MATH 414/514 HOMEWORK 7

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Exercise 1. (10.8.6)

Establish the following properties of the Dirichlet kernel, $D_n(t)$.

Answer.

(a) $D_n(t)$ is a sum of continuous function and is therefore continuous (5.4).

The following shows that $D_n(t)$ is 2π periodic.

$$D_n(t+2\pi) = \frac{1}{2} + \sum_{k=1}^{n} \cos\left(t(t+2\pi)\right) = \frac{1}{2} + \sum_{k=1}^{n} \cos\left(kt + 2k\pi\right)$$

$$= \frac{1}{2} + \sum_{k=1}^{n} \big[\cos\left(kt\right)\cos\left(2k\pi\right) - \sin\left(kt\right)\sin\left(2k\pi\right)\big] = \frac{1}{2} + \sum_{k=1}^{n} \cos\left(kt\right) + 0$$

$$= \frac{1}{2} + \sum_{k=1}^{n} \cos\left(kt\right) = D_n(t).$$

(b) $D_n(t)$ is an even function.

$$D_n(-t) = \frac{1}{2} + \sum_{k=1}^{n} \cos\left(-kt\right) = \frac{1}{2} + \sum_{k=1}^{n} \cos\left(kt\right) = D_n(t).$$

(c) Because of (b) we can calculate the following integral.

$$\frac{1}{\pi}\int_{\pi}^{\pi} D_n(t)dt = \frac{2}{\pi}\int_{0}^{\pi} D_n(t)dt = \frac{2}{\pi}\left[\int_{0}^{\pi} \frac{1}{2}dt + \int_{0}^{\pi} \sum_{k=1}^{n} \cos\left(kt\right)dt\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{2}[t]_0^{\pi} + \left[\frac{\sum_{k=1}^{n} \sin\left(kt\right)}{k}\right]_0^{\pi}\right] = \frac{2}{\pi}\left[\frac{1}{2}[\pi] + 0\right] = 1.$$

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(d) To reveal an alternate form of $D_n(t)$ we multiply through by $2\sin\left(\frac{t}{2}\right)$ and apply trig identities to simplify by recognising a telescoping sum.

$$D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos(kt)$$

$$\implies 2\sin\left(\frac{t}{2}\right) D_n(t) = \sin\left(\frac{t}{2}\right) + \sum_{k=1}^{n} 2\sin\left(\frac{1}{2}\right)\cos(kt)$$

$$= \sin\left(\frac{t}{2}\right) + \sum_{k=1}^{n} \left[\sin\left(\frac{t}{2} + kt\right) + \sin\left(\frac{t}{2} - kt\right)\right]$$

$$= \sin\left(\frac{t}{2}\right) + \sin\left(\frac{3t}{2}\right) - \sin\left(\frac{t}{2}\right) + \sin\left(\frac{5t}{2}\right) - \sin\left(\frac{3t}{2}\right) + \cdots + = \sin\left(t\left(n + \frac{1}{2}\right)\right)$$

$$\implies D_n(t) = \frac{\sin\left(t\left(n+\frac{1}{2}\right)\right)}{2\sin\left(\frac{t}{2}\right)}.$$

(e) Knowing that $|\cos x| \leq 1 \quad \forall \quad x \in \mathbb{R}$

$$D_n(0) = \frac{1}{2} + \sum_{k=1}^{n} \cos(0) = \frac{1}{2} + 1 * n = \frac{1}{2} + n$$

(f)

$$|D_n(t)| = \Big|\frac{1}{2} + \sum_{k=1}^{n} \cos(kt)\Big| \leq \frac{1}{2} + 1 * n = n + \frac{1}{2}$$

(g) Knowing that $|\sin x| \leq 1 \quad \forall \quad x \in \mathbb{R}$, we have that

$$|D_n(t)| = \left|\frac{\sin\left(t\left(n+\frac{1}{2}\right)\right)}{2\sin\left(\frac{t}{2}\right)}\right| \leq \frac{1}{2}.$$

Where $0 < |t| < \pi$ we have $0 < \frac{1}{2} < \frac{\pi}{2|t|}$ which implies that

$$|D_n(t)| \leq \frac{\pi}{2|t|}.$$

$\diamond$

Exercise 2. (10.8.11 + 10.8.14) Knowing that for any continuous function, f, we have a sequence of polynomials converging to f doesn't tell us that there is also a power series that converges to f. A power series must be expressible in terms of one sequence of coefficients. In other words, one infinite polynomial won't do the job of a sequence of polynomials.

We can not apply the WAT to functions that are continuous on an unbounded interval. It works on compact intervals because a function that is continuous on a compact interval is uniformly continuous there. $f(x) = \frac{1}{x}$ is continuous on $(0, 1)$ but is not well behaved enough near 0 to be approximated using polynomials.

Exercise 3. (10.8.15)

Let $\int_0^1 f(x)x^n dx = 0 \; \forall \; n \in \mathbb{N}$ where $f : [0,1] \to \mathbb{R}$ is a continuous function. Prove that $f(x) \equiv 0 \; \forall x \in [0,1]$.

Proof. If $\int_0^1 f(x)x^n dx = 0 \; \forall \; n \in \mathbb{N}$ then by the linear property of integrals(8.5) we must have that

$$\int_0^1 f(x)g_n(x)dx = 0$$

$\forall \; n \in \mathbb{N}$ and where $g_n(x)$ represents the sequence of polynomials of whose existence we are assured of by the WAT (10.37). Knowing this, we sneak this integral integral below as follows:

$$\Big| \int_0^1 (f(x))^2 \Big| = \Big| \int_0^1 (f(x))^2 - \int_0^1 f(x)g_n(x)dx \Big| = \Big| \int_0^1 (f(x))^2 - f(x)g_n(x)dx \Big|.$$

The absolute value property of integrals (8.7) followed by a basic property of absolute values (1.17) gives us the following inequality:

$$\Big| \int_0^1 (f(x))^2 - f(x)g_n(x)dx \Big| \leq \int_0^1 \Big| (f(x))^2 - f(x)g_n(x)dx \Big|$$
$$= \int_0^1 \Big| f(x)\Big(f(x) - g_n(x) \Big) \Big| dx = \int_0^1 \Big| f(x) \Big| \Big| f(x) - g_n(x) \Big| dx$$

Now, because $f(x)$ is continuous on the bounded interval $[0,1]$, we have that there exists an M such that $\Big| f(x) \Big| \leq M \; \forall \; x \in [0,1]$. Furthermore, by the WAT, and letting $\frac{\epsilon}{M} > 0$, combining all of the above results, we have the following:

$$\Big|\int_0^1 (f(x))^2\Big| \leq \int_0^1 \Big|f(x)\Big|\Big|f(x) - g_n(x)\Big| dx \leq 1 * M * \frac{\epsilon}{M} = \epsilon$$
$$\implies \Big|\int_0^1 (f(x))^2\Big| = 0$$
$$\implies f(x) \equiv 0.$$

$\square$