1. Calculating the Two Constants of Motion

The equation of motion of a massive particle follows from the principle of least action $\delta S = \delta\int_{s_1}^{s_2}\sqrt{g_{ik}\dot{x}^i\dot{x}^k}\,ds = 0$. As the integrand is unity (in fact, $\sqrt{g_{ik}\dot{x}^i\dot{x}^k} = \frac{ds}{ds} = 1$) all along the path, there should be

$$\delta I = \delta\int_{s_1}^{s_2} g_{ik}\dot{x}^i\dot{x}^k\,ds = \delta\int_{s_1}^{s_2}\left[\left(1-\frac{2GM}{c^2 r}\right)(c\dot{t})^2 - \left(1-\frac{2GM}{c^2 r}\right)^{-1}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\varphi}^2\right]ds = 0,$$

where the dot denotes the derivative with respect to the interval s.

Take I as the Lagrangian, and define $q_{0,1,2,3} \equiv ct, r, \theta, \varphi$. Using the Euler–Lagrange equation,

$$\frac{\partial I}{\partial q_i} - \frac{d}{ds}\frac{\partial I}{\partial \dot{q}_i} = 0, i = 0,1,2,3,$$

we have $\frac{d}{ds}\left[\left(\frac{2GM}{c^2 r} - 1\right)(c\dot{t})\right] = 0 ...(1.1),$

$\frac{\partial I}{\partial r} - \frac{d}{ds}\frac{\partial I}{\partial \dot{r}} = 0 ...(1.2),$

$\frac{d}{ds}\left(r^2\dot{\theta}\right) = r^2\sin\theta\cos\theta\dot{\varphi}^2 ...(1.3),$

$\frac{d}{ds}\left(r^2\sin^2\theta\dot{\varphi}\right) = 0 ...(1.4),$

In the centrally symmetric case, it is all right to take an arbitrary θ. In order not to violate (1.3), we can take $\theta \equiv \frac{\pi}{2}$, Then (1.1) and (1.4) can at once be integrated to give $\left(1-\frac{2GM}{c^2 r}\right)\dot{t} = \gamma_0 = \text{constant}$ and $r^2\dot{\varphi} = h = \text{costant}.$

1.1. A review of the derivation of the Schwarzschild metric

Consider a gravitational field possessing central symmetry. For all points located at the same distance from the center, the space-time metric, that is, the expression for the interval ds, must be the same. Using the natural units, $c = 1$, a general expression for ds^2 is

$$ds^2 = A(r,t)dt^2 - B(r,t)dr^2 - C(r,t)r^2(d\theta^2 + \sin^2\theta d\varphi^2) + a(r,t)drdt.$$

Suppose that the field is static, then $a(r,t) = 0$ and $A(r,t) = A(r), B(r,t) = B(r),$ $C(r,t) = C(r); A, B, C > 0.$ When $r \to \infty,$ $A, B, C \to 1$ (the Minkowski metric).

For eliminating the parameter C, we can make $\bar{r} = \sqrt{C(r)}r$, then

$$d\overline{r} = \sqrt{C}dr + \frac{r}{2\sqrt{C}}\frac{dC}{dr}dr = (\sqrt{C} + \frac{r}{2\sqrt{C}}\frac{dC}{dr})drds^2$$

$$\Rightarrow dr = \frac{1}{\sqrt{C} + \frac{r}{2\sqrt{C}}\frac{dC}{dr}}d\overline{r} \Rightarrow Bdr^2 = \frac{B}{\left(\sqrt{C} + \frac{r}{2\sqrt{C}}\frac{dC}{dr}\right)^2}d\overline{r}^2 \equiv \overline{B}d\overline{r}^2$$

$$\Rightarrow ds^2 = \overline{A}dt^2 - \overline{B}dr^2 - \overline{r}^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \Rightarrow ds^2 = Adt^2 - Bdr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) ...(1.1).$$

All the non-zero Christoffel symbols corresponding to the metric

$$\begin{pmatrix} A & & & \\ & -B & & \\ & & -r^2 & \\ & & & -r^2\sin^2\theta \end{pmatrix}$$ are

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{2A}\frac{dA}{dr} = \frac{A'}{2A}, \quad \Gamma^1_{00} = \frac{A'}{2B}, \Gamma^1_{11} = \frac{B'}{2B}, \Gamma^1_{22} = -\frac{r}{B}, \quad \Gamma^1_{33} = -\frac{r\sin^2\theta}{B},$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}, \quad \Gamma^2_{33} = -\sin\theta\cos\theta, \quad \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}, \quad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta.$$

For obtaining the Christoffel symbols above, we first create a Lagranian

$$L \equiv g_{ik}\dot{x}^i\dot{x}^k = \left[Adt^2 - Bdr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right],$$

where the dot denotes the derivative with respect to a parameter λ (we use λ instead of s because ds could sometimes goes to zero, which makes $\frac{1}{ds}$ meaningless). Then for example, $\frac{\partial L}{\partial t} - \frac{d}{ds}\frac{\partial L}{\partial \dot{t}} = 0 \Rightarrow \ddot{t} + \frac{A'}{A}\dot{t} = 0 \Rightarrow \Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{2A}\frac{dA}{dr} = \frac{A'}{2A}$, supposing that $x_0 \equiv t$, $x_1 \equiv r, x_2 \equiv \theta, x_3 \equiv \varphi$. Similarly we can obtain the others.

The base of the creation above is that the equation of motion of a massive particle follows from the principle of least action $\delta S = \delta\int_{s_1}^{s_2}\sqrt{g_{ik}\dot{x}^i\dot{x}^k}\,ds = 0$. As the integrand is unity all along the path, there should be (L is the Lagranian)

$$\delta L = \delta\int_{s_1}^{s_2} g_{ik}\dot{x}^i\dot{x}^k\,ds = 0.$$

The next step is to calculate the Ricci tensors

$$R_{ik} = R^m_{imk} = -\Gamma^m_{im,k} + \Gamma^m_{ik,m} - \Gamma^n_{im}\Gamma^m_{nk} + \Gamma^n_{ik}\Gamma^m_{nm} = -\left(\ln\sqrt{-g}\right)_{,i,k} + \Gamma^m_{ik,m} - \Gamma^n_{im}\Gamma^m_{nk} + \Gamma^n_{ik}\left(\ln\sqrt{-g}\right)_{,n}.$$

And all the non-zero ones are

$$R_{00} = \frac{A''}{2B} - \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{A'}{rB} ...(1.2), \quad R_{11} = -\frac{A''}{2A} + \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{A'}{rB} ...(1.3),$$

$$R_{22} = 1 - \frac{1}{B} - \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right) ...(1.4), \quad R_{33} = \left[1 - \frac{1}{B} - \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right)\right]\sin^2\theta = R_{22}\sin^2\theta ...(1.5).$$

In empty space the energy momentum tensor $T_{ik} = 0$, and the equations of the gravitational field $R_{ik} - \frac{1}{2}g_{ik}R = \frac{8\pi k}{c^4}T_{ik}$ reduce to the equation $R_{ik} = 0$. That is

$$R_{00}=\frac{A''}{2B}-\frac{A'}{4B}\left(\frac{A'}{A}+\frac{B'}{B}\right)+\frac{A'}{rB}=0...(1.2),\quad R_{11}=-\frac{A''}{2A}+\frac{A'}{4A}\left(\frac{A'}{A}+\frac{B'}{B}\right)+\frac{A'}{rB}=0...(1.3),$$

$$R_{22}=1-\frac{1}{B}-\frac{r}{2B}\left(\frac{A'}{A}-\frac{B'}{B}\right)=0...(1.4),\quad R_{33}=R_{22}\sin^2\theta=0...(1.5).$$

$(1.2)+(1.3)\Rightarrow\frac{(AB)'}{rB}\Rightarrow AB=\text{const}=\lim_{r\to\infty}AB=1\Rightarrow A=\frac{1}{B}\Rightarrow\frac{A'}{A}=-\frac{B'}{B}...(1.5).$ From (1.5) and (1.4) we have $1-\left(\frac{r}{B}\right)'=0$, that is, $\frac{d}{dr}\left(\frac{r}{B}\right)'=1$ and $\frac{r}{B}=r+C_0...(1.6),$ where C_0 is the constant of integration. Then the equation (1.1) becomes

$$ds^2=\left(1+\frac{C_0}{r}\right)dt^2-\left(1+\frac{C_0}{r}\right)^{-1}dr^2-r^2\left(d\theta^2+\sin^2\theta d\varphi^2\right)...(1.7).$$

To the case of a weak field in the approximation, $g_{00}=1+2\phi=1-\frac{2GM}{r}$ (at large distances, where the field is weak, Newton's law should hold). Then $C_0=-2GM$, the equation (1.7) becomes

$$ds^2=\left(1-\frac{2GM}{r}\right)dt^2-\left(1-\frac{2GM}{r}\right)^{-1}dr^2-r^2\left(d\theta^2+\sin^2\theta d\varphi^2\right)...(1.8).$$

From the natural units to the ordinary units, (1.8) becomes

$$ds^2=\left(1-\frac{2GM}{c^2r}\right)c^2dt^2-\left(1-\frac{2GM}{c^2r}\right)^{-1}dr^2-r^2\left(d\theta^2+\sin^2\theta d\varphi^2\right)...(1.9).$$

2. Physical Meaning and the Explicit Expressions of h and γ_0

In physics, the Newtonian limit is a mathematical approximation applicable to physical systems exhibiting (1) weak gravitation, (2) objects moving slowly compared to the speed of light $(v\ll c)$, and (3) slowly changing (or completely static) gravitational fields; the special relativistic limit is $r\to\infty$ that the space is Minkowski.

Using the two limits, $r\to\infty$ and $v\ll c$, we have $\frac{2GM}{c^2r}=\alpha\to 0$ and $\frac{v}{c}=\beta\to 0.$ Besides, we take $\theta=\frac{\pi}{2}$ to get the constants $h=r^2\dot{\varphi}$ and $\gamma_0=(1-\alpha)\dot{t}$ in Problem 1. In that case, $d\theta=0$, $ds=\sqrt{(1-\alpha)(cdt)^2-(1-\alpha)^{-1}dr^2-r^2d\varphi^2}$. Using the series expansion to calculate functions of α and β, we can write explicitly the expressions of γ_0 and h:

$$\gamma_0=(1-\alpha)\frac{dt}{ds}=\frac{(1-\alpha)dt}{\sqrt{(1-\alpha)(cdt)^2-(1-\alpha)^{-1}dr^2-r^2d\varphi^2}}$$

$$=\frac{1}{\sqrt{(1-\alpha)^{-1}c^2-(1-\alpha)^{-3}\left(\frac{dr}{dt}\right)^2-(1-\alpha)^{-2}(rd\varphi)^2}}=\frac{1}{c\sqrt{(1+\alpha)-(1+3\alpha)(v_r/c)^2-(1+2\alpha)(v_\varphi/c)^2}}$$

$$\overset{*}{=}\frac{1}{c\sqrt{1+\alpha-\left(v_r/c\right)^2-\left(v_\varphi/c\right)^2}}=\frac{1}{c\sqrt{1+\left[\alpha-\left(v/c\right)^2\right]}}\overset{**}{=}\frac{1}{c}\bullet\left[1-\frac{1}{2}\left(\alpha-\left(v/c\right)^2\right)\right]$$

$$=\frac{1}{m_0c^3}\left(m_0c^2+\frac{m_0v^2}{2}-\frac{GMm_0}{r}\right)=\frac{E}{m_0c^3}.$$

(*as α and $v_r/c\sim\beta,\ v_\varphi/c\sim\beta$ are all infinitesimal, we should neglect the two terms $\alpha\left(v_r/c\right)^2$ and $\alpha\left(v_\varphi/c\right)^2$; ** α and $\left(v/c\right)^2$ are both infinitesimal, we can do the series expansion about $\alpha-\beta^2$)

That is, $\gamma_0=\frac{E}{m_0c^3}$, where m_0 is the rest mass of an object and E is its energy. $E=\gamma_0m_0c^3=\text{constant}$ is the conservation of energy.

Similarly, we have

$$h=r^2\frac{d\varphi}{ds}=r^2\frac{d\varphi}{\sqrt{(1-\alpha)(cdt)^2-(1-\alpha)^{-1}dr^2-r^2d\varphi^2}}$$

$$=\frac{r^2\frac{d\varphi}{dt}}{\sqrt{(1-\alpha)c^2-(1-\alpha)^{-1}\left(\frac{dr}{dt}\right)^2-\left(\frac{d\varphi}{dt}\right)^2}}=\frac{rv_\varphi}{c\sqrt{1-\alpha-(1+\alpha)\left(v_r/c\right)^2-\left(v_\varphi/c\right)^2}}$$

$$=\frac{rv_\varphi}{c\sqrt{1-\alpha-\left(v/c\right)^2-\alpha\left(v_r/c\right)^2}}\approx\frac{rv_\varphi}{c\sqrt{1}}=\frac{rv_\varphi}{c}.$$

That is, $h=\frac{rv_\varphi}{c}$. And $m_0rv_\varphi=m_0hc=\text{constant}$ is the conservation of angular momentum.

3. Finding the Circular Orbit of a Photon/Massive Particles

Around a massive object M, the light propagates along a null geodesic, satisfying $ds^2=0$. Thus, we have

$$\frac{ds}{d\lambda}=\left(1-\frac{2GM}{c^2r}\right)\left(\frac{cdt}{d\lambda}\right)^2-\left(1-\frac{2GM}{c^2r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2-r^2\left(\frac{d\theta}{d\lambda}\right)^2-r^2\sin^2\theta\left(\frac{d\varphi}{d\lambda}\right)^2=0,$$

where the parameter λ is an arbitrary independent variable.

In this case, the equation of motion should be

$$\frac{d^2u}{d\varphi^2}+u=\frac{3GMu^2}{c^2},u=1/r\text{...(2.1)}.$$

If the orbit of the photon is a fixed circle, $\frac{d^2u}{d\varphi^2}=\frac{du}{d\varphi}=0$. Substituting it in (2.1), we get $u=c^2/3GM, r=3GM/c^2$, which means that the trajectory must be unique.

In the case of massive particles, the equation of motion is

$$\frac{d^2u}{d\varphi^2}+u-\frac{GM}{h^2}=\frac{3GMu^2}{c^2}, h=r^2\dot{\varphi}=\text{constant}...(2.2).$$

Again, we substitute $\frac{d^2u}{d\varphi^2}=\frac{du}{d\varphi}=0$ in the equation above and get

$$u=\frac{c^2}{6GM}\left(1\mp\sqrt{1-\frac{12G^2M^2}{c^2h^2}}\right), r=\frac{h^2}{2GM}\left(1\pm\sqrt{1-\frac{12G^2M^2}{c^2h^2}}\right), h\geq\frac{2\sqrt{3}GM}{c}.$$

4. Find the Energy-Momentum Tensor $T_{ik}=2\frac{\partial\Lambda}{\partial g^{ik}}-\Lambda g^{ik}$ Corresponding to a Scalar Field φ

If a Lagrangian density Λ does not depend on the derivatives of the metric tensor $g_{ik,l}$, then $\Lambda=\Lambda(g_{ik})$.

Consider a system whose action integral has the form

$$S=\frac{1}{c}\int\Lambda\left(g^{ik},g^{ik}_{,l}\right)\sqrt{-g}\,d\Omega=\frac{1}{c}\int\Lambda\left(g^{ik}\right)\sqrt{-g}\,d\Omega,$$

then the variation of S is

$$\delta S=\frac{1}{c}\int\frac{\partial\left(\Lambda\sqrt{-g}\right)}{\partial g^{ik}}\delta g^{ik}\,d\Omega.$$

We introduce the notation $\frac{\sqrt{-g}}{2}T_{ik}=\frac{\partial\left(\Lambda\sqrt{-g}\right)}{\partial g^{ik}}$ in order to satisfy that $T^k_{i;k}=0$. As we know, $dg=\frac{\partial g}{\partial g_{ik}}dg_{ik}=gg^{ik}dg_{ik}$, then

$$\begin{aligned}T_{ik}&=\frac{2}{\sqrt{-g}}\frac{\partial\left(\Lambda\sqrt{-g}\right)}{\partial g^{ik}}=\frac{2}{\sqrt{-g}}\frac{\partial\Lambda}{\partial g^{ik}}\sqrt{-g}+\frac{2\Lambda}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial g^{ik}}\\&=2\frac{\partial\Lambda}{\partial g^{ik}}+\frac{2\Lambda}{\sqrt{-g}}\left(-\frac{1}{2\sqrt{-g}}\frac{\partial g}{\partial g^{ik}}\right)=2\frac{\partial\Lambda}{\partial g^{ik}}-\frac{\Lambda}{g}\frac{\partial g}{\partial g^{ik}}\\&=2\frac{\partial\Lambda}{\partial g^{ik}}-\frac{\Lambda}{g}gg^{ik}=2\frac{\partial\Lambda}{\partial g^{ik}}-\Lambda g^{ik}.\end{aligned}$$

For $\Lambda=\frac{1}{2}\varphi_{;l}\varphi_{;m}g^{lm}-V(\varphi)=\frac{1}{2}\varphi_{,l}\varphi_{,m}g^{lm}-V$ (for a scalar, $\varphi_{;l}=\varphi_{,l}$), we have

$$T_{ik}=2\frac{\partial\Lambda}{\partial g^{ik}}-\Lambda g^{ik}=\varphi_{,l}\varphi_{,m}-\frac{1}{2}\varphi_{,l}\varphi_{,m}g^{lm}g^{ik}+Vg^{ik}.$$