

# ST 790 Assignment 2

## Instruction

This assignment consists of 4 problems. The assignment is due on **Tuesday, September 29** at 11:59pm EDT. Please submit your assignment electronically through the **Moodle** webpage. The assignment can be done as a group with at most 3 members per group (please include the name of the group members on the front page of the assignment).

## Problem 1.

Let  $\mathbf{D}$  be a  $n \times n$  symmetric dissimilarity matrix, i.e., the entries  $d_{ij}$  of  $\mathbf{D}$  are non-negative for all  $i, j$ ,  $d_{ii} = 0$  for all  $i$ . Let  $\mathbf{B} = -\frac{1}{2}(\mathbf{I} - \mathbf{1}\mathbf{1}^\top/n)\mathbf{D}(\mathbf{I} - \mathbf{1}\mathbf{1}^\top/n)$  where  $\mathbf{I}$  is the  $n \times n$  identity matrix and  $\mathbf{1} \in \mathbb{R}^n$  is the vector of all ones.

Given an integer  $r \leq n$ , the classical multidimensional scaling (CMDS) procedure produces a configuration  $\mathbf{Z}_* \in \mathbb{R}^{n \times r}$  representing  $n$  points in  $\mathbb{R}^r$  that best approximate  $\mathbf{D}$  with respect to the STRAIN criterion, i.e.,

$$\mathbf{Z}_* = \arg \min_{\mathbf{Z} \in \mathbb{R}^{n \times r}} \|\mathbf{B} - \mathbf{Z}\mathbf{Z}^\top\|_F. \quad (1)$$

Consider the following eigendecomposition of  $\mathbf{B}$ .

$$\mathbf{B} = \sum_{i=1}^n \lambda_i u_i u_i^\top; \quad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$$

Then the CMDS procedure returns the  $n \times r$  configuration

$$\mathbf{Z}_* = \left[ (\lambda_1)_+^{1/2} u_1, (\lambda_2)_+^{1/2} u_2, \dots, (\lambda_r)_+^{1/2} u_r \right] \quad (2)$$

where  $(\lambda_i)_+ = \max\{\lambda_i, 0\}$ .

Show that the expression  $\mathbf{Z}_*$  in Eq.(2) is really the minimizer of the STRAIN criterion for  $\mathbf{B}$ . More specifically, show the following result.

**Proposition** Let  $\mathbf{B}$  be a  $n \times n$  symmetric matrix with eigendecomposition

$$\mathbf{B} = \sum_{i=1}^n \lambda_i u_i u_i^\top; \quad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$$

Then the best rank  $r$ , positive semidefinite approximation to  $\mathbf{B}$ , with respect to the Frobenius norm, is the matrix

$$\mathbf{B}_r = \sum_{i=1}^r (\lambda_i)_+ u_i u_i^\top.$$

**Hint:** Let  $\mathbf{M}$  be any  $n \times n$  matrix. Then

$$\|\mathbf{B} - \mathbf{M}\|_F^2 = \|\mathbf{U}\mathbf{\Lambda}\mathbf{U}^\top - \mathbf{V}\mathbf{\Sigma}\mathbf{V}^\top\|_F^2 = \|\mathbf{\Lambda} - \mathbf{W}\mathbf{\Sigma}\mathbf{W}^\top\|_F^2$$

where  $\mathbf{U}\mathbf{\Lambda}\mathbf{U}^\top$  and  $\mathbf{V}\mathbf{\Sigma}\mathbf{V}^\top$  are the eigendecomposition of  $\mathbf{B}$  and  $\mathbf{M}$ , respectively, and  $\mathbf{W}$  is a  $n \times n$  orthogonal matrix.

Now, for fixed diagonal matrices  $\Lambda$  and  $\Sigma$ , show that a minimizer of

$$\min_{\mathbf{W}} \|\Lambda - \mathbf{W}\Sigma\mathbf{W}^\top\|_F^2$$

over the set of orthogonal matrices  $\mathbf{W}$  is given by a permutation matrix. Using the [rearrangement inequality](#), show that this permutation matrix will rearrange the diagonal entries of  $\Sigma$  to coincide with the ordering of the diagonal entries of  $\Lambda$ .

## Problem 2

Download the dataset of games between American College Football teams available [here](#). Now perform community detection using (1) a spectral clustering algorithm using normalized cut (2) spectral clustering using modularity and (3) Louvain algorithm. Evaluate the performance of your clustering (compared to the given ground truth).

## Problem 2.5 (optional)

Now try to do community detection on the Youtube network dataset available [here](#) using any algorithm you want. Now try to evaluate the performance of your clustering compared to the given ground truth (it is not going to be easy :-))

## Problem 3.

Download the USPS digits dataset from [here](#). Next do the following steps.

- Choose the digits for class 4, 7 and 9.
- Randomly sample 800 digits from each class.
- Do an embedding of these 2400 data points into  $\mathbb{R}^d$  for some  $d$ , say  $d = 10$ , using **Laplacian eigenmaps**. You are free to choose your choice of  $\epsilon$  or  $k$ -NN or weights.
- Run a 3-class SVM classifier on the embeddings for these 2400 datapoints. Use a linear kernel for your SVM.
- With the remaining 900 data points (300 from each class), evaluate the performance of your SVM classifier on this holdout data.
- Compare this performance with say SVM classification in the original 256 dimension data using say a linear kernel or a Gaussian kernel.

**Hint** To evaluate the performance on the hold-out data, you will need to do an out-of-sample embedding. Read [this paper](#) to see how to construct an out-of-sample embedding for Laplacian eigenmaps.

## Problem 4.

Complete the proof of Theorem 7 on page 55 of the dimension reduction lecture slides. In particular, show that if  $x_1, x_2, \dots, x_n$  are **distinct** then the matrix  $\mathbf{H} = f(\|x_i - x_j\|^2)$  is positive definite.