MATH 414/514 HOMEWORK 5

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Exercise 1. (9.2.5) Verify that $9.4, 9.5, 9.6$ can be paraphrased as "the order of taking the limit matters."

Answer.

In examples 9.4 and 9.5 we see that the order of taking the limit matters, in that, if we first take the limit of $f_n(x) = \frac{x^n}{n}$ we get 0. Differentiating this gives 0. But if we first take the derivative and then take the limit we end up with example 9.4 which is not 0 for $x \in [0, 1]$

In example 9.6 we see that order matters,in that, integrating $f_n(x)$ after taking the limit isn't necessarily the same as taking the limit first and then integrating.

$\diamond$

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Exercise 2. (9.2.11) Show that the set of convergence points of $\{f_n\}$ can be written as

$$E = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \bigcap_{m=N}^{\infty} \{x : |f_n(x) - f_m(x)| \leq \frac{1}{k}\}.$$

Answer. To show this we observe first that if $\{f_n\}$ is convergent on E then it is Cauchy on E. Or $\forall\ \epsilon > 0\ \exists$ and N such that where $n, m \geq N$ we have

$$|f(_n(x) - f_m(x)| < \epsilon.$$

If this is true for $\epsilon > 0$ then it it is true for $\epsilon = \frac{1}{k}\ \forall\ k \in \mathbb{N}$. So we can write the points where $f_n(x)$ converges as

$$\begin{aligned} E &= \bigcap_{k=1}^{\infty} \{x : \exists N, \forall n, m \geq N, |f(_n(x) - f_m(x)| < \frac{1}{k}\} \\ &= \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n,m \geq N}^{\infty} \{x : |f(_n(x) - f_m(x)| < \frac{1}{k}\} \\ &= \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \bigcap_{m=N}^{\infty} \{x : |f(_n(x) - f_m(x)| < \frac{1}{k}\}. \end{aligned}$$

$\diamond$

Exercise 3. (9.3.4) Prove that if $f_n \to f$ uniformly on E_1 and E_2, then $f_n \to f$ on $E_1 \cup E_2$.

Proof. If $f_n \to f$ uniformly on E_1 and E_2 then we have $N_1, N_2 \in \mathbb{N}$ s.t.

$$|f_n(x) - f(x)| < \epsilon$$
$$\forall x \in E_1 \quad \text{and} \quad n \geq N_1$$

and

$$|f_n(x) - f(x)| < \epsilon$$
$$\forall x \in E_2 \quad \text{and} \quad n \geq N_2.$$

Choose N to be the greater of N_1 and N_2. Now we have

$$|f_n(x) - f(x)| < \epsilon$$
$$\forall x \in E_1 \cup E_2 \quad \text{and} \quad n \geq N.$$

□

Exercise 4. (9.3.5) Prove or disprove that if $f_n \to f$ uniformly on each $E_1, E_2, , E_3 ...$, then $f_n \to f$ uniformly on $\cap_{k=1}^{\infty} E_k$.

Answer. As a counterexample, consider $f_n = x^n$. Let $E_1 = \{1\}$ and let $E_k = [0, 1 - \frac{1}{k}]$ for $k = 2, 3, 4, ...$ Now we see that $f_n = x^n$ converges uniformly on each E_k. However

$$\cap_{k=1}^{\infty} E_k = [0, 1]$$

and from example 9.4 we know that f_n does not uniformly converge on $[0, 1]$. ⋄