1. The equation $A_k = \frac{1}{B^k}$ is not covariant because

$$A'_k = \frac{\partial x^i}{\partial x'^k} A_i = \frac{\partial x^i}{\partial x'^k} \frac{1}{B^i} \neq \frac{\partial x^i}{\partial x'^k} \frac{1}{\frac{\partial x^i}{\partial x'^k} B'^k} = \frac{1}{B'^k},$$

the form of the original equation is changed.

2. (a) Take $dt = du + \frac{dr}{1-\frac{2M}{r}}$ ...(2.1), then

$$\begin{aligned}
ds^2 &= \left(1-\frac{2M}{r}\right)\left(du + \frac{dr}{1-\frac{2M}{r}}\right)^2 - \frac{dr^2}{1-\frac{2M}{r}} - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \\
&= \left(1-\frac{2M}{r}\right)du^2 + 2dudr + \frac{dr}{1-\frac{2M}{r}} - \frac{dr^2}{1-\frac{2M}{r}} - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \\
&= \left(1-\frac{2M}{r}\right)du^2 + 2dudr - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right).
\end{aligned}$$

The so-called Eddington-Finkelstein form of the Schwarzschild metric should then be

$$g^{ik} = \begin{bmatrix} 1-\frac{2M}{r} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{bmatrix}.$$

(b) Take $\theta = \frac{\pi}{2}, c = G = 1$, then for a photon, we have

$$\begin{aligned}
&ds = \left(1-\frac{2M}{r}\right)dt^2 - \left(1-\frac{2M}{r}\right)^{-1} dr^2 - r^2 d\varphi^2 = 0, \\
&{v_r}^2 + {v_\varphi}^2 = v^2 = c^2 = 1,
\end{aligned}$$

That is

$$\begin{aligned}
&\left(1-\frac{2M}{r}\right) - \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{dt}\right)^2 - r^2\left(\frac{d\varphi}{dt}\right)^2 \\
&= \left(1-\frac{2M}{r}\right) - \left(1-\frac{2M}{r}\right)^{-1} {v_r}^2 - {v_\varphi}^2 = \left(1-\frac{2M}{r}\right) - \left(1-\frac{2M}{r}\right)^{-1} {v_r}^2 - \left(1-{v_r}^2\right) \\
&= -\frac{2M}{r} + \frac{{v_r}^2}{1-\frac{r}{2M}} = 0, \\
&{v_r}^2 = \left(\frac{dr}{dt}\right)^2 = \frac{2M}{r} - 1 \text{...(2.2)}
\end{aligned}$$

From (2.1) and (2.2) we have

$$\frac{dr}{dt}=\frac{dr}{du+\dfrac{dr}{1-\dfrac{2M}{r}}}=\frac{1}{\dfrac{du}{dr}+\dfrac{1}{1-\dfrac{2M}{r}}}=\pm\sqrt{\frac{2M}{r}-1},$$

$$\frac{du}{dr}=\frac{1}{\dfrac{2M}{r}-1}+\frac{1}{\pm\sqrt{\dfrac{2M}{r}-1}},$$

$$u=\int du=\int\frac{dr}{\dfrac{2M}{r}-1}+\int\frac{dr}{\pm\sqrt{\dfrac{2M}{r}-1}}=-r\mp\sqrt{r(2M-r)}+2M\left[\pm\arctan\sqrt{\frac{r}{2M-r}}-\ln\left(-2M+r\right)\right].$$

That is the relationship between the coordinates u and r.

3. $\begin{cases} A^i_{;i}=\dfrac{1}{\sqrt{-g}}\dfrac{\partial}{\partial x^i}\left(\sqrt{-g}A^i\right) \\ A^i_{;i}=0 \end{cases} \Rightarrow \dfrac{\partial}{\partial x^0}\left(\sqrt{-g}A^0\right)=-\dfrac{\partial}{\partial x^\alpha}\left(\sqrt{-g}A^\alpha\right), \alpha=1,2,3$. According to Gauss Theorem, $\int_V \frac{\partial}{\partial x^0}\left(\sqrt{-g}A^0\right)dV=\int_V -\frac{\partial}{\partial x^\alpha}\left(\sqrt{-g}A^\alpha\right)dV=\int_\Sigma \sqrt{-g}A^\alpha\, dS=0, \alpha=1,2,3,$ as there is no such current $\mathbf{J}=\sqrt{-g}A^\alpha$ in this assumed case. Then $\int_V \frac{\partial}{\partial x^0}\left(\sqrt{-g}A^0\right)dV=\frac{\partial}{\partial x^0}\int_V\left(\sqrt{-g}A^0\right)dV=0$, $\int_V\left(\sqrt{-g}A^0\right)dV$ must be a constant.

4. $$g_{ik}=\begin{bmatrix} e^\nu & 1 & 0 & 0 \\ 1 & e^\lambda & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{bmatrix}, g_{ik}=\begin{bmatrix} 1/e^\nu & 1 & 0 & 0 \\ 1 & 1/e^\lambda & 0 & 0 \\ 0 & 0 & -1/r^2 & 0 \\ 0 & 0 & 0 & -1/r^2\sin^2\theta \end{bmatrix}.$$

As $g_{ik}, g^{ik}\begin{cases}\neq 0, i=k, \\ =0, i\neq k,\end{cases}$, the Christoffel symbols should then be

$$\Gamma^i_{kl}=\frac{g^{im}}{2}\left(g_{mk,l}+g_{ml,k}-g_{kl,m}\right)=\frac{g^{ii}}{2}\left(g_{ik,l}+g_{il,k}-g_{kl,i}\right)$$
$$=\frac{g^{ii}}{2}\left(g_{ii,i}+g_{ii,i}-g_{ii,i}\right)=\frac{g^{ii}}{2}g_{ii,i}=\Gamma^i_{ii}.$$

All the non-zero Christoffel symbols are

$$\Gamma^0_{00}=\frac{1}{e^\nu}\frac{\partial e^\nu}{\partial ct}=\frac{1}{e^\nu}e^\nu\frac{1}{c}\frac{\partial\nu}{\partial t}=\frac{1}{c}\frac{\partial\nu}{\partial t}, \Gamma^1_{11}=\frac{1}{e^\lambda}\frac{\partial e^\lambda}{\partial r}=\frac{1}{e^\lambda}e^\lambda\frac{\partial\lambda}{\partial r}=\frac{\partial\lambda}{\partial r},$$

$$\Gamma^2_{22}=\frac{1}{r^2}\frac{\partial\left(r^2\right)}{\partial\theta}=\frac{1}{r}\frac{\partial r}{\partial\theta}, \Gamma^3_{33}=\frac{1}{r^2\sin^2\theta}\frac{\partial\left(r^2\sin^2\theta\right)}{\partial\varphi}=\frac{2}{r}\frac{\partial r}{\partial\varphi}+2\cot\theta\frac{\partial\theta}{\partial\varphi}.$$