) + \frac{B(1-\kappa) - A}{F} < \theta_1^* + \frac{B-A}{F},$$

we note that $B - A = R_\mu R_\phi^2 - R_\phi^2 = R_\phi^2(R_\mu - 1) < 0$, so we indeed have $\theta_2^* < \theta_1^*$.

The θ coordinate of the intersection between ℓ_3 and $R_0 = 1$ is

$$\theta_3^* = 1 - \frac{G+H}{J} + \frac{I}{J}(1-\kappa) < 1 - \frac{G+H}{J} + \frac{I}{J}.$$

We note that $\theta_2^* > 1 - \frac{C+D}{F}$ for all $\kappa \in [0,1]$, so we would need

$$1 - \frac{C+D}{F} > 1 - \frac{G+H}{J} + \frac{I}{J},$$

to guarantee $\theta_3^* < \theta_1^*$ for all κ. This is equivalent to $(G+H-I)F > (C+D)J$, and by expanding this expression we get the inequality (2.9) from our statement (see remark below with the calculation details). Note that this region can be obtained independently of κ.

Inequality (2.9) will imply that $\frac{C+D}{F} < 1$ and $\frac{G+H-I}{J} < 1$, and these are readily seen to imply that $\theta_i^* > 0$ for $i = 1, 2, 3$ (see remark below with the calculation details).

This proves that when (2.9) is true, and our polynomial $f(X)$ doesn't have repeated roots, there's a region in the (R_0, θ) plane for which $f(X)$ has three real distinct roots in the interval $[0,1]$. After all of this we are only left to find conditions under which our polynomial indeed doesn't have repeated roots.

Discriminant (?) ... Continue with proof ... almost there ...

$\square$

Remark 2.7. Note that

$$\begin{aligned} G+H-I &= \underbrace{R_\phi R_\mu(1+R_\mu)}_{G} + \underbrace{R_\mu(1-R_\phi) - R_\mu R_\phi}_{H} - \underbrace{R_\mu^2}_{I} \\ &= R_\phi R_\mu + R_\phi R_\mu^2 + R_\mu - R_\mu R_\phi - R_\mu R_\phi - R_\mu^2 \\ &= R_\phi R_\mu^2 + R_\mu - R_\mu R_\phi - R_\mu^2, \end{aligned}$$

and

$$
\begin{aligned}
C + D &= \underbrace{R_\phi R_\mu^2}_{C} + \underbrace{R_\phi(1 - R_\phi) + R_\phi R_\mu}_{D} \\
&= R_\phi R_\mu^2 + R_\phi - R_\phi^2 + R_\mu R_\phi.
\end{aligned}
$$

Then,

$$
\begin{aligned}
&(G + H - I)F > (C + D)J \Leftrightarrow \\
&[R_\phi R_\mu^2 + R_\mu - R_\mu R_\phi - R_\mu^2]R_\phi \underbrace{R_\mu(1 - R_\mu)}_{F} > [R_\phi R_\mu^2 + R_\phi - R_\phi^2 + R_\mu R_\phi] \underbrace{R_\mu(1 - R_\mu)}_{J} \Leftrightarrow \\
&[R_\phi R_\mu^2 + R_\mu - R_\mu R_\phi - R_\mu^2]R_\phi R_\mu(1 - R_\mu) > [R_\mu^2 + 1 - R_\phi + R_\mu]R_\phi R_\mu(1 - R_\mu) \Leftrightarrow \\
&R_\phi R_\mu^2 + R_\mu - R_\mu R_\phi - R_\mu^2 > R_\mu^2 + 1 - R_\phi + R_\mu \Leftrightarrow \\
&R_\phi R_\mu^2 - R_\mu R_\phi + R_\phi > 1 + 2R_\mu^2 \Leftrightarrow \\
&R_\phi > \frac{1 + 2R_\mu^2}{R_\mu^2 - R_\mu + 1}.
\end{aligned}
$$

It's always the case that $G + H - I < J$ because this is equivalent to

$$
\begin{aligned}
&R_\phi R_\mu^2 + R_\mu - R_\mu R_\phi - R_\mu^2 < R_\mu - R_\mu^2 \Leftrightarrow \\
&R_\phi R_\mu^2 - R_\mu R_\phi < 0 \Leftrightarrow \\
&R_\mu^2 < R_\mu,
\end{aligned}
$$

which is true because $R_\mu < 1$. To prove that $C + D < F$, se see that

$$
\begin{aligned}
&C + D < F \Leftrightarrow \\
&R_\phi R_\mu^2 + R_\phi - R_\phi^2 + R_\mu R_\phi < R_\phi R_\mu(1 - R_\mu) \Leftrightarrow \\
&R_\phi R_\mu^2 + R_\phi - R_\phi^2 + R_\mu R_\phi < R_\phi R_\mu - R_\phi R_\mu^2 \Leftrightarrow \\
&2R_\phi R_\mu^2 + R_\phi(1 - R_\phi) < 0 \Leftrightarrow \\
&2R_\mu^2 + 1 < R_\phi.
\end{aligned}
$$

For $0 < R_\mu < 1$ it's simple to see that the polynomial $R_\mu^2 - R_\mu + 1 < 1$ so

$$2R_\mu^2 + 1 < \frac{2R_\mu^2 + 1}{R_\mu^2 - R_\mu + 1} < R_\phi,$$

because of inequality (2.9). Therefore, we end up with $C + D < F$ and thus, $\theta_i^* > 0$ for $i = 1, 2, 3$.

3. Appendix

3.1. **Calculations of coefficients for cubic equilibrium.** Assuming $i > 0$, equations from the model give us

$$C\beta s + \phi z - (\gamma + \mu) = 0,$$

which is

$$C\beta\left(1 - i - \frac{\gamma i}{\phi i + \mu}\right) + \phi\frac{\gamma i}{\phi i + \mu} - (\gamma + \mu) = 0$$

Multiplying by $\phi i + \mu$ this is

$$C\beta\Big((\phi i + \mu)(1 - i) - \gamma i\Big) - \mu(\mu + \gamma + \phi i) = 0.$$

Using that

$$\begin{aligned} C = \frac{C^s C^i}{sC^s + iC^i + zC^z} &= \frac{C^s C^i}{i(C^i - C^s) + z(C^z - C^s) + C^i} \\ &= \frac{C^s C^i(\phi i + \mu)}{i(\phi i + \mu)(C^i - C^s) + \gamma i(C^z - C^s) + C^s} \end{aligned}$$

We have

$$C^sC^i\beta\bigg((\phi i+\mu)(1-i)-\gamma i\bigg)(\phi i+\mu)-\mu(\mu+\gamma+\phi i)\bigg(i(\phi i+\mu)(C^i-C^s)+\gamma i(C^z-C^s)+C^s(\phi i+\mu)\bigg)$$
$$=0$$

If we divide both sides by $(\mu+\gamma)^3$ this expression equals:

$$\underbrace{C^sR_0\bigg((R_\phi i+R_\mu)(1-i)-(1-R_\mu)i\bigg)(R_\phi i+R_\mu)}_{1}$$
$$-\underbrace{R_\mu(1+R_\phi i)\bigg(i(R_\phi i+R_\mu)(C^i-C^s)+(1-R_\mu)i(C^z-C^s)+C^s(R_\phi i+R_\mu)\bigg)}_{2}=0$$

We simplify this expression. Note that

$$\begin{aligned}(R_\phi i+R_\mu)(1-i)-(1-R_\mu)i &= R_\phi i-i^2R_\phi+R_\mu-iR_\mu-i+iR_\mu\\ &= -R_\phi i^2-i(1-R_\phi)+R_\mu.\end{aligned}$$

So the expression 1 becomes:

$$\begin{aligned}&-C^sR_0\bigg(R_\phi i^2+i(1-R_\phi)-R_\mu\bigg)(R_\phi i+R_\mu)\\ &=-C^sR_0\bigg[R_\phi^2i^3+(R_\phi R_\mu+R_\phi(1-R_\phi))i^2+R_\mu(1-R_\phi)i-R_\mu R_\phi i-R_\mu^2\bigg]\\ &=-\bigg[R_\phi^2C^sR_0i^3+(R_\phi R_\mu C^sR_0+R_\phi(1-R_\phi)C^sR_0)i^2+R_\mu(1-R_\phi)C^sR_0i-R_\mu R_\phi C^sR_0i-R_\mu^2C^sR_0\bigg]\end{aligned}$$

Also,

$$\begin{aligned}&i(R_\phi i+R_\mu)(C^i-C^s)+(1-R_\mu)i(C^z-C^s)+C^s(R_\phi i+R_\mu)\\ &=R_\phi(C^i-C^s)i^2+R_\mu(C^i-C^s)i+(1-R_\mu)(C^z-C^s)i+C^sR_\phi i+C^sR_\mu\\ &=R_\phi(C^i-C^s)i^2+\big[R_\mu(C^i-C^s)+(1-R_\mu)(C^z-C^s)+C^sR_\phi\big]i+C^sR_\mu.(*)\end{aligned}$$

Multiplying $(*)$ by $(1+R_\phi i)$ yields:

$$R_\phi(C^i - C^s)i^2 + \big[R_\mu(C^i - C^s) + (1 - R_\mu)(C^z - C^s) + C^s R_\phi\big]i + C^s R_\mu$$
$$+ R_\phi^2(C^i - C^s)i^3 + \big[R_\mu R_\phi(C^i - C^s) + (1 - R_\mu)R_\phi(C^z - C^s) + C^s R_\phi^2\big]i^2 + C^s R_\mu R_\phi i$$

Multiplying by R_μ gives us the full expression for 2 as:

$$R_\phi R_\mu(C^i - C^s)i^2 + \big[R_\mu^2(C^i - C^s) + R_\mu(1 - R_\mu)(C^z - C^s) + C^s R_\phi R_\mu\big]i + C^s R_\mu^2$$
$$+ R_\phi^2 R_\mu(C^i - C^s)i^3 + \big[R_\mu^2 R_\phi(C^i - C^s) + (1 - R_\mu)R_\phi R_\mu(C^z - C^s) + C^s R_\phi^2 R_\mu\big]i^2 + C^s R_\mu^2 R_\phi i$$

Removing the negative signs in front of each expression we get the next expression to equal zero.

$$R_\phi^2 C^s R_0 i^3 + (R_\phi R_\mu C^s R_0 + R_\phi(1 - R_\phi)C^s R_0)i^2 + R_\mu(1 - R_\phi)C^s R_0 i - R_\mu R_\phi C^s R_0 i - R_\mu^2 C^s R_0$$
$$+ R_\phi R_\mu(C^i - C^s)i^2 + \big[R_\mu^2(C^i - C^s) + R_\mu(1 - R_\mu)(C^z - C^s) + C^s R_\phi R_\mu\big]i + C^s R_\mu^2$$
$$+ R_\phi^2 R_\mu(C^i - C^s)i^3 + \big[R_\mu^2 R_\phi(C^i - C^s) + (1 - R_\mu)R_\phi R_\mu(C^z - C^s) + C^s R_\phi^2 R_\mu\big]i^2 + C^s R_\mu^2 R_\phi i$$

The free coefficient is

$$x_0 = C^s R_\mu^2(1 - R_0).$$

The principal coefficient, that is the i^3 coefficient, is

$$\begin{aligned} x_3 &= C^s R_\phi^2 R_0 + R_\phi^2 R_\mu(C^i - C^s) \\ &= R_\phi^2\big(C^s R_0 + \underbrace{R_\mu(C^i - C^s)}_{\textbf{extra}}\big). \end{aligned}$$

The coefficient with i^2 is

$$
\begin{aligned}
&R_\phi R_\mu C^s R_0 + R_\phi(1-R_\phi)C^s R_0 + R_\phi R_\mu(C^i - C^s) + R_\mu^2 R_\phi(C^i - C^s) + (1-R_\mu)R_\phi R_\mu(C^z - C^s) + C^s R_\phi^2 R_\mu \\
&= R_\phi\Big[R_\mu C^s R_0 + (1-R_\phi)C^s R_0 + R_\mu(C^i - C^s) + R_\mu^2(C^i - C^s) + (1-R_\mu)R_\mu(C^z - C^s) + C^s R_\phi R_\mu\Big] \\
&= R_\phi\Big[C^s R_0(1-R_\phi) + R_\mu(C^s R_0 + C^s R_\phi) + R_\mu(1+R_\mu)(C^i - C^s) + (1-R_\mu)R_\mu(C^z - C^s)\Big]
\end{aligned}
$$

Rearranging a little bit,

$$
x_2 = R_\phi\Bigg[C^s\left[R_0(1-R_\phi) + R_\mu(R_0+R_\phi) + R_\mu(1-R_\mu)\left(\frac{C^z}{C^s} - 1\right)\right] + \underbrace{R_\mu(1+R_\mu)(C^i - C^s)}_{\textbf{extra}}\Bigg].
$$

Finally, the term with i equals

$$
\begin{aligned}
&= R_\mu(1-R_\phi)C^s R_0 - R_\mu R_\phi C^s R_0 + R_\mu^2(C^i - C^s) + R_\mu(1-R_\mu)(C^z - C^s) + C^s R_\phi R_\mu + C^s R_\mu^2 R_\phi \\
&= R_\mu\Big[(1-R_\phi)C^s R_0 - R_\phi C^s R_0 + R_\mu(C^i - C^s) + (1-R_\mu)(C^z - C^s) + C^s R_\phi + C^s R_\mu R_\phi\Big] \\
&= R_\mu\Big[C^s R_0(1-R_\phi) + C^s R_\phi(1-R_0) + (1-R_\mu)(C^z - C^s) + C^s R_\mu R_\phi + \underbrace{R_\mu(C^i - C^s)}_{\textbf{extra}}\Big] \\
&= R_\mu\Bigg[C^s\left\{R_0(1-R_\phi) + R_\phi(1-R_0) + (1-R_\mu)\left(\frac{C^z}{C^s} - 1\right) + R_\mu R_\phi\right\} + \underbrace{R_\mu(C^i - C^s)}_{\textbf{extra}}\Bigg]
\end{aligned}
$$

Taking C^s as a factor out of all the coefficients we finally have that +i satisfies a cubic equation $f(i) = x_3 i^3 + x_2 i^2 + x_1 i + x_0 = 0$, where the coefficients are given by

$$
\begin{aligned}
x_3 &= R_\phi^2 R_0 + R_\mu R_\phi^2\left(\frac{C^i}{C^s} - 1\right) \\
x_2 &= R_\phi\left[R_0(1-R_\phi) + R_\mu(R_0+R_\phi) + R_\mu(1-R_\mu)\left(\frac{C^z}{C^s} - 1\right) + R_\mu(1+R_\mu)\left(\frac{C^i}{C^s} - 1\right)\right] \\
x_1 &= R_\mu\left[R_0(1-R_\phi) + R_\phi(1-R_0) + (1-R_\mu)\left(\frac{C^z}{C^s} - 1\right) + R_\mu R_\phi + R_\mu\left(\frac{C^i}{C^s} - 1\right)\right] \\
x_0 &= R_\mu^2(1-R_0).
\end{aligned}
$$

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