MATH 414/514 HOMEWORK 6

JAMES RUSHING

Exercise 1. (9.5.2) Prove that $\int_0^\pi \sum_{n=1}^\infty \frac{\sin nx}{nx} dx = \sum_{n=1}^\infty \frac{2}{(2n-1)^3}$

Proof. As we have seen in example 9.18, because $\sin(nx) \leq 1$ we have that

$$\left|\frac{\sin nx}{n^2}\right| \leq \frac{1}{n^2} \quad \forall x \in \mathbb{R}.$$

Therefore, by the M-test $\sum_{n=1}^\infty \frac{\sin nx}{nx} dx$ converges for all $x \in \mathbb{R}$.

Knowing that the integrand converges we can move the integral through the sum and integrate term-by-term as follows:

$$\int_0^\pi \sum_{n=1}^\infty \frac{\sin nx}{n^2} dx = \sum_{n=1}^\infty \int_0^\pi \frac{\sin nx}{n^2} dx = \sum_{n=1}^\infty \frac{1}{n^2} \int_0^\pi \sin nx dx$$

$$= \sum_{n=1}^\infty -\frac{1}{n^3}\big[\cos nx\big]_0^\pi = \sum_{n=1}^\infty -\frac{1}{n^3}\big[\cos n\pi - \cos 0\big]$$

$$= \sum_{n=1}^\infty -\frac{1}{n^3}\big[(-1)^n - 1\big] = \sum_{n=1}^\infty \frac{(-1)^{n+1}+1}{n^3}$$

$$= \frac{2}{1^3} + 0 + \frac{2}{3^3} + 0 + \frac{2}{5^3} + 0 + \frac{2}{7^3} = \sum_{n=1}^\infty \frac{2}{(2n-1)^3}.$$

□

Date: 03/20/20.

Exercise 2. (10.4.4) Show that

$$f(x) = \int_0^1 \frac{1 - e^{-sx}}{s} ds = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k(k!)}.$$

Answer. To begin, we will show that the kth derivative of $f(x)$ can be written as

$$f^k(x) = \int_0^1 (-1)^{k-1} s^{k-1} e^{-sx} ds.$$

By induction, where $k = 1$ we have

$$f'(x) = \int_0^1 \big(\frac{d}{dx} s^{-1} - \frac{d}{dx} s^{-1} e^{-sx}\big) ds = \int_0^1 e^{-sx} ds = \int_0^1 (-1)^{1-1} s^{1-1} e^{-sx}$$

and assuming the equality hold for k we have for that for $k+1$

$$\begin{aligned} f^{k+1}(x) &= \frac{d}{dx} \int_0^1 (-1)^{k-1} s^{k-1} e^{-sx} ds = \int_0^1 (-s)(-1)^{k-1} s^{k-1} e^{-sx} ds \\ &= \int_0^1 (-1)^{k-1+1} s^{k-1+1} e^{-sx} ds = \int_0^1 (-1)^{(k+1)-1} s^{(k+1)-1} e^{-sx} ds. \end{aligned}$$

By theorem 10.17 we have that the coefficients of the power series around $x = 0$ for $f(x)$ are as follows:

$$a_k = \frac{f^k(0)}{k!} \quad where$$

$$f^k(0) = \int_0^1 (-1)^{k-1} s^{k-1} e^0 ds = (-1)^{k-1} \left[\frac{s^{k-1+1}}{k-1+1} \right]_0^1 = \frac{(-1)^{k-1}}{k}$$

$$\implies a_k = \frac{(-1)^{k-1}}{k(k!)}.$$

Which was to be shown. ◇