1. Adaptive behavior model

We describe the novel technique presented in [2], which gives an adaptive approach to the construction of an epidemiological system. The classical approach is taking the system:

$$
\begin{aligned}
\frac{dS}{dt} &= -C(\cdot)\beta SI/N \\
\frac{dI}{dt} &= C(\cdot)\beta SI/N - \nu I \\
\frac{dZ}{dt} &= \nu I.
\end{aligned}
$$

Where:

- A population of N individuals is divided in three compartiments: $N = S + I + R$. Here S are the susceptible individuals, I are the infected and R are the recovered individuals.
- β represents the likelihood that contact with an infected individual yields infection.
- ν is the rate of recovery.
- $C(\cdot)$ is the rate that susceptibles contact infected, which means that $C(\cdot)\beta$ is the rate that susceptible individuals become infected.

In the classical setting, either $C(\cdot) = c$ (contacts are constant) or $C(\cdot) = cN$ (contacts are proportional to N).

In the adaptive setting, $C(\cdot)$ depends on the incentives different individuals have to vary their number of contacts. The costs and benefits of individual contact vary across health status.

The proposal is then to divide the individuals by health type. Let $Y = \{s, i, z\}$. For $h \in Y$ denote C^h the expected number of contacts made by an individual of type h. For $m, n \in Y$ we define

$$
C^{mn}(\cdot) = C^m C^n N/(SC^s + IC^i + ZC^z).
$$

The rate of contact between individuals of types m and n. In here C^m **is a choice** made by individuals of type m. In the classical model $C(\cdot) = C^{si}$.

People engage in contacts because there is a certain utility to gain from them. The adaptive approach models the utility for an individual of type $h \in Y$ depending on the current time, therefore we have an utility

$$u_t^h = u_t^h(C_t^h),$$

where t is the current time, and C_t^h is the expected number of contacts of an individual of type h made at time t. u_h^t is **the utility of making contacts for an individual of type h at time t.** The utility function should be concave and should have a single peak with respect to the number of contacts, but should decrease with infection.

An example of an utility function provided in [2] is

$$u_t^h = (b^h C_t^h - (C_t^h)^2)^\gamma - a^h,$$

where γ, b^h, a^h are fixed parameters, with $b^s = b^z \geqslant b^i \geqslant 0$, $a^z = a^s = 0$, $\gamma > 0$, $a^i > 0$. **Intiutively, this means that during the infection period, the utility has a term that pauses it's increment. Note that for each state utility has a peak with respect to C_t^h.**

Recovered (and Immune) and Infected individuals: If the individual doesn't think that a change in their contacts will affect their health status, then for a given time t, the best thing would be to choose C_t^h such that u_t^h is maximized. This happens with individuals of types i and z. The optimal choice is $C_h^{t*} = 0.5b^h$.

Susceptible individuals: The number of contacts a susceptible individual engages in might affect their health status, so the optimal choice of contacts C_t^{s*} is subjected to planning towards the future. Here is where the adaptive decision comes in, as a factor of future utility.

Let P_t^i the probability that an s-type individual becomes infected at time t. This depends on the current state of things and the selection of C_t^s, as

$$P_t^i = 1 - e^{-\beta I_t C_t^s C_t^{i*}/(S_t C_t^{s*} + I_t C_t^{i*} + Z_t C_t^{z*})}, \tag{1.1}$$

where C_t^{s*} is the optimal choice of other susceptible individuals the present susceptible individual might encounter.

To find the optimal C_t^{s*} we maximize the value function

$$V_t(s) = \max_{C^s \in X} \left\{ u_t^s(C_t^s) + \delta \left[(1 - P_t^i) V_{t+1}(s) + P_t^i V_{t+1}(i) \right] \right\}, \tag{1.2}$$

where X is the range of possible contacts, δ is a discount factor, $V_{t+1}(s)$ is the present value of expected utility if the individual remains susceptible and $V_{t+1}(i)$ the present value of expected utility if the individual becomes infected.

Solving equation (1.2) gives the first order condition:

$$\frac{\partial u_t}{\partial C_t^s} = \delta(V_{t+1}(s) - V_{t+1}(i)) \left(\frac{\partial P_t^i}{\delta C_t^s} \right). \tag{1.3}$$

The idea on how to select the optimal C_t^{s*} at time t depends on the current state of things and the previsions the individual does for the future. The adaptive approach proposes a continuous update of the selection made across time.

For each $t < \tau - 1$, the individual will solve (1.2). For that they will need to solve (1.3). This requires knowledge of $V_{t+1}(i)$, which it's modeled like:

$$V_{t+1}(i) = u_t^z(C_t^{z*}) \left[\left(\frac{1 - \delta^{\tau+1}}{1 - \delta} \right) - \left(\frac{1 - (\delta(1 - P^z))^{\tau+1}}{1 - \delta(1 - P^z)} \right) \right], \tag{1.4}$$

where τ is the **planning period** and $P^z = 1 - e^{-\nu}$ is the probability of recovery. The adaptive algorithm is performed as follows.

Algorithm 1 Update of C_t^s

1: Input: Planning time frame τ.
2: Susceptible individual chooses C_0^s at time $t = 0$, in order to solve (1.2).
3: Compute $V_{t+1}(i)$ for $t \in [0, \tau]$ using (1.4), and for each choice of C_t^s use (1.3) to compute $V_{t+1}(s)$ using backwards induction. Find C_1^{s*} using (1.2).
4: **for** $t \geqslant 1$ **do**
5: Compute $V_{t+1}(i)$ for $t \in [t, t + \tau]$ using (1.4), and for each choice of C_t^s use (1.3) to compute $V_{t+1}(s)$ using backwards induction. Find C_{t+1}^{s*} using (1.2).
6: Return $\{C_t^{s*}\}_{t \geqslant 0}$.

2. UTILITY CALCULATIONS FOR IMPLEMENTATION

Here we provide some notes on how to implement the adaptive model. First we solve some more explicitly some of the equations presented in the paper.

- In equation (1.1) we have that

$$P_t^i = 1 - \exp\left(-\frac{\beta I_t C_t^s C_t^{i*}}{\phi(t)}\right) = 1 - \exp\left(-\frac{0.5 \cdot \beta b^i \cdot I_t C_t^s}{\phi(t)}\right),$$

where $\phi(t) = S_t C_t^{s*} + I_t C_t^{i*} + Z_t C_t^{z*} = S_t C_t^{s*} + 0.5 b^i \cdot I_t + 0.5 b^z \cdot Z_t$. Therefore

$$\frac{\partial P_t^i}{\partial C_t^s} = \beta I_t C_t^{i*} \cdot \exp\left(-\frac{\beta I_t C_t^s C_t^{i*}}{\phi(t)}\right) = 0.5\beta b^i I_t \cdot \exp\left(-\frac{0.5 \cdot \beta b^i \cdot I_t C_t^s}{\phi(t)}\right)$$

- Given that $u_t^s = (b^s C_t^s - (C_t^s)^2)^\gamma - a^s$, then

$$\frac{\partial u_t^s}{C_t^s} = \gamma (b^s C_t^s - (C_t^s)^2)^{\gamma - 1} + b^s - 2C_t^s.$$

- By definition $V_{t+1}(i) = u_t^z(z_t, C_t^{z*})\xi(\delta, \tau, P^z)$, where

$$\xi(\delta, \tau, P^z) = \left(\frac{1-\delta^{\tau+1}}{1-\delta}\right) - \left(\frac{1-(\delta(1-P^z))^{\tau+1}}{1-\delta(1-P^z)}\right)$$

using the form of u_t^z and the value of C_t^{z*} this equals

$$V_{t+1}(i) = [(0.25 \cdot (b^z)^2)^\gamma - a^z] \cdot \xi(\delta, \tau, P^z).$$

This is for $t \in [t_0, t_0 + \tau - 2]$ for $t = t_0 + \tau - 1$ we have

$$V_{t+1}(i) = V_{t_0+\tau} = u_{t_0+\tau}^i = \left(0.25 \cdot (b^i)^2\right)^\gamma - a^i$$

and for $t = t_0 + \tau$ we have $V_{t+1}(i) = V_{\tau+1}(i) = 0$. So in conclusion

$$V_{t+1}(i) = \begin{cases} [(0.25 \cdot (b^z)^2)^\gamma - a^z] \cdot \xi(\delta, \tau, P^z) & \text{if } t \in [t_0, t_0 + \tau - 2] \\ (0.25 \cdot (b^i)^2)^\gamma - a^i & \text{if } t = t_0 + \tau - 1 \\ 0 & \text{if } t = t_0 + \tau \end{cases} \tag{2.1}$$

- Using the above relations, then equation (1.3) can be written as

$$\gamma (b^s C_t^s - (C_t^s)^2)^{\gamma-1} + b^s - 2c_t^s = \delta(V_{t+1}(s) - V_{t+1}(i)) \cdot 0.5\beta b^i I_t \cdot \exp\left(-\frac{0.5 \cdot \beta b^i \cdot I_t C_t^s}{\phi(t)}\right),$$

thus we can clear $V_{t+1}(s)$ as

$$V_{t+1}(s) = \frac{\gamma(b^s C_t^s - (C_t^s)^2)^{\gamma-1} + b^s - 2C_t^s}{0.5 \cdot \delta\beta b^i \cdot I_t \cdot \exp\left(-\frac{0.5\cdot\beta b^i \cdot I_t C_t^s}{\phi(t)}\right)} + V_{t+1}(i).$$

Then, if we substitute in equation (1.2) we have that

$$V_t(s) = \max_{C^s \in X} \left\{ u_t^s(C_t^s) + \delta \left[(1 - P_t^i) V_{t+1}(s) + P_t^i V_{t+1}(i) \right] \right\}$$

$$= \max_{C^s \in X} \left\{ (b^s C_t^s - C_t^s)^\gamma - a^s - \delta \left[(1 - P_t^i) \left(\frac{\gamma(b^s C_t^s - (C_t^s)^2)^{\gamma-1} + b^s - 2C_t^s}{0.5 \cdot \delta\beta b^i \cdot I_t \cdot \exp\left(-\frac{0.5\cdot\beta b^i \cdot I_t C_t^s}{\phi(t)}\right)} \right) + V_{t+1}(i) \right] \right\}. \tag{2.2}$$

A bit complicated equation but we only made this computation for implementation purposes. For this formula C_t^{s*} is held as a constant (it is present in the definition of $\phi(t)$).

References

[1] J. G. Calvo, A. Hernández, M. A. Porter, and F. Sanchez. A two-patch epidemic model with nonlinear reinfection. *Rev. Mat*, 27(1):23–48, 2020.

[2] E. P. Fenichel, C. Castillo-Chavez, M. G. Ceddia, G. Chowell, P. A. G. Parra, G. J. Hickling, G. Holloway, R. Horan, B. Morin, C. Perrings, M. Springborn, L. Velazquez, and C. Villalobos. Adaptive human behavior in epidemiological models. *Proc. Natl. Acad. Sci. U.S.A.*, 108(15):6306–6311, 2011.