PROBLEM 1 (KRANE #8)

LET $\psi(x) = \begin{cases} b(a^2 - x^2) & , \quad -a \le x \le a \\ 0 & x > 0 \ , \ x < a \end{cases}$

AND $a, b > 0$. THE NORMALIZATION CONDITION IS AS FOLLOWS:

a.) $1 = \int_{-a}^{a} |\psi(x)|^2 dx = \int_{-a}^{a} [b a^2 - b x^2]^2 dx$

$= \int_{-a}^{a} [b^2 a^4 - 2 b^2 a^2 x^2 + b^2 x^4] dx$

$= b^2 a^4 \int_{-a}^{a} dx - 2b^2 a^2 \int_{-a}^{a} x^2 dx + b^2 \int_{-a}^{a} x^4 dx$

$= b^2 a^4 [a + a] - \frac{2b^2 a^2}{3} [a^3 + a^3] + \frac{b^2}{5} [a^5 + a^5]$

$= 2b^2 a^5 - \frac{4}{3} b^2 a^5 + \frac{2}{5} b^2 a^5$

$\Longrightarrow \quad b = \sqrt{\frac{1}{a^5 (2 - 4/3 + \frac{2}{5})}} = \boxed{\sqrt{\frac{1}{1.07 a^5}}}$.

b.) IF WE TAKE $dx = .01a$ THEN

$$P(x)\,dx = |\psi(a)|^2 dx = \left[b\left(a^2 - \left(\frac{a}{2}\right)^2\right)\right]^2 \cdot .01a$$

$$= \left(\sqrt{\frac{1}{1.07a^5}}\right)^2 \cdot \left(a^2 - \left(\frac{a}{2}\right)^2\right)^2 \cdot .01\cancel{a}$$

$$= \frac{.01}{1.07\cancel{a^4}} \cdot \left[a^4 - 2\frac{a^4}{4} + \left(\frac{a}{2}\right)^4\right]$$

$$= \frac{.01\left[1 - \frac{1}{2} + \frac{1}{16}\right]}{1.07} = \boxed{0.0053}.$$

c.) $$\int_{a/2}^{a} |\psi(x)|^2 dx = b^2 a^4 \int_{a/2}^{a} dx - 2b^2a^2 \int_{a/2}^{a} x^2 dx + b^2 \int_{a/2}^{a} x^4 dx$$

$$= b^2 \left[a^4\left[a - \frac{a}{2}\right] - \frac{2a^2}{3}\left[a^3 - \left(\frac{a}{2}\right)^3\right] + \frac{1}{5}\left[a^5 - \left(\frac{a}{2}\right)^5\right]\right]$$

$$= \frac{1}{1.07\cancel{a^5}}\left[a^5\left[\frac{1}{2}\right] - \frac{2}{3}a^5\left[1 - \frac{1}{2^3}\right] + a^5\frac{1}{5}\left[1 - \frac{1}{2^5}\right]\right]$$

$$= \frac{1}{1.07}\left[\frac{1}{2} - \frac{2}{3}\left[\frac{7}{8}\right] + \frac{1}{5}\left[\frac{31}{32}\right]\right] = \boxed{.103}.$$

PROBLEM 2

EQUATION 5.7 GIVES THE PROBABILITY DENSITY BY $P(x)\,dx = |\Psi(x)|^2 dx = \frac{1}{1.07a^5}(a^2 - x^2)^2$, SO THE AVERAGE (EXPECTATION) VALUE OF x, $\langle x \rangle$, CAN BE COMPUTED AS FOLLOWS:

$$\langle x \rangle = \int_{-a}^{a} x \cdot \frac{1}{1.07a^5}(a^2 - x^2)^2 dx = 0$$

WHICH WE CAN SEE BY INSPECTION AS x IS AN ODD FUNCTION, $(a^2 - x^2)^2$ IS AN EVEN FUNCTION. THEREFORE, THE INTEGRAND IS ODD, AND INTEGRATING AN ODD FUNCTION SYMMETRICALLY ACCROSS THE ORIGIN GIVES $\langle x \rangle = 0$.

EQUATION 8 FROM CLASS NOTES GIVES $\langle x^2 \rangle$ AS FOLLOWS:

$$\langle x^2 \rangle = \int_{-a}^{a} x^2 P(x)\,dx = \int_{-a}^{a} x^2 \frac{1}{1.07a^5}(a^2 - x^2)^2 dx = 0.14a^2$$

Wolfram ALPHA

$$\Longrightarrow \quad \sigma = \sqrt{0.14a^2 - 0^2} = \boxed{\sqrt{0.14}\,a}.$$

PROBLEM 3

$$\Delta P_x \sim \frac{\hbar}{\Delta x} = \frac{\hbar}{2a_m} = \frac{1.055 \cdot 10^{-34}\, m^2 kg / s}{2a_m}$$

$$= \boxed{\frac{0.53 \cdot 10^{-34}}{a}\, m \cdot kg \cdot s^{-1}}.$$

$$= \sigma_p .$$

$\boxed{\text{EQUATION } 4.15}$

EQUATION 4.16 $\Rightarrow \langle p \rangle = 0$.

I believe this is a rough order of magnitude estimate as describe in the footnote on page 117. I added a direct calculation if this isn't correct.

PROBLEM 9

WE HAVE THAT $\Psi(x)$, $\frac{\partial \Psi}{\partial x} \to 0$ AS $x \to a, -a$.

THEREFORE, FROM PROBLEM 8,

$$\langle p \rangle = -i\hbar \int_{-a}^{a} \Psi^* \frac{\partial \Psi}{\partial x} dx = -i\hbar \int_{-a}^{a} \Psi \frac{\partial \Psi}{\partial x} dx = 0$$

BECAUSE $\Psi(x)$ AN ODD POLYNOMIAL

$$\Rightarrow \frac{\partial \Psi(x)}{\partial x} \text{ AN EVEN POLYNOMIAL}$$

$$\Rightarrow \frac{\Psi(x)}{\partial x} \cdot \Psi(x) \text{ AN ODD POLYNOMIAL}$$

$$\Rightarrow \int_{-a}^{a} \Psi(x) \cdot \frac{\Psi(x)}{dx} = 0.$$

IF $\langle p \rangle = \int_{-a}^{a} \Psi^* \left(-i\hbar \frac{\partial}{\partial x}\right) \Psi \, dx$ THEN

$$\langle p^2 \rangle = \int_{-a}^{a} \Psi^* \left(-i\hbar \frac{\partial}{\partial x}\right)^2 \Psi \, dx$$

$$= -\hbar^2 \int_{-a}^{a} \Psi^* \frac{\partial^2 \Psi}{\partial x^2} dx$$

WHERE

$$\frac{\partial^2 \psi}{\partial x^2} = -2\left(\sqrt{\frac{1}{1.07a^5}}\right) + \quad \psi^* = \sqrt{\frac{1}{1.07a^5}}\left(a^2 - x^2\right)$$

WOLFRAM

$$\langle p^2 \rangle = -\hbar^2 \int_{-a}^{a} \left(\sqrt{\frac{1}{1.07a^5}}\right)\left(a^2 - x^2\right)\left(2\sqrt{\frac{1}{1.07a}}\right) dx$$

$$= \frac{2\hbar^2}{1.07a^5} \int_{-a}^{a} \left(a^2 - x^2\right) dx$$

$$= \frac{2\hbar^2}{1.07a^5} \cdot \frac{4}{3} a^3 = \frac{2.49\hbar^2}{a^2}$$

WOLFRAM.

$$\Longrightarrow \sigma_p = \sqrt{\frac{2.49\hbar^2}{a^2}} = \frac{1.58\hbar}{a}$$

PROBLEM 4

LET $\psi(x) = Cxe^{-bx}$. THE TISE GIVES

$$\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\,\psi(x).$$ NOW,

$$\frac{d^2\psi(x)}{dx^2} = Ce^{-bx}\left(b^2x - 2b\right)$$

$\boxed{\text{WOLFRAM ALPHA}}$

$$\Rightarrow U(x) = \frac{E\psi(x) - \frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}}{\psi(x)}$$

$$= \frac{E\left(\cancel{C}xe^{-bx}\right) - \frac{\hbar^2}{2m}\cdot\left[\cancel{C}e^{-bx}\left(b^2x - 2b\right)\right]}{\cancel{C}xe^{-bx}}$$

$$= \frac{Ex - \frac{\hbar^2}{2m}\left(b^2x - 2b\right)}{x} = E + \frac{\hbar^2b^2}{2m} - \frac{\hbar^2 b}{mx}$$

IF $E(x)$ is CONSTANT AND $E = U(x) - \frac{\hbar^2 b^2}{2m} + \frac{\hbar^2 b}{mx}$

$$\Rightarrow U(x) + \frac{\hbar^2 b}{mx} = 0 \Rightarrow \underline{U(x) = \frac{-\hbar^2 b}{mx}} \Rightarrow \underline{E = -\frac{\hbar^2 b^2}{2m}}.$$

PROBLEM 5

IF $$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + U\psi = i\hbar\frac{\partial \psi}{\partial t}$$

THEN $$\left(-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + U\psi\right)^* = \left(i\hbar\frac{\partial \psi}{\partial t}\right)^*$$

$$\Rightarrow \quad -\frac{\hbar^2}{2m}\frac{\partial^2 \psi^*}{\partial x^2} + U\psi^* = -i\hbar\frac{\partial \psi^*}{dt}. \qquad \square$$

PROBLEM 6

a.) $$\frac{\partial}{dt}\left(|\Psi|^2\right) = \frac{\partial}{\partial t}\Psi\cdot\Psi^* = \frac{\partial\Psi}{dt}\Psi^* + \frac{\partial\Psi^*}{dt}\Psi$$ Product Rule

b.) IF $$\frac{\partial\Psi^*}{\partial t} = \left(\frac{\hbar^2}{2m}\frac{\partial^2\Psi^*}{\partial x^2} - U\Psi^*\right)\frac{1}{i\hbar}$$

AND $$\frac{\partial\Psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + U\Psi\right)\frac{1}{i\hbar}$$

THEN $$\langle p\rangle = m\int x\left(\frac{\partial\Psi}{\partial t}\Psi^* + \frac{\partial\Psi^*}{\partial t}\Psi\right)dx$$

$$= m\int x\left[\left(-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + U\Psi\right)\frac{1}{i\hbar}\cdot\Psi^* + \left(\frac{\hbar^2}{2m}\frac{\partial^2\Psi^*}{\partial x^2} - U\Psi^*\right)\frac{1}{i\hbar}\cdot\Psi\right]dx$$

$$= \cancel{m}\int x\left[-\frac{\hbar}{2i\cancel{m}}\frac{\partial^2\Psi}{\partial x^2}\Psi^* + \cancel{\frac{U}{i\hbar}\Psi\cdot\Psi^*} + \frac{\hbar}{2\cancel{m}i}\frac{\partial^2\Psi^*}{\partial x^2}\Psi - \cancel{\frac{U}{i\hbar}\Psi^*\cdot\Psi}\right]dx$$

$$= \frac{\hbar}{2i}\int x\left(-\frac{\partial^2\Psi}{\partial x^2}\Psi^* + \frac{\partial^2\Psi^*}{\partial x^2}\Psi\right)dx$$

$$\frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i$$

$$= \frac{-\hbar i}{2}\int x\left(\frac{\partial^2\Psi^*}{\partial x^2} - \frac{\partial^2\Psi}{\partial x^2}\Psi^*\right)dx$$

Problem 7

INTEGRATION BY PARTS

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

$$\frac{dv}{dx} = \frac{d^2F}{dx^2} \Rightarrow v = \frac{dF}{dx}$$

$$u = x\,G(x)$$

a.)

$$\int_{x_1}^{x_2} x\left(\frac{d^2F}{dx^2}\right)G\,dx$$

$$= \int_{x_1}^{x_2} u\,dv = uv\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} v\,du$$

$$= \underbrace{\left[x\,G(x)\cdot\frac{dF}{dx}\right]_{x_1}^{x_2}}_{=0} - \int_{x_1}^{x_2}\left(\frac{dF}{dx}\right)\frac{d}{dx}\left(x\cdot G(x)\right)dx$$

Product Rule

$$= -\int_{x_1}^{x_2}\frac{dF}{dx}\left[1\cdot G(x) + \frac{dG}{dx}\cdot x\right]dx$$

$$= -\int_{x_1}^{x_2}\left[\frac{dF}{dx}\cdot G(x)\right]dx - \int_{x_1}^{x_2} x\left(\frac{dF}{dx}\right)\left(\frac{dG}{dx}\right)dx. \quad \blacksquare$$

b.) AGAIN WITH INTEGRATION BY PARTS

$$-\int_{x_1}^{x_2} \frac{dF}{dx} G \, dx$$

$$\frac{dv}{dx} = \frac{dF}{dx}$$

$$u = G(x)$$

$$= -\left[\underbrace{\left[G \cdot F \right]_{x_1}^{x_2}}_{= 0} - \int_{x_1}^{x_2} \frac{dG}{dx} \cdot F \, dx \right]$$

$$= \int_{x_1}^{x_2} \frac{dG}{dx} \cdot F(x) \, dx \qquad \blacksquare$$

PROBLEM 8

LET $\psi(x),\ \psi^*(x),\ \frac{d\psi}{dx},\ \frac{d\psi^*}{dx} \to 0$ as $x \to x_1, x_2$.

FROM PROBLEM 7 & 6 WE HAVE THAT

$$\langle p \rangle = \frac{-i\hbar}{2}\int_{x_1}^{x_2} x\left(\frac{\partial^2\psi^*}{\partial x^2}\psi - \frac{\partial\psi^2}{\partial x^2}\psi^*\right)dx$$

$$= \frac{-i\hbar}{2}\left\{\int_{x_1}^{x_2} x\cdot\frac{\partial^2\psi^*}{\partial x^2}\cdot\psi\, dx - \int_{x_1}^{x_2} x\cdot\frac{\partial\psi^2}{\partial x^2}\cdot\psi^* dx\right\}$$

$$= \frac{-i\hbar}{2}\left\{\left[\underbrace{-\int_{x_1}^{x_2}\frac{\partial\psi^*}{dx}\cdot\psi\, dx}_{} - \underbrace{\int_{x_1}^{x_2} x\cdot\frac{\partial\psi^*}{dx}\frac{\partial\psi}{dx}dx}_{}\right] - \left[\underbrace{-\int_{x_1}^{x_2}\frac{\partial\psi}{dx}\cdot\psi^* dx}_{} - \underbrace{\int_{x_1}^{x_2} x\cdot\frac{\partial\psi}{\partial x}\cdot\frac{\partial\psi^*}{\partial x}dx}_{}\right]\right\}$$

$= 0$

$$= \frac{-i\hbar}{2}\left\{2\cdot\int_{x_1}^{x_2}\frac{\partial\psi}{dx}\psi^* dx\right\} = -i\hbar\int_{x_1}^{x_2}\frac{\partial\psi}{\partial x}\psi^* dx \quad \blacksquare$$

PROBLEM 9

LET $\psi(x) = A e^{\frac{i}{\hbar}(px - Et)}$ + $E = \frac{p^2}{2m}$. NOW

$$\frac{\partial \psi}{\partial x} = \frac{ip}{\hbar} A e^{\frac{i}{\hbar}(px - Et)} \Rightarrow \frac{\partial^2 \psi}{\partial x^2} = \frac{i^2 p^2}{\hbar^2} A e^{\frac{i}{\hbar}(px - Et)}$$

$$+ \quad \frac{\partial \psi}{\partial t} = -\frac{i}{\hbar} \cdot E A e^{\frac{i}{\hbar}(px - Et)} .$$

IF THIS IS A FREE PARTICLE WE CAN DROP THE $U(x)$ TERM FROM THE TDSE, WITH THAT, THE RHS BECOMES

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} = -\frac{\hbar^2}{2m} \cdot \frac{i^2 p^2}{\hbar^2} A e^{\frac{i}{\hbar}(px - Et)} = -\frac{i^2 p^2}{2m} A e^{\frac{i}{\hbar}(px - Et)}$$

$$= \frac{p^2}{2m} A e^{\frac{i}{\hbar}(px - Et)}$$

AND THE LHS BECOMES

$$i\hbar \frac{\partial \psi}{\partial t} = \cancel{i\hbar}\left(-\frac{i}{\cancel{\hbar}}\right) E \cdot A \cdot e^{\frac{i}{\hbar}(px - Et)} = \left(\frac{p^2}{2m}\right) A e^{\frac{i}{\hbar}(px - Et)} . \quad \boxed{}$$