Wigner's Theorem

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A proof based on *The quantum theory of fields, vol. 1: Foundations* by Weinberg, S. (1995).

QM Assumptions

(1) $\underbrace{\textit{Physical sates}}_{\Phi,\Psi,\text{ etc.}}$ are vectors/rays $\in$ **Hilbert space** $\mathcal{H}$—$\underbrace{\text{some complex vector space}}_{\xi\Phi+\eta\Psi\in\mathcal{H},\,\forall\,\xi,\eta\in\mathbb{C}}$ with a $\underbrace{\text{norm}}_{(\Phi,\Psi)\in\mathbb{C}}$ s.t. (i) $(\Phi,\Psi) = (\Psi,\Phi)^*$ (ii) $(\xi_1\Phi_1+\xi_2\Phi_2,\eta\Psi) = \eta\left[\xi_1^*(\Phi_1,\Psi)+\xi_2^*(\Phi_2,\Psi)\right]$ (iii) $(\Psi\neq 0,\Psi) > (0,0) = 0$.

(2) *Observables* are $\underbrace{\text{Hermitian operators}}_{A^\dagger=A}$ s.t. $\underbrace{A(\xi\Phi+\eta\Psi)=\xi A\Phi+\eta A\Psi}_{\text{an automorphism Aut}(\mathcal{H})}$.

(3) A ray $\mathcal{R}\big|_{\ni\Psi}$ has a *definite* $\underbrace{\text{eigenvalue } \alpha}_{A\Psi=\alpha\Psi}$ $\forall$ operator A.[1] Testing a system in $\mathcal{R}$ brings itself into one of the orthogonal rays $\mathcal{R}_{n=1,2,\dots}\big|_{\ni\Psi_n}$ with transition probability $P(\mathcal{R}\to\mathcal{R}_n) := |(\Psi,\Psi_n)|^2 = P(\mathcal{R}_n\to\mathcal{R})$, s.t. $\sum_n P(\mathcal{R}\to\mathcal{R}_n) = 1$ for a complete set $\{\Psi_n\}$.

NB (i) Pick a $\underbrace{\text{normalised}}_{(\Psi,\Psi)=1}$ vector $\underbrace{\Psi}_{\sim e^{i\theta}\Psi}$ as (a representative of) a *ray* $\in\mathcal{H}$ (ii) define the **adjoint** $A^\dagger$ of a *linear operator* A as $(\Phi,A^\dagger\Psi) := \underbrace{(A\Phi,\Psi)}_{(\Psi,A\Phi)^*}$, or $(\Phi,A^\dagger\Psi) := \underbrace{(A\Phi,\Psi)^*}_{(\Psi,A\Phi)}$ for an *antilinear operator* A.

Wigner's Theorem An $\underbrace{\text{invertible transition-probability-preserving}}_{\text{symmetry}}$ ray-transformation T s.t.

$$\underbrace{P\left(\mathcal{R}'\Big|_{\ni\Psi'=U\Psi}=T\mathcal{R}\leftrightarrow\mathcal{R}'_n\Big|_{\ni\Psi'_n=U\Psi_n}=T\mathcal{R}_n\right)}_{|(\Psi',\Psi'_n)|^2} = \underbrace{P(\mathcal{R}\leftrightarrow\mathcal{R}_n)}_{|(\Psi,\Psi_n)|^2} \tag{1}$$

$\Rightarrow$ a(n) (anti)unitary & (anti)linear operator U on $\mathcal{H}$ s.t.

$$\begin{cases}(U\Phi,U\Psi)=(\Phi,\Psi)\Leftrightarrow U^\dagger U=1\\ U(\xi\Phi+\eta\Psi)=\xi U\Phi+\eta U\Psi\end{cases}\quad\text{or}\quad\begin{cases}(U\Phi,U\Psi)=(\Phi,\Psi)^*\Leftrightarrow U^\dagger U=1\\ U(\xi\Phi+\eta\Psi)=\xi^*U\Phi+\eta^*U\Psi\end{cases}. \tag{2}$$

Pf. A complete $\overbrace{\text{orthogonal set }\{\Psi_n\}}^{(\Psi_{n_1},\Psi_{n_2})=\delta_{n_1,n_2}}$ $\xRightarrow{|(\Psi'_{n_1},\Psi'_{n_2})|^2\overset{(1)}{=\!=}\delta_{n_1,n_2}\xRightarrow{(\Psi'_{n_1},\Psi'_{n_2})\geq 0}(\Psi'_{n_1},\Psi'_{n_2})=\delta_{n_1,n_2}}$ another complete orthogonal set $\{U\Psi_n\}$[2] $\Rightarrow$ the expansions

$$\Psi\Big|_{\in\mathcal{R}} = \sum_{n=1}^{N} C_n\Psi_n \quad \& \quad (U\Psi)\Big|_{\in T\mathcal{R}} = \sum_{n=1}^{N} C'_n U\Psi_n \tag{3}$$

for a given Ψ & its counterpart $U\Psi$. To carry on, we must decide the *relative phases* for the transformed basis $\{U\Psi_n\}$: $\left|\left(\sum_{i=1}^{I=1,\dots,N}\Psi_{n_i},\Psi\right)\right|^2 \overset{(1)}{=\!=} \left|\left(U\sum_{i=1}^{I=1,\dots,N}\Psi_{n_i},U\Psi\right)\right|^2 \xRightarrow[(3)]{\Psi=\Psi_{n_{i=1,\dots,I}}}$

$$U\sum_{i=1}^{I=1,\dots,N}\Psi_{n_i} = \sum_{i=1}^{I=1,\dots,N} e^{i\theta_i}U\Psi_{n_i}, \tag{4}$$

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[1] An order-N Hermitian matrix has N orthogonal eigenvectors with distinct *real* eigenvalues.A

[2] Given a non-zero $\Psi'\notin\{\Psi'_n\}$ but $\perp$ all Ψ'_n, the 1-1 inverse map will take it back to some non-zero $\Psi''\notin\{\Psi_n\}$ s.t. $|(\Psi_n,\Psi'')|^2 = |(\Psi'_n,\Psi')|^2 = 0 \Rightarrow$ impossible, as $\{\Psi_n\}$ is already complete.

& our *convention* is $\theta_{i=1,\dots,I} \equiv 0\ \forall\ I \in \mathbb{N}$. Now we can investigate the relation between the two sets of expansion coefficients, $\{C_n\}$ & $\{C'_n\}$. First, $|(\Psi_n, \Psi)|^2 \overset{(1)}{=\!=} |(U\Psi_n, U\Psi)|^2 \overset{(3)}{\Longrightarrow}$

$$\left|\frac{C'_n}{C_n}\right| = 1. \tag{5}$$

Then $\left|\left(\sum_{i=1}^{2} \Psi_{n_i}, \Psi\right)\right|^2 \overset{(1)}{=\!=} \left|\left(U \sum_{i=1}^{2} \Psi_{n_i} \xlongequal{\text{phase convention}} \sum_{i=1}^{2} U\Psi_{n_i}, U\Psi\right)\right|^2 \overset{(3)}{\Longrightarrow}$

$$\left|\frac{\sum_{i=1}^{2} C'_{n_i}}{\sum_{i=1}^{2} C_{n_i}}\right| = 1 \overset{(5)}{\Longrightarrow} \underbrace{\frac{\left|1 + C'_{n_2}/C'_{n_1}\right|^2 - 1 - \left|C'_{n_2}/C'_{n_1}\right|^2}{\left|1 + C_{n_2}/C_{n_1}\right|^2 - 1 - \left|C_{n_2}/C_{n_1}\right|^2}}_{\operatorname{Re}\left(C'_{n_2}/C'_{n_1}\right)/\operatorname{Re}\left(C_{n_2}/C_{n_1}\right)} = 1 \xRightarrow[(5)]{|C'_{n_2}/C'_{n_1}|=|C_{n_2}/C_{n_1}|} \frac{C_{n_2}}{C_{n_1}} = \frac{C'_{n_2}}{C'_{n_1}} \text{ or } \left(\frac{C'_{n_2}}{C'_{n_1}}\right)^*. \tag{6}$$

Next, think of $\left|\left(\sum_{i=1}^{N\geq 3} \Psi_{n_i}, \Psi\right)\right|^2 \overset{(1)}{=\!=} \left|\left(U \sum_{i=1}^{N\geq 3} \Psi_{n_i} \xlongequal{\text{phase convention}} \sum_{i=1}^{N\geq 3} U\Psi_{n_i}, U\Psi\right)\right|^2$, with $\{C_n\}$ & $\{C'_n\}$ satisfying eq (6) in the way that $C_{n_2}/C_{n_1} \equiv C'_{n_2}/C'_{n_1}$ or $\equiv (C'_{n_2}/C'_{n_1})^*$. To prove this '$\equiv$', let us single out some C_{n_1}, & assume that $C_{n_i}/C_{n_1} = C'_{n_i}/C'_{n_1}\ \forall\ i \in \{2, \dots, M < N\}$ & $C_{n_i}/C_{n_1} = \left(C'_{n_i}/C'_{n_1}\right)^*\ \forall\ i \in \{M+1, \dots, N\}$. Then, from eqs (5) & (6),

$$\begin{aligned} 0 &= \left|1 + \sum_{i=2}^{N} \frac{C'_{n_i}}{C'_{n_1}}\right|^2 - \left|1 + \sum_{i=2}^{N} \frac{C_{n_i}}{C_{n_1}}\right|^2 = \sum_{i,j=2}^{N} \left(\frac{C'_{n_i} C'^*_{n_j} - C_{n_i} C^*_{n_j}}{|C_{n_1}|^2} + i \leftrightarrow j\right) \\ &= \sum_{i=2}^{M} \sum_{j=M+1}^{N} \left[-4\left(\operatorname{Im} \frac{C_{n_i}}{C_{n_1}}\right) \operatorname{Im} \frac{C_{n_j}}{C_{n_1}}\right] = (-4)^{(M-1)(N-M)} \underbrace{\left(\sum_{i=2}^{M} \operatorname{Im} \frac{C_{n_i}}{C_{n_1}}\right)}_{x} \underbrace{\left(\sum_{j=M+1}^{N} \operatorname{Im} \frac{C_{n_j}}{C_{n_1}}\right)}_{y} \end{aligned} \tag{7}$$

$\Rightarrow$ either x or y must $\in \mathbb{R}$, which is an unreasonable stronger statement (transcendental coef-constraint). In brief, $\forall$ symmetric transformation T, the corresponding U satisfies

$$U \sum_n C_n \Psi_n = \sum_n C_n U\Psi_n \quad \text{or} \quad \sum_n C^*_n U\Psi_n. \tag{8}$$

Our final step is to prove that the '=' in eq (8) is actually '$\equiv$', which leads directly to the property (2). In fact, the mixed unitary-antiunitary case with a stronger constraint

$$\begin{cases} \underbrace{U \sum_n A_n \Psi_n}_{\Psi} = \underbrace{\sum_n A_n U\Psi_n}_{\Psi'} \\ \underbrace{U \sum_n B_n \Psi_n}_{\Phi} = \underbrace{\sum_n B^*_n U\Psi_n}_{\Phi'} \end{cases} \xRightarrow[(1)]{|(\Psi,\Phi)|^2 = |(\Psi',\Phi')|^2} \underbrace{\sum_{n_1,n_2} \left(\operatorname{Im} A^*_{n_1} A_{n_2}\right) \operatorname{Im} B^*_{n_1} B_{n_2}}_{\left(|\sum_n A_n B^*_n|^2 - |\sum_n A_n B_n|^2\right)/\prod_{n_1,n_2}(-4)} = 0 \tag{9}$$

is again unreasonable, as the probability preservation (1) is automatically satisfied with the overall (anti)unitary condition '$\equiv$'. □