Finite Sample Properties of the EM Algorithm Applied to Transmission Tomography

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Motivation

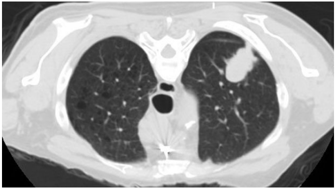
 

Figure: CT machine (https://www.medicalnewstoday.com/articles/153201) and scan (https://scienceblog.cancerresearchuk.org/2018/12/04/lung-cancer-screening-part-1-the-benefits-and-harms-according-to-clinical-trials/).

Jiaxing Sheng's PhD thesis: "An Entropic Bayesian Algorithm for Reconstructing Incomplete Poisson Data."

Simplified Scan

We consider 3 different sections of a scan.

1. The space outside of the scan. Below in gray. This will not be estimated.
2. The space outside the patient but inside the scan. Below in purple. The true θ value for these pixels is 0, but we still need to estimate them.
3. The space inside the patient. Below in yellow. We need to estimate this.

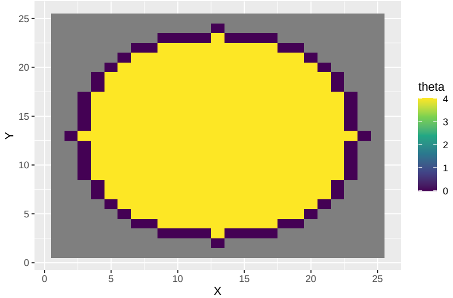


Figure: Scan Example

0 Angles

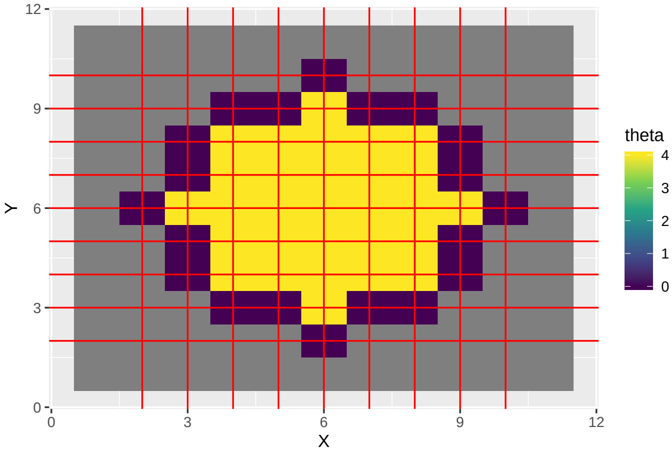


Figure: Horizontal and vertical projections.

2 Angles

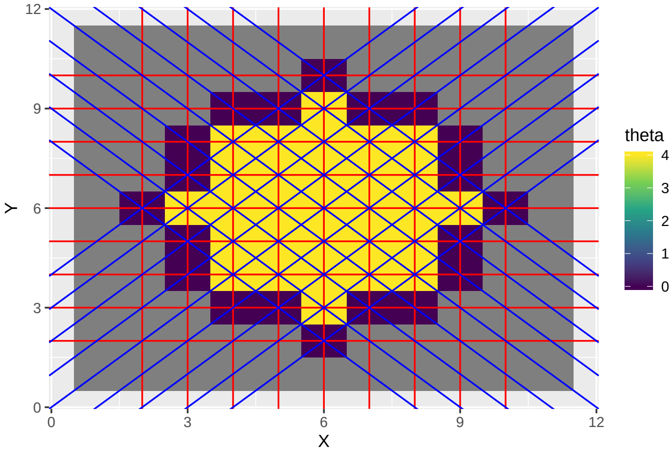


Figure: Horizontal, vertical, and diagonal projections.

6 Angles

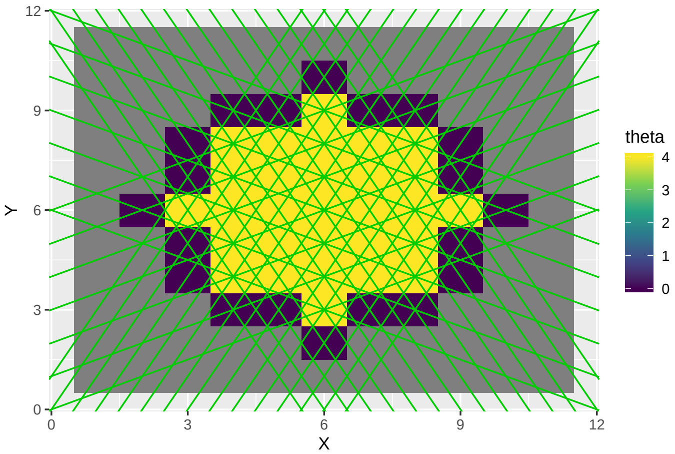


Figure: Projections with slopes of -2, -1/2, 1/2, and 2. Horizontal, vertical, and diagonal projections were omitted.

Illustrative Example

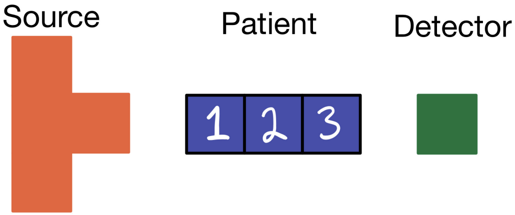
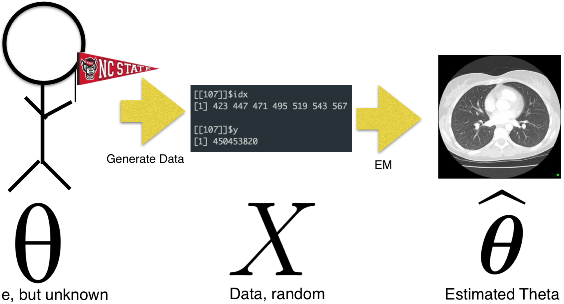


Figure: A simple scan

1. The source emits $X_1 \sim \text{Poisson}(d)$ photons.
2. Pixel 1 has a value of θ_1. When the photons from the emitter intercept, $X_2 | X_1 \sim \text{Binomial}(n = X_1, p = e^{-\theta_1})$ photons pass through.
3. Pixel 2 has a value of θ_2 and $X_3 | X_2 \sim \text{Binomial}(n = X_2, p = e^{-\theta_2})$ pass.
4. Pixel 3 has a value of θ_3 and $X_4 | X_3 \sim \text{Binomial}(n = X_3, p = e^{-\theta_3})$ pass.
5. The detector then receives X_4 photons.

Data

- Data
  - Photons that reach the detector, X_4.
- Missing Data
  - Photons initially emitted, X_1.
  - Photons that pass at each stage, X_2 and X_3.
- Parameters
  - Poisson rate parameter d, known *a priori*.
  - Theta values of each intersected pixel, θ_1, θ_2, and θ_3.



The EM Algorithm

Recall that the EM Algorithm has two steps.

1. E Step: Calculate $Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{\nu})$
2. M Step: Calculate $\boldsymbol{\theta}^{\nu+1}$ that maximizes $Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{\nu})$ with respect to $\boldsymbol{\theta}$.

In general we will have multiple projections, indexed by i, each passing through pixels, indexed by j. We maximize the following Q surrogate function by taking partial derivatives and setting them equal to 0.

$$M_{ij} = E(X_{ij}|Y_i = y_i, \boldsymbol{\theta}^{\nu})$$
$$N_{ij} = E(X_{ij'}|Y_i = y_i, \boldsymbol{\theta}^{\nu})$$

$$Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{\nu}) = \sum_i \sum_j \left[-N_{ij}\ell_{ij} + (M_{ij} - N_{ij}) \log(1 - e^{-\ell_{ij}\theta_j}) \right]$$

$$\frac{\partial Q}{\partial \theta_j} = -\sum_i N_{ij}\ell_{ij} + \sum_i \frac{(M_{ij} - N_{ij})\ell_{ij}}{e^{\ell_{ij}\theta_j} - 1} \overset{\text{set}}{=} 0$$

MC Experimental Design

Degrees of Freedoms and Metrics. $N = 10$ and $d = 10^9$ for all experiments.

- Theta Pattern
  - Three Circles
  - Checkered Pattern
  - Fun Patterns
- Radius
  - 3
  - 5
  - 10
- Number of Non-horizontal/vertical Angles
  - None
  - Two
  - Six
- Metrics
  - Frobenius Norm of the difference between the true and estimated theta matrices (RMSE)
  - Spectral Norm of the difference
  - Iterations

Three Circle Plots, Radius 5, 0 angles

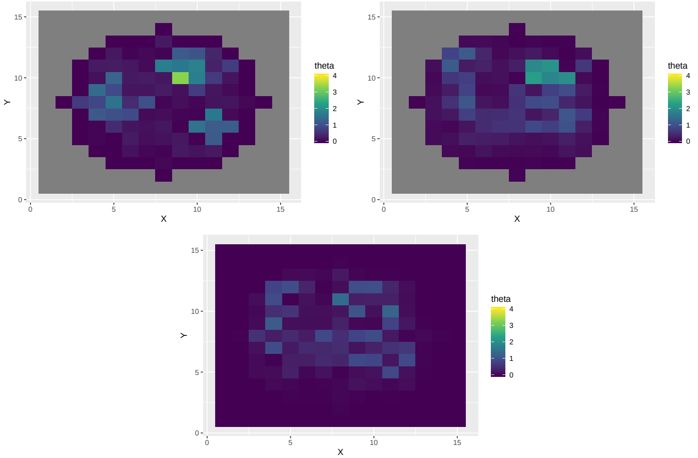


Figure: True theta, estimated theta, absolute difference for three circles, scan radius 5

Three Circle Plots, Radius 5, 2 angles

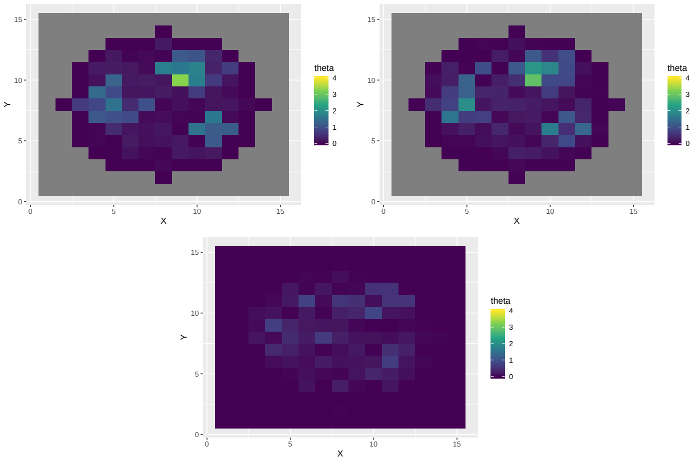


Figure: True theta, estimated theta, absolute difference for three circles, scan radius 5

Three Circle Plots, Radius 5, 6 angles

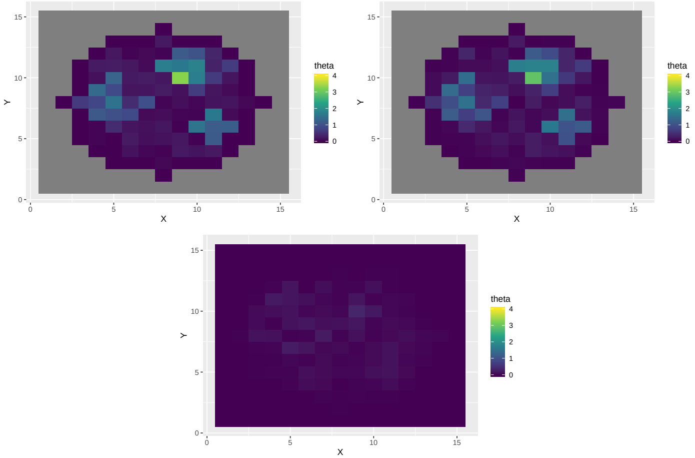


Figure: True theta, estimated theta, absolute difference for three circles, scan radius 5

Three Circle Plots, Radius 10, 6 angles

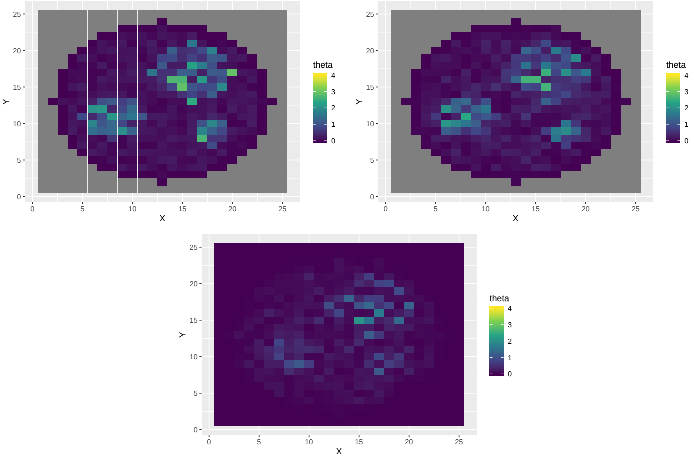


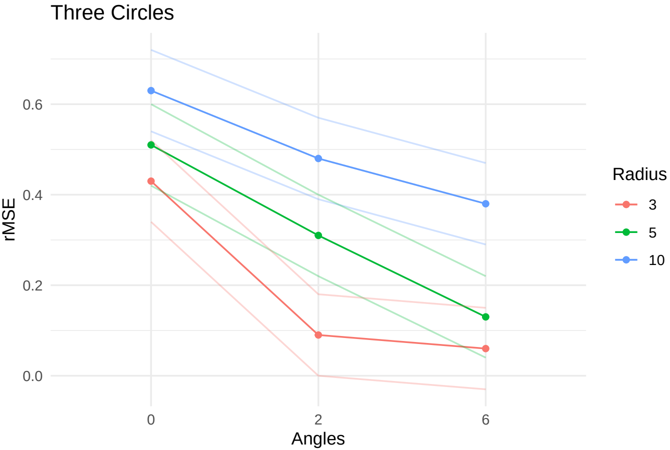
Figure: True theta, estimated theta, absolute difference for three circles, scan radius 10

Three Circle Theta Results

Monte Carlo Simulation Results with $N = 10$. Maximum standard error in () at top of each column.

Radius	Angles	RMSE (0.09)	Spec. Norm (0.7)	Iterations
3	0	0.43	2.7	34 (9)
	2	0.09	0.5	30 (5)
	6	0.06	0.3	30 (5)
5	0	0.51	3.9	50 (10)
	2	0.31	2.1	90 (40)
	6	0.13	0.9	100 (20)
10	0	0.63	7.5	300 (100)
	2	0.48	5.2	210 (40)
	6	0.38	3.9	250 (20)

Three Circles Error Plot



Checker Theta Plots

Monte Carlo Simulation Results with $N = 10$.

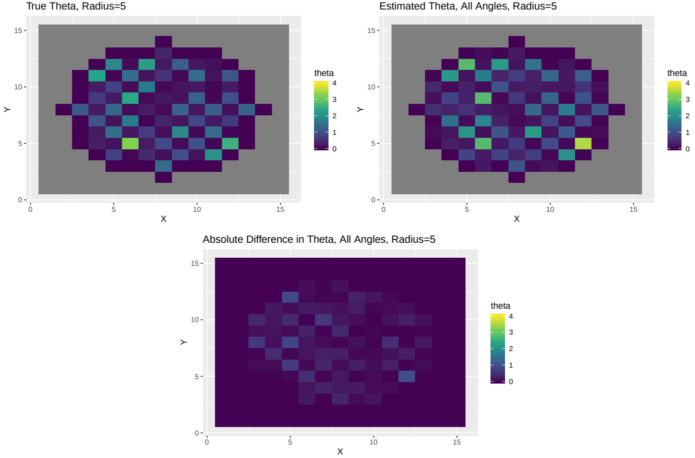


Figure: True theta, estimated theta, absolute difference for checker pattern.

Checker Theta Results

Monte Carlo Simulation Results with $N = 10$. Maximum standard error in () at top of each column.

Radius	Angles	RMSE (0.01)	Spec. Norm (0.7)	Iterations (5)
3	0	0.75	3.6	34
	2	0.35	1.8	74
	6	0.10	0.6	56
5	0	0.75	5.6	100
	2	0.61	4.2	160
	6	0.27	1.7	170

Face Theta Plots

Monte Carlo Simulation Results with $N = 10$.

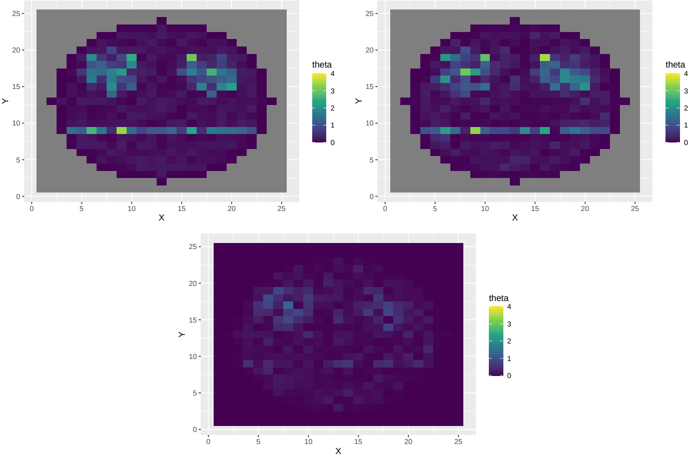


Figure: True theta, estimated theta, absolute difference for face

Face Theta Results

Monte Carlo Simulation Results with $N = 10$. Maximum standard error in () at top of each column.

Radius	Angles	RMSE (< 0.01)	Spec. Norm (< 0.1)	Iterations (10)
3	0	0.62	3.7	80
	2	0.10	0.5	50
	6	0.05	0.3	53
5	0	0.61	4.6	150
	2	0.27	1.9	110
	6	0.12	0.9	97
10	0	0.65	9.5	250
	2	0.37	4.1	250
	6	0.24	2.9	220

Boos Theta Plots

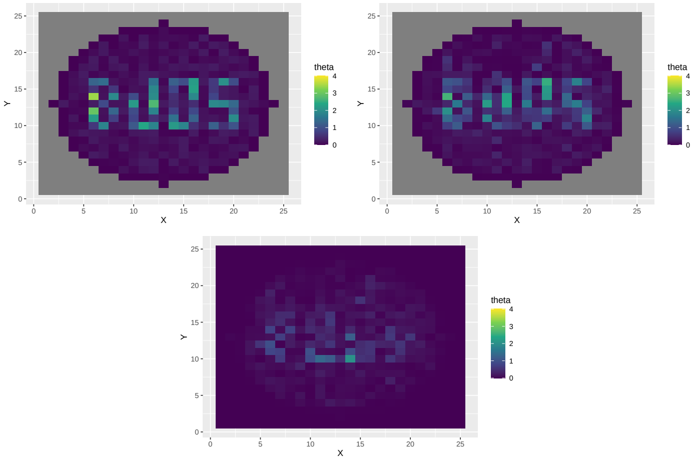


Figure: True theta, estimated theta, absolute difference for Boos

Boos Theta Results

Monte Carlo Simulation Results with $N = 10$.

Radius	Angles	RMSE (< 0.01)	Spec. Norm (0.1)	Iterations (3)
10	0	0.62	8.9	250
	2	0.40	4.8	270
	6	0.30	3.6	280

Discussion

- Visually can see very good $\hat{\theta}$
- RMSE and Spectral Norm: increase as Radius increases but decrease as number of angles increases
- Number of iterations increases as radius and angles increases
- Circle patterns give better results than obscure shapes
- Take home point: **Angles are so important!**
- Outstanding Questions
  - How would the performance change for even more angles?
  - How would errors change for larger N?
  - Determine how error/time scales with N and radius.
  - Compare to MM algorithm.
- Limitations
  - Very slow as radius increases.
  - Some patterns experience convergence issues.

References

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