Elements of Measures

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Abstract

Self-study notes on Axler's *Measure, Integration & Real Analysis*.[i] I share a succinct digest complemented by a bit of my own (naïve) comprehension (in some details for the *measure* part), with the hope of providing a beginner's perspective to fellow learners. Please refer to the original text for much greater interpretations.

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[i]Free copy online! How generous. Thank you, professor Axler.

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1 Measures

1.1 Outer measure on $\mathbb{R}$

1.1.1 Definition & good properties of outer measure

1.1 The *outer measure*[ii][iii]

$$\mathring{\mu}_{A} := \inf\left\{\sum_{k=1}^{\infty}\ell_{I_k}\ \middle|\ \{I_k\}_{k=1}^{\infty} \text{ is an open cover of } A\right\}$$

of $A \subseteq \mathbb{R}$, with

$$\ell_I := \begin{cases} b-a & \text{if } \exists a \,\&\, b \in \mathbb{R} : a < b \wedge I = (a,b) \\ 0 & \text{if } I = \varnothing \\ \infty & \text{if } \exists a \in [-\infty,\infty) : I = \pm(a,\infty) \end{cases}$$

the *length* of an interval $I \subseteq \mathbb{R}$

Properties (outer measure's) 1. $\mathring{\mu}_{\forall \text{countable } C\subseteq\mathbb{R}} = 0$

2. $\mathring{\mu}_{\forall A\subseteq B\subseteq\mathbb{R}} \leq \mathring{\mu}_B$

3. $\mathring{\mu}_{t+A} = \mathring{\mu}_A$ $\forall$*translation* $(t + A)$ of $A \subseteq \mathbb{R}$ by $t \in \mathbb{R}$

4. $\mathring{\mu}_{\bigcup_{k=1}^{\infty}A_k} \leq \sum_{k=1}^{\infty}\mathring{\mu}_{A_k}$ $\forall\{A_k \subseteq \mathbb{R}\}_{k=1}^{\infty}$

Proof. 1. $\forall\epsilon > 0$, an open cover $\{I_k = c_k + {}^{(-\epsilon,\epsilon)}/_{2^k}\}_{k=1}^{\infty}$ of $C = \{c_k\}_{k=1}^{\infty}$
$\Rightarrow \mathring{\mu}_C \leq \sum_{k=1}^{\infty}\left(\ell_{I_k} = {}^{\epsilon}/_{2^{k-1}}\right) = 2\epsilon \xlongequal{\epsilon\text{'s arbitrariness}} 0.$

2. B's every cover covers A.

3. ℓ_I is translational invariant (by any distance t) $\forall$interval I.

4. $\forall\epsilon > 0$, pick an open cover $\left\{I_{j,k}\right\}_{j=1}^{\infty}$ $\forall A_{k\in\mathbb{Z}_{>0}} : \sum_{j=1}^{\infty}\ell_{I_{j,k}} - \mathring{\mu}_{A_k} \in [0, {}^{\epsilon}/_{2^k}]$. Then
$\mathring{\mu}_{\bigcup_{k=1}^{\infty}A_k\subseteq\bigcup_{i=2}^{\infty}\{I_i\equiv\bigcup_{(k,j)\in(\mathbb{Z}_{>0})^2;k+j=i}\{I_{j,k}\}\}=\bigcup_{k=1}^{\infty}\bigcup_{j=1}^{\infty}\{I_{j,k}\}} \leq \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\ell_{I_{j,k}} \leq \sum_{k=1}^{\infty}\mathring{\mu}_{A_k}\underbrace{+\epsilon}_{0}$ □

Remark. • $\mathbb{Q} \subseteq \mathbb{R}$ is countable $\Rightarrow \mathring{\mu}_{\mathbb{Q}} = 0$

• $\mathring{\mu}_{\varnothing} \xlongequal[\forall S\subseteq\mathbb{R},\ \mu_S\geq 0\ \wedge\ \varnothing\leq S]{\text{properties 1–2}} 0.$

1.1.2 Outer measure of compact interval

1.1 $\mathring{\mu}_{[a,b]} = b - a$ $\forall a \,\&\, b \in \mathbb{R} : a < b$

Proof. 1. $\forall\epsilon > 0$, $\mathring{\mu}_{[a,b]\subseteq(a-\epsilon,b+\epsilon)\cup\varnothing\cup\varnothing\cup\cdots=(a-\epsilon,b+\epsilon)} \leq \mathring{\mu}_{(a-\epsilon,b+\epsilon)} = b - a\underbrace{+2\epsilon}_{0}$

2. (a) By Heine-Borel's theorem, every open cover $\{I_k\}_{k=1}^{\infty}$ of a closed bounded $[a,b] \subseteq \mathbb{R}$ has a finite subcover $\{I_k\}_{k=1}^{n}$ (b) Prove $\sum_{k=1}^{n}\ell_{I_k} \geq b - a$ by induction on $n \in \mathbb{Z}_{>0}$. Then $\sum_{k=1}^{\infty}\ell_{I_k} \geq \sum_{k=1}^{n}\ell_{I_k} \geq b - a \Rightarrow \mathring{\mu}_{[a,b]} \geq b - a$ □

Remark. $\mathring{\mu}_{(a,b)\subseteq\mathbb{R}} = \mathring{\mu}_{(a,b]} = \mathring{\mu}_{[a,b]} = \mathring{\mu}_{[a,b)}$.

1.2 *Every nontrivial (i.e. $\exists a \,\&\, b \in I : a < b$) interval $I \subseteq \mathbb{R}$ is uncountable*[iv]

[ii] $\sum_{k=1}^{\infty}t_k := \sum_{k=1}^{n\to\infty}t_k$ $\forall$sequence $\{t_k\}_{k=1}^{\infty} \equiv t_{k=1,2,\ldots}$
[iii] $\overline{\mathbb{R}} \equiv [-\infty,\infty] = \mathbb{R} \uplus \{\pm\infty\}$, with $\mathbb{R} \equiv (-\infty,\infty)$, and $\uplus$ a disjoint union
[iv] $\mathring{\mu}_{I\supseteq[a,b]} \geq \mathring{\mu}_{[a,b]} = b - a > 0$

1.1.3 Nonadditivity of outer meansure

1.3 $\exists A \,\&\, B \subseteq \mathbb{R} : \mathring{\mu}_{A \uplus B} \neq \mathring{\mu}_A + \mathring{\mu}_B$ ■

Proof. Partition $[-1,1]$ into equivalence classes $[a] := \{b \in [-1,1] \mid a - b \in \mathbb{Q}\}$, and pick $V \subseteq [-1,1] : |V \cap [a]| = 1 \; \forall a \in [-1,1]$.[v] Then $\{q_k\}_{k=1}^{\infty} \equiv [-2,2] \cap \mathbb{Q}$
$\Rightarrow [-1,1] \subseteq \biguplus_{k=1}^{\infty} (q_k + V) \subseteq [-3,3]$
$\Rightarrow \underbrace{\mathring{\mu}_{[-1,1]}}_{=2>0} \leq \mathring{\mu}_{\biguplus_{k=1}^{\infty}(q_k+V)} \xlongequal[\text{if } \mathring{\mu}_{A\uplus B} = \mathring{\mu}_A + \mathring{\mu}_B \; \forall A,B \subseteq \mathbb{R}]{\text{mathematical induction}} \sum_{k=1}^{\infty} \underbrace{\mathring{\mu}_{q_k+V}}_{\equiv \mathring{\mu}_V} = \underbrace{\left|\{q_k\}_{k=1}^{\infty}\right|}_{\infty} \cdot \mathring{\mu}_V \leq \underbrace{\mathring{\mu}_{[-3,3]}}_{=6<\infty}$
$\Rightarrow$ contradiction: $0 < \infty \cdot (\mathring{\mu}_V = 0) = 0$ □

1.2 Measurable spaces & maps

1.2.1 Motivation & definition of σ-algebra

1.4 $\underbrace{2^{\mathbb{R}} := \{S\}_{S \subseteq \mathbb{R}}}_{\mathbb{R}\text{'s power set}} \xrightarrow{\nexists \mu} \overline{\mathbb{R}}_{\geq 0}$:

1. $\mu_I = \ell_I \; \forall$ *open interval* $I \subseteq \mathbb{R}$
2. $\mu_{\biguplus_{k=1}^{\infty} A_k} = \sum_{k=1}^{\infty} \mu_{A_k} \; \forall \{A_k \subseteq \mathbb{R}\}_{k=1}^{\infty}$
3. $\mu_{t+A} = \mu_A \; \forall A \subseteq \mathbb{R} \; \forall t \in \mathbb{R}$ ■

Proof. μ has all $\mathring{\mu}$'s properties used to prove theorem 1.3 □

1.2 $\mathcal{S} \subseteq 2^X$ is a *σ-algebra* on a set X if

1. $X \backslash E \in \mathcal{S} \forall E \in \mathcal{S}$
2. $\varnothing \in \mathcal{S}$ ($\Longleftrightarrow X = X \backslash \varnothing \in \mathcal{S}$)
3. $\forall \{E_k \in \mathcal{S}\}_{k=1}^{\infty}, \bigcup_{k=1}^{\infty} E_k \in \mathcal{S}$ ($\xLeftrightarrow{\text{DE MORGAN's laws}} \bigcap_{k=1}^{\infty} E_k = X \backslash \bigcup_{k=1}^{\infty} (X \backslash E_k) \in \mathcal{S}$).

$(X, \mathcal{S})$ is then called a *measurable space*, and $E \in \mathcal{S}$ *measurable sets* ●
E.g. $\{\varnothing, X\}$ and 2^X are σ-algebras on X.
1.5 $\bigcap_{\mathcal{S} \in \{\mathcal{S}' \subseteq 2^X \mid \mathcal{S}' \text{ is a σ-algebra on X containing } \mathcal{A}\}} \mathcal{S}$ *is the smallest σ-algebra on* X *containing* $\mathcal{A} \subseteq 2^X$ ■
Examples (of smallest σ-algebras)

1. $\{E \in X \mid E \text{ countable} \vee X \backslash E \text{ countable}\}$ on X containing $\{\{x\}\}_{x \in X}$.
2. $\{\varnothing, \mathbb{R}, (0,1), \mathbb{R}_{>0}, \mathbb{R}_{\leq 0} \uplus \mathbb{R}_{\geq 1}, \mathbb{R}_{\leq 0}, \mathbb{R}_{\geq 1}, \mathbb{R}_{<1}\}$ on $\mathbb{R}$ containing $\{(0,1), \mathbb{R}_{>0}\}$.

1.2.2 BOREL's subsets of ℝ

1.3 The set $\mathscr{B}$ of *BOREL's* $B \subseteq \mathbb{R}$ is the smallest σ-algebra on $\mathbb{R}$ containing all open $G \subseteq \mathbb{R}$ ●
Examples (of $B \in \mathscr{B}$) • Every closed set, every countable $\{r_k \in \mathbb{R}\}_{k=1}^{\infty}$, and every half-open interval

• $\left\{ r \in \mathbb{R} \;\middle|\; \mathbb{R} \xrightarrow{f} \mathbb{R} \text{ is continuous at } r \right\}$ as an open-set intersection is 'BOREL'.

[v]|V| denotes the order of a set V

1.2.3 Inverse images of measurable maps are measurable

1.4 $X \xrightarrow{f} \mathbb{R}$ is *measurable* on a measurable space $(X, \mathcal{S})$ if $f^{-1}_{\forall B \in \mathscr{B}} \in \mathcal{S}$[vi] ●

E.g. • The only measurable $X \xrightarrow{f} \mathbb{R}$ on the measurable space $(X, \{\varnothing, X\})$ are constant maps.

• Every $X \xrightarrow{f} \mathbb{R}$ is measurable on the measurable space $(X, 2^X)$.

• $\mathbb{R} \xrightarrow{f} \mathbb{R}$ is measurable on the measurable space $(\mathbb{R}, \{\varnothing, \mathbb{R}, \mathbb{R}_{<0}, \mathbb{R}_{\geq 0}\})$ iff f is constant respectively on $\mathbb{R}_{<0}$ and on $\mathbb{R}_{\geq 0}$.

• A *characteristic map* $X \xrightarrow{\chi_E} \mathbb{R}$ of $E \subseteq X$ with $\chi_{E;\forall x \in X} := \left\{ \begin{matrix} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{matrix} \right\}$ is measurable on a measurable space $(X, \mathcal{S})$ iff $E \in \mathcal{S} \Leftarrow \chi^{-1}_{E;B \subseteq \mathbb{R}} = \begin{cases} E & \text{if } 0 \notin B \ni 1 \\ X \backslash E & \text{if } 0 \in B \not\ni 1 \\ X & \text{if } 0 \in B \ni 1 \\ \varnothing & \text{if } 0 \notin B \not\ni 1. \end{cases}$ [vii]

1.6 *$X \xrightarrow{f} \mathbb{R}$ is measurable on a measurable space $(X, \mathcal{S}) \Leftarrow f^{-1}_{(\forall a \in \mathbb{R}, \infty)} \in \mathcal{S}$* ■

Proof. $\{A \subseteq \mathbb{R} \,|\, f^{-1}_A \in \mathcal{S}\}$ is a σ-algebra containing $\mathscr{B}$ □

Remark. The collection $\{\mathbb{R}_{>a}\}_{a \in \mathbb{R}}$ in the condition can be replaced by any $\mathcal{A} \subseteq 2^{\mathbb{R}}$: $\mathscr{B} \subseteq$ the smallest σ-algebra containing $\mathcal{A}$. *E.g.* $\mathcal{A} = \{(p,q]\}_{p,q \in \mathbb{Q}} \vee = \{(q,z]\}_{q \in \mathbb{Q}, z \in \mathbb{Z}}$ $\vee = \{(q, q+1)\}_{q \in \mathbb{Q}} \vee = \left\{\mathbb{R}_{\geq q}\right\}_{q \in \mathbb{Q}}$ *etc.*

1.7 $\{E' \in \mathcal{S}\}_{E' \subseteq X'} = \{E \cap X'\}_{E \in \mathcal{S}}$ *is a σ-algebra on* $X' \in \mathcal{S}$ $\forall$ *σ-algebra* $\mathcal{S} \subseteq 2^X$ ■

1.5 $\forall X \subseteq \mathbb{R}$, $X \xrightarrow{f} \mathbb{R}$ is Borel-*measurable* if $f^{-1}_{\forall B \in \mathscr{B}} \in \mathscr{B}$ ●

1.8 *Every continuous $B \xrightarrow{f} \mathbb{R}$ is $\mathscr{B}$-measurable $\forall B \in \mathscr{B}$* ■

Proof. $f^{-1}_{(\forall a \in \mathbb{R}, \infty)} = \left(\bigcup_{b \in f^{-1}_{\mathbb{R}_{>a}}} (b - \delta_b, b + \delta_b)\right) \cap B \in \mathscr{B}$
$\Leftarrow f_{\forall b \in B} > a \; \exists \delta_b > 0 : f_{\forall x \in (b-\delta_b, b+\delta_b) \cap B} > a$ □

1.9 *Every increasing $B \xrightarrow{f} \mathbb{R}$ is $\mathscr{B}$-measurable $\forall B \in \mathscr{B}$* ■

Proof. $f^{-1}_{(\forall a \in \mathbb{R}, \infty)} \xlongequal{b = \inf f^{-1}_{\mathbb{R}_{>a}}} \mathbb{R}_{>b} \cap B \in \mathscr{B}$ □

1.10 *$X \xrightarrow{g \circ f} \mathbb{R}$ is measurable on a measurable space $(X, \mathcal{S})$ $\forall \mathcal{S}$-measurable $X \xrightarrow{f} \mathbb{R}$ $\forall \mathscr{B}$-measurable $Y \xrightarrow{g} \mathbb{R} : Y \supseteq f_X$* ■

E.g. $X \xrightarrow{f} \mathbb{R}$ is measurable on a measurable space $(X, \mathcal{S}) \Rightarrow$ so are $-f$, $f/_2$, $|f|$, f^2 *etc.*

[vi] $\forall X \xrightarrow{f} Y$, the *inverse image* $f^{-1}_A := \{x \in X \,|\, f_x \in A\} = X \backslash f^{-1}_{Y \backslash A}$ of $A \subseteq Y$. Besides, $f^{-1}_{\bigcirc_{A \in \mathcal{A}} A} \xlongequal{\bigcirc = \cup, \cap} \bigcirc_{A \in \mathcal{A}} f^{-1}_A \; \forall \mathcal{A} \subseteq 2^Y$, $(g \circ f)^{-1}_{\forall A \subseteq Z} = f^{-1}_{g^{-1}_A} \; \forall Y \xrightarrow{g} Z$

[vii] $\forall$ measurable space $(X, \mathcal{S})$ $\forall x \in X$, Dirac's measure (*cf.* definition 1.7) $\mathcal{S} \xrightarrow{\delta_x : E \mapsto \chi_{E;x}} \overline{\mathbb{R}}_{\geq 0}$

1.11 $X \xrightarrow{f\,\&\,g} \mathbb{R}$ *are measurable on a measurable space* $(X, \mathcal{S}) \Rightarrow$ *so are* $f \pm g$, fg *and* $f/_g$ ($g_{\forall x \in X} \neq 0$ *in the quotient*) ■

Proof. $fg = {}^{(f+g)^2 - f^2 - g^2}/_2$, $(f+g)^{-1}_{(\forall a \in \mathbb{R}, \infty)} = \bigcup_{q \in \mathbb{Q}} \left(f^{-1}_{\mathbb{R}_{>q}} \cap g^{-1}_{\mathbb{R}_{>a-q}} \right) \in \mathcal{S}$ □

1.12 $\exists f_{k \to \infty; \forall x \in X}$ *for a sequence* $\left\{ X \xrightarrow{f_k} \mathbb{R} \right\}_{k=1}^{\infty}$ *of measurable maps on a measurable space* $(X, \mathcal{S}) \Rightarrow \mathcal{S}$*-measurable* $X \xrightarrow{f: x \mapsto f_{k \to \infty; x}} \mathbb{R}$. ■

Proof. $f^{-1}_{(\forall a \in \mathbb{R}, \infty)} = \bigcup_{j=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} f^{-1}_{k; \mathbb{R}_{>a+1/j}} \in \mathcal{S}$ □

1.6 $X \xrightarrow{f} \overline{\mathbb{R}}$ is *measurable* on a measurable space $(X, \mathcal{S})$ if $f^{-1}_{\forall B \in \overline{\mathscr{B}}} \in \mathcal{S}$,[viii] where $B \subseteq \overline{\mathbb{R}}$ is Borel's set if $B \cap \mathbb{R} \in \mathscr{B}$ (and $\overline{\mathscr{B}}$ is the collection of all such B) ●

1.13 *A sequence* $\left\{ X \xrightarrow{f_k} \overline{\mathbb{R}} \right\}_{k=1}^{\infty}$ *of measurable maps on a measurable space* $(X, \mathcal{S})$ $\Rightarrow \mathcal{S}$*-measurable* $X \xrightarrow{g\,\&\,h} \overline{\mathbb{R}} : g_{\forall x \in X} := \inf_{\{f_{k;x}\}_{k=1}^{\infty}}$, $h_{\forall x \in X} := \sup_{\{f_{k;x}\}_{k=1}^{\infty}}$ ■

Proof. $g_{\forall x \in X} = -\sup_{\{-f_{k;x}\}_{k=1}^{\infty}}$, $h^{-1}_{(\forall a \in \mathbb{R}, \infty]} = \bigcup_{k=1}^{\infty} f^{-1}_{k; \mathbb{R}_{>a}} \in \mathcal{S}$ □

1.3 Measures & their properties

1.7 $\mathcal{S} \xrightarrow{\mu} \overline{\mathbb{R}}_{\geq 0}$ is a *measure* on a measurable space $(X, \mathcal{S})$ if $\mu_{\biguplus_{k=1}^{\infty} E_k} = \sum_{k=1}^{\infty} \mu_{E_k}$ $\forall \{E_k \in \mathcal{S}\}_{k=1}^{\infty}$. $(X, \mathcal{S}, \mu)$ is then called a *measure space* ●

Remark. $\mu_{E = E \uplus \emptyset \uplus \emptyset \uplus \dots} = \mu_E + \sum_{k=2}^{\infty} \mu_{\emptyset} \Rightarrow \mu_{\emptyset} = 0$.

1.14 $\forall$*measure space* $(X, \mathcal{S}, \mu)$ $\forall \{E_k \in \mathcal{S}\}_{k=1}^{\infty}$

1. $E_1 \subseteq E_2 \Rightarrow \mu_{E_1} \leq \mu_{E_2} \wedge \mu_{E_2 \setminus E_1} \overset{\mu_{E_1} < \infty}{=\!=} \mu_{E_2} - \mu_{E_1}$
2. $\mu_{\bigcup_{k=1}^{\infty} E_k} \leq \sum_{k=1}^{\infty} \mu_{E_k}$
3. $E_{\forall k \in \mathbb{Z}_{>0}} \subseteq E_{k+1} \Rightarrow \mu_{\bigcup_{k=1}^{\infty} E_k} = \mu_{E_{k \to \infty}}$
4. $E_{\forall k \in \mathbb{Z}_{>0}} \supseteq E_{k+1} \wedge \mu_{E_1} < \infty \Rightarrow \mu_{\bigcap_{k=1}^{\infty} E_k} = \mu_{E_{k \to \infty}}$
5. $\mu_{E = E_1 \cap E_2} < \infty \Rightarrow \mu_{E_1 \cup E_2} = \mu_{E_1} + \mu_{E_2} - \mu_E$ ■

Proof. 1. (a) $\mu_{E_2 = E_1 \uplus (E_2 \setminus E_1)} = \mu_{E_1} + \mu_{E_2 \setminus E_1} \geq \mu_{E_1}$

(b) $\mu_{E_1} < \infty \Rightarrow \mu_{E_2} - \mu_{E_1} \geq \mu_{E_1} - \mu_{E_1} = 0$

2. $\mu_{\bigcup_{k=1}^{\infty} E_k = \biguplus_{k=1}^{\infty} (E_k \setminus D_k)} = \sum_{k=1}^{\infty} \left(\mu_{E_k \setminus D_k} \leq \mu_{E_k} \right)$ with $D_{\forall k \in \mathbb{Z}_{>0}} = \bigcup_{j=1}^{k-1} E_k \overset{k=1}{=\!=} \emptyset$

3. Say $\mu_{E_{\forall k \in \mathbb{Z}_{>0}}} < \infty$, as otherwise both sides of the equation are ∞. Let $E_0 = \emptyset$, $\mu_{\bigcup_{k=1}^{\infty} E_k = \biguplus_{j=1}^{\infty} (E_j \setminus E_{j-1})} = \left(\sum_{j=1}^{\infty} \equiv \sum_{j=1}^{k \to \infty} \right) \left(\mu_{E_j \setminus E_{j-1}} = \mu_{E_j} - \mu_{E_{j-1}} \right) = \mu_{E_{k \to \infty}}$

4. $\mu_{E_1} - \mu_{\bigcap_{k=1}^{\infty} E_k} = \mu_{E_1 \setminus \bigcap_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} (E_1 \setminus E_k)} \overset{\text{property 3}}{=\!=\!=} \mu_{E_1 \setminus E_{k \to \infty}} = \mu_{E_1} - \mu_{E_{k \to \infty}}$

5. $\mu_{E_1 \cup E_2 = [\biguplus_{k=1}^{2} (E_k \setminus E)] \uplus E} = \left[\sum_{k=1}^{2} \left(\mu_{E_k \setminus E} = \mu_{E_k} - \mu_E \right) \right] + \mu_E = \mu_{E_1} + \mu_{E_2} - \mu_E$ □

[viii] $X \xrightarrow{f} \overline{\mathbb{R}}$ is measurable on a measurable space $(X, \mathcal{S}) \Leftarrow f^{-1}_{(\forall a \in \overline{\mathbb{R}}, \infty]} \in \mathcal{S}$

1.4 Lebesgue's measure

1.15 $\mathring{\mu}_{A\uplus B} = \mathring{\mu}_{\forall A\subseteq\mathbb{R}} + \mathring{\mu}_{\forall B\in\mathscr{B}}$ ■

Proof. Need to show $\mathring{\mu}_{A\uplus B} \geq \mathring{\mu}_A + \mathring{\mu}_B$.

1. $\mathring{\mu}_{A\uplus B} = \mathring{\mu}_{\forall A\subseteq\mathbb{R}} + \mathring{\mu}_{\forall \text{open } B\subseteq\mathbb{R}}$ Say $\mathring{\mu}_B < \infty$.

(a) If B is an open interval $(a,b)\subseteq\mathbb{R}$, then $\forall$open cover

$$\underbrace{\left\{a+{}^{(-\epsilon,\epsilon)}\!/_4,\, b+{}^{(-\epsilon,\epsilon)}\!/_4\right\}}_{I_0}\cup\{\underbrace{I_k\cap\mathbb{R}_{<a}}_{J_k}\}_{k=1}^{\infty}\uplus\{\underbrace{I_k\cap(a,b)}_{K_k}\}_{k=1}^{\infty}\uplus\{\underbrace{I_k\cap\mathbb{R}_{>b}}_{L_k}\}_{k=1}^{\infty}$$

of $A\uplus B$, $\sum_{k=0}^{\infty}\ell_{I_k} \xlongequal[I_0\cup\{J_k,L_k\}_{k=1}^{\infty}\supseteq A]{\{K_k\}_{k=1}^{\infty}\supseteq B} \underbrace{\overbrace{\ell_{I_0}}^{\epsilon\to 0}+\sum_{k=1}^{\infty}\left(\ell_{J_k}+\ell_{L_k}\right)}_{\geq\mathring{\mu}_A}+\underbrace{\sum_{k=1}^{\infty}\ell_{K_k}}_{\geq\mathring{\mu}_B} \Rightarrow \mathring{\mu}_{A\uplus B}\geq\mathring{\mu}_A+\mathring{\mu}_B$.

(b) If $B=\biguplus_{k=1}^{\infty}I_k$ for some open sequence $\{I_k\subseteq\mathbb{R}\}_{k=1}^{\infty}$, then
$\mathring{\mu}_{A\uplus B}\geq\underbrace{\mathring{\mu}_{A\biguplus_{k=1}^{\forall z\in\mathbb{Z}_{>0}}I_k}=\mathring{\mu}_A+\sum_{i=1}^{z}\ell_{I_k}}_{\text{by property (a) and induction on } z}\Rightarrow\mathring{\mu}_{A\uplus B}\geq\mathring{\mu}_A+\left(\sum_{k=1}^{\infty}\ell_{I_k}\geq\mathring{\mu}_B\right)$.

2. $\mathring{\mu}_{A\uplus B} = \mathring{\mu}_{\forall A\subseteq\mathbb{R}} + \mathring{\mu}_{\forall \text{closed } B\subseteq\mathbb{R}}$ $\forall$open cover $\{I_k\subseteq\mathbb{R}\}_{k=1}^{\infty}$ of $A\uplus B$,
$\sum_{k=1}^{\infty}\ell_{I_k}\geq\mathring{\mu}_{G=\bigcup_{k=1}^{\infty}I_k=(G\setminus B)\uplus B}\xlongequal[G\setminus B=G\cap(\mathbb{R}\setminus B)\text{ is open}]{\text{step 1}}\mathring{\mu}_{G\setminus B\supseteq A}+\mathring{\mu}_B\geq\mathring{\mu}_A+\mathring{\mu}_B$
$\Rightarrow\mathring{\mu}_{A\uplus B}\geq\mathring{\mu}_A+\mathring{\mu}_B$.

3. $\mathscr{L}:=\left\{L\subseteq\mathbb{R}\,\middle|\,\forall\epsilon>0\,\exists\text{ closed }F\subseteq L:\mathring{\mu}_{L\setminus F}<\epsilon\right\}$ is a σ-algebra containing $\mathbb{R}$'s all closed, and thus all open, all Borel's (and all 0-outer-measure) subsets Since $\mathscr{L}$ ($\ni\varnothing$, as $\varnothing$ is both open and closed) ***is closed under***

Countable intersection $L_0=\bigcap_{k=1}^{\infty}L_k\in\mathscr{L}\ \forall\{L_k\in\mathscr{L}\}_{k=1}^{\infty}\Leftarrow\forall\epsilon>0$
$\exists$ closed $F_{\forall k\in\mathbb{Z}_{>0}}\subseteq L_k:\mathring{\mu}_{L_k\setminus F_k}<{}^{\epsilon}\!/_{2^k}\wedge\mathring{\mu}_{L_0\setminus\left(\text{closed }\bigcap_{k=1}^{\infty}F_k\right)=\bigcup_{k=1}^{\infty}(L_0\setminus F_k)\subseteq\bigcup_{k=1}^{\infty}(L_k\setminus F_k)}<\epsilon$.

Complementation $\forall L\in\mathscr{L}\ \forall\epsilon>0$

(a) If $\mathring{\mu}_L<\infty$, then $\exists$ closed $F\subseteq L\subseteq$ open $G:\epsilon$
$>({}^{\epsilon}\!/_2>\mathring{\mu}_G-\mathring{\mu}_L)+\left({}^{\epsilon}\!/_2>\mathring{\mu}_{L\setminus F}=\mathring{\mu}_L-\mathring{\mu}_F\right)=\mathring{\mu}_G-\mathring{\mu}_F$
$=\mathring{\mu}_{G\setminus F\supseteq G\setminus L=(\mathbb{R}\setminus L\supseteq\mathbb{R}\setminus G)\setminus(\mathbb{R}\setminus G)}\geq\mathring{\mu}_{(\mathbb{R}\setminus L)\setminus(\text{closed }\mathbb{R}\setminus G)}$.

(b) If $\mathring{\mu}_L=\infty$, $\mathring{\mu}_{L_{\forall k\in\mathbb{Z}_{>0}}=L\cap[-k,k]\in\mathscr{L}}<\infty$
$\xRightarrow{\text{step (a)}}\mathbb{R}\setminus L_{\forall k\in\mathbb{Z}_{>0}}\in\mathscr{L}\Rightarrow\mathbb{R}\setminus L=\bigcap_{k=1}^{\infty}(\mathbb{R}\setminus L_k)\in\mathscr{L}$.

4. $\forall\epsilon>0\,\exists$ closed $F\subseteq B:\mathring{\mu}_{B\setminus F}<\epsilon\wedge\mathring{\mu}_{A\uplus B}\geq\mathring{\mu}_{A\uplus F}=\mathring{\mu}_A+\left(\mathring{\mu}_F=\mathring{\mu}_B-\mathring{\mu}_{B\setminus F}\geq\mathring{\mu}_B\right)$ □

1.16 $\exists B\subseteq\mathbb{R}:\mathring{\mu}_B<\infty\wedge B$ *is not* Borel*'s set* ■

Proof. By theorems 1.3, 1.15 □

1.17 $(\mathbb{R},\mathscr{B},\mathring{\mu})$ *is a measure space* ■

Proof. $\forall\{B_k\in\mathscr{B}\}_{k=1}^{\infty}$, $\mathring{\mu}_{\biguplus_{k=1}^{\infty}B_k}\geq\underbrace{\mathring{\mu}_{\biguplus_{k=1}^{\forall z\in\mathbb{Z}_{>0}}B_k}=\sum_{k=1}^{z}\mathring{\mu}_{B_k}}_{\text{by theorem 1.15 and induction on } z}\Rightarrow\mathring{\mu}_{\biguplus_{k=1}^{\infty}B_k}\geq\sum_{k=1}^{\infty}\mathring{\mu}_{B_k}$ □

1.8 $A\subseteq\mathbb{R}$ is <u>Lebesgue-*measurable*</u>

$\iff$ $\exists B^-\in\mathscr{B}:B^-\subseteq A\wedge\mathring{\mu}_{A\setminus B^-}=0$

$\iff$ $\forall\epsilon>0\,\exists$ closed $F\subseteq A:\mathring{\mu}_{A\setminus F}<\epsilon$

$\iff \exists\{\text{closed } F_k \subseteq A\}_{k=1}^{\infty} : \mathring{\mu}_{A \setminus \bigcup_{k=1}^{\infty} F_k} = 0$

$\iff \exists\{\text{open } G_k \supseteq A\}_{k=1}^{\infty} : \mathring{\mu}_{\bigcap_{k=1}^{\infty} G_k \setminus A} = 0$

$\iff \forall \epsilon > 0 \; \exists \text{ open } G \supseteq A : \mathring{\mu}_{G \setminus A} < \epsilon$

$\iff \exists B^+ \in \mathscr{B} : B^+ \supseteq A \wedge \mathring{\mu}_{B^+ \setminus A} = 0$

$\iff \mathring{\mu}_{(-n,n) \cap A} + \mathring{\mu}_{(-n,n) \setminus A} = 2n \; \forall n \in \mathbb{Z}_{>0}$ ■

Proof. $\mathring{\mu}_{A \setminus B^-} = 0 = \mathring{\mu}_{B^+ \setminus A} \xRightarrow[(A \setminus B^-) \uplus B^- = A = B^+ \cap \left[\mathbb{R} \setminus (B^+ \setminus A)\right]]{B^{\pm} \in \mathscr{B} \subseteq \mathscr{L}} A \setminus B^- \;\&\; B^+ \setminus A \;\&\; A \;\&\; \mathbb{R} \setminus A \in \mathscr{L}$

$\xRightarrow{\exists F \subseteq \bigcup_{k=1}^{\infty} F_k \subseteq A \subseteq \bigcap_{k=1}^{\infty} G_k \subseteq G} \mathring{\mu}_{A \setminus F} \;\&\; \mathring{\mu}_{G \setminus A = (\mathbb{R} \setminus A \supseteq \mathbb{R} \setminus G) \setminus (\mathbb{R} \setminus G)} < \epsilon \to 0^+$ *etc.* □

Remark. The σ-algebra $\mathscr{L}$ in theorem 1.15.3 is the collection of $\mathbb{R}$'s all $\mathscr{L}$-measurable subsets.

1.18 *$(\mathbb{R}, \mathscr{L}, \mathring{\mu})$ is a measure space (dubbed* LEBESGUE*'s)* ■

Proof. $\forall \{L_k \in \mathscr{L}\}_{k=1}^{\infty} \; \exists \{B_k \in \mathscr{B} \mid L_k = B_k \uplus (L_k \setminus B_k)\}_{k=1}^{\infty} : \mathring{\mu}_{L_{\forall k \in \mathbb{Z}_{>0}} \setminus B_k} = 0$

$\wedge \; \mathring{\mu}_{\biguplus_{k=1}^{\infty} L_k} \geq \mathring{\mu}_{\biguplus_{k=1}^{\infty} B_k} \xlongequal{\text{theorem 1.17}} \sum_{k=1}^{\infty} \left(\mathring{\mu}_{B_k} \geq \mathring{\mu}_{L_k}\right)$ □

Remark. $\forall A \subseteq \mathbb{R}$ with $\mathring{\mu}_A < \infty$, $A \in \mathscr{L} \iff \forall \epsilon > 0 \; \exists G = \biguplus_{k=1}^{n < \mathbb{Z}_{>0}} G_k$ with $G_{k=1,\ldots,n}$ bounded open intervals: $\mathring{\mu}_{A \setminus G} + \mathring{\mu}_{G \setminus A} < \epsilon$. Practically, this means that every $B \in \mathscr{B}$ with $\mathring{\mu}_B < \infty$ is almost a finite disjoint union of bounded open intervals.

1.5 Convergence of measurable maps

1.5.1 Pointwise convergence is almost uniform convergence

1.9 $\left\{X \xrightarrow{f_k} \mathbb{R}\right\}_{k=1}^{\infty}$ *converges* to $X \xrightarrow{f} \mathbb{R}$

***Pointwise** (on X)* if $f_{k \to \infty; \forall x \in X} = f_x$

Uniformly if $\forall \epsilon > 0 \; \exists n \in \mathbb{Z}_{>0} : \left| f_{\forall k \geq n; \forall x \in X} - f_x \right| < \epsilon$ ●

E.g. $\left\{[-1,1] \xrightarrow{f_k} \mathbb{R} \,\middle|\, f_{k;x} = \begin{cases} 1 - k|x| & \text{if } |x| \in [0, 1/k] \\ 0 & \text{if } |x| \in (1/k, 1] \end{cases} \right\}_{k=1}^{\infty}$ converges pointwise but not uniformly to $[-1,1] \xrightarrow{f : x \mapsto \delta_{0,x}} \mathbb{R}$.

1.19 $\left\{X \xrightarrow{f_k} \mathbb{R} \,\middle|\, f_{\forall j \in \mathbb{Z}_{>0}} \text{ continuous at } x \in X\right\}_{k=1}^{\infty}$ *converges uniformly to* $X \xrightarrow{f} \mathbb{R}$ $\Rightarrow f$ *continuous at* x ■

Proof. $\forall \epsilon > 0 \; \exists \delta > 0 : \left| f_{\forall x' \in (x-\delta, x+\delta) \cap X} - f_x \right| < \epsilon$, because

$\left| f_{x'} - f_x \right| \leq \left| f_{x'} - f_{j;x'} \right| + \left| f_{j;x'} - f_{j;x} < \epsilon' \right| + \left| f_{j;x} - f_x \right| \; \forall j \in \mathbb{Z}_{>0} \; \forall \epsilon' \in (0, \epsilon)$

$\xRightarrow{\left| f_{\exists n \in \mathbb{Z}_{>0}; \forall x'' \in X} - f_{x''} \right| < (\epsilon - \epsilon')/2} \left| f_{x'} - f_x \right| < \left| f_{x'} - f_{n';x'} \right| + \epsilon' + \left| f_{n;x} - f_x \right| < \epsilon$ □

Theorem (EGOROV's) $\forall$*measure* $\mathcal{S} \xrightarrow{\mu} \mathbb{R}_{\geq 0}$ *on a measurable space* $(X, \mathcal{S})$ $\exists E \subseteq X$: $\mu_{X \setminus E} \in [0, \forall \epsilon > 0) \wedge \left\{\mathcal{S}\text{-measurable } X \xrightarrow{f_k} \mathbb{R}\right\}_{k=1}^{\infty}$ *converges to* $X \xrightarrow{f : x \mapsto f_{k \to \infty; x}} \mathbb{R}$ *uniformly on* E ■

Proof. $f_{k\to\infty;\forall x\in X} = f_x \xRightarrow{g_k \equiv f_k - f} \bigcup_{m=1}^{\infty}\left(A_{m;\forall n\in\mathbb{Z}_{>0}} := \bigcap_{k=m}^{\infty} g^{-1}_{k;\left(-1/n,\,1/n\right)}\right) = X$, where $A_{m;n}\in\mathcal{S}$ as $X \xrightarrow{g_{\forall k\in\mathbb{Z}}} \mathbb{R}$ is $\mathcal{S}$-measurable (by theorems 1.12, 1.11), and $\{A_{m,n}\}_{m=1}^{\infty}$ is an increasing sequence $\xRightarrow{\text{theorem 1.14.3}} \mu_X = \mu_{A_{m\to\infty;n}}$; *i.e.* $\mu_X - \mu_{\exists m_n \in\mathbb{Z}_{>0}} < \epsilon/2^n$. Thus $\mu_{X\setminus\left(E=\bigcap_{n=1}^{\infty} A_{m_n;n}\right)=\bigcup_{n=1}^{\infty}\left(X\setminus A_{m_n;n}\right)} \le \sum_{n=1}^{\infty}\mu_{X\setminus A_{m_n;n}} < \epsilon$, and $\{f_k\}_{k=1}^{\infty}$ converges to f uniformly on $E\subseteq A_{m_n;\forall n\in\mathbb{Z}_{>0}}$, as $\forall\epsilon'>0\ \exists n\in\mathbb{Z}_{>0} : \left|g_{\forall k\in\mathbb{Z}_{>0};\forall x\in E}\right| < 1/n < \epsilon'$ □

1.5.2 Approximation by simple maps

1.10 A map is *simple* if it takes only finitely many values ●

E.g. A simple $X \xrightarrow{f=\sum_{k=1}^{n} c_k\chi_{E_k}} \mathbb{R}$ (measurable) on a measurable space $(X,\mathcal{S})$, with $c_{k=1,\dots,n}$ the distinct values $\in\mathbb{R}_{\neq 0}$ of f, and $E_{k=1,\dots,n} = f^{-1}_{\{c_k\}}\in\mathcal{S}$.

1.20 $\forall$ *measurable* $X\xrightarrow{f}\overline{\mathbb{R}}$ *on a measurable space* $(X,\mathcal{S})$ $\exists\left\{\text{simple } \mathcal{S}\text{-measurable} X \xrightarrow{f_k}\mathbb{R}\ \middle|\ \left|f_{\forall j\in\mathbb{Z}_{>0};\forall x\in X}\right| \le \left|f_{j+1;x}\right| \le \left|f_x\right|\right\}_{k=1}^{\infty}$ *converging pointwise (uniformly for bounded f) to f* ■

E.g. $\left\{f_{k;\forall x\in X} = \left(\left|f_{k;x}\right| = \begin{cases} m/2^k & \text{if } \exists m\in\mathbb{Z} : \left|f_x\right|\in[0,k]\cap [m,m+1)/2^k \\ k & \text{if } \left|f_x\right|\in(k,\infty)\end{cases}\right)\text{sign}_{f_x}\right\}_{k=1}^{\infty}$ is a desired sequence of simple $\mathcal{S}$-measurable ($f^{-1}_{[0,k]\cap [m,m+1)/2^k}\in\mathcal{S}$ *etc.* $\Leftarrow$ $\mathcal{S}$-measurable f) maps: $\left|f_{\forall k\in\mathbb{Z}_{>0};\forall x\in X} - f_x\right| \le 1/2^k$ if $\left|f_x\right|\in[0,k]$.

1.21 $\forall$ *continuous* $F\xrightarrow{f}\mathbb{R}$ *on a closed* $F\subseteq\mathbb{R}$ $\exists$ *continuous* $\mathbb{R}\xrightarrow{\bar f}\mathbb{R} : \bar f\big|_F = f$ ■

E.g. $\exists\{\text{open interval } I_k\}_{k=1}^{\infty} : \mathbb{R}\setminus F = \biguplus_{k=1}^{\infty} I_k$. $\bar f\big|_{I_k} := f_a \ \vee :=$ linear map connecting f_b & f_c for $I_{k\in\mathbb{Z}_{>0}} = \pm(a,\infty) \vee = (b,c)$.

1.5.3 Borel's measurability is almost continuity

Theorem (Lusin's) $\mathscr{B}$-*measurable* $E\xrightarrow{f}\mathbb{R} \Rightarrow \forall\epsilon>0\ \exists$

- *closed* $F\subseteq\mathbb{R} : \mathring{\mu}_{E\setminus F} < \epsilon$
- *continuous* $\mathbb{R}\xrightarrow{\bar f}\mathbb{R} : \bar f\big|_F = f\big|_F$ ■

Proof. 1. Prove the theorem for $\left(E\xrightarrow{f}\mathbb{R}\right) = \left(\mathbb{R}\xrightarrow{f}\mathbb{R}\right)$ 1st.

(a) Say $f = \sum_{k=1}^{n} c_k\chi_{B_k} \xlongequal[B_0=\mathbb{R}\setminus\bigcup_{k=1}^{n} B_k\in\mathscr{B}]{c_0=0} \sum_{k=0}^{n} c_k\chi_{B_k}$ of distinct $c_{k=1,\dots,n}\in\mathbb{R}_{\neq 0}$ and disjoint $B_{k=0,\dots,n}\in\mathscr{B}$. $\forall\epsilon>0$, theorem 1.8 $\Rightarrow$ '$\forall k\in\{1,\dots,n\}\ \exists$ closed $F_k\subseteq B_k\subseteq$ open $G_k : \mathring{\mu}_{G_k\setminus B_k} < \epsilon/2n > \mathring{\mu}_{B_k\setminus F_k} \wedge \mathring{\mu}_{G_k\setminus F_k=(G_k\setminus B_k)\uplus(B_k\setminus F_k)} < \epsilon/n$'

$\Rightarrow$ closed $F \xlongequal{F_0=\mathbb{R}\setminus\bigcup_{k=1}^{n} G_k} \biguplus_{k=0}^{n} F_k : \mathring{\mu}_{\mathbb{R}\setminus F\subseteq\bigcup_{k=1}^{n}(G_k\setminus F_k)} < \epsilon$

$\wedge\ f\big|_F$ continuous (as $f\big|_{F_{\forall k\in\{0,\dots,n\}}\subseteq B_k} \equiv c_k$ is continuous)

(b) $\forall\mathscr{B}$-measurable $\mathbb{R}\xrightarrow{f}\mathbb{R}$

i. Theorem 1.20 $\Rightarrow \exists\left\{\text{simple } \mathscr{B}\text{-measurable}\mathbb{R}\xrightarrow{f_k}\mathbb{R}\right\}_{k=1}^{\infty} : f_{k\to\infty;\forall x\in X} = f_x$. $\forall\epsilon>0$, step 1.(a) $\Rightarrow$ '$\forall k \in \mathbb{Z}_{>0}\ \exists$ closed $C_k \subseteq \mathbb{R} : \mathring{\mu}_{\mathbb{R}\setminus C_k} < {}^{\epsilon}/_{2^{k+1}} \wedge f_k\big|_{C_k}$ continuous'

$\Rightarrow f_{\forall k\in\mathbb{Z}_{>0}}\big|_{C=\bigcap_{j=1}^{\infty} C_j}$ continuous: $\mathring{\mu}_{(\mathbb{R}\setminus C)=\bigcup_{k=1}^{\infty}(\mathbb{R}\setminus C_k)} < {}^{\epsilon}/_{2}$.

ii. $\forall n \in \mathbb{Z}_{>0}$, $\quad f_{k\to\infty;\forall x\in(n,n+1)} = f_x \xRightarrow{\text{Egorov's theorem}} \exists\, B_n \in \mathscr{B} : \quad \mathring{\mu}_{(n,n+1)\setminus(B_n\subseteq(n,n+1))}$ $< {}^{\epsilon}/_{2^{|n|+3}} \wedge \left\{f_k\big|_{B_n}\right\}_{k=1}^{\infty}$ converges to $f\big|_{B_n}$ uniformly on $C \cap B_n$.

iii. $f_{\forall k\in\mathbb{Z}_{>0}}\big|_{(C\cap B_{\forall n\in\mathbb{Z}_{>0}})\subseteq C\subseteq C_k}$ continuous $\xRightarrow{\text{theorem 1.19}} (f = f_{k\to\infty})\big|_{C\cap B_n}$ continuous

$\Rightarrow f\big|_{D=\bigcup_{n\in\mathbb{Z}_{>0}}(C\cap B_n)}$ continuous, where theorem 1.8

$\Rightarrow \mathring{\mu}_{D\setminus\exists \text{ closed } F\subseteq D\in\mathscr{L}} < \underbrace{\epsilon - \mathring{\mu}_{\mathbb{R}\setminus D=(\mathbb{R}\setminus C)\cup\left[\mathbb{R}\setminus\left(\bigcup_{n\in\mathbb{Z}_{>0}} B_n\right)\subseteq\mathbb{Z}_{>0}\cup\left(\bigcup_{n\in\mathbb{Z}_{>0}}(n,n+1)\setminus B_n\right)\right]}}_{>0}$

$\wedge\ \mathring{\mu}_{\mathbb{R}\setminus F=(\mathbb{R}\setminus D)\uplus(D\setminus F)} = \mathring{\mu}_{\mathbb{R}\setminus D} + \mathring{\mu}_{D\setminus F} < \epsilon \wedge f\big|_{F\subseteq D}$ continuous.

2. $\forall \epsilon > 0$, consider an extension $\mathbb{R} \xrightarrow{\tilde{f}:=\chi_E\cdot f} \mathbb{R}$ of $E \xrightarrow{f} \mathbb{R}$, then step 1

$\Rightarrow$ '$\exists$ closed $C \subseteq \mathbb{R} : \mathring{\mu}_{\mathbb{R}\setminus C} < \epsilon \wedge \tilde{f}\big|_C$ continuous'

$\Rightarrow$ '$\exists$ closed $F \subseteq C \cap E : \mathring{\mu}_{(C\cap E)\setminus F} < \underbrace{\epsilon - \mathring{\mu}_{\mathbb{R}\setminus C}}_{>0} \wedge \mathring{\mu}_{E\setminus F=\left[(C\cap E)\setminus F\right]\uplus[(E\setminus C)\subseteq(\mathbb{R}\setminus C)]} < \epsilon$

$\wedge\ \tilde{f}\big|_{F\subseteq E} = f\big|_F$ continuous'

$\xRightarrow{\text{theorem 1.21}} \exists$ continuous $\mathbb{R} \xrightarrow{\bar{f}} \mathbb{R} : \bar{f}\big|_F = f$ $\square$

Remark. $\biguplus_{k=1,\dots,n} F_k \xrightarrow{f} \mathbb{R}$ with closed $F_{k=1,\dots,n} \subseteq \mathbb{R}$ and continuous $f\big|_{F_{k=1,\dots,n}}$ is continuous.

1.5.4 Lebesgue's measurability is almost Borel's measurability

1.11 $\forall X \subseteq \mathbb{R}$, $X \xrightarrow{f} \mathbb{R}$ is Lebesgue-*measurable* if $f^{-1}_{\forall B\in\mathscr{B}} \in \mathscr{L}$

1.22 $\forall \mathscr{L}$-*measurable* $\mathbb{R} \xrightarrow{f} \mathbb{R}\ \exists\ \mathscr{B}$-*measurable* $\mathbb{R} \xrightarrow{g} \mathbb{R} : \mathring{\mu}_{\{x\in\mathbb{R}\,|\,g_x\neq f_x\}} = 0$

Proof. $\mathscr{L}$-measurable $\mathbb{R} \xrightarrow{f} \mathbb{R} \xRightarrow{\text{theorem 1.20}} \exists\left\{\text{simple } \mathscr{L}\text{-measurable} \mathbb{R} \xrightarrow{f_k} \mathbb{R}\right\}_{k=1}^{\infty}$: $f_{k\to\infty;\forall x\in X} = f_x \wedge f_{\forall k\in\mathbb{Z}_{>0}} = \sum_{j=1}^{\exists n\in\mathbb{Z}_{>0}} c_j\chi_{A_j}$ of distinct $c_{j=1,\dots,n} \in \mathbb{R}_{\neq 0}$ and disjoint $A_{j=1,\dots,n} \in \mathscr{L}$. Theorem 1.8 $\Rightarrow$ '$\forall j \in \{1,\dots,n\}\ \exists\, B_j \in \mathscr{B} : \mathring{\mu}_{A_j\setminus(B_j\subseteq A_j)} = 0$'

$\Rightarrow \mathscr{B}$-measurable $g_{\forall k\in\mathbb{Z}_{>0}} = \sum_{j=1}^{n} c_j\chi_{B_j} : \mathring{\mu}_{\epsilon_k=\{x\in\mathbb{R}\,|\,g_{k;x}\neq f_{k;x}\}} = 0$. Thus $g_{k\to\infty;\forall x\in E} = f_x$ with $\mathring{\mu}_{\mathbb{R}\setminus\left(E=\{x\in\mathbb{R}\,|\,\exists g_{k\to\infty;x}\}\right)\subseteq\bigcup_{k=1}^{\infty}\epsilon_k} = 0 \Rightarrow \exists g_{\forall x\in\mathbb{R}} = (\chi_E\cdot g_{k\to\infty})_x$

$\xRightarrow[\text{theorem 1.12}]{\mathscr{B}\text{-measurable } (\chi_E\cdot g_{\forall k\in\mathbb{Z}_{>0}})} \mathscr{B}$-measurable $g : \mathring{\mu}_{\{x\in\mathbb{R}\,|\,g_x\neq f_x\}\subseteq\bigcup_{k=1}^{\infty}\epsilon_k} = 0$ $\square$

2 Integration

2.1 Integration with respect to a measure

2.1.1 Integration of nonnegative maps

2.1 $\left\{A_j \in \mathcal{S} \,\middle|\, \biguplus_{k=1}^m A_k = X\right\}_{j=1}^{m\in\mathbb{Z}_{>0}}$ is an *$\mathcal{S}$-partition* of a measurable space $(X,\mathcal{S})$

2.2 The *integral* $\int f \,d\mu := \sup_{\mathcal{P}}\underbrace{\left\{\mathcal{L}_{f,\mathcal{P}=\{A_j\}_{j=1}^m} := \sum_{j=1}^m \mu_{A_j} \inf_{A_j} f\right\}}_{\text{Lebesgue's lower sum}}{}_{\mathcal{S}\text{-partition } \mathcal{P} \text{ of } X}$ of a measurable $X \xrightarrow{f} \overline{\mathbb{R}}_{\geq 0}$ on the measure space $(X,\mathcal{S},\mu)$[ix]

2.1 $\int \chi_E \,d\mu = \mu_E$ $\forall$ *measure space* $(X, \mathcal{S} \ni E, \mu)$

Proof. $\int \chi_E \,d\mu \geq \mathcal{L}_{\chi_E, \exists\mathcal{S}\text{-partition } \{E, X\backslash E\} \text{ of } X} = \mu_E \geq \mu_{\biguplus_{j=1, A_j\subseteq E}^m A_j} = \sum_{j=1, A_j\subseteq E}^m \mu_{A_j}$

$$= \sum_{j=1}^m \left(\mu_{A_j} \inf_{A_j} \chi_E = \begin{cases} \mu_{A_j} & \text{if } A_j \subseteq E \\ 0 & \text{if } A_j \backslash E \neq \varnothing \end{cases}\right) = \mathcal{L}_{\chi_E, \forall \mathcal{S}\text{-partition } \{A_j\}_{j=1}^m \text{ of } X} \qquad \square$$

E.g. $\forall$ Lebesgue's measure $\mathring{\mu}$ on X, $\int \chi_{\mathbb{Q}} \,d\mathring{\mu} = \mathring{\mu}_{\mathbb{Q}} = 0$, $\int \chi_{[0,1]\backslash\mathbb{Q}} \,d\mathring{\mu} = \mathring{\mu}_{[0,1]\backslash\mathbb{Q}} = 1$.

E.g. $\int b \,d\mu = \sum_{k=1}^\infty b_k$, with $\mathbb{Z}_{>0} \xrightarrow{b:k\mapsto b_k} \mathbb{R}_{\geq 0}$, and μ the counting measure on $\mathbb{Z}_{>0}$.[x]

2.2 $\int \left(\sum_{k=1}^n c_k \chi_{E_k}\right) d\mu \overset{\forall\{c_k \in \overline{\mathbb{R}}_{\geq 0}\}_{k=1}^n}{\underset{\forall \text{disjoint sequence } \{E_k \in \mathcal{S}\}_{k=1}^n}{=\!=\!=}} \sum_{k=1}^n c_k \mu_{E_k}$ $\forall$ *measure space* $(X,\mathcal{S},\mu)$

Proof. Define $f := \sum_{k=0}^n c_k \chi_{E_k}$, with $\{E_0 := X \backslash \biguplus_{k=1}^n E_k\} \cup \{E_k\}_{k=1}^n$ an $\mathcal{S}$-partition of X, and set $c_0 \equiv 0$. Then

$$\sum_{k=1}^n c_k \mu_{E_k} \equiv \sum_{k=0}^n c_k \mu_{E_k} = \mathcal{L}_{f,\{E_k\}_{k=0}^n} \leq \int \left(f \equiv \sum_{k=1}^n c_k \chi_{E_k}\right) d\mu$$
$$= \mathcal{L}_{f, \exists\mathcal{S}\text{-partition } \left\{A_j = \left[B_j = \biguplus_{k=1}^n \left(B_{j,k} = A_j \cap E_k\right)\right] \uplus \left[A_j \backslash B_j\right]\right\}_{j=1}^m \text{ of } X}$$
$$= \sum_{j=1}^m \left\{\left[\mu_{A_j} \overset{\mu_\varnothing = 0}{=\!=\!=} \begin{cases} \sum_{k=1, B_{j,k}\neq\varnothing}^n \mu_{B_{j,k}} + \mu_{A_j\backslash B_j} & \text{if } A_j \backslash B_j \neq \varnothing \\ \sum_{k=1, B_{j,k}\neq\varnothing}^n \mu_{B_{j,k}} & \text{if } A_j \backslash B_j = \varnothing \end{cases}\right]\right.$$
$$\left.\times\left[\inf_{A_j} f \overset{(A_j\backslash B_j)\cap\left(E_{\forall k\in\{1,\ldots,n\}} = \biguplus_{j=1}^m B_{j,k}\right) = \varnothing}{=\!=\!=\!=\!=\!=\!=} \begin{cases} 0 & \text{if } A_j \backslash B_j \neq \varnothing \\ \min_{i\in\{1,\ldots,n\}, B_{j,i}\neq\varnothing} c_i & \text{if } A_j \backslash B_j = \varnothing \end{cases}\right]\right\}$$
$$= \sum_{j=1, A_j\backslash B_j = \varnothing}^m \left[\left(\sum_{k=1, B_{j,k}\neq\varnothing}^n \mu_{B_{j,k}}\right) \min_{i\in\{1,\ldots,n\}, B_{j,i}\neq\varnothing} c_i \leq \sum_{k=1, B_{j,k}\neq\varnothing}^n \mu_{B_{j,k}} c_k\right]$$
$$\leq \sum_{j=1}^m \sum_{k=1}^n \mu_{B_{j,k}} c_k = \sum_{k=1}^n c_k \left(\sum_{j=1}^m \mu_{B_{j,k}} = \mu_{\biguplus_{j=1}^m B_{j,k} = E_k}\right) \qquad \square$$

2.3 $\int f \,d\mu \leq \int g \,d\mu$ $\forall X \xrightarrow{f,g} \overline{\mathbb{R}}_{\geq 0}$ *measurable on a measure space* $(X,\mathcal{S},\mu)$: $f_{\forall x\in X} \leq g_x$ ($\Rightarrow \inf_{A_{\forall j\in\{1,\ldots,m\}}} f \leq \inf_{A_j} g \Rightarrow \mathcal{L}_{f,\mathcal{P}} \leq \mathcal{L}_{g,\mathcal{P}}$, $\forall \mathcal{S}$*-partition* $\mathcal{P} = \{A_j\}_{j=1}^m$ *of* X)

2.1.2 Monotone convergence theorem about limits & integrals

2.4 $\int f \,d\mu = \sup_{S = \left\{\sum_{j=1}^m (c_j \in \mathbb{R}_{\geq 0}) \mu_{A_j \in \mathcal{S}} \,\middle|\, A_{j=1,\ldots,m} \text{ are disjoint} \wedge f_{\forall x\in X} \geq \sum_{j=1}^m c_j \chi_{A_j; x}\right\}}$

[ix] $\infty \cdot 0 := 0 =: 0 \cdot \infty$

[x] The *counting measure* μ on a measurable space $(X,\mathcal{S})$ counts the number of elements in $E \in \mathcal{S}$; *i.e.* $\mu_E := |E|$

$\forall$ *measurable* $X \xrightarrow{f} \overline{\mathbb{R}}_{\geq 0}$ *on a measure space* $(X, \mathcal{S}, \mu)$ ■

Proof. 1. $\underbrace{\int f \, d\mu \geq \int \Big(\sum_{j=1}^m c_j \chi_{A_j}\Big)}_{\text{theorem 2.3}} \overbrace{d\mu = \sum_{j=1}^m c_j \mu_{A_j}}^{\text{theorem 2.2}}$.

2. *C.f.* definition 2.1. (a) $\inf_{\forall A \in \mathcal{S}: \mu_A > 0} f < \infty \Rightarrow$ '$\forall \mathcal{S}$-partition $\mathcal{P} = \Big\{A_j \in \mathcal{S} \backslash \{\varnothing\}\Big\}_{j=1}^m$ of X, taking $c_j = \inf_{A_j} f$ shows that $\mathcal{L}_{f,\mathcal{P}} \in S \xRightarrow{\text{definition of } \int f d\mu} \sup_S \geq \int f \, d\mu$'.
(b) $\inf_{\exists A \in \mathcal{S}: \mu_A > 0} f = \infty \Rightarrow$ '$\forall t \in \mathbb{R}_{>0}$, taking $\Big\{A_j\Big\}_{j=1}^{m=1} = \{A\}$ and $c_1 = t$ shows that $\sup_S \geq t\mu_A = \infty \geq \int f \, d\mu$' □

Theorem (monotone convergence) $\forall \Big\{X \xrightarrow{f_k} \overline{\mathbb{R}}_{\geq 0} \,\Big|\, f_k \leq f_{k+1} \wedge f_{k;\forall x \in X} \xrightarrow{k \to \infty} f_x\Big\}_{k=1}^{\infty}$ *of measurable maps on a measure space* $(X, \mathcal{S}, \mu)$, $\int f_k \, d\mu \xrightarrow{k \to \infty} \int f \, d\mu$ ■

Proof. 1. Theorem 1.13 $\Rightarrow X \xrightarrow{f} \overline{\mathbb{R}}_{\geq 0}$ is measurable $\xRightarrow[\text{theorem 2.3}]{f_{\forall k \in \mathbb{Z}_{>0}; \forall x \in X} \leq f_x} \int f_{\forall k \in \mathbb{Z}_{>0}} \, d\mu \leq \int f \, d\mu \Rightarrow \lim_{k \to \infty} \int f_k \, d\mu \leq \int f \, d\mu$

2. $\forall \Big\{c_j \in \mathbb{R}_{\geq 0}\Big\}_{j=1}^m \forall \Big\{A_j \in \mathcal{S} \,\Big|\, A_{j=1,\ldots,m}$ are disjoint $\wedge\, f_{\forall x \in X} \geq \sum_{j=1}^m c_j \chi_{A_j;x}\Big\}_{j=1}^m$
$\forall t \in (0,1)$, $E_{k \in \mathbb{Z}_{>0}} := \Big\{x \in X \,\Big|\, f_{k;x} > t \sum_{j=1}^m c_j \chi_{A_j;x} \wedge \bigcup_{j \in \mathbb{Z}_{>0}} E_j = X\Big\} \subseteq E_{k+1} \in \mathcal{S}$
$\xRightarrow{\text{theorem 1.14}} \mu_{A_j \cap E_k} \xrightarrow{k \to \infty} \mu_{A_j}$. Then $f_{\forall k \in \mathbb{Z}_{>0}; \forall x \in X} \geq t \sum_{j=1}^m c_j \chi_{A_j \cap E_k;x}$
$\xRightarrow{\text{theorem 2.4}} \int f_{\forall k \in \mathbb{Z}_{>0}} \, d\mu \geq t \sum_{j=1}^m c_j \mu_{A_j \cap E_k} \xRightarrow{k \to \infty} \lim_{k \to \infty} \int f_k \, d\mu \geq t \sum_{j=1}^m c_j \mu_{A_j}$
$\xrightarrow{t \to 1} \sum_{j=1}^m c_j \mu_{A_j} \xrightarrow{\text{taking supremum over S in theorem 2.4}} \int f \, d\mu$ □

2.5 $\forall$ *measure space* $(X, \mathcal{S}, \mu)$, $f = \sum_{j=1}^m \Big(a_j \in \overline{\mathbb{R}}_{\geq 0}\Big) \chi_{A_j \in \mathcal{S}} = \sum_{k=1}^n \Big(b_k \in \overline{\mathbb{R}}_{\geq 0}\Big) \chi_{B_k \in \mathcal{S}} = g$
$\Rightarrow \sum_{j=1}^m a_j \mu_{A_j} = \sum_{k=1}^n b_k \mu_{B_k}$ ■

Proof. 1. Say $\bigcup_{j=1}^m A_j = X$.[xi] $\forall$nondisjoint pairs $A'_{k=1,2} \in \Big\{A_j\Big\}_{j=1}^m$, repeat the decomposition
$$\begin{cases} \bigcup_{j=1}^2 A'_j = \underbrace{(A'_1 \backslash A'_2) \uplus (A'_1 \cap A'_2)}_{A'_1} \uplus (A'_2 \backslash A'_1), \quad \overbrace{(A'_1 \cap A'_2) \uplus (A'_2 \backslash A'_1)}^{A'_2} \\ \sum_{j=1}^2 a_j \chi_{A'_j} = a_1 \chi_{A'_1 \backslash A'_2} + (a_1 + a_2) \chi_{A'_1 \cap A'_2} + a_2 \chi_{A'_2 \backslash A'_1} \\ \sum_{j=1}^2 a_j \mu_{A'_j} = a_1 \mu_{A'_1 \backslash A'_2} + (a_1 + a_2) \mu_{A'_1 \cap A'_2} + a_2 \mu_{A'_2 \backslash A'_1} \end{cases}$$
for finite steps, one can convert the initial sets A into disjoint ones with modified coefficients a but unchanged value of '$\sum a\mu_A$'.

2. Replace the sets A corresponding to each modified a from step 1 by $\bigcup A$, μ's finite additivity $\Rightarrow$ '$\sum a\mu_A$'s value remains unchanged when making the coefficients a distinct.

3. Drop any terms for which $A = \varnothing$, getting f's standard[xii] representation with '$\sum a\mu_A$'s value unchanged. Finally, applying the same procedure to g shows that $f = g$ iff $\sum a\mu_A = \sum b\mu_B$. □

[xi]Otherwise add the term $0 \cdot \chi_{X \backslash \bigcup_{j=1}^m A_j}$ to the simple map $X \xrightarrow{f} \overline{\mathbb{R}}_{\geq 0}$

[xii]The representation $\sum_{k=1}^n c_k \chi_{E_k}$ $\forall$simple map $X \xrightarrow{h} \overline{\mathbb{R}}_{\geq 0}$ on a measurable space $(X, \mathcal{S})$ is *standard* if $c_{k=1,\ldots,n} \in \overline{\mathbb{R}}_{\geq 0}$ are disjoint $\wedge \Big\{E_k = h^{-1}_{\{c_k\}} \neq \varnothing\Big\}_{k=1}^n$ is an $\mathcal{S}$-partition of X

2.6 $\int\left(\sum_{k=1}^{n} c_k \chi_{E_k}\right) d\mu \xlongequal[\forall\{E_k\in\mathcal{S}\}_{k=1}^{n}]{\forall\{c_k\in\overline{\mathbb{R}}_{\geq 0}\}_{k=1}^{n}} \sum_{k=1}^{n} c_k \mu_{E_k}$ *$\forall$ measure space* $(X,\mathcal{S},\mu)$ ■

Proof. Apply theorems 2.2 & 2.5 on the standard representation of $\sum_{k=1}^{n} c_k\chi_{E_k}$ □

2.7 $\int (f+g)\, d\mu = \int f\, d\mu + \int g\, d\mu$ *$\forall$ measurable* $X \xrightarrow{f,g} \overline{\mathbb{R}}_{\geq 0}$ *on a measure space* $(X,\mathcal{S},\mu)$ ■

Proof. Theorem 1.20 $\Rightarrow \exists$ increasing sequences $\left\{X \xrightarrow{f_k} \overline{\mathbb{R}}_{\geq 0}\right\}_{k=1}^{\infty}$ & $\left\{X \xrightarrow{g_k} \overline{\mathbb{R}}_{\geq 0}\right\}_{k=1}^{\infty}$ of simple maps measurable on $(X,\mathcal{S},\mu)$: $f_{\forall x\in X} = f_{k\to\infty;x}$ & $g_{\forall x\in X} = g_{k\to\infty;x}$. Then $\int (f+g)\, d\mu \xleftarrow[\text{monotone convergence theorem}]{\infty\leftarrow k} \int (f_k+g_k)\, d\mu \xlongequal{\text{theorem 2.6}} \int f_k\, d\mu + \int g_k\, d\mu$
$\xrightarrow[\text{monotone convergence theorem}]{k\to\infty} \int f\, d\mu + \int g\, d\mu$ □

2.1.3 Integration of real-valued maps

2.3 Define $X \xrightarrow{f^\pm} \overline{\mathbb{R}}_{\geq 0}$ by $f^\pm_{\forall x\in X} := \max\{f_x, 0\}$ $\forall X \xrightarrow{f} \overline{\mathbb{R}}$. f is measurable on a measure space $(X,\mathcal{S},\mu)$ with at least one of $\int f^\pm\, d\mu < \infty \Rightarrow \int f\, d\mu := \int f^+\, d\mu - \int f^-\, d\mu$ ●

Remark. • $\int\left(\left|f = f^+ - f^-\right| = f^+ + f^-\right) d\mu < \infty$ iff $\int \left|f^\pm\right| d\mu < \infty$.

• $\int f\, d\mu$ is defined $\Rightarrow$ measurable f with at least one of $\int f^\pm\, d\mu < \infty$.

E.g. $\int \operatorname{sgn} d\mu$ is not defined $\forall$ LEBESGUE's measure μ on $\mathbb{R}$ because $\int \operatorname{sgn}^\pm d\mu = \infty$.

2.8 *$\forall$ measurable* $X \xrightarrow{f} \overline{\mathbb{R}}$ *on a measure space* $(X,\mathcal{S},\mu)$, $\int f\, d\mu$ *is defined*

$\Rightarrow \int cf\, d\mu \xlongequal{\forall c\in\mathbb{R}} c\int f\, d\mu \wedge \left|\int f\, d\mu\right| \leq \int |f|\, d\mu$ ■

Proof. 1. Without loss of generality, say $c \geq 0$. Then $\int cf\, d\mu = \sum_{s=\pm} s\int (cf)^s\, d\mu$
$\xlongequal[\forall X\xrightarrow{g}\overline{\mathbb{R}}_{\geq 0}\ \forall \text{partition } \mathcal{P} \text{ of } X]{\mathcal{L}_{cg,\mathcal{P}}=c\mathcal{L}_{g,\mathcal{P}}\Rightarrow\int cg\, d\mu = c\int g\, d\mu} \sum_{s=\pm} s\left(\int cf^s\, d\mu = c\int f^s\, d\mu\right) = c\int f\, d\mu$.

2. $\left|\int f\, d\mu\right| = \left|\sum_{s=\pm} s \underbrace{\int f^s\, d\mu}_{<\infty \text{ for at least one of } s}\right| \leq \sum_{s=\pm} s\int f^s\, d\mu = \int |f|\, d\mu$ □

2.9 *$\forall$ measurable* $X \xrightarrow{f_{k=1,2}} \overline{\mathbb{R}}$ *on a measure space* $(X,\mathcal{S},\mu)$

• $\int \left|f_{k=1,2}\right| d\mu < \infty \Rightarrow \int \left(\sum_{k=1}^{2} f_k\right) d\mu = \sum_{k=1}^{2} \int f_k\, d\mu$ *(c.f. theorem 2.7)*

• $f_{1;\forall x\in X} \leq f_{2;x} \Rightarrow \int f_1\, d\mu \leq \int f_2\, d\mu$ *(c.f. theorem 2.3)* ■

2.2 Limits of integrals & integrals of limits

2.2.1 Bounded convergence theorem

2.10 $\left|\int_E f\, d\mu := \int \chi_E f\, d\mu\right| \leq \int \chi_E\left(|f| \leq \sup_E |f|\right) d\mu = \mu_E \sup_E |f|$ *$\forall$ measurable* $X \xrightarrow{f} \overline{\mathbb{R}}$ *on a measure space* $(X,\mathcal{S}\ni E,\mu)$ ■

Theorem (bounded convergence) $\forall \left\{X \xrightarrow{f_k} \mathbb{R} \,\middle|\, f_{k;\forall x\in X} \xrightarrow{k\to\infty} f_x\right\}_{k=1}^{\infty}$ *of measurable maps on a measure space* $(X,\mathcal{S},\mu)$ *with* $\mu_X < \infty$, $\int f_k\, d\mu \xrightarrow{k\to\infty} \int f\, d\mu$ *if* $\exists c \in \mathbb{R}_{>|f_{\forall k\in\mathbb{Z}_{>0};\forall x\in X}|}$ ■

Proof. Theorem 1.12 $\Rightarrow$ measurable $X \xrightarrow{f} \mathbb{R} \xRightarrow{\text{EGOROV's theorem}} \forall \epsilon > 0\, \exists E \in \mathcal{S} : \mu_{X\setminus E} < \epsilon/_{4c}$
$\wedge \{f_k\}_{k=1}^{\infty}$ converges to f uniformly[xiii] on E
$\Rightarrow \lim_{k\to\infty} \left| \int f_k \,d\mu - \int f \,d\mu = \int_{X\setminus E} f_k \,d\mu - \int_{X\setminus E} f \,d\mu + \int_E (f_k - f)\,d\mu \right|$
$\leq \lim_{k\to\infty} \overbrace{\int_{X\setminus E} \left|f_k\right| d\mu}^{\leq \mu_{X\setminus E} c < \epsilon/_4} + \overbrace{\int_{X\setminus E} \left|f\right| d\mu}^{\leq \mu_{X\setminus E} c < \epsilon/_4} + \underbrace{\lim_{k\to\infty} \int_E \left|f_k - f\right| d\mu}_{\leq (\mu_E < \infty)\left(\sup_E \left|f_{k\to\infty} - f\right| < \epsilon/_{2\mu_E}\right)} < \epsilon \xrightarrow{\text{arbitrariness of } \epsilon} 0$

Remark. EGOROV's theorem is crucial for interchanging limits and integrals in proofs.

2.2.2 0-measure sets in integration theorems

2.4 $\forall$measure space $(X, \mathcal{S}, \mu)$, $E \in \mathcal{S}$ contains *almost every* $x \in X$ (denote $\underset{\sim}{\forall} x \in X$) if $\mu_{X\setminus E} = 0$ ●

Remark 1. Integration theorems can almost always be relaxed to hold for almost everywhere instead of everywhere. *E.g.* relax in the bounded convergence theorem '$f_{k;\forall x \in X} \xrightarrow{k\to\infty} f_x$' to '$f_{k;\underset{\sim}{\forall} x \in X} \xrightarrow{k\to\infty} f_x$'; *i.e.* $\exists E \in \mathcal{S} : \mu_{X\setminus E} = 0 \wedge f_{k;\forall x \in E} \xrightarrow{k\to\infty} f_x$, then $\int f_k \,d\mu = \int_E f_k \,d\mu \equiv \int \chi_E \left(f_k \xrightarrow{k\to\infty} f\right) d\mu \equiv \int_E f \,d\mu = \int f \,d\mu$.

2.2.3 Dominated convergence theorem

2.11 $\forall$*measurable* $X \xrightarrow{g} \overline{\mathbb{R}}_{\geq 0}$ *on a measure space* $(X, \mathcal{S}, \mu)$ *with* $\int g \,d\mu < \infty$ $\forall \epsilon > 0$

1. $\exists \delta > 0 : \int_{\forall B \in \mathcal{S} : \mu_B < \delta} g \,d\mu < \epsilon$

2. $\exists E \in \mathcal{S} : \int_{X\setminus E : \mu_E < \infty} g \,d\mu < \epsilon$ ■

Proof. 1. Theorem 2.4 $\Rightarrow$ '$\exists$simple $\mathcal{S}$-measurable $X \xrightarrow{h \in [0,g]} \mathbb{R}_{\geq 0} : \underbrace{\int g \,d\mu}_{<\infty} - \int h \,d\mu$ $\in [0, \epsilon/_2)$' $\Rightarrow \exists \delta > 0 : \delta \underbrace{\max_{\{h_x \mid x \in X\}}}_{H} < \epsilon/_2 \wedge \int_{B : \mu_B < \delta} g \,d\mu = \underbrace{\int_B (g - h)\,d\mu}_{\leq \int (g-h) d\mu < \epsilon/_2} + \underbrace{\int_B h \,d\mu}_{\leq H\mu_B < H\delta < \epsilon/_2} < \epsilon$.

2. '$\exists \mathcal{S}$-measurable partition $\mathcal{P} = \left\{A_j\right\}_{j=1}^{m}$ of X : $\underbrace{\int g \,d\mu}_{<\infty} - \mathcal{L}_{g,\mathcal{P}} \in [0, \epsilon) \wedge \mu_{E = \bigcup_{\substack{j=1,\ldots,m \\ \inf_{A_j} g > 0}} A_j}$ $< \infty$ ($\Leftarrow \mathcal{L}_{g,\mathcal{P}} < \infty$) $\wedge \inf_{\forall A \in \mathcal{P} : A \not\subseteq E} g = 0$ ($\Rightarrow \mathcal{L}_{g,\mathcal{P}} = \mathcal{L}_{\chi_E g, \mathcal{P}}$)'
$\Rightarrow \int_{X\setminus E} g \,d\mu = \int g \,d\mu - \int \chi_E g \,d\mu - \int \chi_E g \,d\mu < \epsilon + \mathcal{L}_{g,\mathcal{P}} - \mathcal{L}_{\chi_E g, \mathcal{P}} = \epsilon$ □

Theorem (dominated convergence) $\forall$*measurable* $X \xrightarrow{f} \overline{\mathbb{R}}$ *on a measure space* $(X, \mathcal{S}, \mu)$ $\forall \left\{ \text{measurable } X \xrightarrow{f_k} \overline{\mathbb{R}} \,\middle|\, f_{k;\underset{\sim}{\forall} x \in X} \xrightarrow{k\to\infty} f_x \right\}_{k=1}^{\infty}$, $\int f_k \,d\mu \xrightarrow{k\to\infty} \int f \,d\mu$ *if* $\exists$*measurable* $X \xrightarrow{g} \overline{\mathbb{R}}_{\geq 0} : \int g \,d\mu < \infty \wedge \left| f_{\forall k \in \mathbb{Z}_{>0}; \underset{\sim}{\forall} x \in X} \right| \leq g_x$ ■

Proof. $\left| \int f_k \,d\mu - \int f \,d\mu \right| \overset{\forall E \in \mathcal{S}}{=\!=} \left| \int_{X\setminus E} f_k \,d\mu - \int_{X\setminus E} f \,d\mu + \int_E f_k \,d\mu - \int_E f \,d\mu \right|$
$\leq \left(\left| \int_{X\setminus E} f_k \,d\mu \right| + \left| \int_{X\setminus E} f \,d\mu \right| \leq \left| 2 \int_{X\setminus E} g \,d\mu \right| \right) + \left| \int_E (f_k - f)\,d\mu \right|$

1. $\mu_X < \infty \xRightarrow{\text{EGOROV's theorem}} \exists E \in \mathcal{S} : \mu_{X\setminus E \in \mathcal{S}} < \infty$ ($\xRightarrow{\text{theorem 2.11.1}} \int_{X\setminus E} g \,d\mu < \epsilon/_4$) $\wedge \{f_k\}_{k=1}^{\infty}$ converges uniformly on E to f ($\Rightarrow \left| \int_E (f_k - f)\,d\mu \right| < \epsilon/_2$ for large enough k). Thus

[xiii]*I.e.* $\left|f_k - f\right|$ arbitrarily small for large enough k

$\left|\int f_k \,\mathrm{d}\mu - \int f \,\mathrm{d}\mu\right| \xrightarrow{k\to\infty} 0$

2. For $\mu_X = \infty$, theorem 2.11.2 $\Rightarrow \exists \mathrm{E} \in \mathcal{S} : \mu_\mathrm{E} < \infty \wedge \int_{X\setminus E} g \,\mathrm{d}\mu < \epsilon/_4$. Besides, $\left|\int_\mathrm{E} f_k \,\mathrm{d}\mu - \int_\mathrm{E} f \,\mathrm{d}\mu\right| < \epsilon/_2$ for large enough k by case 1 as applied to $\left\{f_k\big|_\mathrm{E}\right\}_{k=1}^{\infty}$. Thus $\left|\int f_k \,\mathrm{d}\mu - \int f \,\mathrm{d}\mu\right| \xrightarrow{k\to\infty} 0$ □

2.2.4 Riemann's & Lebesgue's integrals

2.12 *A bounded* $[a,b] \xrightarrow{f} \mathbb{R}$ *is* Riemann-*integrable iff* $\mathring{\mu}_{\{x\in[a,b]\,|\,f \text{ is discontinuous at } x\}} = 0$ *(say* $-\infty < a < b < \infty$*); besides,* f *is measurable on the measure space* $(\mathbb{R}, \mathscr{L}, \mathring{\mu})$*, with* Riemann*'s integral* $\int_a^b f = \int_{[a,b]} f \,\mathrm{d}\mathring{\mu}$[xiv] ■

Proof. $\forall$partition $\mathcal{P}_{\forall n\in\mathbb{Z}_{>0}}$ dividing $[a,b]$ into 2^n subintervals $\mathrm{I}_{j=1,\dots,2^n}$ of equal size $^{(b-a)}/_{2^n}$, Riemann's lower sum $\mathrm{L}_{f,\mathcal{P}_n,[a,b]} = \int_{[a,b]}\left(g_n = \sum_{j=1}^{2^n} \chi_{\mathrm{I}_j} \inf_{\mathrm{I}_j} f\right) \mathrm{d}\mathring{\mu}$ & upper sum $\mathrm{U}_{f,\mathcal{P}_n,[a,b]} = \int_{[a,b]}\left(h_n = \sum_{j=1}^{2^n} \chi_{\mathrm{I}_j} \sup_{\mathrm{I}_j} f\right) \mathrm{d}\mathring{\mu}$,[xv] Then $g_1 \le \cdots \le g_{n\to\infty} \le f \le h_{n\to\infty} \le \cdots \le h_1 \xRightarrow[\text{(if applicable; c.f. remark 1)}]{\text{bounded convergence theorem}}$ Riemann's lower & upper integrals $\mathrm{L}_{f,[a,b]} = \lim_{n\to\infty} \mathrm{L}_{f,\mathcal{P}_n,[a,b]} = \int_{[a,b]} g_{n\to\infty} \,\mathrm{d}\mathring{\mu}$ & $\mathrm{U}_{f,[a,b]} = \lim_{n\to\infty} \mathrm{U}_{f,\mathcal{P}_n,[a,b]} = \int_{[a,b]} h_{n\to\infty} \,\mathrm{d}\mathring{\mu}$.

Thus Riemann-integrable $f \xLeftrightarrow{\text{by definition}} \mathrm{L}_{f,[a,b]} \equiv \mathrm{U}_{f,[a,b]}$

$\iff \int_{[a,b]} (h_{n\to\infty} - g_{n\to\infty} \ge 0) \,\mathrm{d}\mathring{\mu} = 0$

$\iff 0 = \left\{x \in [a,b] \,\middle|\, g_{n\to\infty;x} \ne h_{n\to\infty;x}\right\} = \left\{x \in [a,b] \,\middle|\, f \text{ is discontinuous at } x\right\}$ □

2.2.5 Appoximation by nice maps

2.5 $\forall$measurable $X \xrightarrow{f} \overline{\mathbb{R}}$ on a measure space $(X, \mathcal{S}, \mu)$, f's $\underline{\mathcal{L}^1\textit{-norm}}$ $\|f\|_1 := \int |f| \,\mathrm{d}\mu$; $\underline{\text{Lebesgue}\textit{'s space}}$ $\mathcal{L}^1_\mu := \left\{\mathcal{S}\text{-measurable } X \xrightarrow{f} \mathbb{R} \,\middle|\, \|f\|_1 < \infty\right\}$ ●

E.g. $\forall$measure space $(X, \mathcal{S}, \mu)$, $f \xlongequal[\mathrm{E}_{k=1,\dots,n}\in X \text{ disjoint}]{a_{k=1,\dots,n}\in\mathbb{R}_{\ne 0} \text{ distinct}} \sum_{k=1}^n a_k \chi_{\mathrm{E}_k} \in \mathcal{L}^1_\mu$ iff $\mu_{\mathrm{E}_{\forall k\in\{1,\dots,n\}}\in\mathcal{S}} < \infty$, with $\|f\|_1 = \sum_{k=1}^n |a_k| \mu_{\mathrm{E}_k}$.

E.g. $\mathcal{L}^1_\mu$ is ^{1} if μ is the counting measure on the measurable space $(\mathbb{Z}_{>0}, 2^{\mathbb{Z}_{>0}})$. Say $\mathbb{Z}_{>0} \xrightarrow{a:k\mapsto a_k} \mathbb{R}$, then $\|a \in {}^1\|_1 = \sum_{k=1}^\infty |a_k| < \infty$.

Properties ($\mathcal{L}^1$-norm's) $\forall$measure space $(X, \mathcal{S}, \mu)$ $\forall f$ & $g \in \mathcal{L}^1_\mu$

- $\|f\|_1 \ge 0$
- $\|f\|_1 = 0$ iff $f_{\underset{\sim}{\forall} x\in X} = 0$
- $\|cf\|_1 \xlongequal{\forall c\in\mathbb{R}} |c| \cdot \|f\|_1$
- $\|f+g\|_1 \le \|f\|_1 + \|g\|_1$

[xiv]Say $-\infty \le a < b < c \le \infty$, $(a,b) \xrightarrow{f} \mathbb{R}$ is measurable on $(\mathbb{R}, \mathscr{L}, \mathring{\mu})$, then $-\int_b^a f = \int_a^b f \equiv \int_a^b f_x \,\mathrm{d}x \equiv \int_{(a,b)} f \,\mathrm{d}\mathring{\mu} = \int_a^c f + \int_c^b f$

[xv]For aesthetically pleasing form of mathematics, at each of the endpoints (other than a & b) that is in two of the subintervals, change g_n's value to be f's infimum over the two subintervals, and h_n's value to be f's supremum over the two subintervals.

- $\forall \epsilon > 0\, \exists$simple $h \in \mathcal{L}^1_\mu : \left\|f - h\right\|_1 < \epsilon$ ■

2.6 Denotes $\mathcal{L}^1_{\mathring{\mu}}$ by $\mathcal{L}^1_{\mathbb{R}}$ for the measure space $(\mathbb{R}, \mathscr{F} \in \{\mathscr{B}, \mathscr{L}\}, \mathring{\mu})$, with $\left\|f\right\|_1 = \int_{\mathbb{R}} |f|\, d\mathring{\mu}$ ●

2.7 $\mathbb{R} \xrightarrow{\vartheta = \sum_{k=1}^{n} a_k \chi_{I_k}} \mathbb{R}$ with intervals $I_{k=1,\ldots,n} \subseteq \mathbb{R}$ and $a_{k=1,\ldots,n} \in \mathbb{R}_{\neq 0}$ is a ***step map*** ●

Remark. • $\|\vartheta\|_1 = \sum_{k=1}^{n} |a_k| \mathring{\mu}_{I_k}$ if $I_{k=1,\ldots,n}$ are disjoint.

- $\vartheta \in \mathcal{L}^1_{\mathbb{R}}$ iff $\mathring{\mu}_{I_{\forall k \in \{1,\ldots,n\}}} < \infty$.

- The intervals in ϑ's definition can be open or closed, or half-open; including/excluding interval endpoints does not matter when using ϑ in integrals.

2.13 $\forall f \in \mathcal{L}^1_{\mathbb{R}}\, \forall \epsilon > 0$

- $\exists$*step* $\vartheta \in \mathcal{L}^1_{\mathbb{R}} : \left\|f - \vartheta\right\|_1 < \epsilon$

- $\exists$*continuous* $\mathbb{R} \xrightarrow{g} \mathbb{R} : \left\|f - g\right\|_1 < \epsilon \wedge \mathring{\mu}_{\{x \in \mathbb{R} \,|\, g_x \neq 0\}} < \infty$ ■

3 Differentiation

3.1 Hardy-Littlewood's maximal map

Inequality (Markov's) $\mu_{\{x\in X\,|\,|h(x)|>c\}} \leq {}^{\|h\in\mathcal{L}^1_\mu\|_1}\!/_{(c>0)}$ $\forall$ *measure spaces* $(X,\mathcal{S},\mu)$ ■

Lemma (Vitali's covering) *Every sequence* $\{I_k \subseteq \mathbb{R}\}_{k=1}^n$ *of bounded nonempty open intervals has a disjoint subsequence* $\left\{I_{k_j}\right\}_{j=1}^m : \bigcup_{k=1}^n I_k \subseteq \bigcup_{j=1}^m 3I_{k_j}$, *with* $3I$ *the open interval with the same centre as* I *and* $\mathring{\mu}_{3I} = 3\mathring{\mu}_I$ ■

Inequality (Hardy-Littlewood's maximal) $\mathring{\mu}_{\{b\in\mathbb{R}\,|\,h^*_b>c\}} \leq {}^{3\|h\in\mathscr{L}^1_{\mathbb{R}}\|_1}\!/_{(c>0)}$, *with* $\mathbb{R} \xrightarrow{h^* : b \mapsto \sup_{t>0} \left(\int_{b-t}^{b+t}|h|\right)/_{2t}} \overline{\mathbb{R}}_{\geq 0}$ <u>Hardy-Littlewood's *maximal map*</u>[xvi] $\forall \mathscr{L}$*-measurable* $\mathbb{R} \xrightarrow{h} \mathbb{R}$ ■

3.2 Derivatives of integrals

3.1 $(I \xrightarrow{g} \mathbb{R})$'s <u>*derivative*</u> $g'_b := \lim_{t\to 0} {}^{(g_{b+t}-g_b)}\!/_t$ (if the limit exists; g is then dubbed <u>*differentiable*</u>) at $b \in I$ $\forall$open interval $I \subseteq \mathbb{R}$ ●

Fundamental theorem of calculus $f \in \mathscr{L}^1_{\mathbb{R}}$ *is continuous at* $b \in \mathbb{R} \Rightarrow g'_b = f_b$ *with* $\mathbb{R} \xrightarrow{g:x\mapsto\int_{-\infty}^x f} \mathbb{R}$

Theorem (Lebesgue's differentiation) $f \in \mathscr{L}^1_{\mathbb{R}} \Rightarrow \underset{\sim}{\forall} b \in \mathbb{R}$

- $\lim_{t\downarrow 0} \left(\int_{b-t}^{b+t}|f-f_b|\right)\Big/_{2t} = 0$
- $g'_b = f_b$ *with* $\mathbb{R} \xrightarrow{g:x\mapsto\int_{-\infty}^x f} \mathbb{R}$ ■

3.1 $\nexists \mathscr{L}$*-measurable* $E \subseteq [0,1] : \mathring{\mu}_{E\cap[0,b]} = b/2\ \forall b \in [0,1]$ ■

Proof. $\exists$such $E \Rightarrow g_{b\in\mathbb{R}} = \int_{-\infty}^b \chi_E \overset{\forall b\in[0,1]}{=\!=\!=} b/2$

$\Rightarrow 1/2 \overset{\forall b\in(0,1)}{=\!=\!=} g'_b \underset{\text{Lebesgue's differentiation theorem}}{\overset{\underset{\sim}{\forall} b\in\mathbb{R}}{=\!=\!=\!=\!=\!=\!=\!=\!=}} \chi_{E;b} \in \{0,1\}$ □

3.2 $f_{\underset{\sim}{\forall} b\in\mathbb{R}} = \lim_{t\downarrow 0} \left(\int_{b-t}^{b+t} f\right)\Big/_{2t}\ \forall f \in \mathcal{L}^1_{\mathbb{R}}$ ■

3.2 $\rho_{E\subseteq\mathbb{R};b\in\mathbb{R}} := \lim_{t\downarrow 0} \left(\mathring{\mu}_{E\cap(b-t,b+t)}\right)\Big/_{2t}$ is E's <u>*density*</u> at b ●

E.g. $\rho_{[0,1];b} = \begin{cases} 1 & \text{if } b \in (0,1) \\ 1/2 & \text{if } b \in \{0,1\} \\ 0 & \text{otherwise} \end{cases}$.

Theorem (Lebesgue's density) $\rho_{\forall E\in\mathscr{L};b} = \begin{cases} 1 & \underset{\sim}{\forall} b \in E \\ 0 & \underset{\sim}{\forall} b \in \mathbb{R}\setminus E \end{cases}$ ■

3.3 $\exists E \in \mathscr{B} : 0 < \mathring{\mu}_{E\cap I} < \mathring{\mu}_I$ $\forall$*nonempty bounded open interval* I ■

[xvi]*E.g.* $\left(\chi_{[-1,1]/2}\right)^*_b = \begin{cases} {}^1\!/_{(1+2|b|)} & \text{if } 2|b| \geq 1 \\ 1 & \text{if } 2|b| < 1 \end{cases}$

4 Product Measures

4.1 Product of measure spaces

4.1.1 Product σ-algebras

4.1 $A \times B$ is a *rectangle* in $X \times Y$ $\forall (A, B) \in 2^{X \times Y}$ ●

4.2 The *product* $\mathcal{S} \otimes \mathcal{T}$ is the smallest σ-algebra on $X \times Y$ containing all rectangles $A \times B$ (dubbed *measurable*) with $(A, B) \in \mathcal{S} \times \mathcal{T}$ $\forall$measurable spaces $(X, \mathcal{S})$ & $(Y, \mathcal{T})$ ●

4.3 $[E]_{a \in X} := \{y \in Y \mid (a, y) \in E\}$ and $[E]^{b \in Y} := \{x \in X \mid (x, b) \in E\}$ are the *cross sections* of $E \subseteq X \times Y$ ●

Example 4.1 $[A \times B]_{a \in X} = \begin{cases} B & \text{if } a \in A \\ \varnothing & \text{if } a \notin A \end{cases}$ & $[A \times B]^{b \in Y} = \begin{cases} A & \text{if } b \in B \\ \varnothing & \text{if } b \notin B \end{cases}$ $\forall (A, B) \in 2^{X \times Y}$.

4.1 $([E]^{b \in Y}, [E]_{a \in X}) \in \mathcal{S} \times \mathcal{T}$ $\forall E \in \mathcal{S} \otimes \mathcal{T}$ $\forall$*measurable spaces* $(X, \mathcal{S})$ *&* $(Y, \mathcal{T})$ ■

Proof. $A \times B \in \mathcal{E} = \{E \subseteq X \times Y \mid ([E]^{b \in Y}, [E]_{a \in X}) \in \mathcal{S} \times \mathcal{T}\}$ $\forall (A, B) \in \mathcal{S} \times \mathcal{T}$ by example 4.1, with $\mathcal{E}$ closed under complementation and countable unions as $[(X \times Y) \backslash E]_a = Y \backslash [E]_a$, $[\bigcup_{k \in \mathbb{Z}_{>0}} (E_k \subseteq X \times Y)]_a = \bigcup_{k \in \mathbb{Z}_{>0}} [E_k]_a$ $\forall a \in X$ *etc.* Hence $\mathcal{E}$ is a σ-algebra on $X \times Y$ containing all $A \times B \in \mathcal{S} \otimes \mathcal{T}$; *i.e.* $\mathcal{S} \otimes \mathcal{T} \subseteq \mathcal{E}$ □

4.4 $Y \xrightarrow{[f]_{\forall a \in X}: y \mapsto f_{a,y}} \mathbb{R}$ & $X \xrightarrow{[f]^{\forall b \in Y} x \mapsto f_{x,b}} \mathbb{R}$ are the *cross sections* of $X \times Y \xrightarrow{f} \mathbb{R}$ ●

4.2 $[f]_{\forall a \in X}$ *is* $\mathcal{T}$*-measurable on* Y *and* $[f]^{\forall b \in Y}$ *is* $\mathcal{S}$*-measurable on* X $\forall \mathcal{S} \otimes \mathcal{T}$*-measurable* $X \times Y \xrightarrow{f} \mathbb{R}$ $\forall$*measurable spaces* $(X, \mathcal{S})$ *&* $(Y, \mathcal{T})$ ■

Proof. $\forall B \in \mathscr{B}$, $\mathcal{S} \otimes \mathcal{T}$-measurable $f \Rightarrow f_B^{-1} \in \mathcal{S} \otimes \mathcal{T} \xRightarrow{\text{theorem 4.1}} [f_B^{-1}]_a \in \mathcal{T}$; besides, $y \in ([f]_a)_B^{-1} \iff f_{a,y} = ([f]_a)_y \in B \iff (a, y) \in f_B^{-1} \iff y \in [f_B^{-1}]_a$. Thus $([f]_a)_{\forall B \in \mathscr{B}}^{-1} = [f_B^{-1}]_a \in \mathcal{T}$; *i.e.* $[f]_a$ is $\mathcal{T}$-measurable. Similarly, $[f]^b$ is $\mathcal{S}$-measurable. □

4.1.2 Monotone class theorem

4.5 $\mathcal{A} \subseteq 2^X$ is an *algebra* on X if

- $\varnothing \in \mathcal{A}$
- $E \in \mathcal{A} \Rightarrow X \backslash E \in \mathcal{A}$
- $E_{k=1,2} \in \mathcal{A} \Rightarrow \bigcup_{k=1}^{2} E_k \in \mathcal{A}$ ●

4.3 $\forall$*measurable spaces* $(X, \mathcal{S})$ *and* $(Y, \mathcal{T})$*, the set* $\mathcal{A}$ *of finite unions of rectangles in* $\mathcal{S} \otimes \mathcal{T}$ *is an algebra on* $X \times Y$*, each such union equals a finite union of disjoint measurable rectangles in* $\mathcal{S} \otimes \mathcal{T}$ ■

Proof. 1. (a) Obviously $\mathcal{A}$ is closed under finite unions.

(b) $\forall A_{1,\dots,n}$ & $C_{1,\dots,m} \in \mathcal{S}$ $\forall B_{1,\dots,n}$ & $D_{1,\dots,m} \in \mathcal{T}$, $\left(\bigcup_{j=1}^{n} A_j \times B_j\right) \bigcap \left(\bigcup_{k=1}^{m} C_k \times D_k\right) = \bigcup_{j=1}^{n} \bigcup_{k=1}^{m} \left[\left(A_j \times B_j\right) \cap (C_k \times D_k) = \left(A_j \cap C_k\right) \times \left(B_j \cap D_k\right)\right]$; intersection of two rectangles is a rectangle, implying that $\mathcal{A}$ is closed under finite intersections.

(c) $(X \times Y) \backslash (A \times B) = [(X \backslash A) \times Y] \cup [X \times ((Y \backslash B))]$ $\forall (A, B) \in \mathcal{S} \times \mathcal{T}$. Hence the complement of each $\mathcal{S} \otimes \mathcal{T}$-measurable rectangle is in $\mathcal{A}$. Thus the complement of a finite union of $\mathcal{S} \otimes \mathcal{T}$-measurable rectangles is in $\mathcal{A}$ (use de Morgan's laws and step (b) that $\mathcal{A}$ is closed under finite intersections). *I.e.* $\mathcal{A}$ is closed under complementation.

2. $[A\times B]\cup[C\times D]=[A\times B]\uplus[C\times(D\backslash B)]\uplus[(C\backslash A)\times(B\cap D)]$ $\forall\mathcal{S}\otimes\mathcal{T}$-measurable rectangles $A\times B$ & $C\times D$. Hence $\forall$finite union of $\mathcal{S}\otimes\mathcal{T}$-measurable rectangles, if it is not a disjoint union, choose any nondisjoint pair of measurable rectangles in the union and replace them with the union of three disjoint measurable rectangles as above. Iterate this process until obtaining a disjoint union of measurable rectangles. □

4.6 $\mathcal{M}\subseteq 2^X$ is a *monotone class* on X if

- $\left\{E_k\in\mathcal{M}\,\middle|\,E_{\forall j\in\mathbb{Z}_{>0}}\subseteq E_{j+1}\right\}_{k\in\mathbb{Z}_{>0}}\Rightarrow\bigcup_{k\in\mathbb{Z}_{>0}}E_k\in\mathcal{M}$
- $\left\{E_k\in\mathcal{M}\,\middle|\,E_{\forall j\in\mathbb{Z}_{>0}}\supseteq E_{j+1}\right\}_{k\in\mathbb{Z}_{>0}}\Rightarrow\bigcap_{k\in\mathbb{Z}_{>0}}E_k\in\mathcal{M}$ ●

Theorem (monotone class) *The smallest σ-algebra $\mathcal{S}$ containing an algebra $\mathcal{A}$ on X is the smallest monotone class $\mathcal{M}$ containing $\mathcal{A}$* ■

Proof. 1. Every σ-algebra is a monotone class $\Rightarrow\mathcal{M}\subseteq\mathcal{S}$.

2. (a) $A\in\mathcal{A}\Rightarrow\mathcal{A}\subseteq$ monotone class $\mathcal{E}=\{E\in\mathcal{M}\mid A\cup E\in\mathcal{M}\}$ (as $\mathcal{A}\subseteq\mathcal{M}$ is closed under finite union) $\Rightarrow A\cup E\in\mathcal{M}\subseteq\mathcal{E}\ \forall E\in\mathcal{M}\Rightarrow$

(b) $\mathcal{A}\subseteq$ monotone class $\mathcal{D}=\{D\in\mathcal{M}\mid D\cup E\in\mathcal{M}\ \forall E\in\mathcal{M}\}\Rightarrow\mathcal{M}\subseteq\mathcal{D}$ is closed under finite union $\Rightarrow$

(c) i. $F_k=\bigcup_{j=1}^{k}\left(E_j\in\mathcal{M}\right)\in\mathcal{M}\Rightarrow F_{k\to\infty}=\bigcup_{k=1}^{\infty}F_k\subseteq\mathcal{M}$ (as $\mathcal{M}$ is a monotone class) $\Rightarrow\mathcal{M}$ is closed under countable union.

ii. $\mathcal{A}$ is closed under complementation $\Rightarrow\mathcal{A}\subseteq$ monotone class $\mathcal{M}'$ $=\{E\in\mathcal{M}\mid X\backslash E\in\mathcal{M}\}\Rightarrow\mathcal{M}\subseteq\mathcal{M}'$ is closed under complementation.

Hence $\mathcal{M}$ is an σ-algebra containing $\mathcal{A}$, and thus $\mathcal{M}\supseteq\mathcal{S}$ □

4.1.3 Products of measures

4.7 A measure μ on a measurable space $(X,\mathcal{S})$ is dubbed

Finite if $\mu_X<\infty$.

σ-finite if $X=\bigcup_{k\in\mathbb{Z}_{>0}}(X_k\in\mathcal{S})$ with $\mu_{X_{\forall k\in\mathbb{Z}_{>0}}}<\infty$ ●

E.g. • Lebesgue's measure on $[0,1]$ is finite.

- Lebesgue's measure on $\mathbb{R}$ is not finite but σ-finite.
- Counting measure on $\mathbb{R}$ is not σ-finite (because the countable union of finite sets is countable).

4.4 $\forall$σ-*finite measure spaces* $(X,\mathcal{S},\mu)$ *&* $(Y,\mathcal{T},\nu)$

1. $x\mapsto\nu_{[E]_{x\in X}}$ *is* $\mathcal{S}$*-measurable on* X *and* $y\mapsto\mu_{[E]^{y\in Y}}$ *is* $\mathcal{T}$*-measurable on* Y $\forall E\in\mathcal{S}\otimes\mathcal{T}$.

2. *the product* $\mathcal{S}\otimes\mathcal{T}\xrightarrow{\mu\times\nu:E\mapsto\int_X\int_Y\chi_{E;x,y}d\nu_y d\mu_x}(\mu\times\nu)_{\mathcal{S}\otimes\mathcal{T}}$ *is a measure on* $(X\times Y,\mathcal{S}\otimes\mathcal{T})$ ■

Proof. 1. Without lose of generality, one just need to prove that $x\mapsto\nu_{[E]_x}$ (well-defined, as $[E\in\mathcal{S}\otimes\mathcal{T}]_{\forall x\in X}\in\mathcal{T}\Leftarrow$ theorem 4.1) is $\mathcal{S}$-measurable on X.

(a) If ν is finite, one need to prove that

$$\mathcal{S}\otimes\mathcal{T}=\mathcal{M}=\left\{E\in\mathcal{S}\otimes\mathcal{T}:x\mapsto\nu_{[E]_x}\text{ is }\mathcal{S}\text{-measurable on X}\right\}.$$

By example 4.1, $(A,B)\in\mathcal{S}\times\mathcal{T}\Rightarrow\nu_{[A\times B]_x}=\nu_B\chi_{A;x}\ \forall x\in X$; *i.e.* $x\mapsto\nu_{[A\times B]_x}$ equals the $\mathcal{S}$-measurable map $\nu_B\chi_A$ on X. Hence $\mathcal{M}$ contains all measurable rectangles in $\mathcal{S}\otimes\mathcal{T}$.

By theorem 4.3, $E\in$ algebra $\mathcal{A}$ of all finite unions of measurable rectangles in $\mathcal{S}\otimes\mathcal{T}\Rightarrow\exists$measurable rectangles $E_{k=1,\dots,n}:\nu_{\left[E=\biguplus_{k=1}^{n}E_k\right]_x=\biguplus_{k=1}^{n}[E_k]_x}=\sum_{k=1}^{n}\nu_{[E_k]_x}$.

I.e. $x \mapsto \nu_{[E]_x}$ is a finite sum of $\mathcal{S}$-measurable maps and is thus $\mathcal{S}$-measurable. Hence $E \in \mathcal{M}$, and $\mathcal{A} \subseteq \mathcal{M}$.

The next is to show that $\mathcal{M}$ is a monotone class on $X \times Y$. $\forall$increasing sequence $\{E_k \in \mathcal{M}\}_{k=1}^{\infty}$, $\nu_{[\bigcup_{k=1}^{\infty} E_k]_x = \bigcup_{k=1}^{\infty}[E_k]_x} \xleftarrow{\infty \leftarrow k} \nu_{[E_k]_x}$. Hence $x \mapsto \nu_{[\bigcup_{k=1}^{\infty} E_k]_x}$ is $\mathcal{S}$-measurable,[xvii] $\bigcup_{k=1}^{\infty} E_k \in \mathcal{M}$, and $\mathcal{M}$ is closed under countable increasing unions. $\forall$decreasing sequence $\{E_k \in \mathcal{M}\}_{k=1}^{\infty}$, $\nu_{[\bigcap_{k=1}^{\infty} E_k]_x = \bigcap_{k=1}^{\infty}[E_k]_x} \xleftarrow{\infty \leftarrow k} \nu_{[E_k]_x}$ for finite ν. Hence $x \mapsto \nu_{[\bigcap_{k=1}^{\infty} E_k]_x}$ is $\mathcal{S}$-measurable, $\bigcap_{k=1}^{\infty} E_k \in \mathcal{M}$, and $\mathcal{M}$ is closed under countable decreasing intersections.

Finally, monotone class theorem $\Rightarrow$ the monotone class $\mathcal{M}$ containing $\mathcal{A}$ contains the smallest σ-algebra containing $\mathcal{A}$; *i.e.* $\mathcal{M} \supseteq \mathcal{S} \otimes \mathcal{T}$.

(b) If ν is a σ-finite, $\exists \{Y_k \in \mathcal{T}\}_{k=1}^{\infty} : \bigcup_{k=1}^{\infty} Y_k = Y \wedge \nu_{Y_{\forall k \in \mathbb{Z}_{>0}}} < \infty$. Replacing each Y_k by $Y_1 \cup \cdots \cup Y_k$, one can assume that $Y_1 \subseteq Y_2 \subseteq \cdots$. $\forall E \in \mathcal{S} \otimes \mathcal{T}$, $\nu_{[E]_x} \xleftarrow{\infty \leftarrow k} \nu_{[E \cap (X \times Y_k)]_x}$, with $x \mapsto \nu_{[E \cap (X \times Y_k)]_x}$ $\mathcal{S}$-measurable on X (by step (a), with ν considered finite when restricted to the σ-algebra on Y_k consisting of $\mathcal{T}$-measurable sets $E \subseteq Y_k$). Hence $x \mapsto \nu_{[E]_x}$ is $\mathcal{S}$-measurable on X.

2. Clearly $(\mu \times \nu)_{\varnothing} = 0$, and $\mu \times \nu$ is the countably additive as $(\mu \times \nu)_{\biguplus_{k=1}^{\infty}(E_k \in \mathcal{S} \otimes \mathcal{T})}$
$= \int_X \left(\nu_{[\biguplus_{k=1}^{\infty} E_k]_x = \biguplus_{k=1}^{\infty}[E_k]_x} = \sum_{k=1}^{\infty} \nu_{[E_k]_x} \right) d\mu_x \overset{\text{monotone convergence theorem}}{=\!=\!=} \sum_{k=1}^{\infty} \int_X \nu_{[E_k]_x} d\mu_x$
$= \sum_{k=1}^{\infty} (\mu \times \nu)_{E_k}$ □

E.g. $(\mu \times \nu)_{A \times B} = \mu_A \nu_B \ \forall (A, B) \in \mathcal{S} \times \mathcal{T}$

4.2 Iterated integrals

Theorem (TONELLI's) $\int_{X \times Y} f \, d(\mu \times \nu) = \int_X \underbrace{\int_Y f_{x,y} \, d\nu_y}_{\mathcal{S}\text{-measurable on } X} d\mu_x = \int_Y \overbrace{\int_X f_{x,y} \, d\mu_x}^{\mathcal{T}\text{-measurable on } Y} d\nu_y$ $\forall \mathcal{S} \otimes \mathcal{T}$-*measurable* $X \times Y \xrightarrow{f} \overline{\mathbb{R}}_{\geq 0}$ *on σ-finite measure spaces* $(X, \mathcal{S}, \mu)$ & $(Y, \mathcal{T}, \nu)$ ■

E.g. Consider $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \xrightarrow{x:(j,k) \mapsto x_{j,k}} \overline{\mathbb{R}}_{\geq 0}$ and σ-finite counting measure spaces $(\mathbb{Z}_{>0}, 2^{\mathbb{Z}_{>0}}, \mu)$, then $\int_{\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}} x \, d(\mu \times \mu) = \left(\sum_{j \in \mathbb{Z}_{>0}} \sum_{k \in \mathbb{Z}_{>0}} = \sum_{k \in \mathbb{Z}_{>0}} \sum_{j \in \mathbb{Z}_{>0}} \right) x_{j,k}$.

Theorem (FUBINI's) $\int_{X \times Y} f \, d(\mu \times \nu) = \int_X \underbrace{\int_Y f_{x,y} \, d\nu_y}_{\mathcal{S}\text{-measurable on } X} d\mu_x = \int_Y \overbrace{\int_X f_{x,y} \, d\mu_x}^{\mathcal{T}\text{-measurable on } Y} d\nu_y$ $\forall \mathcal{S} \otimes \mathcal{T}$-*measurable* $X \times Y \xrightarrow{f} \overline{\mathbb{R}}$ *on σ-finite measure spaces* $(X, \mathcal{S}, \mu)$ & $(Y, \mathcal{T}, \nu)$: $\int_{X \times Y} |f| \, d(\mu \times \nu) < \infty$ *(and thus* $\int_Y \left| f_{\underset{\sim}{\forall x \in X}, y} \right| d\nu_y < \infty > \int_X \left| f_{x, \underset{\sim}{\forall y \in Y}} \right| d\mu_x$*)* ■

4.5 $U_f := \{(x, t) \in X \times \mathbb{R}_{>0} \mid 0 < t < f_x\}$ *is the* <u>*region under the graph*</u> *of* $X \xrightarrow{f} \overline{\mathbb{R}}_{\geq 0}$. *Then measurable* f *on σ-finite measure space* $(X, \mathcal{S}, \mu) \Rightarrow U_f \in \mathcal{S} \otimes \mathcal{B}$
$\wedge \ (\mu \times \mathring{\mu})_{U_f} = \int_X f \, d\mu = \int_{\mathbb{R}_{>0}} \mu_{\{x \in X \mid t < f_x\}} d\mathring{\mu}_t$ $\forall$ LEBESGUE*'s measure space* $(\mathbb{R}_{>0}, \mathcal{B}, \mathring{\mu})$ ■

4.3 LEBESGUE's integrals on $\mathbb{R}^n$

4.6 $\bigtimes_{k=1}^{2} G_k \subseteq \mathbb{R}^{\sum_{k=1}^{2} n_k}$ *is open* $\forall$ *open* $G_{k=1,2} \subseteq \mathbb{R}^{n_k}$ ■

[xvii]Recall that pointwise limit of $\mathcal{S}$-measurable functions is $\mathcal{S}$-measurable

4.8 BOREL's $B \subseteq \mathbb{R}^n$ is an element of the smallest σ-algebra on $\mathbb{R}^n$ containing all open $G \subseteq \mathbb{R}^n$; denote the σ-algebra of all BOREL's $B \subseteq \mathbb{R}^n$ by $\mathscr{B}_n$ ●

4.7 • $G \subseteq \mathbb{R}^n$ *is open* $\Longleftrightarrow G = \bigcup_{k \in \mathbb{Z}_{>0}} C_k$ *with* $C_{\forall k \in \mathbb{Z}_{>0}}$ *open cubes* $\subseteq \mathbb{R}^n$.

• $\mathscr{B}_n$ *is the smallest* σ*-algebra on* $\mathbb{R}^n$ *containing all open cubes* $\subseteq \mathbb{R}^n$ ■

4.8 $\mathscr{B}_{\sum_{k=1}^{2} n_k} = \otimes_{k=1}^{2} \mathscr{B}_{n_k}$ ■

4.9 Define inductively LEBESGUE's measure $\mathring{\mu}_n = \mathring{\mu}_{n-1} \times \mathring{\mu}_1$ on measurable spaces $(\mathbb{R}^n, \mathscr{B}_n)$ with $\mathring{\mu}_1$ LEBESGUE's measure on $(\mathbb{R}, \mathscr{B}_1)$ ●

4.9 $\forall E \in \mathscr{B}_n\ \forall t \in \mathbb{R}_{>0},\ tE \in \mathscr{B}_n \wedge \mathring{\mu}_{n;tE} = t^n \mathring{\mu}_{n;E}$ ■

4.10 $D_1(D_2 f) = D_2(D_1 f)\ \forall G \xrightarrow{f} \mathbb{R} : \exists$ *continuous* $D_1 f$ & $D_2 f$ & $D_1(D_2 f)$ & $D_2(D_1 f)$ *on the open* $G \subseteq \mathbb{R}^2$, *where the <u>partial derivzates</u>* $(D_1 f)_{x,y} := \lim_{t \to 0} {(f_{x+t,y} - f_{x,y})}\big/_{t}$ *&* $(D_2 f)_{x,y} := \lim_{t \to 0} {(f_{x,y+t} - f_{x,y})}\big/_{t}\ \forall (x,y) \in G$ *etc.* ■

A Riemann's integration

A.1 Riemann integral

A.1 A *partition* of $[a,b] \subseteq \mathbb{R}$ is a finite list $\{x_i\}_{i=0}^n$ with $a = x_0 < x_1 < \cdots < x_n = b$

Remark. Use the partition to think of $[a,b] = \bigcup_{i=1}^n [x_{i-1}, x_i]$.

A.2 $\inf_{\mathsf{A}} f = \inf f_{\mathsf{A}}$ & $\sup_{\mathsf{A}} f = \sup f_{\mathsf{A}}$ $\forall \mathsf{A} \subseteq$ domain of a real-valued map f

A.3 $\forall$bounded map $[a,b] \xrightarrow{f} \mathbb{R}$ $\forall$partition $\mathsf{P} = \{x_i\}_{i=0}^n$ of $[a,b]$, Riemann*'s lower & upper sums* are

$$\mathsf{L}_{f,\mathsf{P},[a,b]} = \sum_{i=1}^n (x_i - x_{i-1}) \inf_{[x_{i-1},x_i]} f \quad \& \quad \mathsf{U}_{f,\mathsf{P},[a,b]} = \sum_{i=1}^n (x_i - x_{i-1}) \sup_{[x_{i-1},x_i]} f$$

Remark. Riemann's sums approximate the signed area under f's graph.

A.1 *$\forall$bounded map $[a,b] \xrightarrow{f} \mathbb{R}$ $\forall$partitions P, P' of $[a,b]$ with the list defining P a subset of the list defining P', $\mathsf{L}_{f,\mathsf{P},[a,b]} < \mathsf{L}_{f,\mathsf{P}',[a,b]} < \mathsf{U}_{f,\mathsf{P}',[a,b]} < \mathsf{U}_{f,\mathsf{P},[a,b]}$*

A.2 *$\forall$bounded map $[a,b] \xrightarrow{f} \mathbb{R}$ $\forall$partitions P, P' of $[a,b]$, $\mathsf{L}_{f,\mathsf{P},[a,b]} \leq \mathsf{U}_{f,\mathsf{P}',[a,b]}$*

A.4 $\forall$bounded map $[a,b] \xrightarrow{f} \mathbb{R}$, Riemann*'s lower & upper integrals* are

$$\mathsf{L}_{f,[a,b]} := \sup_{\mathsf{P}} \mathsf{L}_{f,\mathsf{P},[a,b]} \quad \& \quad \mathsf{U}_{f,[a,b]} := \inf_{\mathsf{P}} \mathsf{U}_{f,\mathsf{P},[a,b]}$$

A.3 *$\forall$bounded map $[a,b] \xrightarrow{f} \mathbb{R}$, $\mathsf{L}_{f,[a,b]} \leq \mathsf{U}_{f,[a,b]}$*

A.5 A bounded map on a closed bounded interval is Riemann *integrable* if its lower and upper Riemann integrals are equal. *E.g.* Riemann*'s integral* $\int_a^b f = \mathsf{L}_{f,[a,b]} = \mathsf{U}_{f,[a,b]}$ of a Riemann integrable map $[a,b] \xrightarrow{f} \mathbb{R}$

Example A.1 $\forall [0,1] \xrightarrow{f:x\mapsto x^2} \mathbb{R}$ $\forall \mathsf{P}_n = \{i/n\}_{i=0}^n$,

$$\mathsf{L}_{f,\mathsf{P}_n,[0,1]} = \sum_{i=1}^n \frac{1}{n}\left(\frac{i-1}{n}\right)^2 = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2},$$

$$\mathsf{U}_{f,\mathsf{P}_n,[0,1]} = \sum_{i=1}^n \frac{1}{n}\left(\frac{i}{n}\right)^2 = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2};$$

$$\mathsf{U}_{f,[0,1]} \leq \inf_{n\in\mathbb{Z}_{>0}} \mathsf{U}_{f,\mathsf{P}_n,[0,1]} = \boxed{\int_0^1 f = \frac{1}{3}} = \inf_{n\in\mathbb{Z}_{>0}} \mathsf{L}_{f,\mathsf{P}_n,[0,1]} \leq \mathsf{L}_{f,[0,1]}.$$

A.4 *Every continuous real-valued map on a closed bounded interval (and thus the map is uniformly continuous) is* Riemann *integrable*

A.5 *$\forall$*Riemann *integrable map $[a,b] \xrightarrow{f} \mathbb{R}$,*

$$(b-a) \inf_{[a,b]} f \leq \int_a^b f \leq (b-a) \sup_{[a,b]} f$$

A.2 RIEMANN's integral is not good enough

RIEMANN's integration does not

- handle maps with many discontinuities or maps unbounded
- work well with limits

Example A.2 $f_{x\in[0,1]} = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$ has many discontinuities, and

$$\inf_{[a,b]} f = 0 \neq 1 = \sup_{[a,b]} f \xLeftarrow{\forall [a,b]\subseteq[0,1]} \exists r \in (\mathbb{R}\backslash\mathbb{Q})^{\in[a,b]} \wedge \exists q \in \mathbb{Q}_{\in[a,b]}.$$

Thus $L_{f,P,[0,1]} = 0 \neq 1 = U_{f,P,[0,1]}$ $\forall$partition P of $[0,1]$, $L_{f,[0,1]} = 0 \neq 1 = U_{f,[0,1]}$, and $[0,1] \xrightarrow{f} \mathbb{R}$ not RIEMANN integrable.

Example A.3 $f_x = \begin{cases} 1/\sqrt{x} & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0 \end{cases}$ is unbounded, and $\sup f_{[x_0,x_1]} = \infty$ $\forall$partition $P = \{x_i\}_{i=0}^n \Rightarrow U_{f,P,[0,1]} = \infty$ by definition. However, we may redefine $\int_0^1 f$ as $\lim_{a\downarrow 0} \int_a^1 f$, for the area under f's graph is $\lim_{a\downarrow 0}\left(\int_a^1 f = 2 - 2\sqrt{a}\right) = 2$.

Example A.4 Given a sequence $r_1, r_2, \ldots$ that includes each $q \in \mathbb{Q}_{\in[0,1]}$ exactly once but no other numbers, and $f_{k\in\mathbb{Z}_{>0},x\in[0,1]} = \begin{cases} 1/\sqrt{x-r_k} & \text{if } x > r_k \\ 0 & \text{if } x \leq r_k, \end{cases}$ then $f_x = \sum_{k=1}^\infty {f_{k;x}}/{2^k}$ is unbounded on every non-empty open subinterval $I \subseteq [0,1]$ because $I \ni q \in \mathbb{Q}$, and f's RIEMANN integral is thus undefined on I, although the area (< 2) under f's graph seems reasonable.

Example A.5 RIEMANN's integration does not work well with pointwise limits. *E.g.* given a sequence $r_1, r_2, \ldots$ that includes each $q \in \mathbb{Q}_{\in[0,1]}$ exactly once but no other numbers, then each $f_{k\in\mathbb{Z}_{>0},x\in[0,1]} = \begin{cases} 1 & \text{if } x \in \{r_i\}_{i=1}^k \\ 0 & \text{otherwise} \end{cases}$ is RIEMANN integrable and $\int_0^1 f_k = 0$. However, $f_x = \lim_{k\to\infty} f_{k;x} = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$ is not RIEMANN integrable (*cf.* example A.2).

A.6 *$\forall$sequence $f_1, f_2, \ldots$ of* RIEMANN *integrable maps on $[a,b]$ with* $\left|f_{k\in\mathbb{Z}_{>0},x\in[a,b]}\right| \leq M \in \mathbb{R}$, $\int_a^b f = \lim_{k\to\infty} \int_a^b f_k$ *if*

1. $\forall x \in [a,b]\, \exists f_x = \lim_{k\to\infty} f_{k,x}$
2. *f is* RIEMANN *integrable on $[a,b]$* ■

Remark. The undesirable hypothesis 2 and the difficulty in finding a simple RIEMANN-integration-based proof suggest that RIEMANN's integration is not the ideal integration theory.

B Complete ordered fields

B.1 A *field* is a set $\mathbb{F}$ with two binary operations symbolised as addition and multiplication: $\forall a \,\&\, b \,\&\, c \in \mathbb{F}$

Commutativity $a + b = b + a \wedge ab = ba$

Associativity $(a+b)+c = a+(b+c) \wedge (ab)c = a(bc)$

Multiplicative distributivity over addition $a(b+c) = ab + ac$

Additive identity $\exists ! \mathbf{0}_{\mathbb{F}} \in \mathbb{F} : a + \mathbf{0} = a$

Multiplicative identity $\exists ! \mathbf{1}_{\mathbb{F}} \in \mathbb{F} : a\mathbf{1} = a$

Additive inverse $\exists ! -a \in \mathbb{F} : a + (-a) = \mathbf{0}$

Multiplicative inverse $\exists ! a^{-1} \in \mathbb{F} : aa^{-1} = \mathbf{1}$

Remark 2. $-(-a = -\mathbf{1} \cdot a) = a \overset{\neq \mathbf{0}}{=\!=} (a^{-1})^{-1}\ \forall a \in \mathbb{F}$.

E.g. The set $\mathbb{Q}$ of rationals under usual addition and multiplication.
E.g. The set $\{0, 1\}$ under usual addition and multiplication except that $1 + 1 := 0$.
B.1 $a\mathbf{0} = \mathbf{0}\ \forall a \in$ *field* $\mathbb{F}$
B.2 $\forall a, b \in$ field $\mathbb{F}$, their

Difference $a - b := a + (-b)$

Quotient $a/b := ab^{-1}$ for $b \neq \mathbf{0}$

B.3 A field $\mathbb{F}$ is *ordered* if $\exists$ *positive* $\mathrm{P} \subset \mathbb{F}$:

- $a \in \mathbb{F} \Rightarrow a \in \mathrm{P} \veebar a = \mathbf{0} \veebar -a \in \mathrm{P}$
- $a \,\&\, b \in \mathrm{P} \Rightarrow a + b \in \mathrm{P} \wedge ab \in \mathrm{P}$

B.2 *A positive* $\mathrm{P} \subset$ *ordered field* $\mathbb{F}$ *is closed under multiplicative inverse; i.e.* $a^{-1} \in \mathrm{P}$ $\forall a \in \mathrm{P}$, *with* $\mathbf{1} \in \mathrm{P}$
B.4 $\forall a \,\&\, b \in$ ordered field $\mathbb{F} \supset$ positive P

- $a < b \iff b - a \in \mathrm{P} \iff b > a$
- $a \leq b \iff a < b \vee a = b \iff b \geq a$

Remark 3. $\mathbf{0} < b$ iff $b \in \mathrm{P}$.
B.3 *The ordering* $<$ *on an ordered field* $\mathbb{F}$ *is* *transitive; i.e.* $a < b < c \xrightarrow{\forall a,b,c \in \mathbb{F}} a < c$

B.5 The *absolute value* $|b| := \begin{Bmatrix} b & \text{if } b \geq \mathbf{0} \\ -b & \text{if } b < \mathbf{0} \end{Bmatrix}$ of $b \in$ ordered field $\mathbb{F}$

Remark 4. $|b| \geq b, -b$.
B.4 $|a + b| \leq |a| + |b|\ \forall a \,\&\, b \in$ *ordered field* $\mathbb{F}$
B.5 *Every ordered field* $\mathbb{F} \supseteq \mathbb{Q}$*; i.e.* $\exists$ *injection*[xviii] $\mathbb{Q} \xrightarrow{\varphi} \mathbb{F}$, *such that*

$$\varphi_{\pm m/n} := (\underbrace{\pm\mathbf{1} \pm \cdots \pm \mathbf{1}}_{m \text{ times}})(\underbrace{\mathbf{1} + \cdots + \mathbf{1}}_{n \text{ times}})^{-1} \overset{m=0}{=\!=} \mathbf{0} =: \varphi_0$$

$\forall m \in \mathbb{Z}_{\geq 0} := \{z \in \mathbb{Z} | z \geq 0\}\ \forall n \in \mathbb{Z}_{>0}$, *preserving all ordered field properties.*[xix]
B.6 $q^2 = 2 \Rightarrow q \notin \mathbb{Q}$
B.6 $b \in$ ordered field $\mathbb{F}$ is an *upper bound* of $\mathrm{A} \subseteq \mathbb{F}$ if $a \leq b \in \mathbb{F}\ \forall a \in \mathrm{A}$
E.g. For both $\mathbb{Q}_{\leq 3}$ and $\mathbb{Q}_{<3}$, every $b \in \mathbb{Q}_{\geq 3}$ is an upper bound, and 3 is the *least* upper bound.
Remark 5. A least upper bound of a set, if it exists, is unique.
Example B.1 $\mathbb{Q}_{<\sqrt{2}} = \left\{q \in \mathbb{Q} \middle| q^2 < 2\right\}$ has no least upper bound $b \in \mathbb{Q}$. The idea is that

- $b \in \mathbb{Q}_{<\sqrt{2}} \Rightarrow \exists b'\ \left(= \left[b + {}^{(2-b^2)}\!/_{5}\right] \text{ for example}\right) \in \mathbb{Q}_{<\sqrt{2}}$ slightly bigger than b
- $b \in \mathbb{Q}_{>\sqrt{2}} \Rightarrow \mathbb{Q}_{<\sqrt{2}}$ has an upper bound ($\left[b - {}^{(b^2-2)}\!/_{2b}\right]$ for example) slightly smaller than b
- So $b = \sqrt{2} \notin \mathbb{Q}$.

[xviii] *I.e.* $\varphi_{m/n} = \varphi_{p/q} \xleftrightarrow{\forall m,n,p,q \in \mathbb{Z}_{>0}} m/n = p/q$

[xix] *Viz.*, $\forall a \,\&\, b \in \mathbb{Q}$, $\varphi_{a+b} = \varphi_a + \varphi_b$, $\varphi_{ab} = \varphi_a \varphi_b$, $\varphi_a > 0 \iff a > 0$ *etc.* (with $a \neq 0$ for the multiplicative inverse condition)

B.7 An ordered field is *complete* if every its non-empty subset *bounded above* has a least upper bound; denote the field by $\mathbb{R}$ and call it the field of *real numbers*
B.8 $\widetilde{r}$ is *Dedekind's cut* if

- $\varnothing \subset \widetilde{r} \subset \mathbb{Q}$
- $q \in \mathbb{Q}_{<r\in\widetilde{r}} \Rightarrow q \in \widetilde{r}$
- $\widetilde{r}$ has no largest element

Denote the set of all Dedekind's cuts by $\widetilde{\mathbb{R}}$
Remark 6. Intuitively, $\widetilde{r} = \mathbb{Q}_{<r} \approx r \in \mathbb{R} \approx \widetilde{\mathbb{R}}$.
B.9 $S\backslash A := \{s \in S | s \notin A\}$ is the *set difference* from A to S. If $A \subseteq S$, then $S\backslash A$ is A's *complement* in S
B.10 ***Make*** $\widetilde{\mathbb{R}}$ ***a field*** $\forall \widetilde{r}_{i=1,2} \in \widetilde{\mathbb{R}}, \widetilde{\mathbb{R}} \ni$

- $\sum_{i=1,2} \widetilde{r}_i := \left\{\sum_{i=1,2} r_i \middle| r_{j=1,2} \in \widetilde{r}_j\right\}$
- $\widetilde{0} := \mathbb{Q}_{<0}$
- $-\widetilde{r} := \left\{r \in \mathbb{Q} \middle| (\mathbb{Q}\backslash\widetilde{r})^{<-r} \neq \varnothing\right\}$
- $\prod_{i=1}^{2} \widetilde{r}_i := \left\{\begin{array}{ll} \left\{\prod_{i=1}^{2} r_i \middle| r_{j=1,2} \in \widetilde{r}_j^+\right\} \cup \mathbb{Q}_{\leq 0} & \text{if } \widetilde{r}^+_{j=1,2} \neq \varnothing \\ \left\{\prod_{i=1}^{2} r_i \middle| r_j \in \widetilde{r}_j, r_{3-j} \in \mathbb{Q}\backslash\widetilde{r}_{3-j}\right\} & \text{if } \widetilde{r}^+_{j\in\{1,2\}} = \varnothing \neq \widetilde{r}^+_{3-j} \\ \left\{q \in \mathbb{Q} \middle| \exists r_{i=1,2} \in \widetilde{r}_i^- : q < \prod_{i=1}^{2} r_i\right\} & \text{if } \widetilde{r}^+_{j=1,2} = \varnothing \end{array}\right\}$ with $\widetilde{r}^+ := \widetilde{r}^{>0}$[xx]

and $\widetilde{r}^- := (\mathbb{Q}\backslash\widetilde{r})^{\leq 0}$

- $\widetilde{1} := \mathbb{Q}_{<1}$
- $\widetilde{r}^{-1} := \left\{r \in \mathbb{Q} \middle| (\mathbb{Q}\backslash\widetilde{r})^{<r^{-1}} \neq \varnothing\right\}$

Make field $\widetilde{\mathbb{R}}$ ***ordered*** define $\widetilde{r} \in \widetilde{\mathbb{R}}$ to be *positive* if $\exists b \in \widetilde{r} : b > 0$[xxi]
B.7 *The ordered field* $\widetilde{\mathbb{R}}$ *is complete; i.e.* $\varnothing \subset \widetilde{R} \subset \widetilde{\mathbb{R}} \wedge \widetilde{R}$ *bounded above* $\Rightarrow \widetilde{R}$ *has a least upper bound* $\bigcup_{\widetilde{r}\in\widetilde{R}} \widetilde{r}$

C Supremum & infimum

Property (Archimedian) $\forall r \in \mathbb{R}\ \exists z \in \mathbb{Z}_{>0} : r < z$. *I.e.* $\forall r \in \mathbb{R}^{>0}\ \exists z \in \mathbb{Z}_{>0} : z^{-1} < r$
C.1 $\forall a \in \mathbb{R}^{<b\in\mathbb{R}}\ \exists q \in \mathbb{Q}_{\in(a,b)}$
C.1 $b \in \mathbb{R}$ is a *lower bound* of $A \subseteq \mathbb{R}$ if $b \leq a\ \forall a \in A$
E.g. For both $\mathbb{R}^{>3}$ and $\mathbb{R}^{\geq 3}$, every $b \in \mathbb{R}^{\leq 3}$ is a lower bound, and 3 is the *greatest* lower bound.
Remark 7. A greatest lower bound of $A \subseteq \mathbb{R}$, if it exists, is unique.
C.2 *Every non-empty* $A \subseteq \mathbb{R}$ *bounded below has a greatest lower bound*
C.2 $\forall A \subseteq \mathbb{R}$, its *supremum & infimum* are respectively

$$\sup_A := \begin{cases} \text{A's least upper bound} & \text{if A bounded above} \wedge A \neq \varnothing \\ \infty & \text{if A has no upper bound} \\ -\infty & \text{if } A = \varnothing \end{cases}$$

[xx]Think of the condition $\widetilde{r}^+ \neq \varnothing$ as equivalent to $\widetilde{r} > \widetilde{0}$
[xxi]$\widetilde{r}_1 \subset \widetilde{r}_2 \iff \widetilde{r}_1 < \widetilde{r}_2 \overset{\text{definition}}{\iff} (\widetilde{r}_2 - \widetilde{r}_1)$ positive

$$\& \ \mathrm{inf}_A := \begin{cases} \text{A's greatest lower bound} & \text{if A bounded below} \wedge A \neq \varnothing \\ -\infty & \text{if A has no lower bound} \\ \infty & \text{if } A = \varnothing \end{cases}$$ ■

C.3 $r \in \mathbb{R}$ is *irrational* if $r \notin \mathbb{Q}$; *i.e.* $r \in \mathbb{R} \backslash \mathbb{Q}$ ●
C.3 $\exists r \in \mathbb{R}^{>0} : r^2 = 2$. *I.e.* $\exists r = \sqrt{2} \in \mathbb{R} \backslash \mathbb{Q}$ ●
C.4 $\forall a \in \mathbb{R}^{<b \in \mathbb{R}} \ \exists r \in (\mathbb{R} \backslash \mathbb{Q})^{\in (a,b)}$ ■
C.4 $(-\infty, \infty) := \mathbb{R}$, with

- the ordering $>$ on $\mathbb{R}$ extended to $[-\infty, \infty] := \mathbb{R} \cup \{\pm\infty\}$ as
  - $a < \infty \ \forall a \in [-\infty, \infty) := \mathbb{R} \cup \{-\infty\}$
  - $-\infty < a \ \forall a \in (-\infty, \infty] := \mathbb{R} \cup \{\infty\}$
- $\forall a, b \in [-\infty, \infty]$
  - $a < b \iff b > a$
  - $a \leq b \iff a < b \vee a = b \iff b \geq a$ ●

C.5 $I \in [-\infty, \infty]$ is an *interval* if $(a, b) \subseteq I \ \forall a, b \in I$ ●
C.5 *$\forall$interval $I \in [-\infty, \infty]$ $\exists a \ \& \ b \in [-\infty, \infty]$: $(a, b) \subseteq I \subseteq [a, b]$. So $I = (a, b) \vee [a, b] \vee (a, b] \vee [a, b)$* ■

D Open & closed subsets of $\mathbb{R}^n$

D.1 $\mathbb{R}^n := \left\{ (x_1, \ldots, x_n) \equiv (x_i)_{i=1}^n \middle| x_{j=1,\ldots,n} \in \mathbb{R} \right\}$ is the set of all ordered n-tuples of real numbers ●
D.2 $\forall x = (x_i)_{i=1}^n \in \mathbb{R}^n$, $\|x\| := \sqrt{\sum_{i=1}^n |x_i|^2}$, $\|x\|_\infty := \max\{|x_i|\}_{i=1}^n$ ●
D.3 A sequence $a_1, a_2, \cdots \in \mathbb{R}^n$ *converges* to a *limit* $L = \lim_{k \to \infty} a_k$ if $\forall \epsilon > 0 \ \exists m \in \mathbb{Z}_{>0}$: $\|a_{\forall k \geq m} - L\|_\infty < \epsilon$ ●
Remark 8. $\lim_{k \to \infty} a_k = L \xLeftrightarrow{\text{definition D.3}} \lim_{k \to \infty} \|a_k - L\|_\infty = 0$
$\xLeftrightarrow{\|x\|_\infty \leq \|x\| \leq \sqrt{n}\|x\|_\infty \ \forall x \in \mathbb{R}^n} \lim_{k \to \infty} \|a_k - L\|$.
D.1 *A convergent sequence $a_1, a_2, \cdots \in \mathbb{R}^n$ converges coordinate-wise; i.e. $\lim_{k \to \infty} \left(a_k = (a_{k,j})_{j=1}^n \right) = L = (L_j)_{j=1}^n$ iff $\lim_{k \to \infty} a_{k, \forall j \in \{i\}_{i=1}^n} = L_j$* ■
D.4 $\forall x \in \mathbb{R}^n \ \forall \delta > 0$, the *open cube* $B_{x,\delta} := \left\{ y \in \mathbb{R}^n \middle| \|y - x\|_\infty < \delta \right\}$ ●
D.5 An *open interval* $I = (a, b) \subseteq \mathbb{R}$ for some $a, b \in [-\infty, \infty]$ ●
D.6 $X \subseteq \mathbb{R}^n$ is

Open if $B_{\forall x \in X, \exists \delta > 0} \subseteq X$

Closed if its complement in $\mathbb{R}^n$ is open ●
Remark 9. Instead of open cubes, open sets could have been equivalently defined using open balls $\left\{ y \in \mathbb{R}^n \middle| \|y - x\| < \delta \right\} \subseteq B_{x,\delta} \subseteq \left\{ y \in \mathbb{R}^n \middle| \|y - x\| < \sqrt{n}\delta \right\}$.
D.7 $\forall$collection $\mathcal{A}$ of a set S's subsets, the *union* $\bigcup_{E \in \mathcal{A}} E := \{x \in S | \exists E \in \mathcal{A} : x \in E\}$ and the *intersection* $\bigcap_{E \in \mathcal{A}} E := \{x \in S | x \in E \ \forall E \in \mathcal{A}\}$ ■
E.g. $\bigcup_{k=1}^\infty [1/k, 1 - 1/k] = (0, 1)$, $\bigcap_{k=1}^\infty (-1/k, 1/k) = \{0\}$.
D.2 *The union of every collection of open subsets of $\mathbb{R}^n$ is open in $\mathbb{R}^n$; so as the intersection of every finite collection of open subsets of $\mathbb{R}^n$* ■
D.8 A set C is *countable* if $C = \varnothing \vee C = \{c_1, c_2, \ldots\}$ for some sequence $c_1, c_2, \ldots$ of elements of C ●
Remark. Every finite set is countable. If C is infinite countable, then it can be written as $\{b_1, b_2, \ldots\}$ of distinct elements.

D.3 *$\mathbb{Q}$ is countable* ■

Proof. Start with the list $\{-1, 0, 1\}$ at step 1, adjoin to the list in increasing order the rationals $\in [-n,, n]$ that can be written in the form m/n for some $m \in \mathbb{Z}$ at step n, and continue in this fashion to produce a sequence containing each rational □

D.9 A sequence $E_1, E_2, \ldots$ of sets is <u>*disjoint*</u> if $E_{\forall j \neq k} \cap E_k = \varnothing$ ●
D.4 *$A \subseteq \mathbb{R}$ open iff A the countable disjoint union of open intervals* ■
D.5 *$A \subseteq \mathbb{R}^n$ closed iff $A \ni$ limit of every convergent sequence of elements of A* ■
Laws (de Morgan's) *$\forall$collection $\mathcal{A}$ of subsets of some set X,* $X \backslash \bigcup_{E \in \mathcal{A}} E = \bigcap_{E \in \mathcal{A}} (X \backslash E)$, $X \backslash \bigcap_{E \in \mathcal{A}} E = \bigcup_{E \in \mathcal{A}} (X \backslash E)$ ■
D.6 *The intersection of every collection of closed subsets of $\mathbb{R}^n$ is closed in $\mathbb{R}^n$; so as the union of every finite collection of closed subsets of $\mathbb{R}^n$* ■
D.7 *The only subsets of $\mathbb{R}^n$ that are both open and closed are $\varnothing$ and $\mathbb{R}^n$* ■

E Sequences & continuity

E.1 A sequence $a_1, a_2, \cdots \in \mathbb{R}$ is

Increasing if $a_{\forall k \in \mathbb{Z}_{>0}} \leq a_{k+1}$

Decreasing if $a_{\forall k \in \mathbb{Z}_{>0}} \geq a_{k+1}$

Monotone if it is either increasing or decreasing ●

E.2 • $A \subseteq \mathbb{R}^n$ is <u>*bounded*</u> if $\sup\{\|a\|_\infty\}_{a \in A} < \infty$

• A map into $\mathbb{R}^n$ is <u>*bounded*</u> if its range is a bounded subset of $\mathbb{R}^n$. Particularly, a sequence $a_1, a_2, \cdots \in \mathbb{R}^n$ is bounded if $\sup\{\|a_k\|_\infty\}_{k \in \mathbb{Z}_{>0}} < \infty$ ●
E.1 *Every bounded monotone sequence of real numbers converges* ■
E.3 $a_{k_1}, a_{k_2}, \ldots$, with $k_{i=1,2,\ldots} \in \mathbb{Z}_{>0}$ and $k_1 < k_2 < \cdots$, is a <u>*subsequence*</u> of a sequence $a_1, a_2, \ldots$ ●
E.2 *Every sequence of real numbers has a monotone subsequence* ■
E.3 **(Bolzano-Weierstrass's)** *Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence* ■
E.4 *Every sequence of elements of a closed bounded $F \subseteq \mathbb{R}^n$ has a subsequence that converges to an element of F* ■
E.4 $A \xrightarrow{f} \mathbb{R}^n \ \forall A \subseteq \mathbb{R}^m$ is <u>*continuous*</u>

At $b \in A$ if $\forall \epsilon > 0 \ \forall a \in A \ \exists \delta > 0 : \|a - b\|_\infty < \delta \Rightarrow \left\|f_a - f_b\right\|_\infty < \epsilon$

On A if it is continuous at every $b \in A$ ●
E.5 *$A \xrightarrow{f} \mathbb{R}^n \ \forall A \subseteq \mathbb{R}^m$ is continuous at $b \in A$ iff $f_{b_k} \xrightarrow{k \to \infty} f_b$ $\forall$sequence $b_{k=1,2,\ldots} \in A$ that converges at b* ■
E.5 $A \xrightarrow{f} \mathbb{R}^n \quad \forall A \subseteq \mathbb{R}^m$ is <u>*uniformly continuous*</u> if $\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall a, b \in A :$ $\|a - b\|_\infty < \delta \Rightarrow \left\|f_a - f_b\right\|_\infty < \epsilon$ ●
Example E.1 $\mathbb{R} \xrightarrow{f : x \mapsto x^2} \mathbb{R}$ is continuous but not uniformly continuous.
E.6 *Every continuous $\mathbb{R}^n$-valued map on a closed bounded subset of $\mathbb{R}^m$ is uniformly continuous* ■
E.7 *Every continuous real-valued map of a closed bounded subset of $\mathbb{R}^m$ attains its maximum and minimum* ■
E.6 $\forall S \xrightarrow{f} T$ between sets S and T, $f_X := \{f_x\}_{x \in X}$ is the <u>*image*</u> of $X \subseteq S$ under f ●

E.8 *A continuous* $\mathsf{F} \xrightarrow{f} \mathbb{R}^n$ *of a closed bounded* $\mathsf{F} \subseteq \mathbb{R}^m$ *is a closed bounded subset of* $\mathbb{R}^n$ ■