Experiment 1

Measurements and Uncertainties

The density of a solid metal cylinder is found from measurements made of its mass and volume. A double pan balance is used to find the mass, and a meter stick and a vernier caliper are alternately used to measure the height and diameter of the cylinder in order to emphasize the relative differences in the uncertainties of the two measuring instruments. These uncertainties are shown to propagate through the calculation resulting in an uncertainty associated with the determination of density.

Theory

The density of a homogeneous material is defined as the mass per unit volume, or

$$\rho = \frac{mass}{volume} = \frac{m}{v} \qquad (1)$$

The units associated with the density are kg/m^3, g/cm^3, and $slug/ft^3$.

Whenever measurements are performed, two factors contribute to the total uncertainty associated with those measurements. First, an uncertainty exists which is associated with the instrument itself and its construction. This uncertainty is generally included in the specifications of the instrument and expressed as a plus or minus value or as a percentage. For example, Figure 1 shows the measurement to the right edge of a book to be 25.7$\underline{4}$ cm. (The edge of the book is between 25.7 and 25.8 cm, and by estimating where the edge lies between these two positions, one gets 25.7$\underline{4}$ cm. The 4 is an estimated value that can be different for each observer.) Because an uncertainty exists due to the limitation of the measuring instrument, the measurement made with the meter stick might be expressed as 25.74 ± 0.02 cm, or if a percentage uncertainty such as 0.1% is given, then the uncertainty in the reading is 0.1% times 25.74 or ± 0.03 (0.02574) cm. The uncertainty described above is the *instrumental uncertainty.*

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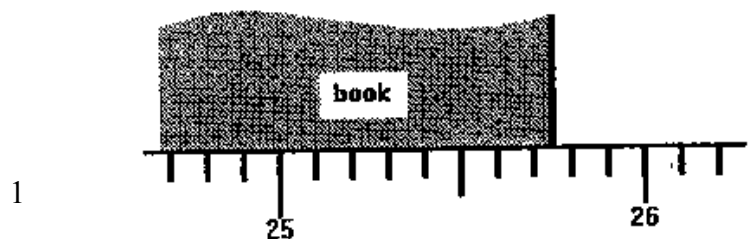
The second uncertainty is an uncertainty associated with the ability of the experimenter to perform the measurement accurately due to the characteristics of the object to be measured. For example, if the edge of the book is damaged and not sharply defined, the experimenter must estimate the position of the edge and this estimate introduces an additional uncertainty that must be evaluated subjectively by the experimenter. Note that this uncertainty must be assigned by the experimenter. This is referred to as the *experimental uncertainty*. (Other methods of error analysis that require multiple measurements result in a more objective evaluation of the uncertainty.)

When the instrumental uncertainty is combined with the experimenter's estimated uncertainty, the result is the *total uncertainty* associated with the measurement.

This type of uncertainty is a form of *random error*. Random errors can be minimized (and always should be) but cannot be entirely eliminated, and they are equally likely to be too high or too low (hence "random"). The only recourse available to the experimenter in this case is to assess the uncertainty as carefully as possible and to report it honestly. This is the point of this experiment!

The other major category of experimental error is *systematic error*. Systematic errors may be caused by faulty equipment (ie. failure to zero a balance) or because incorrect, simplifying assumptions have been made about the experiment (ie. that a surface is frictionless). Unlike random errors, systematic errors are biased in one direction (always too high or too low). Also, unlike random errors, systematic errors can be eliminated (and always should be) by repairing the equipment or by correcting the assumptions about the experiment. A proper experiment should never have systematic errors. We will learn more about this kind of error in subsequent experiments.

Propagation of Errors

Two methods are shown that demonstrate how the uncertainties associated with measurements are propagated through the calculations to result in an uncertainty in the final result.

Suppose a physical quantity A is to be found where

$$A = C\frac{X^2 Y}{Z}$$

(3)

C is assumed to be a constant and X, Y, and Z are measured quantities with corresponding uncertainties $\pm\delta X, \pm\delta Y, and \pm\delta Z$. We assume that these are the total random uncertainties.

The first method requires the substitution of the uncertainties into the expression for A. That is,

$$A \pm \delta A = C\frac{(X \pm \delta X)2(Y \pm \delta Y)}{(Z \mp \delta Z)}.$$

The two terms, $A + \delta A$ and $A - \delta A$ represent the *maximum and minimum values A* can have when the uncertainties are taken into account. The maximum value, $A + \delta A$, is found by choosing either the plus or minus sign preceding the uncertainties so as to maximize *A*. In this case, the plus signs are chosen for the terms in the numerator and the minus sign in the denominator. The maximum value of *A* is then

$$A + \delta A = C\frac{(X + \delta X)2(Y + \delta Y)}{(Z - \delta Z)} \tag{4}$$

In a similar fashion, the minimum value of A is

$$A - \delta A = C\frac{(X - \delta X)2(Y - \delta Y)}{(Z + \delta Z)} \tag{5}$$

The value of *A* is calculated from (3), and δA is found either by subtracting (3) from (4) or (5).

The <u>second method</u> of finding how uncertainties are propagated utilizes differentials. Recall that differentials represent infinitesimal changes, and that small finite changes are approximately equal to the infinitesimal changes. That is,

$$dQ \approx \delta Q$$

as long as δQ is small.

To find the uncertainty δA, the differential *dA* is taken, then all the differentials are replaced by their appropriate small finite changes. In order to simplify the taking of the differential, the logarithm is taken first, then the differential. This produces the same result. First, take the natural log of *A*,

$$\ln A = \ln C + 2\ln X + \ln Y - \ln Z.$$

Now take the differential of this expression,

$$\frac{dA}{A} = \frac{dC}{C} + 2\frac{dX}{X} + \frac{dY}{Y} - \frac{dZ}{Z}$$

Multiply both sides of this equation by *A* and recall that the term *dC/C* is zero because *C* is constant, then

$$dA = A\left[2\frac{dX}{X} + \frac{dY}{Y} - \frac{dZ}{Z}\right]$$

The infinitesimal uncertainties are replaced with the corresponding small finite uncertainties $\pm\delta X, \pm\delta Y, and \pm\delta Z$ (the values of $\delta X, \delta Y, and \delta Z$ are positive), then

$$\delta A = A\left[2\frac{\delta X}{X} + \frac{\delta Y}{Y} + \frac{\delta Z}{Z}\right]$$

(6)

In the above expression, the plus or minus sign associated with $\delta X, \delta Y,$ and δZ is chosen in order to maximize δA.

Now that A from (3) and δA. from (6) are found, the range within which the experimental value of A should fall is defined. This range is from $A - \delta A$ to $A + \delta A$. The range can be indicated on a one-dimensional graph using error bars. (See Figure 2.)
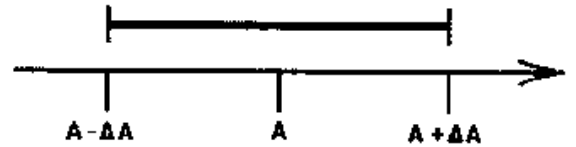

Figure 2. A one-dimensional graph showing the range of the experimental value of A.

The meaning of this range is that the "true" value of A must lie within the range $\pm\delta A$. provided the uncertainties $\delta X, \delta Y, and \delta Z$ have been correctly estimated and the measurements and calculations have been performed correctly. If the "true" or best known value of A lies outside the range, then this indicates that a measurement or computational error may have been made and/or the uncertainties in the measured values of X, Y, and Z were not correctly estimated. Anytime this occurs, _the experimenter should reexamine the calculations, measurements, and the estimations of uncertainties_.

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Apparatus

- o meter stick, ± 0.05 cm
- o vernier caliper, ± 0.005 cm
- o double pan balance, ± 0.2 gm
- o solid metal cylinder

Procedure

1) Find the mass of the solid cylinder, m, using the double pan balance. Record m and its uncertainty δm. Remember that the uncertainty consists of two parts: the instrumental uncertainty and the uncertainty associated with the experimenter's ability to accurately perform the measurement.

2) Use the meter stick to measure the height, h, and the diameter, D, of the cylinder. Record h and D and their uncertainties δh and δD. Again remember that the uncertainties consist of two parts.

3) Use the vernier caliper to measure the height, h, and the diameter, D, of the cylinder. Record h and D and their uncertainties, δh and δD.

Analysis

The density of a sample of material is given by (1), that is,

$$\rho = \frac{m}{v} \tag{1}$$

Where m is the mass of the sample and V is the volume. The volume of a cylinder is

$$V = \frac{\pi D2h}{4}, \tag{7}$$

where D is the diameter and h is the height of the cylinder. The density can then be written as

$$\rho = \frac{4m}{\pi D2h}, \tag{8}$$

and the uncertainty in the density as (using the second method described earlier)

$$\delta\rho = \rho\left[\frac{\delta m}{m} + 2\frac{\delta D}{D} + \frac{\delta h}{h}\right]. \tag{9}$$

Use (8) and (9) and the meter stick measurements to calculate the density of the sample and its uncertainty. Graph these results on a one-dimensional error graph similar to Figure 2. Label the error bar as the "meter stick determination" of the density.

Use (8) and (9) and the vernier caliper measurements to calculate the density and its uncertainty. Graph these results on the same one-dimensional graph described above with the error bar displaced vertically above the previous results. Label the error bar as the "vernier caliper determination" of the density.

Ask the instructor for the type of metal from which the cylinder is made, and look up the theoretical value of the density from your text, the Handbook of Chemistry and Physics, or from some other source. Plot this value on the same graph described above but displace it vertically above the previous results. Label this value as the "best known value."

Report the experimental values of the density, their corresponding uncertainties, and the book value of the density in a table of results.

Conclusions

Look carefully at your results. Were the random errors estimated adequate to account for the difference between your obtained results and the known density values? In other words, did your experimental result fall within the expected "uncertainty range"?

If your results were outside the expected uncertainty range, describe the major sources of systematic error that could account for the differences. Explain how the sources of error would have affected the experimental value.

Extensions & Questions

1) The basic "rule of significant digits" states that the total number of significant digits in a calculated quantity is equal to the least number of significant digits in the measured quantities that are used to calculate a result. Did your results reflect this simple rule?
2) Did you observe evidence of a *systematic error* in your results, with either the ruler of the calipher-derived density?
3) How would the little hole in the cylinder affect the results? Would this be a systematic or a random error? Did you see evidence for this type of error in your results?