# Obtaining the Upper Mass Limit of Neutron Stars

## 1. Obtaining the expressions for the energy-momentum tensor $T_i^k$

### (1) From $ds^2 = A(r,t)c^2dt^2 - B(r,t)dr^2 - C(r,t)r^2(d\theta^2 + \sin^2\theta d\varphi^2) + a(r,t)drdt$ to $ds^2 = e^v c^2 dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)$.

Consider a gravitational field possessing central symmetry. For all points located at the same distance from the centre, the space-time metric, that is, the expression for the interval $ds$, must be the same. A general expression for $ds^2$ is

$$ds^2 = A(r,t)c^2dt^2 - B(r,t)dr^2 - C(r,t)r^2(d\theta^2 + \sin^2\theta d\varphi^2) + a(r,t)drdt,$$

where $A, B, C > 0$ and $A, B, C \to 1, r \to \infty$ (the Minkowski metric). Suppose that the field is static, then $a(r,t) = 0$ and $A(r,t) = A(r), B(r,t) = B(r), C(r,t) = C(r)$.

For eliminating the parameter $C$, we take $\bar{r} = \sqrt{C(r)}r$, then

$$d\bar{r} = \sqrt{C}dr + r\frac{d\sqrt{C}}{dr}dr \Rightarrow dr = \frac{1}{\sqrt{C} + r\frac{d\sqrt{C}}{dr}}d\bar{r} \Rightarrow Bdr^2 = \frac{B}{\left(\sqrt{C} + r\frac{d\sqrt{C}}{dr}\right)^2}d\bar{r}^2 \equiv \bar{B}d\bar{r}^2$$

$$\Rightarrow ds^2 = \bar{A}c^2dt^2 - \bar{B}d\bar{r}^2 - \bar{r}^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \Rightarrow ds^2 = Ac^2dt^2 - Bdr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \ldots (1.1).$$

**Question: Is it true that $\bar{B} \to 1, r \to \infty$?**

**Answer: Yes. Because when $r \to \infty$, $\sqrt{C(r)} \to 1$. And $\frac{dC}{dr}$, or equivalently, $\frac{d\sqrt{C}}{dr}$ must be smaller than any positive infinitesimal, e.g. $\frac{d\sqrt{C(r)}}{dr} \le \varepsilon(r) = \frac{1}{r^2}$. Thus, $r\frac{d\sqrt{C}}{dr} \le r\varepsilon(r) = \frac{1}{r} \to 0, r \to \infty$. And $\bar{B} \to B \to 1, r \to \infty$.**

And for further simplification, we can take $A(r) = e^{v(r)}, B(r) = e^{\lambda(r)}$. Thus, (1.1) becomes

$$ds^2 = e^v c^2 dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \ldots (1.2).$$

### (2) Obtaining $T_i^k$

With the corresponding metric, we can calculate the corresponding $\Gamma_{ik}^l$, $R^{ik}$ and the Ricci scalar $R$. Using the Einstein equation,

$$R_i^k - \frac{1}{2}\delta_i^k R = \frac{8\pi G}{c^4}T_i^k,$$

we have (the prime means differentiation with respect to $r$, while a dot on a symbol means differentiation with respect to $ct$):

$$\frac{8\pi G}{c^4}T_0^0 = \frac{e^{-\lambda}\lambda'}{r} + \frac{1-e^{-\lambda}}{r^2} \quad ...(1.3)$$

$$\frac{8\pi G}{c^4}T_1^1 = -\frac{e^{-\lambda}\nu'}{r} + \frac{1-e^{-\lambda}}{r^2} \quad ...(1.4)$$

$$\frac{8\pi G}{c^4}T_2^2 = \frac{8\pi G}{c^4}T_3^3 = -\frac{e^{-\lambda}}{2}\left(\nu'' + \frac{\nu'^2}{2} + \frac{\nu'-\lambda'}{r} - \frac{\nu'\lambda'}{2}\right) + \frac{e^{-\nu}}{2}\left(\ddot{\lambda} + \frac{\dot{\lambda}^2}{2} - \frac{\dot{\lambda}\dot{\nu}}{2}\right)$$

$$= -\frac{e^{-\lambda}}{2}\left(\nu'' + \frac{\nu'^2}{2} + \frac{\nu'-\lambda'}{r} - \frac{\nu'\lambda'}{2}\right) \quad ...(1.5)$$

$$\frac{8\pi G}{c^4}T_0^1 = -\frac{e^{-\lambda}\dot{\lambda}}{r} = 0 \quad ...(1.6)$$

## 2. Obtaining $M_{\max}$ $\left(c = G = 1, x(r) = \int_0^r re^{\frac{\lambda}{2}}\,dr, M(r) = \frac{4}{3}\pi r^3\bar{\rho} = 4\pi\int_0^r \rho(r)r^2\,dr\right)$

If $T_i^k$ take the form that

$$T_i^k = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix},$$

then

$$\frac{e^{-\lambda}\lambda'}{r} + \frac{1-e^{-\lambda}}{r^2} = \frac{8\pi G\rho}{c^2} = 8\pi\rho \quad ...(2.1)$$

$$\frac{e^{-\lambda}\nu'}{r} + \frac{e^{-\lambda}-1}{r^2} = \frac{8\pi Gp}{c^4} = 8\pi p \quad ...(2.2)$$

$$\frac{e^{-\lambda}}{2}\left(\nu'' + \frac{\nu'^2}{2} + \frac{\nu'-\lambda'}{r} - \frac{\nu'\lambda'}{2}\right) = \frac{8\pi Gp}{c^4} = 8\pi p \quad ...(2.3)$$

Replacing (2.3) by $T_{i;k}^k = 0$, we can further derive

$$p' = -\left(\rho c^2 + p\right)\frac{\nu'}{2} \quad ...(2.4).$$

(1) From $1 - \frac{1}{r}\int_0^r (2.1) r^2\,dr$ we have

$$\text{the left side} = 1 - \frac{1}{r}\int_0^r \left(\frac{e^{-\lambda}\lambda'}{r} + \frac{1-e^{-\lambda}}{r^2}\right) r^2\,dr = 1 - \frac{1}{r}\left[\int_0^r e^{-\lambda}\lambda' r\,dr + \int_0^r \left(1-e^{-\lambda}\right)dr\right]$$

$$= 1 - \frac{1}{r}\left[\left(-e^{-\lambda}r\right)\Big|_0^r + \int_0^r e^{-\lambda}\,dr + \int_0^r \left(1-e^{-\lambda}\right)dr\right] = 1 - \frac{1}{r}\left(-e^{-\lambda}r + \int_0^r dr\right)$$

$$= 1 - \frac{1}{r}\left(-e^{-\lambda}r + r\right) = e^{-\lambda} > 0,$$

and

$$\text{the right side} = 1 - \frac{1}{r}\int_0^r \rho r^2\, dr = 1 - \frac{2GM}{c^2 r},$$

$$\text{the left side} = \text{the right side} = e^{-\lambda} = 1 - \frac{2GM}{c^2 r} > 0.$$

(2) From $(2.3) + \frac{1}{2}(2.1) - (2.2)$ we have

$$e^{-\frac{\nu+\lambda}{2}} r \frac{d}{dr}\left[\frac{e^{-\frac{\lambda}{2}}}{r}\frac{d}{dr}\left(e^{\frac{\nu}{2}}\right)\right] = 4\pi(\rho + 3p) - \frac{3e^{-\lambda}\nu'}{2r} \ldots (2.5).$$

(3) From $\frac{1}{r^3}\int (2.1) r^2\, dr + (2.2)$ we have:

$$\text{the left side} = \frac{1}{r^3}\int_0^r \left(\frac{e^{-\lambda}\lambda'}{r} + \frac{1-e^{-\lambda}}{r^2}\right) r^2\, dr + \frac{e^{-\lambda}\nu'}{r} + \frac{e^{-\lambda}-1}{r^2}$$

$$= \frac{1}{r^3}\left(-e^{-\lambda} r + r\right) + \frac{e^{-\lambda}\nu'}{r} + \frac{e^{-\lambda}-1}{r^2} = \frac{e^{-\lambda}\nu'}{r}$$

and

$$\text{the right side} = \frac{8\pi}{r^3}\int_0^r \rho(r) r^2\, dr + 8\pi p = 8\pi\left(\frac{\bar{\rho}}{3} + p\right),$$

$$\text{the left side} = \text{the right side} = \frac{e^{-\lambda}\nu'}{r} = 8\pi\left(\frac{\bar{\rho}}{3} + p\right) \ldots (2.6).$$

(4) Eliminating $\frac{e^{-\lambda}\nu'}{r}$ and $p$ in (2.5) and (2.6), we get

$$e^{-\frac{\nu+\lambda}{2}} r \frac{d}{dr}\left[\frac{e^{-\frac{\lambda}{2}}}{r}\frac{d}{dr}\left(e^{\frac{\nu}{2}}\right)\right] = 4\pi(\rho - \bar{\rho}) \ldots (2.7);$$

As $x(r) = \int_0^r r e^{\frac{\lambda}{2}}\, dr$, $\frac{d}{dr} = \frac{d}{dx}\frac{dx}{dr} = r e^{\frac{\lambda}{2}}\frac{d}{dx}$, (2.7) can be rewritten into

$$e^{-\frac{\nu+\lambda}{2}} r \frac{d}{dr}\left[\frac{e^{-\frac{\lambda}{2}}}{r}\frac{d}{dr}\left(e^{\frac{\nu}{2}}\right)\right] = e^{-\frac{\nu}{2}} r^2 \frac{d}{dx}\left[\frac{d}{dx}\left(e^{\frac{\nu}{2}}\right)\right] = e^{-\frac{\nu}{2}} r^2 \frac{d^2}{dx^2}\left(e^{\frac{\nu}{2}}\right) = 4\pi(\rho - \bar{\rho}) \le 0.$$

Then

$$\frac{d^2}{dx^2}\left(e^{\frac{\nu}{2}}\right) = \frac{4\pi}{r^2} e^{\frac{\nu}{2}}(\rho - \bar{\rho}) \le 0 \ldots (2.8).$$

To prove that $(\rho - \bar{\rho}) \le 0$, we shall take a look at the TOV equation

$$\frac{dp}{dr} = -\frac{G}{r^2}\left(\rho + \frac{p}{c^2}\right)\left(M + \frac{4\pi r^3 p}{c^2}\right)\underbrace{\left(1 - \frac{2GM}{c^2 r}\right)^{-1}}_{e^{\lambda}>0} < 0,$$

where $p = p(r), \rho = \rho(r), M = M(r)$. We have known that $\frac{dp}{d\rho} = \frac{dp}{dr}\frac{dr}{d\rho} \geq 0$, then $\frac{d\rho}{dr} \leq 0$. And because

$$M(R) = 4\pi\int_0^R \rho(r) r^2\, dr = 4\pi\int_0^R \bar{\rho}(R) r^2\, dr,$$

we get $\rho(R) \leq \bar{\rho}(R)$, or, more generally, $(\rho - \bar{\rho}) \leq 0$.

We can derive from $\frac{d^2}{dx^2}\left(e^{\frac{\nu}{2}}\right) \leq 0$ that $\frac{d}{dx}\left(e^{\frac{\nu}{2}}\right) \leq \frac{e^{\frac{\nu}{2}}}{x}$ *, or equivalently,

$$\frac{e^{-\frac{\lambda}{2}}\nu'}{2r} \leq \frac{1}{x} ...(2.9).$$

Besides,

$$x(r) = \int_0^r r e^{\frac{\lambda}{2}}\, dr = \int_0^r r\left(1 - \frac{2M(r)}{r}\right)^{-\frac{1}{2}} dr \geq \int_0^r r\, dr = \frac{r^2}{2},\ \ x \geq r^2 ...(2.10),$$

With (2.4), (2.9), (2.10) and the TOV equation, we have

$$\begin{aligned}\frac{e^{-\frac{\lambda}{2}}}{2r} \bullet \left(-\frac{2p'}{\rho + p}\right) &= \frac{\left(1 - \frac{2M}{r}\right)^{\frac{1}{2}}}{2r} \bullet \left[\frac{2}{\rho + p} \bullet \frac{1}{r^2}(\rho + p)\left(M + 4\pi r^3 p\right)\left(1 - \frac{2M}{r}\right)^{-1}\right] \\ &= \frac{M + 4\pi r^3 p}{r^3\left(1 - \frac{2M}{r}\right)^{\frac{1}{2}}} \leq \frac{1}{x} \leq \frac{2}{r^2}.\end{aligned}$$

Therefore,

$$\begin{aligned}&15\left(\frac{M}{r}\right)^2 - 2(k+8)\frac{M}{r} + 4 - k^2 \leq 0, k = 4\pi p r^2 \\ &\Rightarrow \frac{M}{r} \geq \frac{k + 8 + |4k - 2|}{15} \text{ or } \frac{M}{r} \leq \frac{k + 8 - |4k - 2|}{15}.\end{aligned}$$

If $p(R) = 0$, $\frac{M(R)}{R} \leq \frac{2}{5}$.

# How to Derive $\frac{d}{dx}\left(e^{\frac{\nu}{2}}\right) \le \frac{e^{\frac{\nu}{2}}}{x}$ From $\frac{d^2}{dx^2}\left(e^{\frac{\nu}{2}}\right) \le 0$ *

Let $y \equiv \frac{d\nu}{dx}$, then

$$\frac{d^2}{dx^2}\left(e^{\frac{\nu}{2}}\right) = \frac{d}{dx}\left(\frac{1}{2}e^{\frac{\nu}{2}}y\right) = \frac{1}{4}e^{\frac{\nu}{2}}y^2 + \frac{1}{2}e^{\frac{\nu}{2}}\frac{dy}{dx} = e^{\frac{\nu}{2}}\left(\frac{1}{4}y^2 + \frac{1}{2}\frac{dy}{dx}\right) \le 0 \Rightarrow \frac{y}{2} \le \sqrt{-\frac{1}{2}\frac{dy}{dx}}.$$

Our purpose is to demonstrate that $\frac{y}{2} \le \frac{1}{x}$. If $\sqrt{-\frac{1}{2}\frac{dy}{dx}} \le \frac{1}{x}$, or, $\frac{dy}{dx} + \frac{2}{x^2} \ge 0$, then

$$\frac{y}{2} \le \sqrt{-\frac{1}{2}\frac{dy}{dx}} \le \frac{1}{x} \Rightarrow \frac{e^{-\frac{\lambda}{2}}}{2r}\nu' = \frac{e^{\frac{-\lambda}{2}}}{2r}\frac{d\nu}{dx}\frac{dx}{dr} = \frac{y}{2} \le \frac{1}{x}.$$

To prove that $\frac{dy}{dx} + \frac{2}{x^2} = \frac{d}{dx}\left(y - \frac{2}{x}\right) \ge 0$, we only have to prove that

$$y - \frac{2}{x} = \frac{d\nu}{dx} - \frac{2}{x} = \frac{d}{dx}(\nu - 2\ln x) \ge 0.$$

Again,

$$\begin{aligned}
&\frac{d}{dx}(\nu - 2\ln x) \ge 0 \\
&\Leftrightarrow \frac{\nu}{2} - \ln x \ge 0 \\
&\Leftrightarrow e^{\frac{\nu}{2}} - x \ge 0 \Leftrightarrow \frac{\nu'}{2}e^{\frac{\nu}{2}} - x' \ge 0
\end{aligned}$$

We have known that $\frac{e^{-\lambda}\nu'}{r} = 8\pi\left(\frac{\overline{\rho}}{3} + p\right)$ and $x' = re^{\frac{\lambda}{2}}$. Eliminating $\nu'$ and $x'$ in the inequality above, we get

$$\begin{aligned}
&\frac{\nu'}{2}e^{\frac{\nu}{2}} - x' = 4\pi r\left(\frac{\overline{\rho}}{3} + p\right)e^{\frac{\nu}{2}+\lambda} - re^{\frac{\lambda}{2}} \ge 0 \\
&\Leftrightarrow 4\pi\left(\frac{\overline{\rho}}{3} + p\right)e^{\frac{\nu+\lambda}{2}} - 1 \ge 0,
\end{aligned}$$

which can certainly be satisfied.