ST 790 Assignment 2

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Instruction

This assignment consists of 4 problems. The assignment is due on **Tuesday, September 29** at 11:59pm EDT. Please submit your assignment electronically through the **Moodle** webpage. The assignment can be done as a group with at most 3 members per group (please include the name of the group members on the front page of the assignment).

Problem 1.

Let $\mathbf{D}$ be a $n \times n$ symmetric dissimilarity matrix, i.e., the entries d_{ij} of $\mathbf{D}$ are non-negative for all i, j, $d_{ii} = 0$ for all i. Let $\mathbf{B} = -\frac{1}{2}(\mathbf{I} - 11^\top/n)\mathbf{D}(\mathbf{I} - 11^\top/n)$ where $\mathbf{I}$ is the $n \times n$ identity matrix and $1 \in \mathbb{R}^n$ is the vector of all ones.

Given an integer $r \leq n$, the classical multidimensional scaling (CMDS) procedure produces a configuration $\mathbf{Z}_* \in \mathbb{R}^{n \times r}$ representing n points in $\mathbb{R}^r$ that best approximate $\mathbf{D}$ with respect to the STRAIN criterion, i.e.,

$$\mathbf{Z}_* = \arg\min_{\mathbf{Z} \in \mathbb{R}^{n \times r}} \|\mathbf{B} - \mathbf{Z}\mathbf{Z}^\top\|_F. \tag{1}$$

Consider the following eigendecomposition of $\mathbf{B}$.

$$\mathbf{B} = \sum_{i=1}^{n} \lambda_i u_i u_i^\top; \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

Then the CMDS procedure returns the $n \times r$ configuration

$$\mathbf{Z}_* = \left[(\lambda_1)_+^{1/2} u_1, (\lambda_2)_+^{1/2} u_2, \ldots, (\lambda_r)_+^{1/2} u_r\right] \tag{2}$$

where $(\lambda_i)_+ = \max\{\lambda_i, 0\}$.

Show that the expression $\mathbf{Z}_*$ in Eq.(2) is really the minimizer of the STRAIN criterion for $\mathbf{B}$. More specifically, show the following result.

Proposition Let $\mathbf{B}$ be a $n \times n$ symmetric matrix with eigendecomposition

$$\mathbf{B} = \sum_{i=1}^{n} \lambda_i u_i u_i^\top; \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

Then the best rank r, positive semidefinite approximation to $\mathbf{B}$, with respect to the Frobenius norm, is the matrix

$$\mathbf{B}_r = \sum_{i=1}^{r} (\lambda_i)_+ u_i u_i^\top.$$

Hint: Let $\mathbf{M}$ be any $n \times n$ matrix. Then

$$\|\mathbf{B} - \mathbf{M}\|_F^2 = \|\mathbf{U}\Lambda\mathbf{U}^\top - \mathbf{V}\Sigma\mathbf{V}^\top\|_F^2 = \|\Lambda - \mathbf{W}\Sigma\mathbf{W}^\top\|_F^2$$

where $\mathbf{U}\Lambda\mathbf{U}^\top$ and $\mathbf{V}\Sigma\mathbf{V}^\top$ are the eigendecomposition of $\mathbf{B}$ and $\mathbf{M}$, respectively, and $\mathbf{W}$ is a $n \times n$ orthogonal matrix.

Now, for fixed diagonal matrices Λ and Σ, show that a minimizer of

$$\min_{\mathbf{W}} \|\Lambda - \mathbf{W}\Sigma\mathbf{W}^\top\|_F^2$$

over the set of orthogonal matrices $\mathbf{W}$ is given by a permutation matrix. Using the rearrangement inequality, show that this permutation matrix will rearrange the diagonal entries of Σ to coincide with the ordering of the diagonal entries of Λ.

Problem 2

Download the dataset of games between American College Football teams available here. Now perform community detection using (1) a spectral clustering algorithm using normalized cut (2) spectral clustering using modularity and (3) Louvain algorithm. Evaluate the performance of your clustering (compared to the given ground truth).

```
g<-read.graph("football.gml",format=c("gml"))

g2 <- as.undirected(g)
## Largest connected component
lcc <- decompose.graph(g2, min.vertices = max(components(g2)$csize))[[1]]
Xhat2 <- embed_laplacian_matrix(lcc, type = "I-DAD", which = "sa", no = 12, scaled = FALSE)

Xhat2$D ## The three smallest eigenvalue of the normalized Laplacian
```

```
##  [1] 0.0000000 0.1368043 0.1829191 0.2250875 0.2396260 0.2823248 0.2998659
##  [8] 0.3247005 0.3773143 0.4099849 0.4581212 0.5512367
```

```
table(vertex_attr(lcc)$value) ## Ground truth
```

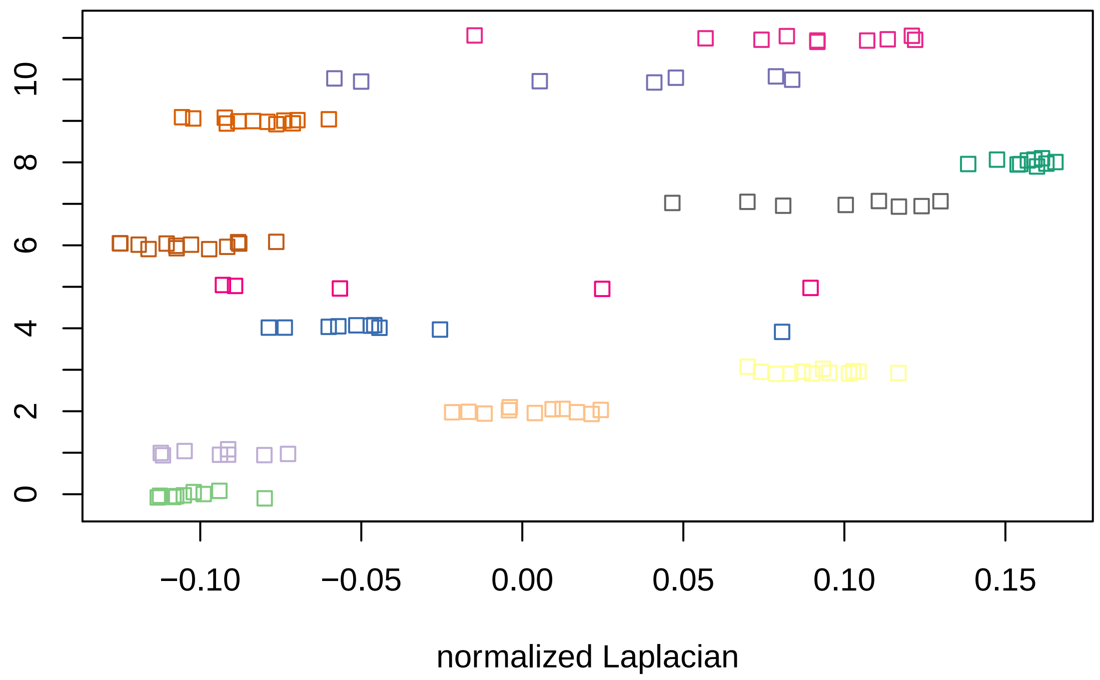
```
##
##  0  1  2  3  4  5  6  7  8  9 10 11
##  9  8 11 12 10  5 13  8 10 12  7 10
```

```
gt <- vertex_attr(lcc)$value
gt
```

```
##   [1]  7  0  2  3  7  3  2  8  8  7  3 10  6  2  6  2  7  9  6  1  9  8  8  7 10
##  [26]  0  6  9 11  1  1  6  2  0  6  1  5  0  6  2  3  7  5  6  4  0 11  2  4 11
##  [51] 10  8  3 11  6  1  9  4 11 10  2  6  9 10  2  9  4 11  8 10  9  6  3 11  3
##  [76]  4  9  8  8  1  5  3  5 11  3  6  4  9 11  0  5  4  4  7  1  9  9 10  3  6
## [101]  2  1  3  0  7  0  2  3  8  0  4  8  4  9 11
```

```r
### Getting a vector of distinct colors for use in normalized cut
n <- 12
qual_col_pals = brewer.pal.info[brewer.pal.info$category == 'qual',]
col_vector = unlist(mapply(brewer.pal, qual_col_pals$maxcolors, rownames(qual_col_pals)))
col_vector<- col_vector[1:12]


stripchart(Xhat2$X[,2] ~ vertex_attr(lcc)$value, method = "jitter",
col = col_vector[1:12], xlab = "normalized Laplacian")
```



```r
clusters2 <- kmeans(Xhat2$X, centers = 12)

table(clusters2$cluster, vertex_attr(lcc)$value)
```

```
##
##       0  1  2  3  4  5  6  7  8  9 10 11
##   1   0  0  0  0  0  0  0  0 10  0  0  0
##   2   0  0  0  0  0  1  0  0  0  0  3  1
##   3   9  0  0  0  0  0  0  0  0  0  0  0
##   4   0  0  3  0  0  0  0  0  0  0  0  0
##   5   0  0  8  0  0  0  0  0  0  0  0  0
##   6   0  0  0  0  0  1  0  0  0  0  4  1
##   7   0  0  0 12  0  0  0  0  0  0  0  0
##   8   0  0  0  0  0  0  0  0  0 12  0  0
##   9   0  0  0  0  9  0  0  0  0  0  0  0
##   10  0  0  0  0  1  0  0  8  0  0  0  8
##   11  0  0  0  0  0  1 13  0  0  0  0  0
##   12  0  8  0  0  0  2  0  0  0  0  0  0
```

```
###spectral with modularity
c2 = cluster_leading_eigen(g2)
# as per the documentation this performs clustering by calculating the
#leading non-negative eigenvector of the modularity matrix of the graph.
# c2$membership is the membership based on the results of modularity spectral clustering



###louvain
cc <- cluster_louvain(g2)

table(cc$membership)
```

```
##
##  1  2  3  4  5  6  7  8  9 10
## 12 11  9 15  9 14 10 16  9 10
```

```
###Normal cut comparison
## Compare the clustering using normalized mutual information
compare(clusters2$cluster, gt, method = c("nmi"))
```

```
## [1] 0.8909241
```

```
## Compare the clustering using Rand index
compare(clusters2$cluster, gt, method = c("rand"))
```

```
## [1] 0.9700992
```

```
## Compare the clustering using adjusted Rand index
compare(clusters2$cluster, gt, method = c("adjusted.rand"))
```

```
## [1] 0.8055414
```

```
###modularity comparison
## Compare the clustering using normalized mutual information
compare(c2$membership, gt, method = c("nmi"))
```

```
## [1] 0.6986702
```

```
## Compare the clustering using Rand index
compare(c2$membership, gt, method = c("rand"))
```

```
## [1] 0.8930587
```

```r
## Compare the clustering using adjusted Rand index
compare(c2$membership, gt, method = c("adjusted.rand"))
```

```
## [1] 0.4640505
```

```r
###Louvain comparison
## Compare the clustering using normalized mutual information
compare(cc$membership, gt, method = c("nmi"))
```

```
## [1] 0.8903166
```

```r
## Compare the clustering using Rand index
compare(cc$membership, gt, method = c("rand"))
```

```
## [1] 0.9688787
```

```r
## Compare the clustering using adjusted Rand index
compare(cc$membership, gt, method = c("adjusted.rand"))
```

```
## [1] 0.8069409
```

For normalized mutual information, Louvain was the largest, followed by normal cut and modularity. For Rand index, normal cut was the largest, followed by modularity and Louvain. For adjusted Rand index, Louvain was largest, followed by normal cut and modularity. Based on this, it appears to be the case that Louvain is doing the best job of correctly assesisng the data, as for adjusted Rand and NMI, Louvain appears to be closest to the truth. In the Rand index caclulation, all 3 comparison measures are pretty large, indicating that for this metric all 3 methods of clustering perform well.

Problem 2.5 (optional)

Now try to do community deteciton on the Youtube network dataset available here using any algorithm you want. Now try to evaluate the performance of your clustering compared to the given ground truth (it is not going to be easy :-))

Problem 3.

Download the USPS digits dataset from here. Next do the following steps.

- Choose the digits for class $4, 7$ and 9.
- Randomly sample 800 digits from each class.
- Do an embedding of these 2400 data point into $\mathbb{R}^d$ for some d, say $d = 10$, using **Laplacian eigenmaps**. You are free to choose your choice of ϵ or k-NN or weights.
- Run a 3-class SVM classifier on the embeddings for these 2400 datapoints. Use a linear kernel for your SVM.
- With the remaining 900 data points (300 from each class), evaluate the performance of your SVM classifier on this holdout data.
- Compare this performance with say SVM classification in the original 256 dimension data using say a linear kernel or a Gaussian kernel.

Hint To evaluate the performance on the hold-out data, you will need to do an out-of-sample embedding. Read this paper to see how to construct an out-of-sample embedding for Laplacian eigenmaps.

```
usps<-readMat(file.path("usps_all.mat"))$data
usps_sub <- usps[,,c(5,8,10)]

usps_4 <- usps_sub[,,1]
usps_7 <- usps_sub[,,2]
usps_9 <- usps_sub[,,3]

usps_4[, sample(ncol(usps_4), 1)]
```

```
##   [1]   0   0   0   0   0   0   0   0  23  95 139  83   6   0   0   0   0   0
##  [19]   0   0   0   0   0   0 116 182 171 234 216  91   1   0   0   0   0   0
##  [37]   0   0   0   0   0   0   0  29 184 255  91   0   0   0   0   0   0   0
##  [55]   0   0   0   0   0   0  31 228 244   3   0   0   0   0   0   0   0   0
##  [73]   0   0   0   0   0  81 255  80   0   0   0   0   0   0   0   0   0   0
##  [91]   0   0   0   1 246 168   0   0   0   0   0   0   0   0   0   0   0   0
## [109]   0   0 246 170   0   0   0   0   0   0   0   0   0   0   0   0   0  35
## [127] 250 170   0   0  26 135 101  55  11   0   0   0   0   0   0  92 255 100
## [145]   0  56 240 255 255 255 197 111  11   0   0   0   0 180 255  38  86 252
## [163] 243  31  44 136 252 255 206  96  85  33 128 254 123   0  89 255  97   0
## [181]   0   0  35 180 245 255 255 255 255 213   5   0 152 255  84   0   0   0
## [199]   0   0  31 103 140 120  60   3   0   0  87 255 172   0   0   0   0   0
## [217]   0   0   0   0   0   0   0   0   4 235 243  17   0   0   0   0   0   0
## [235]   0   0   0   0   0   0   0 121 175  46   0   0   0   0   0   0   0   0
## [253]   0   0   0   0
```

Problem 4.

Complete the proof of Theorem 7 on page 55 of the dimension reduction lecture slides. In particular, show that if $x_1, x_2, \ldots, x_n$ are **distinct** then the matrix $\mathbf{H} = f(\|x_i - x_j\|^2)$ is positive definite.