MATH 414/514 HOMEWORK 3

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Exercise 1. For what values of p does $\int_1^\infty x^{-p}dx$ converge?

Answer.

To avoid dividing by 0 we treat the case where $p = 1$ separately. By Definition 8.11 we have

$$\int_1^\infty \frac{1}{x}dx = \lim_{y\to\infty}\int_1^y \frac{1}{x}dx = \lim_{y\to\infty}(\ln(y) - \ln(1)) \to \infty.$$

In general we have the following:

$$\int_1^\infty x^{-p}dx = \lim_{y\to\infty}\int_1^y x^{-p}dx = \lim_{y\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^y$$

$$= \lim_{y\to\infty}\left(\frac{1}{1-p}\right)\left[y^{1-p} - 1\right].$$

We see from this that this integral converges for values p where $1 - p < 0 \implies p > 1$. $\diamond$

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Exercise 2. Show that $\int_0^\infty x^n e^{-x} dx = n!$.

Proof. By induction, Where $n = 1$ we have

$$\int_0^\infty x^1 e^{-x} dx = \int_0^\infty x(-e^{-x})' dx = [-xe^{-x}]_0^\infty + \int_0^\infty e^{-x} dx$$
$$= [0 - 0] + [-e^{-x}]_0^\infty = -[0 - \frac{1}{e^0}] = 1 = 1!.$$

Now assuming the condition holds for n we show that it holds for $n + 1$:

$$\int_o^\infty x^{n+1} e^{-x} dx = \int_0^\infty x^{n+1}(-e^{-x})' dx = [-x^{n+1}e^{-x}]_0^\infty + (n + 1) \int_0^\infty x^n e^{-x} dx$$
$$= [0 - 0] + (n + 1)n! = (n + 1)!.$$

□

Exercise 3. If f is Riemann integrable on $[a,b]$ show that for every $\epsilon > 0$ there are two step functions where

$$L(x) \leq f(x) \leq U(x)$$

such that

$$\int_a^b (U(x) - L(x))dx < \epsilon.$$

Answer.

Because f is Riemann integrable on $[a,b]$ we have a partition $[x_0, x_1], ..., [x_{n-1}, x_n]$ of $[a,b]$ such that for every $\epsilon > 0$

$$\sum_{k=1}^{n} \omega f([x_{k-1}, x_k])(x_k - x_{k-1}) < \epsilon.$$

Now for that partition we take $\{a_0, ..., a_n\}$ to be the points where $a_k \in [x_{k-1}, x_k]$ and $f(a_k) = \inf_{x \in [x_{k-1}, x_k]} f(x)$ and the points $\{b_0, ..., b_n\}$ to be the points where $b_k \in [x_{k-1}, x_k]$ and $f(b_k) = \sup_{x \in [x_{k-1}, x_k]} f(x)$. This provides the following step functions:

$$L(x) = \begin{cases} f(a_1) & x \in [x_0, x_1] \\ f(a_2) & x \in [x_1, x_2] \\ . & . \\ . & . \\ . & . \\ f(a_n) & x \in [x_{n-1}, x_n] \end{cases}$$

$$U(x) = \begin{cases} f(b_1) & x \in [x_0, x_1] \\ f(b_2) & x \in [x_1, x_2] \\ . & . \\ . & . \\ . & . \\ f(b_n) & x \in [x_{n-1}, x_n] \end{cases}$$

where $L(x) \leq f(x) \leq U(x)$. Moreover,

$$\sum_{k=1}^{n} \omega f([x_{k-1}, x_k])(x_k - x_{k-1}) = \left| \sum_{k=1}^{n} \Big[\sup_{x \in [x_{k-1}, x_k]} f(x) - \inf_{x \in [x_{k-1}, x_k]} f(x) \Big] (x_k - x_{k-1}) \right|$$
$$= \left| \sum_{k=1}^{n} U(x) - L(x)(x_k - x_{k-1}) \right| = \left| \sum_{k=1}^{n} U(x)(x_k - x_{k-1}) - \sum_{k=1}^{n} L(x)(x_k - x_{k-1}) \right| < \epsilon.$$

Because step functions are integrable(def 8.18) the above inequality meets the Cauchy criteria and that we can write $\sum_{k=1}^{n} L(x)(x_k - x_{k-1}) = \int_a^b L(x)dx$ and similarly $\sum_{k=1}^{n} U(x)(x_k - x_{k-1}) = \int_a^b U(x)dx$. Keeping in mind that we can drop the absolute value because $L(x) \leq f(x) \leq U(x)$ we get the following desired result:

$$\sum_{k=1}^{n} U(x)(x_k - x_{k-1}) - \sum_{k=1}^{n} L(x)(x_k - x_{k-1}) = \int_a^b U(x)dx - \int_a^b L(x)dx$$
$$= \int_a^b (U(x) - L(x))dx < \epsilon.$$

$\diamond$