A Practical Guide to Interest Rate Curve Building Validations (w/ Excel Replica of Bloomberg Libor @ GitHub)

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Abstract

Curve building is an information extraction process defined by a set of bootstrapping instruments $\mathcal{B}$, bootstrapping procedures $\mathcal{P}$, interpolation technique $\mathcal{I}$, and extrapolation technique $\mathcal{E}$. The bootstrapping procedures allow one to deduce/bootstrap information about risk factors (such as future Libor rates) that can be used to derive a fair value of a financial instrument with payoff function dependent on these risk factors (such as interest rate swap). The bootstrapping procedures deduce information about either expected values of the risk factors (such as forward rate curve) or their distributions (such as volatility surface). The curve building process involves a number of steps that include instrument selection, market data prioritization, bootstrapping, interpolation, and extrapolation. The paper provides information about market conventions, common curve building issues, validation techniques, standard tests, and validation criteria. We will focus on the bootstrapping of the expected values of interest rates (such as future forward rates). Some of the validation techniques described in this paper (such as round-trip testing) can be extended onto other markets and other types of the risk factors (including volatility surfaces). The paper has a reference to the published Excel spreadsheet that demonstrates how to replicate Bloomberg Libor curve. The spreadsheet can be downloaded from GitHub repository.

1 Which Curves?

Let's start our discussion with interest rate curves. Interest rate curves reflect time value of money concept. Time value of money needs to be recorded in a way so that different market participants can have a consensus. Interest rate curves are built for this purpose. Market participants express their views of rates (such as forward rates, discount factors, etc.) via market quotes of different financial instruments (such as forward rate agreements, Eurodollar futures, etc.). Hence, curve building can be viewed as a process of extracting market information.

Different market segments view the time value of money differently. Hence, this information can be recorded using different types of curves. The curve families are usually classified by the underlying index (e.g., LIBOR, Prime, SIFMA, etc.) and rate type (e.g., zero, forward, discount, etc.). Since different forms of curves are conveying the same information (time value of money), they can be converted into each other under certain conventions.

Interest rates curves are not directly observable in the markets, they are usually built (bootstrapped) by utilizing a discrete set of market quotes of rate locking instruments. Curve

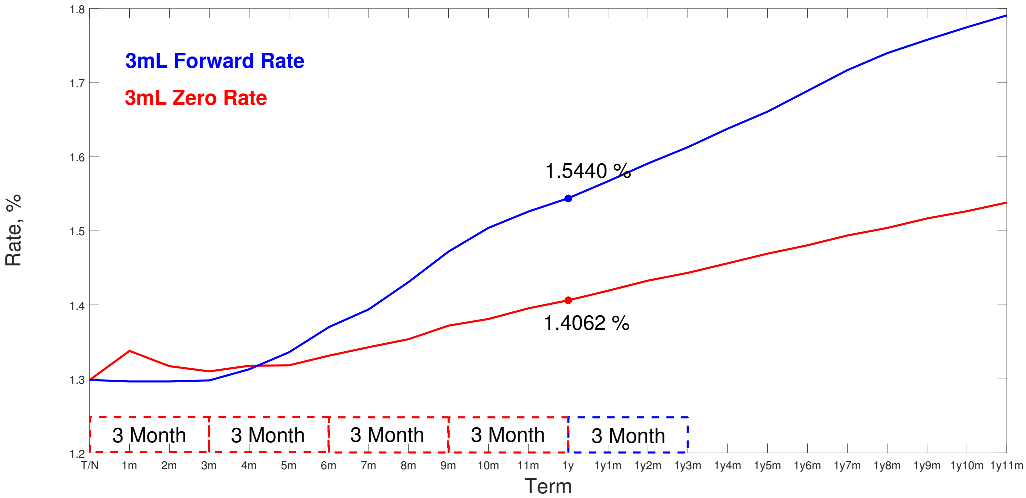

Figure 1: Comparison of the 3-month LIBOR Zero and Forward Rates

building process often needs to supplement the missing information via some sort of interpolation assumptions. This topic will be discussed in more details later. Let's take a step back and review different curve types:

- Discount Factor Curve: representing current price of zero coupon bond for that curve index with notional of $1
- Zero Rate Curve: the one rate to summarize the holding period return by assuming money is invested in the curve index and rolled based on index term until the end of holding period
- Forward Rate Curve: current expectation of the future index rate starting from certain period from now
- Swap Rate Curve: the fixed rate to equate the series of floating rate payments indexed to the floating rate for the life of the contract

It should be noted that rate indexes are always associated with time, i.e. tenors. The index rate tenor and the curve tenors (or nodes) are different. The index rate tenor (such as 3-month LIBOR, 1-month LIBOR, etc.) reflects the liquidity and credit risk embedded in the underlying index, it is fixed for any given index. The curve tenors (or nodes) can vary, one can build a more granular or less granular curves by adding or removing the bootstrapping instruments. These nodes should be interpreted as either the total investment period or the time between current and future effective dates. Let's consider 3-month LIBOR (3mL) curve as an example: based on Figure 1, current market implied 1-year 3-month LIBOR forward rate is 1.544% and 1-year 3-month LIBOR spot (zero) rate is 1.4062%.

The market implied information can be interpreted in the following manner.

- The expected annualized rate to be offered on a 3-month term deposit starting 1 year from today is 1.544%. We will denote this rate as $f_{3mLibor}(0;1,1.25)$.

- Average annualized funding cost/investment yield by rolling short term 3 month time deposit contract for 1 year starting from today is 1.4062%. We will denote this rate as $r_{3mLibor}(0; 0, 1)$.

Generally speaking, a curve at time t can be viewed as an array of a form $\{r(t; t_i, T_i\}_{i=1}^{n}$ under market convention $\mathcal{C}$ (market conventions include zero rate, forward rate, etc.) with $[t_i, T_i]$ representing investment periods. Spot or zero rates have starting points t_is fixed at the current time $t = 0$ ($r_i = r(0; 0, T_i)$), while forward rates have fixed investment period length $\delta = T_i - t_i$ ($f(0; t_i, t_{i+1}) = r(0; t_i, t_i + \delta)$).

As stated above, different curves can be converted into each other. Let's demonstrate how zero rates can be converted to discount factors. Given zero rate r_t, the discount zero-coupon bond price $Z(0, t)$ (i.e. discount factors $Z(0, t)$) can be calculated as follows under continuous compounding:

$$Z(0, t) = e^{-r_t t}$$

The zero rate r_t and forward rate ($f(0; t_i, t_{i+1})$) are connected in the following manner.

$$f(0; t_i, t_{i+1}) = \frac{r_{t_{i+1}} t_{i+1} - r_{t_i} t_i}{t_{i+1} - t_i}$$

Market conventions begin to play a role as the determination of $(t_{i+1} - t_i)$ (measured in years) depends on the way one deals with holidays and day counts. The day counting rules vary by jurisdiction and instrument type. Normally, 30/360 or Actual methods are used for day counting. This calculation should take into consideration holiday calendars. In Bloomberg terminal, the swap manager function (SWPM <Go>) provides market convention details for different curve indexes (please, refer to Figure 7 in Appendix A).

So far, we have talked about interest rates market. The same logic can be applied to other markets where forward rates play an important role. This includes: commodity, foreign exchange, and equity markets. On the contrary, credit markets deal with credit spreads or par spreads that reflect expected default rate. One can use different instruments to imply market participants' expectations of the forward rates. Some instruments (such as interest rate swaps, commodity swaps, foreign exchange swaps) can be priced off of these market-implied forward rates. However, option-embedded instruments require information about distribution of forward rates rather than just their expected values. The distributional information is typically compressed into the implied volatility parameter that implicitly assumes log-normal distribution. As stated earlier, in this paper we will focus on the curve building associated with expected rates rather than distributions.

Summarizing the information outlined above, in order to specify a curve one needs to specify the following items.

- the reference rate (LIBOR, SIFMA, etc,)
- the term associated with the reference rate (LIBOR 1M, LIBOR 3M, etc.)
- the "as of time" t
- investment periods $[t_i, T_i]$
- type of rates and/or market conventions $\mathcal{C}$

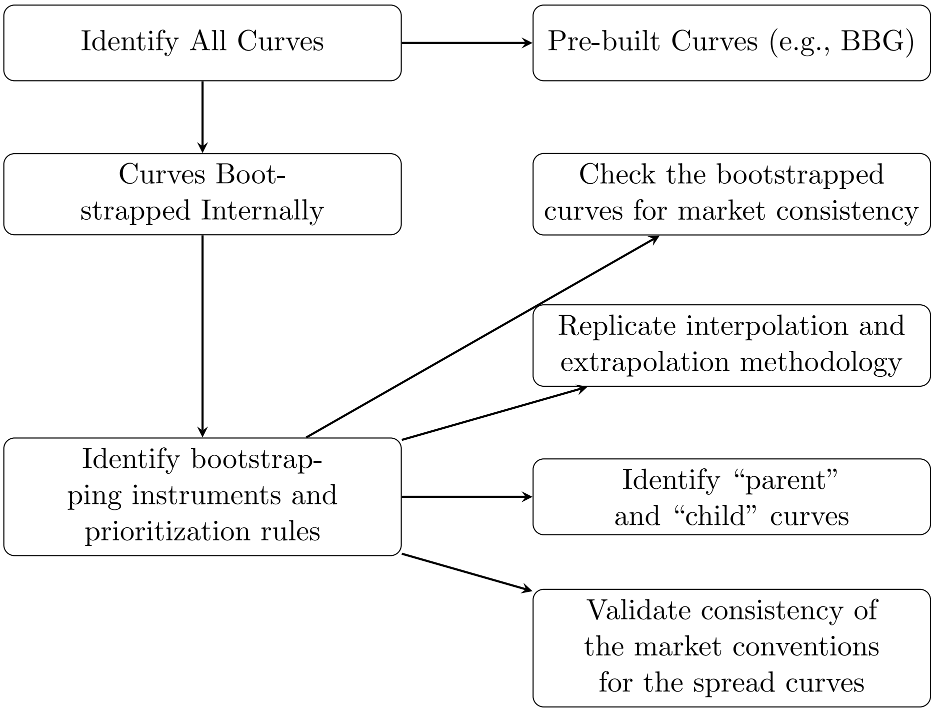

Figure 2: Curve Building Validation Process

2 Validation Process

Let's describe a curve building validation process outlined in Figure 2. The first step is to identify all the curves. As described in the first section, there is a number of curve attributes that need to be identified. There will be a number of curves corresponding to different markets, products, underlying entities, etc. Once the curves are identified, they can be grouped into two buckets - curves that are built/bootstrapped internally and curves that are pulled from a third party (such as Bloomberg, Reuters, etc.). It is very common for market risk management teams to pull pre-bootstrapped curves from a third party. In Appendix C we give examples of Bloomberg tickers corresponding to different pre-bootstrapped curves. Now we will focus on the validation procedures associated with internally built curves. It is the best practice, to apply the same validation standards to the pre-bootstrapped curves. However, there will be certain limitations that may limit validator's ability to fully replicate third party's curve building process.

One needs to start with the identification of the bootstrapping set $\mathcal{B}$ and bootstrapping procedures $\mathcal{P}$. What are they? The bootstrapping set is a set of market-traded instruments. The payoff function of these instruments should depend on the index rate tenor (such as 3-month LIBOR). In order to price these instruments one needs to estimate future values of the corresponding index rate tenor. A bootstrapping procedure allows one to invert the pricing process to derive assumptions about the future values of index rate tenor from market prices of the instruments. This inversion problem is typically under-determined and requires additional assumptions (in the form of interpolation method) about implied rates that are not dictated/observed by/in the market. This interpolation method should be viewed as part of the bootstrapping procedures $\mathcal{P}$. It is important to check the bootstrapping instruments

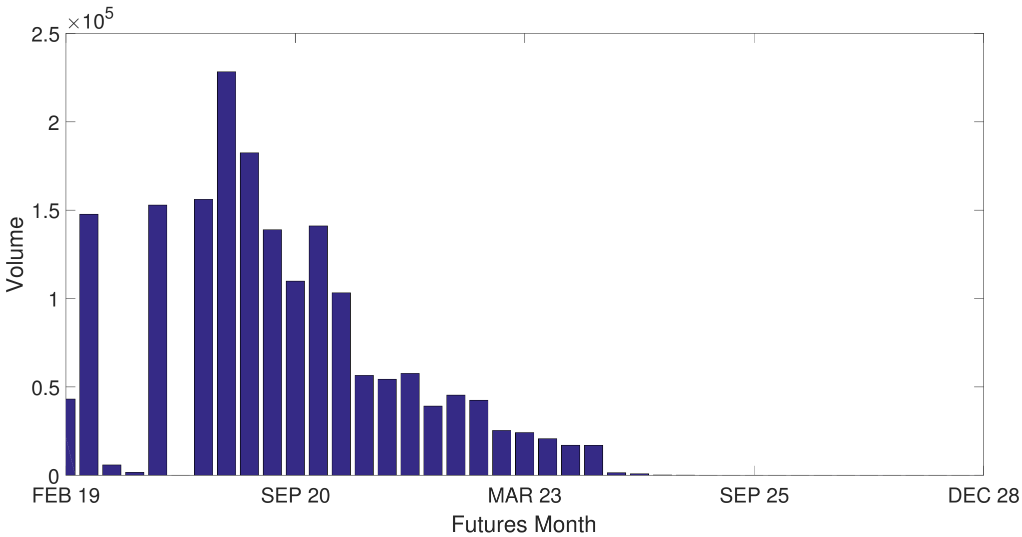

Figure 3: EDF Trading Volume on 1/29/2019
Source: CME Group (reporting available here)

and index rate tenor for consistency. For example, one can't use Eurodollar futures (EDF) to build 1-month LIBOR curve since EDFs are linked to the 3-month LIBOR rates. However, one could still utilize them by incorporating 3s1s (3-month vs 1-month) basis spread. Furthermore, different instruments may imply contradictory rates. This can be caused by the market illiquidity. In this case, one need to define prioritization rules that would allow a bootstrapping mechanism to overcome market anomaly. These prioritization rules are part of the bootstrapping procedures as well. It is important to keep in mind that liquidity plays an important role when defining a bootstrapping basket. It is well known that EDF marks tend to be less liquid for longer tenors. The trading volume corresponding to different EDFs is displayed in Figure 3. The trading volume beyond 4 year tenors appear to be very minimal. It is interesting to note that there are some liquidity gaps even on the short end.

Now, when all the inputs and outputs are clearly defined, it is time to discuss the validation testing activities. In the testing phase, it is important to replicate the bootstrapping process. Different bootstrapping procedures have different pros and cons that we will discuss later. It is important to confirm the curve's consistency with the market. In other words, a theoretical price of any instrument from the bootstrapping set should match the market price when priced off of the bootstrapped curve. This is a so-called round trip test.

The bootstrapped curve gives rates corresponding to a discrete set of time points. When pricing instruments that do not belong to the bootstrapping set (such as over-the-counter trades), one needs to use some sort of interpolation and extrapolation techniques. A model validator needs to replicate and assess these techniques.

In addition to the standard "parent" curves (LIBOR, EURIBOR, Treasury, etc.), model developers build "child" curves. The "child" curves are built as a spread over "parent" curve. Here are some examples of the "child" curves. The tenor basis curves (1-month, 6-month, and 12-month tenors curves), product basis curves (prime, commercial paper, bma), etc. One needs to make sure that basis spreads that are used to build these "child" curves are being incorporated in accordance with the market conventions. All these topics will be discussed in more details in the following sections.

3 Instrument Selection

3.1 Interest Rate Instruments

Curve bootstrapping starts with market inputs. The bootstrapping instruments are a discrete set of rate locking instruments traded in the market. For validation purpose, it's critical to make sure that the curve bootstrapping inputs are pulled and interpreted correctly.

A validator needs to understand the market conventions of the bootstrapping instruments. For example, for LIBOR curves, Eurodollar futures have moving term points while FRAs can provide fixed term points. Another example is, for basis curves, some basis spreads are quoted as an add-on spread curve (e.g., LIBOR 1M, LIBOR 6M, LIBOR 12M) while other basis spreads are quoted as a ratio of the base curve(e.g., BMA).

Another important aspect is the liquidity of the market instruments. As different rate locking instruments have different liquidity profiles, bootstrapping the curve beyond the liquid term point requires some data manipulation. A good reference to the liquidity profile of common bootstrapping instruments can be found in the following research paper [11].

A more recent example on OIS curve bootstrapping can be found in the Bloomberg white paper [12]. The paper points out some of the challenges concerning instruments availability for the long tenors of the OIS curve. "Currently for USD, OIS rates are not widely available in the marketplace beyond the 10-year maturity. In order to support OIS/DC stripping, the OIS curve is extended beyond the 10-year maturity by harnessing USD Fed Funds (FF) basis swap quotes, that are available to 30-year maturity, since both OIS and FF basis are stripped to project forwards of the FF Effective Rate, FEDL01, and sensible inferences can be made from each other in an arbitrage-free environment. Two methods have been developed to imply OIS rates from FF basis swaps." Hence, when validating the OIS curve, a validator should verify how the system extends the OIS raw input curve beyond 10 year-maturity and replicate the process by taking into consideration conventional difference between the OIS and LIBOR curves (such as day count, compounding, averaging method, etc.).

Common bootstrapping instruments and their corresponding Bloomberg tickers have been provided in Appendix A. In order to have a fixed term points curves, FRA contracts rather than Eurodollar futures are selected. The four curves captured in Appendix A are standalone discounting curve (OIS), base index curve (LIBOR 3M), add-on basis curve (LIBOR 1M), and ratio basis curve (BMA). The curves can be located in Bloomberg terminal using the following commands.

- OIS curve: ICVS 42 <Go>
- LIROB 3M Curve: ICVS 23 <Go>
- LIBOR 1M Curve: ICVS 50 <Go>

- BMA curve: ICVS 357 <Go>

3.2 Fixed Income Instruments

The most straightforward bootstrapping process is the Treasury curve bootstrapping. Treasury curve serves as the base curve in the fixed income world. The other fixed income instruments will trade based on the spread over treasury (e.g. muni curves, agency curves, etc.). Taking on-the-run treasury instruments, we can derive the daily par yield published by the Treasury department (Table 1). Since treasury bonds are coupon bearing instruments, directly using the yield for discounting can't generate arbitrage free results. And in order to obtain the zero rate, one needs to perform the par yield to zero rate conversion.

Table 1: Daily Constant Maturity Treasury Quote
Source: The Treasury's Website (available here)

Date	1M	2M	3M	6M	1Y	2Y	3Y	5Y	7Y	10Y	20Y	30Y
1/2/2019	2.4	2.4	2.42	2.51	2.6	2.5	2.47	2.49	2.56	2.66	2.83	2.97
1/3/2019	2.42	2.42	2.41	2.47	2.5	2.39	2.35	2.37	2.44	2.56	2.75	2.92
1/4/2019	2.4	2.42	2.42	2.51	2.57	2.5	2.47	2.49	2.56	2.67	2.83	2.98
1/7/2019	2.42	2.42	2.45	2.54	2.58	2.53	2.51	2.53	2.6	2.7	2.86	2.99
1/8/2019	2.4	2.42	2.46	2.54	2.6	2.58	2.57	2.58	2.63	2.73	2.88	3

The bootstrapping process gets more complicated when one deals with corporate bonds. There may not be enough name specific bonds in the market in order to build a name specific yield curve. Hence, modelers use bond information from similar (in terms of credit rating and industry sector) companies. This may lead to conflicting information since bonds from different companies may imply different information. In this case, the non-exact bootstrapping methods are used. We will discuss these methods in the Bootstrapping section.

4 Bootstrapping

Curve building is both an art and science because one needs to make educated guesses about rates not directly observed in the market. Hence, there is no one correct curve building method. One can talk about the quality of a bootstrapping method with respect to a number of desired characteristics (e.g., smoothness). In the following subsections, we will discuss a number of bootstrapping topics. Please, refer to [5] for a thorough review of the bootstrapping methods. We have also published an Excel spreadsheet [6] that demonstrates how to replicate Bloomberg LIBOR curve. The spreadsheet can be downloaded from GitHub repository here.

From a very high level, one can break bootstrapping methods into two major categories - exact methods and non-exact methods. Exact methods allow modeler to replicate market exactly, but require additional assumptions regarding rates that would need to be interpolated using some sort of parametric approach (e.g., cubic spline interpolation). On the other hand, non-exact methods start with a functional form that represent a curve. Consequently, this functional form can be tied to the market data via calibration. In this case, a modeler is not guaranteed to replicate the market exactly. Figure 4 outlines dependencies between exact (E) and non-exact (NE) methods with parametric (P) and non-parametric (NP) representations

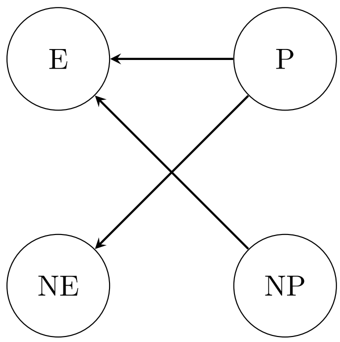

Figure 4: Bootstrapping Approaches

of the curve.

4.1 Exact Method

In mathematical terms, exact bootstrapping can be interpreted as solving a system of equations [5].

$$Cd = p,$$

where C is an instruments' cash flow matrix, d is a discount factor vector, and p is a price vector. This system of equations is usually under-determined and, therefore, allows for infinitely many solutions. In other words, the number of rates that we need to imply is larger than the number of liquid instruments that are quoted in the market. A typical way to deal with this problem, is to complement a system of equations with artificial/interpolated ones. These artificial equations represent our belief (i.e. interpolation assumption) about rates not observable in the markets.

In our Excel spreadsheet [6], there are three instrument types in the calibration basket - cash deposits, Eurodollar futures, and interest rate swaps. The cells corresponding to different instrument types are marked in different colors. The data is pulled from Bloomberg as of June 4th of 2018. The EDF dates were pulled from CBOE page here.

4.2 Smoothing Methods

In mathematical terms, non-exact bootstrapping can be interpreted as optimization.

$$\min_{\alpha_1,\ldots,\alpha_n} \sum_{i=1}^{N} (y_i - f(T_i))^2$$

Notice, that there can be variations of this general approach based on the object of modeling and object of optimization. In the formula above, both the object of modeling and the object of optimization is yield. In the famous Nelson-Siegel approach, the object of modeling is forward rate rather than yield. In other words, function $f(u)$ represents instantenouos forward rate. The yield in this case can be calculated as follows.

$$y_i = \frac{1}{T_i} \int_0^{T_i} f(u)du$$

The optimization problem can be rewritten as follows.

$$\min_{\alpha_1,\ldots,\alpha_n} \sum_{i=1}^{N} \left(T_i y_i - \int_0^{T_i} f(u)du \right)^2$$

In the optimization problem above, the functional form of the curve is pre-defined. But how good that assumption is? What functional form is the "best?" The answer to this question is given by the Lorimier's theorem [7]. In this theorem, the author solved the following optimization problem.

$$\min_{f \in H} \int_0^T (f'(u))^2 du + \alpha \sum_{i=1}^{N} \left(T_i y_i - \int_0^{T_i} f(u)du \right)^2,$$

where H is a Hilbert space of absolutely continuous function on interval $[0, T]$. The functional above consists of two terms - the first term represents the smoothness while the second one represents yield accuracy. The parameter α reflects the importance of the accuracy term. The solution to this problem is given by quadratic splines.

4.3 Excel Replica of Bloomberg Libor

In this section we will describe our Excel replication of Bloomberg Libor curve [6]. The spreadsheet can be downloaded from GitHub. There are six tabs.

- BBG Replica
- BBG Data Table
- BBG Input
- BBG Input Tickers
- BBG Output
- BBG Output Tickers

On the "BBG Input" tab, one can see Bloomberg screen shots of the input data. Input tab screen shot has three instrument groups - Cash Rates, Contiguous Futures, and Swap Rates. Instruments that are used for bootstrapping are highlighted in orange. The Zero Rates Chart at the bottom is the output curve. This curve is broken into three segments - short end, middle, and long end. Notice, that Contiguous Futures quotes contain prices and convexity adjustments that can be converted into forward rates. The day counting conventions, maturity dates, and terms are also captured on this tab. We have recorded input data on the "BBG Data Table" tab. The data from this tab will be referenced on the main calculation tab, the "BBG Replica" tab.

On the "BBG Output" tab, one can see Bloomberg output - zero rates and discount factors. In our calculations, we will focus on the discount factors. Our calculated discount factors are compared to these Bloomberg output rates on the "BBG Replica" tab. Bloomberg tickers corresponding to the input and output rates are captured on the "BBT Input Tickers" and "BBG Output Tickers" tabs.

Finally, the discount factors are calculated on the "BBG Replica" tab. Rows corresponding to different instrument types are filled with different colors - red for cash, blue for EDF, and white for swap. Instrument specific discount factor formulas are captured at the bottom of this tab. The calculated discount factors are compared to the Bloomberg output in the last two columns. There is also a check on the swap value in rows 22-23, columns E-G. The value of the interest rate swap with fixed rate set to the spot swap rate should be 0.

4.4 Prioritizing Market Data

Due to the market imperfections, it is natural to see cases when the same or similar rates implied from different market instruments do not agree with each other. For example, the front end LIBOR forward rate can be implied from either Euro-dollar Futures(EDF) contract or Forward Rate Agreement (FRA). Due to market supply and demand and the trading mechanism difference (exchange versus OTC), these two rate locking instruments may indicate different rates for the same underlying time window. Basically, the liquidity profile difference between different instruments leads to the transactional cost differences.

The Eurodollar contract is much more liquid for the front end LIBOR curves, thus the hedging of interest rate exposure by Eurodollar is more cost efficient. However, market risk system needs to generate scenarios which is ideally free from interpolation noise. Generating scenarios using FRA contracts will be preferable as the historical market movements are consistent due to the fixed term points.

The underlying position will also play a role in the market data selection consideration. The exposure concentration of the underlying position will lead to the model user to prefer certain term points to be smooth or at least not subject to interpolation noise. This consideration will justify the addition of instruments pointing to a certain tenor in the bootstrapping set.

5 Market Developments

5.1 OIS Discounting

OIS discounting is a new market trend after the 2008 financial crisis. Prior to the crisis, one LIBOR curve(3M) has been used for projecting future floating rate reset and discounting the future value into present, implying market agrees that the LIBOR curve is a good proxy for risk free rate.

During the financial crisis, however, the LIBOR OIS spread spiked to historical high level, forcing the market to rethink about a better proxy for risk free rate, at least a better rate index so that it's credit risk free under market stress. OIS becomes a market standard for derivatives pricing as it can better capture the risk free rate change without worrying the credit component.

A very good example can be found in the Bloomberg white paper section-Extension of USD OIS Curves [8].

5.2 Basis Spreads

During the financial crisis, previous convergence of market rates that refer to the same time interval does not hold any more. LIBOR rates quotes for different terms cannot neglect the liquidity and credit risk premium demanded by market participants. Two options for the market to adapt to the new regime: develop model which can incorporate the credit and liquidity premium under the original framework or begin the market segmentation so that different terms of the same curve (LIBOR) will be treated as different risk factors.

Obviously, after crisis market embraces for the second option, basis spreads become a new critical market risk factor in bank's market risk system.

From model validation perspective, the market convention for how the basis spreads are actually used needs to be examined. For add-on basis curves, the quoted basis spreads are the difference added to the reference floating index so that the two floating legs can be priced the same. And the spread is usually added to the shorter term leg:

For example, for the LIBOR 1M v.s 3M basis swap, the two legs are:

- 3M leg: reset and pay quarterly on 3M LIBOR rate
- 1M leg: reset monthly on 1-month LIBOR compounded quarterly + basis spread

While for LIBOR 3M v.s. 6M basis spread, the two legs are:

- 3M leg: reset and pay quarterly on 3M LIBOR rate compounded to 6M + basis spread
- 6M leg: reset and pay semi-annually on 6M LIBOR rate

5.3 CME and LCH spread

Around mid-May 2015, bank swap books suffered as CME-LCH basis explodes: the relative cost of clearing interest rate swaps at CME versus LCH experienced a 13-fold increase. The basis has been quoted around 0.15 basis point, but the basis for 10-year US dollar swap surged to 2 bps peak. So for a swap trade with notional of 1 billion, a difference over the life of the trade will amount to 2 million.CME has released some official comments as:

"Recently, a pricing differential, or basis, has developed in the interest rate swap market. The most common explanation given for the cause of this pricing differential is a structural imbalance between the supply and demand for pay-fixed and receive-fixed swaps in the marketplace as a result of the 2013 clearing mandate for interest rate swaps. The issuers of corporate debt are exempt end-users who hold their receive-fixed swaps bilaterally outside of clearing, but the buy side parties who purchase this debt must clear their pay-fixed swaps. With swap dealers acting as the counterparty for both sides of these trades, it is believed that this has caused a large accumulation of client pay-fixed swaps in clearing, where the dealer receives fixed. And CME is trying to build the CME specific swap curve for dealers to better estimate the clearing costs."

The basis between exchanges has some implication to the market participants. A business entity should manage this basis risk by either dealing with the same exchange for all the derivatives trades or hedging the basis risk as in the case for Libor 1M vs 3M.

6 Interpolation and Extrapolation

6.1 Linear Interpolation

Linear interpolations assume a linear relationship between the unknown rate(discount factor) and the known rates(discount factor). As the same rate can be represented in different forms (e.g., spot, forward, discount factor, etc.), the linear relationship can also be built on each different form for any t in (t_{i-1}, t_i).

Table 2: Linear Interpolation Methods

Linear on the zero rate	$r(t) = \frac{t-t_{i-1}}{t_i-t_{i-1}} r(t_i) + \frac{t_i-t}{t_i-t_{i-1}} r(t_{i-1})$
Linear on the log zero rate	$\ln(r(t)) = \frac{t-t_{i-1}}{t_i-t_{i-1}} \ln(r(t_i)) + \frac{t_i-t}{t_i-t_{i-1}} \ln(r(t_{i-1}))$
Linear on the discount factor	$Z(t) = \frac{t-t_{i-1}}{t_i-t_{i-1}} Z(t_i) + \frac{t_i-t}{t_i-t_{i-1}} Z(t_{i-1})$
Linear on the log of discount factor	$r(t)t = \frac{t-t_{i-1}}{t_i-t_{i-1}} r(t_i)t_i + \frac{t_i-t}{t_i-t_{i-1}} r(t_{i-1})t_{i-1}$
Linear on the swap rate	$s(t) = \frac{t-t_{i-1}}{t_i-t_{i-1}} s(t_i) + \frac{t_i-t}{t_i-t_{i-1}} s(t_{i-1})$
Linear on the forward rate	$f(t) = \frac{t-t_{i-1}}{t_i-t_{i-1}} f(t_i) + \frac{t_i-t}{t_i-t_{i-1}} f(t_{i-1})$

Special attention should be paid to the method- linear on the log of discount factor, which is a benchmark method for yield curve bootstrapping process. The method is easy to implement and very stable. The log of discount factor equals to $-tr(t)$. The interpolation method is essentially interpolating in the $tr(t)$ space. A noticeable feature of this interpolation method is that all instantaneous forward rates are positive.

A common problem with the linear interpolation is the continuity of the forward rates. Even the benchmark method-linear on log of discount factor has jumps in the transition points. Linear on swap rates is not common, since the method makes assumption on observable market inputs. Linear on forward rate is known to be undesirable either, on one hand, the continuity of forward rates will be addressed; on the other hand, the method will introduce zig-zag behavioral [3].

6.2 Non-Linear Interpolation

In this section we will consider one of the most common interpolation techniques - cubic splines. Cubic splines can be defined as follows.

$$r(t) = a_i + b_i(t - t_i) + c_i(t - t_i)^2 + d_i(t - t_i)^3$$

Spline is supposed to alleviate the problem of the zig-zag behavior of the single polynomial fitting. Cubic splines are a set of cubic functions trying to connect the discrete nodes corresponding to traded market instruments. Several desired properties are pursued by the practitioners.

- The splines pass all the nodes derived from the traded market instruments
- The first order derivative is continuous at the transition points
- The second order derivative is continuous at the transition points

Continuity can be defined via right and left limits that should give the same value. For cubic splines, the right and left derivatives can be expressed in the following manner.

$$\partial_+ r(t_i) = b_i + 2c_i(t_i - t_i) + 3d_i(t_i - t_i)^2 = b_i$$

$$\partial_- r(t_i) = b_{i-1} + 2c_{i-1}(t_i - t_{i-1}) + 3d_{i-1}(t_i - t_{i-1})^2$$

In order to ensure continuity/smoothness, one needs to make both right and left derivatives equal. These "continuity/smothness" equations can be constructed for all transition points. In order to ensure continuity in the first place, one needs to match the left and right limits at each transition point. The left and right limits can be expressed as follows.

$$\lim_{t \to t_i^+} r(t) = a_i + b_i(t_i - t_i) + c_i(t_i - t_i)^2 + d_i(t_i - t_i)^3 = a_i$$

$$\lim_{t \to t_i^-} r(t) = a_{i-1} + b_{i-1}(t_i - t_{i-1}) + c_{i-1}(t_i - t_{i-1})^2 + d_{i-1}(t_i - t_{i-1})^3$$

One can consider additional properties when interpolating. Following [3], monotone convexity is a highly desirable property in order to build a "good" curve. Hagan demonstrates that this method is the best choice among various yield curve bootstrapping methods in terms of forward rates positivity, smoothness, stability, localness, and hedge's localness. Please, refer to Hagan's paper [3] for complete details.

6.3 Extrapolation

The rates need to be extrapolated (rather than interpolated) for instruments with maturity dates going beyond the last available point on the discount curve. For example, FinCAD function "aaDFCurve_Extend" allows one to obtain extended discount rates. The curve is extended by assuming a constant par swap rates for the extended dates.

6.4 Reconciliation of Interpolation Terminology Among Vendors

Let's take a look at how different vendors define the bootstrapping and interpolation methods.

FinCAD defines bootstrapping as follows [4]:

The process of building a discount factor curve from market quotes is known as bootstrapping.

Bloomberg provides the following definition [8] of bootstrapping:

The process of building of the IR curve a.k.a. curve stripping is a process of creating a curve object which would correctly price a set of N given instruments, e.g. produces correct discount factors and forward rates used in these instruments.

These definitions appear to be very similar, however, there are some differences in the implementation. In FinCAD, the curve building process can be broken down into two steps - bootstrapping and interpolation.

As discussed in the Bootstrapping section, the bootstrapping problem is underdetermined and requires additional assumptions about rates. Hagan stated in [3], "the interpolation method is intimately connected to the bootstrap, as the bootstrap proceeds with incomplete information. This information is "completed" using the interpolation scheme." FinCAD offers multiple methods to deal with this step - the linear swap rate (LSR), linear forward rate (LFR and LFR2), and quadratic forward rate (QFR). These methods are, in essence, dealing with interpolation required for converting prices of the bootstrapping instruments into rates. Once this step is completed, one can start using these rates to price other instruments. However, the other instruments may require information corresponding to the time points that were not captured in the bootstrapping instruments. Hence, additional interpolation is required. FinCAD offers a number of interpolation methods for the final rate output (i.e. discount factors) - the linear discount factors, linear spot rates, cubic spline discount factor, and exponential discount factors. The two interpolations discussed above can be named as bootstrapping interpolation and pricing interpolation.

It is important to note that interpolation objects may vary as well. In FinCAD, the bootstrapping interpolation is performed on the forward rates and pricing interpolation is performed on the discount factors. While Bloomberg allows the user to choose 4 bootstrapping interpolation methods - piecewise linear performed on simple-compounded zero rate and piecewise quadratic, step-function, and piecewise linear performed on continuously-compounded forward rate.

It should be noted that the so-called "raw method", which is the constant forward rate method in FinCAD and step-function forward method in Bloomberg actually mean the same thing. Hence, different vendors can name the same method differently, a validator can easily form nature benchmarks based on the understanding of this fact. Secondly, interpolation methods allow for alternative definitions in terms of different interpolation objects. For example, linear interpolation performed on the product of zero rate and time is equivalent to the exponential interpolation performed on the discount factor. In order to facilitate the benchmarking of different vendors' bootstrapping methodologies, Table 3 groups different bootstrapping and pricing interpolation methods based on the vendor, interpolation object, and linearity of the interpolation method.
It is worth mentioning that a model validator should assess whether bootstrapping interpolation and pricing interpolation "agree" with each other. Combination of certain methods may lead to the curve anomalies. This topic has been discussed in [4]. The results and recommendations of the FinCAD analysis can be found in Figure 4. In addition to the interpolation assumptions, one may use some sort of smoothing techniques. In FinCAD, this can be achieved by applying post-smoothing. In Bloomberg, this can be achieved by using a global method

Table 3: Reconciliation of the Interpolation Terminology

	Linear	Non-linear
Zero Rate	FinCAD (linear spot rate)	
	Bloomberg (piecewise linear simple)	
	Bloomberg (piecewise linear continuous)	
Forward	FinCAD (linear forward)	Bloomberg (quadratic forward)
	FinCAD (enhanced linear forward)	FinCAD (quadratic forward)
	Bloomberg (step function forward)	
Discount Factor	FinCAD (Linear discount factor)	FinCAD (exponential discount factor)
		FinCAD (cubic spline discount factor)
Swap	FinCAD (linear swap)	

rather than a bootstrap method.

Table 4: FinCAD Recommendations Concerning Bootstrapping and Pricing Interpolations

Bootstrapping Interpolation	Post-smoothing Required?	Recommended Pricing Interpolation
Linear Swap Rates (LSF)	Yes	Any
Constant Forward Rates (CFR)	No	Exponential
Original Linear Forward Rates (LFR)	Yes	Any
Enhanced Linear Forward Rates (LFR2)	Yes	Any
Quadratic Forward Rates (QFR)	No	Cubic Spline

Summarizing the topics discussed in this section, a model validator needs to identify a number of curve building parameters in order to compare different interpolation methods. These parameters include bootstrapping instruments set, bootstrapping interpolation method along with interpolation object rates type, pricing interpolation method along with interpolation object type, and equilibrium assumptions.

7 Desired Curve Characteristics

The desired curve characteristics actually depend on the purpose or use of the bootstrapped curve.

For example, on the front desk, it is quite likely that the best curve bootstrapping method is subject to many different conditions. As noted by Amir Sadr,

“When selecting a method, one has to decide on the right tradeoff between smoothness and the sensibility of the prescribed hedges. Because each trader may have a different opinion on where the right tradeoff is, there has yet not emerged a universally accepted curve build method.”

While for risk management purpose, the liquidity of the instruments may not be the most critical consideration, the curve which offers less interpolation noise and easy scenario generation will be more favored.

Though different users have different preferences to the curve characteristics, some common desirable properties can be identified, such as:

- Forward rates are positive
- Forward rates are continuous
- Forward rates are smooth

A validator is concerned about how the curve looks like now based on current market inputs and how the curve will look like given some shock to the inputs. The current curve is desired to be positive, continuous, and smooth. The shocked curve is desired to have stability and localness. No method possesses all the merits at once, one has to sacrifice some for the others. For validation purposes, there is good reference [3] for the pros and cons associated with various bootstrapping options. For the sake of completeness, we present Hagan's summary table here (Table 5). Depending on the users, the best curve is the one which can maximize his/her utility function.

Table 5: Hagan's Summary [3]

Yield curve type	**Fwd > 0?**	**Smoothness**	**Localness?**	**Stablity?**	**Hedge Localness?**
Linear on discount	No	Not Continuous	Excellent	Excellent	Very Good
Linear on rates	No	Not Continuous	Excellent	Excellent	Very Good
Raw(linear on log of discount)	Yes	Not Continuous	Excellent	Excellent	Very Good
Linear on the log of rates	No	Not Continuous	Excellent	Excellent	Very Good
Piecewise linear forward	No	Continuous	Poor	Very Poor	Very Poor
Quadratic	No	Continuous	Poor	Very Poor	Very Poor
Natural Cubic	No	Smooth	Poor	Good	Poor
Hermite/Bessel	No	Smooth	Very Good	Good	Poor
Financial	No	Smooth	Poor	Good	Poor
Quadratic Nature	No	Smooth	Poor	Good	Poor
Hermite/Bessel on rt function	No	Smooth	Very Good	Good	Poor
Monotone Piecewise Cubic	No	Continuous	Very Good	Good	Good
Quartic	No	Smooth	Poor	Very Poor	Very Poor
Monotone Convex(unamerliorated)	Yes	Continuous	Very Good	Good	Good
Monotone Convex(amerliorated)	Yes	Continuous	Good	Good	Good
Minimal	No	Continuous	Poor	Good	Very Poor

7.1 Positivity and Continuity

The condition for positive forward rates is another way of saying discount curve should be monotonically decreasing. Forward curve positivity can be guaranteed through the design of interpolation method. For example, in a normal market(where yield curve are upwardly sloping), the raw method can guarantee that instantaneous forward rate is always positive. The positivity of forward rates is unique to financial problem as it is consistent with the time value of money concept.

The positivity of forward curve is also a check point for scenario designs. For example, during the stress testing process, the market curve will be shocked by some artificial market scenarios to identify the portfolio's vulnerability. One should keep in mind that in this scenario design process, the shocked market should also preserve certain properties.

The condition for continuous forward rates helps to guarantee that no jumps happen at the connecting points where bootstrapping instruments switch.

7.2 Curve Smoothness

Curve smoothness is a desired feature for curve building, however, a measure for the smoothness is not a straightforward concept. One reference can be found in the research work by Ken Adams, in which, he referred to the strain energy concept ($g(x)$ represents the curve shape function):

$$\int_a^b \left(\frac{\partial^2 g(x)}{\partial x^2}\right)^2 dx$$

The measure is trying to quantify the "bending" of the elastic spline, also the smaller this measure of smoothness, the smoother the interpolated curve.

Curve smoothness is a step further from the curve continuity. Some conclusions drawn by Adam:

For forward rate: "The interpolating function that guarantees the smoothest continuously compounded instantaneous forward-rate curve is a quartic spline."

For zero rate: "A financial cubic spline denotes a cubic spline that is constrained so that its derivative at its right-hand end is zero, and its second derivative at the left-hand end is also zero. No other interpolation function that is subject to the same constraints as the financial cubic spline, and which fits the given data, is smoother than the financial cubic spline that interpolates that data."

7.3 Curve Stability

Per Hagan's definition, the stability of the interpolation method is measured by "given a change in one of the inputs, how much can the output interpolated curve be changed." Hagan developed a norm for both the yield curve(r) and forward curve(f).

$$\|M(r)\| = \sup_t \max_i \left|\frac{\partial r(t)}{\partial r_i}\right|$$

$$\|M(f)\| = \sup_t \max_i \left|\frac{\partial f(t)}{\partial f_i^d}\right|$$

However, Hagan also notes that the norms usually cannot be determined analytically but estimated empirically. The practical approach for estimating the norm is to measure the maximum difference between original bootstrapped curves and any of the curves which arise when any of the original bootstrapping nodes are changed by 1 basis point.

While in FinCAD, the stability is the way how curve looks. For example, the exponential interpolation (constant forward rate) approach is deemed as a good example for curve stability.

7.4 Localness

By definition, some methods are global method and some methods are local method. The localness measures the sensitivity of the curve output to the input change. If we view the curve stability as the magnitude of the output change, localness can be viewed as the direction of the output change. So a curve which is stable and have small spill-over when input rate changes should be preferred. The interval for yield curve values change is denoted as (t_{i-l}, t_{i+u}).

The localness concept has also been expanded to hedging side. Per Hagan's definition, if assuming the admissible hedging instruments are exactly those used to bootstrap the yield curve, it's important to note whether most of the delta risk get assigned to the hedging instruments that have maturities close to the given tenors. For the sake of completeness, we present Hagan's localness indices for various curve interpolation methods in Table 6.

Table 6: Hagan's Localness Indices [3]

Interpolation Method	**l**	**u**
Linear on discount	1	1
Linear on rates	1	1
Raw(linear on log of discount)	1	1
Linear on the log of rates	1	1
Natural Cubic	$i-1$	$n-i$
Hermite/Bessel	2	2
Financial	$i-1$	$n-i$
Quadratic Nature	$i-1$	$n-i$
Bessel on cap function	2	2
Monotone Piecewise Cubic	2	1
Quartic	$i-1$	$n-i$
Monotone Convex(unamerliorated)	2	2
Monotone Convex(amerliorated)	3	3
Minimal	$i-1$	$n-i$

8 Common Validation Tests

Validation tests for curve bootstrapping process provide the assurance that the curves coming out of the bootstrapping process is reasonable in terms matching current market and predicting future market. One needs to pay special attention to the order of the validation activities. As one wants to make sure the building blocks are validated first.

For interest rates curves, it's important to keep in mind the validation sequence should be consistent with the curve building sequence. For example, under dual curve framework, the OIS curve is usually built first as a standalone discount curve. Then, LIBOR 3M curve can be built based on the LIBOR indexed instruments and the OIS discount curve. After OIS curve and LIBOR 3M curve are ready, other spread curves such as: LIBOR 1M, BMA curve can be built based on the previously bootstrapped OIS and LIBOR 3M curves.

This validation sequence will help identify any issue more effectively. The following sections will demonstrate various kinds of tests applicable to the bootstrapped curves.

8.1 Round-trip Consistency Test

Round-trip consistency test refers to the process of using the bootstrapped curve to reprice the exact bootstrapping instruments. Round-trip test confirms that the curve output is consistent with the market bootstrapping instruments. As we know that bootstrapping input is a set of discrete market instruments, when we mention market consistency achieved based on round-trip results, we only refer to matching those instruments.

Round-trip result is neither a necessary or sufficient condition for a "good" curve. As if one opts for exact method, round-trip results will guarantee the match but it may run into the problems of over-fitting, or if one opts for non-exact method, round-trip results will not be guaranteed.

Round-trip testing should be conducted in the same system where the curve is bootstrapped. Taking Bloomberg as an example, SWPM function can be used to perform the round-trip testing for the curve built under ICVS function. And one has to be aware what exactly the round-trip test tries to tie back to the market.

For example, the following table summarizes sample instruments for several curve indices.

Table 7: Market Instruments for Round-trip Testing

Curve Index	OIS	LIBOR 3M	LIBOR 1M	BMA
Front	Cash Point	Cash Point	Cash Points	Cash Point
Mid and Long	OIS Swaps	EDF/FRA/Swaps	Basis Swap Spreads	Basis Swap Spreads

Taking LIBOR curve round-trip test as an example, the cash point, EDF, or swaps should be priced the same as market quotes by the bootstrapped curve. Cash point or spot LIBOR should match the forward rate 2 business day from today. Referring back to the concepts mentioned in the first section, the spot LIBOR quote should match $f_{3mLibor}(0; 0+2D, 0+2D+3M)$. For the middle portion, the Eurodollar contract adjusted by convexity should match the forward rate terms which are the effective dates of the ED contracts. For the long end, the fixed rate of the swap indexed on 3M LIBOR curve should match the market quotes for the swap contract.

If the curve building is under dual curve framework, one needs to keep in mind that the front end of of the curves (the portion prior to the swap segment) should be exactly the same as that under single curve environment since no discounting is needed (up to the first swap point).

It should be noted that round-trip test does not guarantee that the curve output from the bootstrapping process is reasonable. As the main purpose of the curve bootstrapping process is to be able to price instruments which are not directly observable in the market (instruments not captured in the bootstrapping set).

Another reminder is that under dual curve framework, round-trip test only confirms that the combined OIS curve and OIS adjusted LIBOR forward curve can reprice market instruments from the bootstrapping set, but we cannot infer whether we have fully achieved the desirable results for standalone LIBOR curve or OIS curve.

Thus, round-trip consistency test should be accompanied by other tests to further assess the

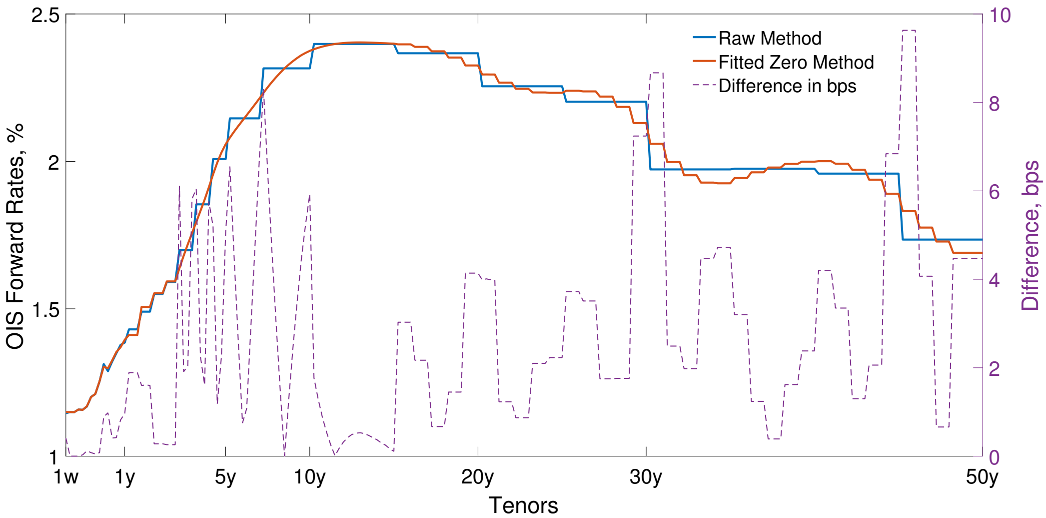

Figure 5: OIS Forward Rate Curve Benchmarking

quality of the bootstrapped curves.

8.2 Benchmarking Tests

It is always important to benchmark the curves bootstrapped from one method to another method or from one system to another system. The curve benchmarking is an effective way to confirm the pros and cons embedded in each bootstrapping method. For example, curves built from advanced interpolation methods are usually benchmarked to the raw method as a starting point.

An example can be seen in Figure 5 for a benchmarking analysis between "raw" method and a "fitted zero" method for OIS forward rate curve.

The benchmarking can also be used for the dual curve discounting. As noted in the Bloomberg white paper, "while effects on the forward rates from using dual-curve stripping are convoluted, the formula below can be used to provide a helpful estimate. Let

$$\delta_r(T) = r_{sw}(T) - r_{OIS}(T)$$

be the spread in bp between the swap rate and the OIS rate for maturity T, $d_{sw}(T)$ be the annual rate of change in bp of the swap rate curve for maturity T (a measure of curve steepness). Then the forward rate change in bp, $\delta_f(T)$ can be approximated in the following manner."

$$\delta_f(T) \approx -\frac{0.5 * \delta_r(T) * d_{sw}(T) * T^2}{10000}$$

In this test, the forward curve difference due to discounting can be approximated and quantified. If one uses the BBG function and tests different smoothing methods, the benchmark result can be drawn as in Figure 6.

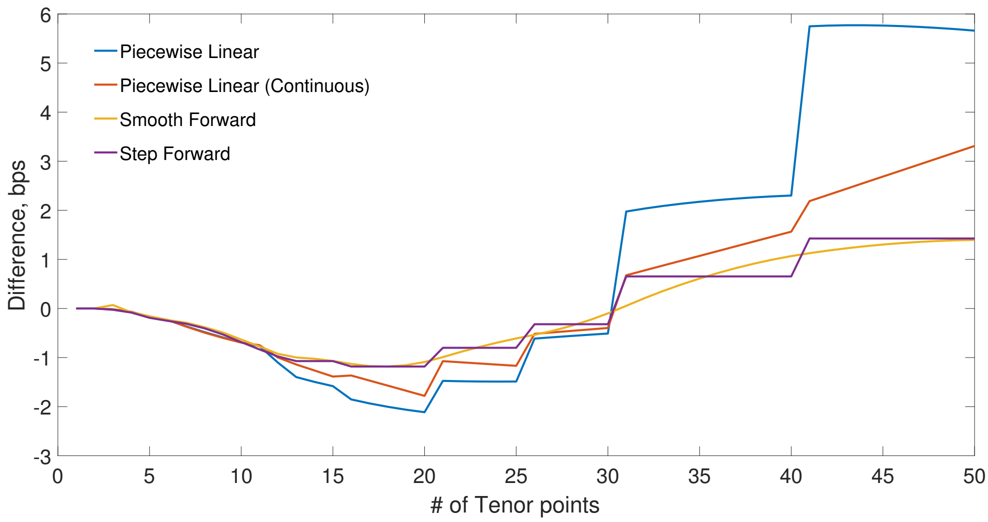

Figure 6: Forward Rate Difference Benchmarking due to OIS and LIBOR Discounting

A quick take away from the formula above is: the OIS adjusted LIBOR forward rate will not necessarily be smaller than the original LIBOR forward curve. One may have an impression that since the OIS rate is generally lower than LIBOR, to value the OIS based swaps the same as under LIBOR based discounting, the OIS adjusted LIBOR forward rates need to be lower to compensate for the smaller discount factor. But the reality is the actual sign of the forward rates difference depends on the shape of the forward curve. That's why the difference can be either positive or negative.

8.3 Sensitivity Tests

Sensitivity tests are connected with curve stability and curve localness. The test can help the validator gain confidence how the model reacts to shocks to the inputs.

For localness test, validator could shock the input instruments by small amount and see the range of the output curve gets impacted. Though, based on Hagan's table, the localness of different bootstrapping methods have been quantified, however, still, the validator could verify whether the model output is consistent with Hagan's table(Table 6). Also, the fact that simple methods have a better localness performance over complex methods can be checked as well.

For the stability test, the validator could rely on the similar shock performed for the localness test, but focusing on the maximum change along the output curve. Since it's hard to have a closed-form result for this test, the validator should measure the relative performance from different options.

9 Conventions and Maths

In the BBG function SWPM, the curve conventions have been established as the default setting for different curve indexes. The screen can also be used for the round-trip benchmarking for the swap trades.

The attention should be paid to the fields such as: reset frequency, day count, pay frequency, etc for both legs. As any discrepancy with market convention could result in inaccurate round-trip result.

Sometimes, the validator should also look at the conversion transformation process, as different indexes having different conventions may need to be combined. The workaround is to make sure the market quotes get transformed in a way different instruments can be combined. A good example will be for the OIS swap rates conversions, since the market only quotes LIBOR swap and OIS-LIBOR basis swap. The Bloomberg white paper "OIS Discounting and Dual-Curve Stripping Methodology at Bloomberg" has good details about this process.

10 Outlook

Alternative Reference Rate Committee[1]

The Financial Stability Oversight Council (FSOC) recommended in its 2014 Annual Report that U.S. regulators cooperate with foreign regulators, international bodies, and market participants to promptly identify alternative interest rate benchmarks anchored in observable transactions and supported by appropriate governance structures, and to develop a plan to accomplish a transition to new benchmarks while such alternative benchmarks were being identified. Greater reliance on alternative reference interest rates will make financial markets more robust and thus enhance the safety and soundness of individual institutions, make financial markets more resilient, and support financial stability in the United States. The Financial Stability Board (FSB) has also called for the development of alternative, nearly risk-free reference rates.

In response to the FSOC's recommendations and the objectives of the FSB, the Federal Reserve convened the Alternative Reference Rates Committee (ARRC) on November 17, 2014 in a meeting with representatives of major over-the-counter (OTC) derivatives market participants and their domestic and international supervisors and central banks. The ARRC was convened to identify a set of alternative reference interest rates that are more firmly based on transactions from a robust underlying market and that comply with emerging standards such as the IOSCO Principles for Financial Benchmarks and to identify an adoption plan with means to facilitate the acceptance and use of these alternative reference rates. The ARRC was also asked to consider the best practices related to robust contract design that ensure that contracts are resilient to the possible cessation or material alteration of an existing or new benchmarks.

New benchmark rate selection process has considered the following aspects [16].

- **Benchmark Quality** The degree to which the benchmark design ensures the integrity and continuity of the rate. The underlying market was evaluated according to its liquidity, transaction volume, and resilience.

[1] https://www.newyorkfed.org/arrc/index.html

- **Methodology Quality** The degree to which the benchmark construction could satisfy the IOSCO Principles for soundness and robustness, including standardized terms, transparency of data, and availability of historic data.

- **Accountability** Evidence of a process that ensures compliance with the IOSCO Principles.

- **Governance** Evidence of governance structures that promote the integrity of the benchmark.

- **Ease of Implementation** Anticipated demand for and relevance to hedging/trading and Existence of, or potential for a term market in the underlying rate.

Based on the criteria above, the two favored candidates are: Overnight Unsecured Lending Rates (OBFR) and Secured Lending Rates (Treasury Repo).

SOFR- the LIBOR Replacement

In June 2017, the ARRC identified SOFR(Secured Overnight Financing Rate) as its preferred alternative to USD LIBOR index. The index represents a volume weighted median rate of repurchase agreements(Repo) that are secured by treasuries for overnight term. Unlike the LIBOR index which is an arithmetic average of the participating banks' offer rates, the SOFR is a transactions based volume weighted index. SOFR index has been published since April 2nd 2018. [2]

Soon after the daily fixing of SOFR is published, the one-month and three-month SOFR futures came to CME listing in May 2018. CME SOFR futures are the leading source for price discovery for the forward SOFR fixing. [3]

Cash instruments indexed to SOFR index has entered the market since July 2018. As of Jan 2019, over $40 billion SOFR indexed securities have been issued in the market place since Fannie Mae's first issuance in July 2018.

The major challenge facing the SOFR index is the lack of term rate setting. Market is witnessing higher and higher futures volumes for SOFR index but still the open interests beyond one year are quite small. With more and more longer term cash instruments are traded in the market, the open interests for longer term SOFR futures are expected to pick up.

On July 18th 2018, LCH announces its first clearance of SOFR swaps. Credit Suisse, Goldman Sachs, and JP. Morgan are among the first participants. On Oct 9th 2018, the CME group announced its clearance of the first OTC SOFR swaps: five market participants with trades notional over $200 million. The cleared trades include SOFR OIS and basis swaps against LIBOR and Fed Funds. Though the market is still unclear how the term structure of SOFR will be determined, the proliferation of SOFR linked derivatives undoubtedly helps to pave a smoother market transition from LIBOR regime to the new era.

[2]https://apps.newyorkfed.org/markets/autorates/sofr
[3]https://www.cmegroup.com/trading/interest-rates/secured-overnight-financing-rate-futures.html

11 Appendix A: Swap Manager

Figure 7: BMA Swap Conventions (SWPM <Go>)

12 Appendix B: Bloomberg Tickers for Bootstrapping Instruments

Table 8: Sample BBG tickers for Bootstrapping OIS curve

OIS Rate Term	BBG Tickers	
O/N	FDFD Index	Cash
1w	USSO1Z Curncy	Swap
2w	USSO2Z Curncy	Swap
3w	USSO3Z Curncy	Swap
1m	USSOA Curncy	Swap
2m	USSOB Curncy	Swap
3m	USSOC Curncy	Swap
4m	USSOD Curncy	Swap
5m	USSOE Curncy	Swap
6m	USSOF Curncy	Swap
9m	USSOI Curncy	Swap
1y	USSO1 Curncy	Swap
1y6m	USSO1F Curncy	Swap
2y	USSO2 Curncy	Swap
3y	USSO3 Curncy	Swap
4y	USSO4 Curncy	Swap
5y	USSO5 Curncy	Swap
6y	USBG6 Index	Swap
7y	USBG7 Index	Swap
8y	USBG8 Index	Swap
9y	USBG9 Index	Swap
10y	USBG10 Index	Swap
12y	USBG12 Index	Swap
15y	USBG15 Index	Swap
20y	USBG20 Index	Swap
25y	USBG25 Index	Swap
30y	USBG30 Index	Swap

Table 9: Sample BBG tickers for bootstrapping LIBOR 3M curve

LBIRO 3M Rate Term	BBG Tickers	
O/N	US00O/N Index	Cash
3m	USFR00C PREB Curncy	FRA
4m	USFR0AD PREB Curncy	FRA
5m	USFR0BE PREB Curncy	FRA
6m	USFR0CF PREB Curncy	FRA
7m	USFR0DG PREB Curncy	FRA
8m	USFR0EH PREB Curncy	FRA
9m	USFR0FI PREB Curncy	FRA
10m	USFR0GJ PREB Curncy	FRA
11m	USFR0HK PREB Curncy	FRA
1y	USFR0I1 PREB Curncy	FRA
13m	USFR0J1A PREB Curncy	FRA
14m	USFR0K1B PREB Curncy	FRA
15m	USFR011C PREB Curncy	FRA
16m	USFR1A1D PREB Curncy	FRA
17m	USFR1B1E PREB Curncy	FRA
18m	USFR1C1F PREB Curncy	FRA
19m	USFR1D1G PREB Curncy	FRA
20m	USFR1E1H PREB Curncy	FRA
21m	USFR1F1I PREB Curncy	FRA
22m	USFR1G1J PREB Curncy	FRA
23m	USFC1H1K PREB Curncy	FRA
2y	USFR1I2 PREB Curncy	Swap
3y	USSWAP3 BGN Curncy	Swap
4y	USSWAP4 BGN Curncy	Swap
5y	USSWAP5 BGN Curncy	Swap
6y	USSWAP6 BGN Curncy	Swap
7y	USSWAP7 BGN Curncy	Swap
8y	USSWAP8 BGN Curncy	Swap
9y	USSWAP9 BGN Curncy	Swap
10y	USSWAP10 BGN Curncy	Swap
11y	USSWAP11 BGN Curncy	Swap
12y	USSWAP12 BGN Curncy	Swap
15y	USSWAP15 BGN Curncy	Swap
20y	USSWAP20 BGN Curncy	Swap
25y	USSWAP25 BGN Curncy	Swap
30y	USSWAP30 BGN Curncy	Swap
40y	USSWAP40 BGN Curncy	Swap

Table 10: Sample BBG tickers for bootstrapping LIBOR 1M curve

LBIRO 1M Rate Term	BBG Tickers	
1m	LIBT1M Index	Cash
3m	USBAAC CMPN Curncy	Basis Swap
6m	USBAAF CMPN Curncy	Basis Swap
9m	USBAAI CMPN Curncy	Basis Swap
1y	USBA1 CMPN Curncy	Basis Swap
1y6m	USBA1F CMPN Curncy	Basis Swap
2y	USBA2 CMPN Curncy	Basis Swap
3y	USBA3 CMPN Curncy	Basis Swap
4y	USBA4 CMPN Curncy	Basis Swap
5y	USBA5 CMPN Curncy	Basis Swap
7y	USBA7 CMPN Curncy	Basis Swap
10y	USBA10 CMPN Curncy	Basis Swap
12y	USBA12 CMPN Curncy	Basis Swap
15y	USBA15 CMPN Curncy	Basis Swap
20y	USBA20 CMPN Curncy	Basis Swap
30y	USBA30 CMPN Curncy	Basis Swap

Table 11: Sample BBG tickers for bootstrapping BMASFIMA curve

BMA Rate Term	BBG Tickers	
1w	MUNIPSA Index	Cash
1y	USSML1 Curncy	Basis Swap
2y	USSML2 Curncy	Basis Swap
3y	USSML3 Curncy	Basis Swap
4y	USSML4 Curncy	Basis Swap
5y	USSML5 Curncy	Basis Swap
7y	USSML7 Curncy	Basis Swap
10y	USSML10 Curncy	Basis Swap
12y	USSML12 Curncy	Basis Swap
15y	USSML15 Curncy	Basis Swap
20y	USSML20 Curncy	Basis Swap
30y	USSML30 Curncy	Basis Swap

13 Appendix C: Bloomberg Tickers for Pre-Bootstrapped Curves

Table 12: Sample BBG tickers for pre-bootstrapped LIBOR 3M zero curve

Tenor	BBG Tickers
1D	S0023Z 1D BLC2 Curncy
1W	S0023Z 1W BLC2 Curncy
1M	S0023Z 1M BLC2 Curncy
2M	S0023Z 2M BLC2 Curncy
3M	S0023Z 3M BLC2 Curncy
6M	S0023Z 6M BLC2 Curncy
9M	S0023Z 9M BLC2 Curncy
1Y	S0023Z 1Y BLC2 Curncy
18M	S0023Z 18M BLC2 Curncy
2Y	S0023Z 2Y BLC2 Curncy
3Y	S0023Z 3Y BLC2 Curncy
4Y	S0023Z 4Y BLC2 Curncy
5Y	S0023Z 5Y BLC2 Curncy
7Y	S0023Z 7Y BLC2 Curncy
10Y	S0023Z 10Y BLC2 Curncy
15Y	S0023Z 15Y BLC2 Curncy
20Y	S0023Z 20Y BLC2 Curncy
25Y	S0023Z 25Y BLC2 Curncy
30Y	S0023Z 30Y BLC2 Curncy
40Y	S0023Z 40Y BLC2 Curncy
50Y	S0023Z 50Y BLC2 Curncy

References

[1] Sadr, A., *Interest Rate Swaps and Their Derivatives. A Practitioner's Guide*, Wiley Finance, 2009.

[2] Adams, K., *Smooth Interpolation of Zero Curves*, Algo Research Quarterly, Vol. 4, NOs. 1/2, June 2011.

[3] Hagan, P., & West, G., *Interpolation Methods for Yield Curve Construction*, Applied Mathematical Finance, Vol. 13, No. 2, 89129, June 2006.

[4] FinCAD, *Revisiting "the Art and Science of Curve Building"*, 2011. Available: here.

[5] Filipović, D., *Term-Structure Models: A Graduate Course*, Springer, 2009.

[6] Abramov, V., Zhou, X., *Bloomberg Curve Building Replication.xlsx*, 2019. Available: here.

[7] Lorimier, S., *Interest Rate Term Structure Estimation Based on the Optimal Degree of Smoothness of the Forward Rate Curve*, Ph.D. Thesis, University of Antwerp, 1995.

[8] Bloomberg, *Building the Bloomberg Interest Rate Curve Definitions and Methodology*, March 2016.

[9] OpenGamma, *Interest Rate Instruments and Market Conventions*, 2013.

[10] FitchSolutions, *Charting a Course Through the CDS Big Bang*, 2009.

[11] Credit Suisse, *A Guide to the Front-End and Basis Swap Markets*, Fixed Income Research, Credit Suisse, 2010.

[12] Bloomberg, *OIS Discounting and Dual-Curve Stripping Methodology at Bloomberg*, 2012.

[13] FRB, *Capital Planning at Large Bank Holding Companies: Supervisory Expectations and Range of Current Practice*, August 2013. Available: here.

[14] The Department of The Treasury, *Daily Treasury Yield Curve Rates*, Daily Report. Available: here.

[15] FHFA, *Generating Historically-Based Stress Scenarios Using Parsimonious Factorization*, Working Pater 13-2, August 2014. Available: here.

[16] ARRC, *Rate Evaluation Process*, ARRC Presentation Materials, June 16, 2017. Available: here.