1. As it is shown in Figure 1, the radiation source is moving right towards the observer. If the source is not in line with the observer, and its velocity $\mathbf{v}$ forms an angle $\theta$ with the observer, we can illustrate it by Figure 2.

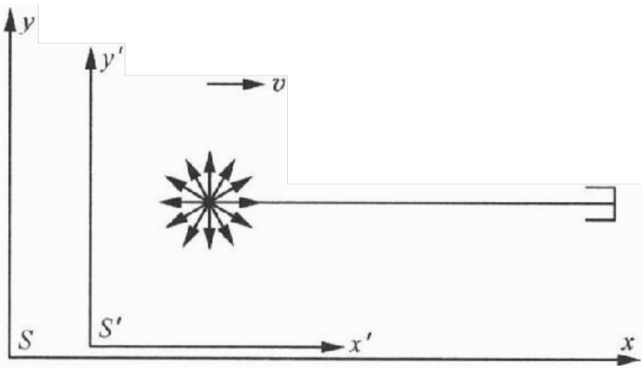


Figure 1

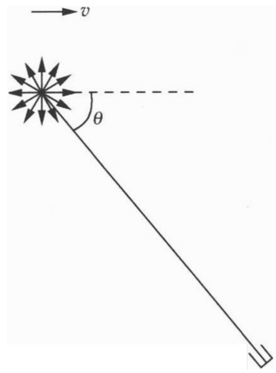


Figure 2

a) Shown as follows, Point $A$ and point $B$ are where the first two emissions take place, respectively. The observer is at Point $O$. Then in S, it takes the source $T$ to travel from $A$ to $B$. Then the distance $|\mathbf{AB}| = vT$. Suppose that $|\mathbf{AO}| = x$ ,$|\mathbf{BO}| = x+\Delta x$, then the relation between the arrival of the first two pulses in S should be $\Delta x = |\mathbf{BO}| - |\mathbf{AO}| = \sqrt{|\mathbf{AO}|^2 + |\mathbf{AB}|^2 - 2|\mathbf{AO}||\mathbf{AB}|\cos\theta_0} - |\mathbf{AO}| =$ $\frac{VT(VT-2x\cos\theta_0)}{\sqrt{x^2+(VT)^2-2xVT\cos\theta_0}+x}$. If $T$ is such a short time that s $VT \ll x$, then $\Delta x = -VT\cos\theta_0 = -V_{r0}T$.

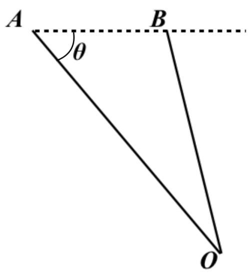


Figure 3

b) The time observed in S would last longer, so $T = \frac{T_0}{\sqrt{1-\beta^2}}, \beta = V/c$.

c) The wavelength observed in S should be compressed for the shifting of the source, that is $\lambda = cT - V\cos\theta T = (c - V_r)T$ (Figure 3). Then $v = \frac{c}{\lambda} = \frac{c}{c - V\cos\theta} \bullet \frac{\sqrt{1-\beta^2}}{T_0}$ $\frac{\sqrt{1-\beta^2}}{1-\beta\cos\theta} v_0$ , $= \frac{v}{v_0} = \frac{\sqrt{1-\beta^2}}{1-\beta\cos\theta}$.

d) From above, we could easily conclude that if $V_t = 0, V_r = V$, then $v = \frac{c}{\lambda} = \frac{c}{c-V_r} \bullet$ $\frac{\sqrt{1-\beta^2}}{T_0} = \frac{c}{c-V} \bullet \frac{\sqrt{1-\beta^2}}{T_0} = \sqrt{\frac{1+\beta}{1-\beta}} v_0$.

e) Similarly, if $V_t = V, V_r = 0$, then $v = \frac{c}{\lambda} = \frac{c}{c-V_r} \bullet \frac{\sqrt{1-\beta^2}}{T_0} = \frac{c}{c-0} \bullet \frac{\sqrt{1-\beta^2}}{T_0} = \sqrt{1-\beta^2} v_0$.

2. Under the Lorentz transformation, $\left.\begin{array}{l} x = \gamma(x' + \beta c t') \\ t = \gamma(\beta x'/c + t') \end{array}\right\}, \beta = v/c, \gamma = 1/\sqrt{1-\beta^2}$.

In that case, $\frac{\partial}{\partial x'}=\frac{\partial}{\partial x}\frac{\partial x}{\partial x'}+\frac{\partial}{\partial t}\frac{\partial t}{\partial x'}=\gamma\left(\frac{\partial}{\partial x}+\frac{\beta}{c}\frac{\partial}{\partial t}\right)$,

$$\frac{\partial^2}{\partial x'^2}=\frac{\partial}{\partial x'}\left(\frac{\partial}{\partial x'}\right)=\gamma\frac{\partial}{\partial x'}\left(\frac{\partial}{\partial x}+\frac{\beta}{c}\frac{\partial}{\partial t}\right)=\gamma^2\left(\frac{\partial}{\partial x}+\frac{\beta}{c}\frac{\partial}{\partial t}\right)^2=\gamma^2\left(\frac{\partial^2}{\partial x^2}+2\frac{\beta}{c}\frac{\partial}{\partial x\partial t}+\frac{\beta^2}{c^2}\frac{\partial^2}{\partial t^2}\right).$$

Similarly, $\frac{\partial^2}{\partial t'^2}=\gamma^2(\frac{\partial^2}{\partial t^2}+2\beta c\frac{\partial}{\partial x\partial t}+\beta^2c^2\frac{\partial^2}{\partial x^2})$.

Therefore, $\frac{\partial^2}{\partial x'^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t'^2}=\gamma^2(1-\beta^2)\left(\frac{\partial^2}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)=\frac{\partial^2}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}$.

It is obvious that $(\frac{\partial^2}{\partial x'^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t'^2})\bullet\Psi'(x,t)=(\frac{\partial^2}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2})\bullet\Psi(x,t)=0$.

3. The matrix of moduli of Lorentz transformation $a^{ik}=\begin{pmatrix}\gamma & -\gamma\beta & 0 & 0\\ -\gamma\beta & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$.

Then under the Lorentz transformation, a second-order four-tensor $A^{ij}$ should satisfy this equation: $A'^{ij}=a^{ik}a^{jl}A^{kl}$.

Or we could use another method to do the transformation—by regarding $A^{ij}$ as a product of two four-vectors : $A^{ij}\sim B^iC^j$ (not saying that $A^{ij}$ is actually equal to $B^iC^j$).

In that case, $A'^{ij}\sim B'^iC'^j$, where

b $\quad B'^0=\gamma(B^0-\beta B^1), B'^1=\gamma(B^1-\beta B^0), C'^0=\gamma(C^0-\beta C^1), C'^1=\gamma(C^1-\beta C^0)$.

Then if $A^{ij}$ is symmetric, we have

$$A'^{00}\sim B'^0C'^0=\gamma^2(B^0-\beta B^1)(C^0-\beta C^1)=\gamma^2(B^0C^0+\beta^2B^1C^1-\beta B^0C^1-\beta B^1C^0),$$
$$A'^{00}=\gamma^2(A^{00}+\beta^2A^{11}-\beta A^{01}-\beta A^{10})=\gamma^2(A^{00}+\beta^2A^{11}-2\beta A^{01}).$$

Likewise, we can derive from $A'^{ij}=a^{ik}a^{jl}A^{kl}$ the other components of $A'^{ij}$. And it would be the same case when $A^{ij}$ is anti-symmetric. (For instance, $A'^{00}=\gamma^2\bullet$ $(A^{00}+\beta^2A^{11}--\beta A^{01}-\beta A^{10})=\gamma^2(0+0-\beta A^{01}+\beta A^{01})=0$ )

4. The momentum four-vector of a free particle is $p^2=(\frac{E}{c},\mathbf{p})=(\frac{E}{c},p_x,p_y,p_z)$. From the general formulas for transformation of four-vectors, we immediately have the transformation of momentum and energy from one inertial system to another:

$$p_x=\gamma(p_x'+\beta E'/c), p_y=p_y', p_z=p_z', E=\gamma(E'+\beta cp_x').$$

5. $\left(\frac{\partial}{\partial x^i}\right)^2=g_{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}{}^2=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2$.

6. The matrix of moduli of Lorentz transformation $a^{ik}=\begin{pmatrix}\gamma & -\gamma\beta & 0 & 0\\ -\gamma\beta & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$.

Then under the Lorentz transformation, $T^{ij}$ should satisfy this equation: $T'^{ij} = a^{ik}a^{jl}T^{kl}$. Therefore, $T'^{00} = \gamma^2(T^{00} + \beta^2 T^{11} - \beta T^{01} - \beta T^{10}) == \gamma^2(T^{00} + \beta^2 T^{11} - 2\beta T^{01})$. For instance, if the energy-momentum tensor for the given portion of the body (in the reference system in which it is at rest) has the form:

$$T^{ik} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix},$$

where $\varepsilon$ is the energy density and $p$ (equally transmitted in all directions) the pressure applied to the body, then the expressions for $T'^{00}$ (the energy density $W$) is $T'^{00} = \gamma^2(\varepsilon + \beta^2 p)$.

7.

a) $M^{\alpha\beta\gamma} = x^\alpha T^{\beta\gamma} - x^\beta T^{\alpha\gamma} = -(x^\beta T^{\alpha\gamma} - x^\alpha T^{\beta\gamma}) = -M^{\beta\alpha\gamma}$.

b) Noting that $\frac{\partial x^\alpha}{\partial x^\gamma} = \delta^\alpha_\gamma$ and, we could then find:

$$\frac{\partial}{\partial x^\gamma}\left(M^{\alpha\beta\gamma}\right) = \delta^\alpha_\gamma T^{\beta\gamma} - \delta^\beta_\gamma T^{\alpha\gamma} = T^{\beta\alpha} - T^{\alpha\beta} = 0.$$

c) Generally, if

$$\mathbf{A} = A^{\alpha\beta}\mathbf{e}_\alpha\mathbf{e}_\beta, \nabla = \mathbf{e}_\alpha \frac{\partial}{\partial x^\alpha}(\alpha,\beta = 0,1,2,3)\,; \mathbf{e}_\alpha \bullet \mathbf{e}_\beta = \delta_{\alpha\beta} = \begin{cases} 1, \alpha = \beta \\ 0, \alpha \neq \beta \end{cases} (\alpha,\beta = 0,1,2,3),$$

then

$$\nabla \bullet \mathbf{A} = \left(\mathbf{e}^i \frac{\partial}{\partial x^i}\right) \bullet \left(A^{\alpha\beta}\mathbf{e}_\alpha\mathbf{e}_\beta\right) = \frac{\partial A^{\alpha\beta}}{\partial x^i}\left(\mathbf{e}_i \bullet \mathbf{e}_\alpha\right)\mathbf{e}_\beta = \frac{\partial A^{i\beta}}{\partial x^i}\mathbf{e}_\beta.$$

Therefore,

$$\begin{aligned}\nabla \bullet M^{\alpha\beta\gamma} &= \left(\mathbf{e}_i \frac{\partial}{\partial x^i}\right) \bullet \left(x^\alpha T^{\beta\gamma}\mathbf{e}_\beta\mathbf{e}_\gamma - x^\beta T^{\alpha\gamma}\mathbf{e}_\alpha\mathbf{e}_\gamma\right) \\ &= \frac{\partial\left(x^\alpha T^{i\gamma}\right)}{\partial x^i}\mathbf{e}_\gamma - \frac{\partial\left(x^\beta T^{i\gamma}\right)}{\partial x^i}\mathbf{e}_\gamma = \left(\frac{\partial x^\alpha}{\partial x^i} - \frac{\partial x^\beta}{\partial x^i}\right)T^{i\gamma}\mathbf{e}_\gamma \\ &= \left(\delta^\alpha_i - \delta^\beta_i\right)T^{i\gamma}\mathbf{e}_\gamma = \left(T^{\alpha\gamma} - T^{\beta\gamma}\right)\mathbf{e}_\gamma.\end{aligned}$$

That is,

$$\nabla \bullet M^{\alpha\beta\gamma} = \left(T^{\alpha\gamma} - T^{\beta\gamma}\right)\mathbf{e}_\gamma,$$

$$L^{\alpha\beta} = \int_V \left(T^{\alpha\gamma} - T^{\beta\gamma}\right)\mathbf{e}_\gamma\, dV(\alpha,\beta,\gamma,i = 0,1,2,3; dV = dx^1dx^2dx^3).$$

From the Gauss theorem, we first obtain $\oint \nabla \bullet M^{\alpha\beta\gamma}\, dS_\gamma = \int_{Spacetime} \frac{\partial M^{\alpha\beta\gamma}}{\partial x^\gamma} d\Omega = 0$.

Now we take $S$ consisting of any two space-like surfaces, $S_1$ and $S_2$, whose connection point is the zero. Then $\oint \nabla \bullet M^{\alpha\beta\gamma}\, dS_\gamma = \int_{S_2} \nabla \bullet M^{\alpha\beta\gamma}\, dS_\gamma - \int_{S_1} \nabla \bullet M^{\alpha\beta\gamma}\, dS_\gamma = 0$.

Hence, $\int_S \nabla \bullet M^{\alpha\beta\gamma}\, dS_\gamma$ is independent of the space-like surface $S$. For example, by choosing the hyper-surface $x^0 = 0$, we have $L^{\alpha\beta} = \int_S \nabla \bullet M^{\alpha\beta 0}\, dS_0 = \int_V \nabla \bullet M^{\alpha\beta 0}\, dV = const.$

d) The antisymmetric second-order tensor $L^{\alpha\beta}$ has only six independent nonzero components. Each component of $L^{\alpha\beta}$ is a energy four-vector.