QFT HW 01

Jimmy Wu
Dept. of Physics
Sun Yat-sen University

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Contents

1 The Coherent States $|z\rangle$

1.1 Introduction

1.1.1 One Expression of $|z\rangle$

Essentially any potential well can be approximated by a simple harmonic oscillator. The basic Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}. \tag{1}$$

where m is the particle's mass, ω the angular frequency of the oscillator, q the position operator and $p \equiv \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x}$ the momentum operator. The correspondent **annihilation and creation operators**, along with the **number operator**, can be defined as:

$$a \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(q + \frac{\mathrm{i}p}{m\omega}\right), \qquad a^\dagger \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(q - \frac{\mathrm{i}p}{m\omega}\right), \tag{2}$$

$$N \equiv a^\dagger a = \frac{H}{\hbar\omega} - \frac{1}{2}, \tag{3}$$

where $\left[a, a^\dagger\right] = 1$ and we denote an energy eigenket of N by its eigenvalue n:

$$N\left|n\right\rangle = n\left|n\right\rangle. \tag{4}$$

Note that $Na = [N, a] + aN$, $Na^\dagger = \left[N, a^\dagger\right] + a^\dagger N$, then

$$\begin{cases} [N, a] = -a \\ \left[N, a^\dagger\right] = a^\dagger \end{cases} \Rightarrow \begin{cases} Na\left|n\right\rangle = (n-1)a\left|n\right\rangle \\ Na^\dagger\left|n\right\rangle = (n+1)a\left|n\right\rangle \end{cases} \Rightarrow \begin{cases} a\left|n\right\rangle = c\left|n-1\right\rangle \\ a^\dagger\left|n-1\right\rangle = c^*\left|n\right\rangle \end{cases},$$

where c is a constant. In requirement of normalization, there should be

$$\left\langle n\right| a^\dagger a \left|n\right\rangle = \left\langle n\right| N \left|n\right\rangle = |c|^2 = n.$$

Taking c to be real and positive by convention, we finally get

$$a\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle, \qquad a^\dagger\left|n-1\right\rangle = \sqrt{n}\left|n\right\rangle. \tag{5}$$

Suppose that $|z\rangle$ are eigenkets of a:

$$a\left|z\right\rangle = z\left|z\right\rangle. \tag{6}$$

As $\left(a^\dagger\right)^2\left|0\right\rangle = \sqrt{n!}\left|n\right\rangle \Rightarrow \left\langle n\right| = \left\langle 0\right| \frac{a^n}{\sqrt{n!}} \Rightarrow \left\langle n|z\right\rangle = \left\langle 0\right| \frac{a^n}{\sqrt{n!}}\left|z\right\rangle = \frac{z^n}{\sqrt{n!}}\left\langle 0|z\right\rangle$,

$$\left|z\right\rangle = \sum_n \left|n\right\rangle\left\langle n|z\right\rangle = \sum_n \frac{z^n}{\sqrt{n!}}\left|n\right\rangle\left\langle 0|z\right\rangle. \tag{7}$$

In requirement of normalization, there must be $\langle z|z\rangle = 1$. Thus

$$\begin{aligned}
\langle z|z\rangle =& \left(\sum_n \frac{z^{*n}}{\sqrt{n!}} \langle z|0\rangle \langle n|\right)\left(\sum_n \frac{z^n}{\sqrt{n!}} |n\rangle \langle 0|z\rangle\right) \\
=& \left|\langle 0|z\rangle\right|^2 \sum_n \frac{|z|^{2n}}{\sqrt{n!}} = \left|\langle 0|z\rangle\right|^2 \exp\left(|z|^2\right) = 1.
\end{aligned}$$

That is,

$$|\langle 0|z\rangle| = \exp\left(-\frac{|z|^2}{2}\right),$$

$$|z\rangle = \sum_n \frac{z^n}{\sqrt{n!}} |n\rangle \langle 0|z\rangle = \exp\left(-\frac{|z|^2}{2}\right) \sum_n \frac{z^n}{\sqrt{n!}} |n\rangle . \tag{8}$$

From Eq. (8), we could tell that the so-called **coherent states**, $|z\rangle$, are superposition of $|n\rangle$. As for $z = 0$, we have

$$|0\rangle = \exp\left(-\frac{|0|^2}{2}\right) \sum_n \frac{0^n}{\sqrt{n!}} |n\rangle = 1 \cdot \frac{0^0}{\sqrt{0!}} |0\rangle = |0\rangle . \tag{9}$$

1.1.2 Another Expression

Note that $|n\rangle = \frac{\left(a^\dagger\right)^n}{\sqrt{n!}} |0\rangle$, then from Eq. (8) we have

$$\begin{aligned}
|z\rangle =& \exp\left(-\frac{|z|^2}{2}\right) \sum_n \frac{z^n}{\sqrt{n!}} \left[\frac{\left(a^\dagger\right)^n}{\sqrt{n!}} |0\rangle\right] \\
=& \exp\left(-\frac{|z|^2}{2}\right) \sum_n \frac{\left(za^\dagger\right)^n}{n!} |0\rangle \\
=& \exp\left(-\frac{|z|^2}{2}\right) \exp\left(za^\dagger\right) |0\rangle .
\end{aligned} \tag{10}$$

1.1.3 $\langle z_1|z_2\rangle$

$$\begin{aligned}
\langle z_1|z_2\rangle =& \exp\left(-\frac{|z_1|^2}{2}\right)\left(\sum_n \frac{z_1^{*n}}{\sqrt{n!}} \langle n|\right) \exp\left(-\frac{|z_2|^2}{2}\right)\left(\sum_n \frac{z_2^n}{\sqrt{n!}} |n\rangle\right) \\
=& \exp\left(-\frac{|z_1|^2 + |z_2|^2}{2}\right) \sum_n \frac{\left(z_1^* z_2\right)^n}{n!} = \exp\left(-\frac{|z_1|^2 + 2z_1^* z_2 + |z_2|^2}{2}\right).
\end{aligned} \tag{11}$$

1.1.4 Physical Meaning of $|z|^2$

From Eq. (8), $|z\rangle = \exp\left(-\frac{|z|^2}{2}\right)\sum_n \frac{z^n}{\sqrt{n!}}|n\rangle$, we know that $\left[\exp\left(-\frac{|z|^2}{2}\right)\right]^2 = \exp\left(-|z|^2\right) \approx 1-|z|^2$ is the probability of the system being in state $|z\rangle$. Thus $|z|^2$ is the probability of the system *not* being in state $|z\rangle$.

1.1.5 Transition Operator $D(z)$

① $D^\dagger(z) = D^{-1}(z)$
Consider two operators A and B which, by hypothesis, both commute with their commutator $[A,B]$. In this case, we shall have **Glauber's formula**:

$$e^{A+B} = e^A e^B e^{-[A,B]/2}. \tag{12}$$

According to definition,

$$D(z) = \exp\left(za^\dagger - z^*a\right) = \sum_n \frac{\left(za^\dagger - z^*a\right)^n}{n!}. \tag{13}$$

Thus

$$\begin{aligned} D^\dagger(z) = &\sum_n \frac{\left[\left(za^\dagger - z^*a\right)^n\right]^\dagger}{n!} = \sum_n \frac{\left[\left(za^\dagger - z^*a\right)^\dagger\right]^n}{n!} \\ = &\sum_n \frac{\left(z^*a - za^\dagger\right)^n}{n!} = \exp\left(z^*a - za^\dagger\right) \end{aligned}$$

$$\Rightarrow D^\dagger(z)D(z) = \underbrace{\exp\left(z^*a - za^\dagger\right)\exp\left(za^\dagger - z^*a\right)}_{\text{Glauber's formula}} = 1 \tag{14}$$

$$\Rightarrow D^\dagger(z) = D^{-1}(z). \tag{15}$$

② $D^{-1}(z)aD(z) = a + zI$
Baker-Hausdorff lemma: for any two arbitrary matrices A and B, there is

$$e^B A e^{-B} = A + [B,A] + \frac{1}{2!}[B,[B,A]] + \frac{1}{3!}[B,[B,[B,A]]] + \cdots. \tag{16}$$

Take $z^*a - za^\dagger = B$, then

$$D^{-1}(z)aD(z) = e^B a e^{-B} = a + [B,a] + \frac{1}{2!}[B,[B,a]] + \cdots. \tag{17}$$

$$
\begin{aligned}
&[a, a^\dagger] = 1 = I, \quad \text{where } I \text{ is the } \textbf{identity operator} \\
\Rightarrow &Ba = z^* aa - z a^\dagger a = z^* aa - z\left(aa^\dagger - I\right) = aB + zI \\
\Rightarrow &[B, a] = Ba - aB = zI, \\
\Rightarrow &[B, [B, a]] = [B, zI] = 0, \\
\Rightarrow &[B, [B, [B, a]]] = [B, 0] = 0, \\
\Rightarrow &\cdots .
\end{aligned}
$$

Finally, we get $D^{-1}(z) a D(z) = a + zI + 0 + \cdots = a + zI$.

3 $D(z_1)D(z_2) = D(z_1 + z_2)$
i) $|z\rangle = D(z)|0\rangle$
Since

$$
D(z) = \exp\left(z a^\dagger - z^* a\right) \xlongequal{\text{Glauber's formula}} \exp\left(-\frac{|z|^2}{2}\right)\exp\left(z a^\dagger\right)\exp\left(-z^* a\right)
$$

and

$$
\exp\left(-z^* a\right)|0\rangle = \sum_n \frac{\left(-z^* a\right)^n}{n!}|0\rangle = \frac{\left(-z^* a\right)^0}{0!}|0\rangle + \sum_{n=1} 0 = |0\rangle ,
$$

then

$$
D(z)|0\rangle = \exp\left(-\frac{|z|^2}{2}\right)\exp\left(z a^\dagger\right)\exp\left(-z^* a\right)|0\rangle = \exp\left(-\frac{|z|^2}{2}\right)\exp\left(z a^\dagger\right)|0\rangle = |z\rangle . \quad (18)
$$

ii) $D(z_1)D(z_2) \sim D(z_1 + z_2)$

$$
D(z_1)|z_2\rangle = |z_1 + z_2\rangle \iff D(z_1)D(z_2)|0\rangle = D(z_1 + z_2)|0\rangle . \quad (19)
$$

To prove that $D(z_1)|z_2\rangle = |z_1 + z_2\rangle$, we only need to demonstrate that

$$
D(z_1)D(z_2) = D(z_1 + z_2). \quad (20)
$$

This is true when z_1 and z_2 are collinear in the complex plane:

$$
\begin{aligned}
D(z_1)D(z_2) = &\underbrace{\exp\left(z_1 a^\dagger - z_1^* a\right)\exp\left(z_2 a^\dagger - z_2^* a\right)}_{\text{Glauber's formula}} \\
= &\exp\left(z_1 a^\dagger - z_1^* a + z_2 a^\dagger - z_2^* a\right)\exp\left(\left[z_1 a^\dagger - z_1^* a, z_2 a^\dagger - z_2^* a\right]/2\right) \\
= &\exp\left(\frac{z_1 z_2^* - z_2 z_1^*}{2}\right)\exp\left[(z_1 + z_2)a^\dagger - (z_1 + z_2)^* a\right] \\
= &\exp\left(\frac{z_1 z_2^* - z_2 z_1^*}{2}\right) D(z_1 + z_2) = D(z_1 + z_2).
\end{aligned} \quad (21)
$$

Generally, $\exp\left(\frac{z_1 z_2^* - z_2 z_1^*}{2}\right) \neq 0$. Thus

$$D(z_1)D(z_2)\left|0\right\rangle = \exp\left(\frac{z_1 z_2^* - z_2 z_1^*}{2}\right) D(z_1 + z_2)\left|0\right\rangle . \tag{22}$$

However, in quantum mechanics, we are dealing with "rays" rather than vectors. Hence, it is passable here to take $D(z_1+z_2)\left|0\right\rangle$ and $\exp\left(\frac{z_1 z_2^* - z_2 z_1^*}{2}\right) D(z_1+z_2)\left|0\right\rangle$ as the same. That is,

$$D(z_1)D(z_2) \sim D(z_1+z_2), \text{ or } D(z_1)\left|z_2\right\rangle \sim \left|z_1+z_2\right\rangle . \tag{23}$$

1.2 Physical Meaning o f $|z\rangle$

1.2.1 $|z\rangle$ at a Later Time

For $t > 0$, we now denote $|\psi(t)\rangle$, $U(t) = \exp\left(\frac{Ht}{\mathrm{i}\hbar}\right)$ and $D(z) = \exp\left(za^\dagger - z^* a\right)$ by $|\psi\rangle$, U and D, respectively:

$$|\psi(t=0)\rangle = |z\rangle \Rightarrow |\psi\rangle = U|z\rangle = UD|0\rangle = UDU^{-1}U|0\rangle .$$

As $H = \hbar\omega\left(N + \frac{1}{2}\right)$, $H|0\rangle = \frac{\hbar\omega}{2}|0\rangle$, it is easy to see that $U|0\rangle = \exp\left(\frac{Ht}{\mathrm{i}\hbar}\right)|0\rangle = \exp\left(\frac{\omega t}{2\mathrm{i}}\right)|0\rangle$. Thus

$$|\psi\rangle = UDU^{-1}U|0\rangle = \exp\left(\frac{\omega t}{2\mathrm{i}}\right) UDU^{-1}|0\rangle . \tag{24}$$

To further simplify Eq. (24), we take $za^\dagger - z^*a = A$, $HT/\mathrm{i}\hbar = B$, then

$$UDU^{-1} = U\sum_n \frac{A^n}{n!}U^{-1} = \sum_n U\frac{A^n}{n!}U^{-1} = \sum_n \frac{(UAU^{-1})^n}{n!} = \exp\left(UAU^{-1}\right); \tag{25}$$

$$UAU^{-1} = e^B A e^{-B} = A + [B,A] + \frac{1}{2!}[B,[B,A]] + \cdots . \tag{26}$$

And so our next step is to calculate all the "$[B,A]$", "$[B,[B,A]$", ... Given $[N,a] = -a$, $[N,a^\dagger] = a^\dagger$ and $H = \hbar\omega\left(N+\frac{1}{2}\right)$, it follows that

$$[H,a] = -\hbar\omega a, \qquad [H,a^\dagger] = \hbar\omega a^\dagger. \tag{27}$$

Hence

$$[B,A] = [Ht/\mathrm{i}\hbar, za^\dagger - z^*a] = \frac{t}{\mathrm{i}\hbar}\left(z\left[H,a^\dagger\right] - z^*[H,a]\right) = \frac{\omega t}{\mathrm{i}\hbar}\left(za^\dagger + z^*a\right), \tag{28}$$

$$[B,[B,A]] = [Ht/\mathrm{i}\hbar, [B,A]] = \frac{\omega t}{\mathrm{i}\hbar}[Ht/\mathrm{i}\hbar, za^\dagger + z^*a] = \left(\frac{\omega t}{\mathrm{i}\hbar}\right)^2\left(za^\dagger - z^*a\right), \tag{29}$$

$$[B,[B,[B,A]]] = \left(\frac{\omega t}{\mathrm{i}\hbar}\right)^3\left(za^\dagger + z^*a\right), \tag{30}$$

$$\vdots$$

then

$$
\begin{aligned}
UAU^{-1} &= A + [B, A] + \frac{1}{2!}[B, [B, A]] + \cdots \\
&= \left(za^\dagger - z^* a\right) + \frac{\omega t}{\mathrm{i}\hbar}\left(za^\dagger + z^* a\right) + \frac{1}{2!}\left(\frac{\omega t}{\mathrm{i}\hbar}\right)^2 \left(za^\dagger - z^* a\right) + \cdots \\
&= \sum_n \frac{(\omega t/\mathrm{i}\hbar)^n}{n!}\left[za^\dagger - (-1)^n z^* a\right] \\
&= za^\dagger \sum_n \frac{(\omega t/\mathrm{i}\hbar)^n}{n!} - z^* a \sum_n \frac{(\mathrm{i}\omega t/\hbar)^n}{n!} \\
&= \exp(\frac{\omega t}{\mathrm{i}\hbar}) za^\dagger - \exp(\frac{\mathrm{i}\omega t}{\hbar}) z^* a
\end{aligned} \tag{31}
$$

and

$$
\begin{aligned}
UDU^{-1} &= \exp\left(UAU^{-1}\right) \exp\left[\exp(\tfrac{\omega t}{\mathrm{i}\hbar}) za^\dagger - \exp(\tfrac{i\omega t}{\hbar}) z^* a\right] \\
&\overset{\text{Glauber's formula}}{=\!=\!=} \exp\left(-\tfrac{|z|^2}{2}\right) \exp\left[\exp(\tfrac{\omega t}{\mathrm{i}\hbar}) za^\dagger\right] \exp\left[-\exp(\tfrac{\mathrm{i}\omega t}{\hbar}) z^* a\right]
\end{aligned} \tag{32}
$$

Substitute it into Eq. (24):

$$
|\psi\rangle = \exp\left(\frac{\omega t}{2\mathrm{i}} - \frac{|z|^2}{2}\right) \exp\left[\exp(\frac{\omega t}{\mathrm{i}\hbar}) za^\dagger\right] \exp\left[-\exp(\frac{\mathrm{i}\omega t}{\hbar}) z^* a\right] |0\rangle \,. \tag{33}
$$

Seeing that $\exp\left[-\exp(\frac{\mathrm{i}\omega t}{\hbar}) z^* a\right] |0\rangle = \sum_n \frac{\left[-\exp(\frac{\mathrm{i}\omega t}{\hbar}) z^* a\right]^n}{n!} |0\rangle = |0\rangle$, then finally,

$$
|\psi\rangle = \exp\left(\frac{\omega t}{2\mathrm{i}} - \frac{|z|^2}{2}\right) \exp\left[\exp(\frac{\omega t}{\mathrm{i}\hbar}) za^\dagger\right] |0\rangle \,. \tag{34}
$$

1.2.2 Evolution of a Classical Harmonic Oscillator

The basic Hamiltoninian of a classical harmonic oscillator is

$$
H = \frac{p(t)^2}{2m} + \frac{m\omega^2 q(t)^2}{2}, \tag{35}
$$

where m is the particle's mass, ω the angular frequency of the oscillator, $q(t)$ the position, and $p(t)$ the momentum; H is invariant due to the **conservation of energy** and the phase ϕ_0 which determines the starting point, $t = 0$, on the sine wave, can be of any arbitrary value. Consequently, the evolution of such a system can be seen as follows:

$$
\left\{
\begin{aligned}
q(0) &= \sqrt{2mH}\cos(\phi_0) \\
p(0) &= \sqrt{\tfrac{2H}{m\omega^2}}\sin(\phi_0)
\end{aligned}
\right.
\Longrightarrow
\left\{
\begin{aligned}
q(t) &= \sqrt{2mH}\cos(\omega t + \phi_0) \\
&= q(0)\cos(\omega t) + \sqrt{2m}\omega p(0)\sin(\omega t) \\
p(t) &= \sqrt{\tfrac{2H}{m\omega^2}}\sin(\omega t + \phi_0) \\
&= \tfrac{q(0)}{\sqrt{2m\omega}}\sin(\omega t) + p(0)\cos(\omega t)
\end{aligned}
\right. . \tag{36}
$$

Compare it with the evolution of $|z\rangle$,

$$
\begin{aligned}
|\psi(0)\rangle &= |z\rangle = \exp\left(-\frac{|z|^2}{2}\right)\exp\left[za^\dagger\right]|0\rangle \\
\Longrightarrow |\psi(t)\rangle &= \exp\left(\frac{\omega t}{2\mathrm{i}} - \frac{|z|^2}{2}\right)\exp\left[\exp(\frac{\omega t}{\mathrm{i}\hbar})za^\dagger\right]|0\rangle\,,
\end{aligned}
\tag{37}
$$

we could tell that both the two processes are non-linear with respect to time, and the "non-linear parts" are both some terms of "$\exp(\mathrm{i}\omega t)$".

1.2.3 Evolution of a Driven Harmonic Oscillator

Consider a Hamiltonian,

$$
H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2} - qF(t) = H_0 + V(t), \tag{38}
$$

where $H_0 = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$ does not contain time explicitly and $V(t) = -\sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right)F(t)$ is the time-dependent potential. The energy eigenkets and eigenvalues are defined by

$$
H_0|n\rangle = E_n|n\rangle\,. \tag{39}
$$

(1) Conversion to the Interaction Picture and Expansion of $U_I(t)$

In the **interaction picture**, we define the state kets, the observables and the **time-evolution operator** by

$$
|\psi_I(t)\rangle \equiv e^{\mathrm{i}H_0 t/\hbar}|\psi_S(t)\rangle\,, \tag{40}
$$

$$
A_I \equiv e^{\mathrm{i}H_0 t/\hbar}A_S e^{H_0 t/\mathrm{i}\hbar}, \tag{41}
$$

$$
|\psi_I(t)\rangle = U_I(t)|\psi_I(0)\rangle\,,\quad t > 0. \tag{42}
$$

where "I" and "S" (usually omitted) refer to the interaction and the Schrödinger picture, respectively. In particular,

$$
\begin{aligned}
V_I(t) &= e^{\mathrm{i}H_0 t/\hbar}Ve^{H_0 t/\mathrm{i}\hbar} \\
&= \exp\left[\mathrm{i}\omega t\left(a^\dagger a + \frac{1}{2}\right)\right]f(t)\left(a + a^\dagger\right)\left[\mathrm{i}\omega t\left(a^\dagger a + \frac{1}{2}\right)\right] \\
&= f(t)\left(ae^{-\mathrm{i}\omega t} + a^\dagger e^{\mathrm{i}\omega t}\right),
\end{aligned}
\tag{43}
$$

where $f(t) = -\sqrt{\frac{\hbar}{2m\omega}}F(t)$. It is easy to prove that

$$
\mathrm{i}\hbar\frac{\partial}{\partial t}|\psi_I(t)\rangle = V_I|\psi_I(t)\rangle\,, \tag{44}
$$

$$
\mathrm{i}\hbar\frac{dA_I}{dt} = [A_I, H_0]\,. \tag{45}
$$

Note that $\frac{\partial}{\partial t}|\psi(0)\rangle = 0$ and $|\psi_I(0)\rangle = |\psi(0)\rangle$ can be arbitrary, then

$$\begin{aligned} \mathrm{i}\hbar\frac{\partial}{\partial t}|\psi_I(t)\rangle &= \mathrm{i}\hbar\left[\frac{\partial}{\partial t}U_I(t)|\psi_I(0)\rangle + U_I(t)\frac{\partial}{\partial t}|\psi_I(0)\rangle\right] = \mathrm{i}\hbar\frac{\partial}{\partial t}U_I(t)|\psi_I(0)\rangle \\ &= V_I|\psi_I(t)\rangle = V_I U_I(t)|\psi_I(0)\rangle, \end{aligned}$$

$$\Longrightarrow \mathrm{i}\hbar\frac{\partial U_I(t)}{\partial t} = V_I(t)U_I(t). \tag{46}$$

This differential equation along with the initial condition,

$$U_I(0) = 1, \tag{47}$$

is equivalent to the integral equation

$$U_I(t) = 1 + \frac{1}{\mathrm{i}\hbar}\int_0^t V_I(t_1)U_I(t_1)dt_1. \tag{48}$$

We can obtain an approximation solution to the equation by iteration:

$$\begin{aligned} U_I(t) &= 1 + \frac{1}{\mathrm{i}\hbar}\int_0^t V_I(t_1)\left[1 + \frac{1}{\mathrm{i}\hbar}\int_0^{t_1} V_I(t_2)U_I(t_2)dt_2\right]dt_1 \\ &= 1 + \sum_{n=1}^{\infty}\left(\frac{1}{\mathrm{i}\hbar}\right)^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n V_I(t_1)\cdots V_I(t_n). \end{aligned} \tag{49}$$

This perturbation expansion is known as the **Dyson series**.

At this point it is convenient to introduce the **time-ordered product** of operators, which is denoted by T. For two operators, the explicit definition of T gives:

$$\mathcal{T}\{A(x)B(y)\} = \theta(\tau_x - \tau_y)A(x)B(y) \pm \theta(\tau_y - \tau_x)B(y)A(x), \tag{50}$$

where x and y are spacetime locations and θ is the **Heaviside step function**; τ_x and τ_y denote the *invariant* scalar time-coordinates of x and y, the choice of the sign depends on the **bosonic** ("+") or **fermionic** ("−") nature of the operaters. Here for $V(t_m)$ and $V(t_n)$, we have

$$\mathcal{T}\{V(t_m)V(t_n)\} = \theta(t_m - t_n)V(t_m)V(t_n) + \theta(t_n - t_m)V(t_n)V(t_m). \tag{51}$$

Now rearrange the following integral using the above identity:

$$\frac{1}{2!}\int_0^t dt_1\int_0^t dt_2\mathcal{T}\{V(t_1)V(t_2)\} = \frac{1}{2!}\int_0^t dt_1\int_0^{t_2} dt_2 V_I(t_1)V_I(t_2) + \frac{1}{2!}\int_0^t dt_2\int_0^{t_1} dt_1 V_I(t_2)V_I(t_1). \tag{52}$$

The two terms on the right-hand side are equal to each other, which can be seen by simply switching the two integration variables, t_1 and t_2. Thus

$$\frac{1}{2!}\int_0^t dt_1\int_0^t dt_2\mathcal{T}\{V(t_m)V(t_n)\} = \int_0^t dt_1\int_0^{t_1} dt_2 V_I(t_1)V_I(t_2). \tag{53}$$

Similarly, one can show that for all positive number n, there is

$$\frac{1}{n!}\int_0^t dt_1\cdots\int_0^t dt_n T\left\{V_I(t_1)\cdots V_I(t_n)\right\} = \int_0^t dt_1\cdots\int_0^{t_{n-1}} dt_n V_I(t_1)\cdots V_I(t_n) \tag{54}$$

Returning to Eq. (49), the Dyson series can now be rewritten in a most compact form:

$$\begin{aligned} U_I(t) &= 1+\sum_{n=1}^{\infty}\left(\frac{1}{\mathrm{i}\hbar}\right)^n\int_0^t dt_1\cdots\int_0^{t_{n-1}} dt_n V_I(t_1)\cdots V_I(t_n) \\ &= 1+\sum_{n=1}^{\infty}\frac{(1/\mathrm{i}\hbar)^n}{n!}\int_0^t dt_1\cdots\int_0^t dt_n T\left\{V_I(t_1)\cdots V_I(t_n)\right\} \\ &\equiv \mathcal{T}\left\{\exp\left[\frac{1}{\mathrm{i}\hbar}\int_0^t V_I(\tau)d\tau\right]\right\}. \end{aligned} \tag{55}$$

For the last step in Eq. (55), we have introduced the identical "τ" to denote "$t_1, t_2, \cdots, t_n$" as they are mathematically equivalent; the time-ordered exponential is just a compact way of writing and remembering the correct expression.

(2) $z(t)$ **and the Translation Operator** $D\left[z(t)\right]$

Before really plugging in the exact form of potential V_I, we can develop an even nicer form of $U_I(t)$ starting from Eq. (55). We begin by breaking up the integral into the sum of integrals over infinitesimal lengths of time $\epsilon = \frac{t}{N}$ with $N \in \mathbb{N}$ and $N \to +\infty$, defining

$$V_n = \frac{1}{\mathrm{i}\hbar}\int_{(n-1)\epsilon}^{n\epsilon} V_I(\tau)d\tau, \quad n = 1, 2, \cdots, N. \tag{56}$$

Noting that even if the individual V_n's do not commute, their time intervals are infinitesimal and hence are not subject to internal time ordering.[1] As a result, Eq. (55) can be rewritten as

$$U_I(t) = U_I(t, t-\epsilon)U_I(t-\epsilon, t-2\epsilon)\cdots U_I(2\epsilon,\epsilon)U_I(\epsilon) = \lim_{N\to\infty}\prod_{n=1}^{N}\mathcal{T}\left\{e^{V_n}\right\}. = \lim_{N\to\infty}\prod_{n=1}^{N} e^{V_n}. \tag{57}$$

Eq. (43) gives the unequal time commutation of the potential:

$$\begin{aligned} \left[V_I(t), V_I(t')\right] &= f(t)f(t')\left[ae^{-\mathrm{i}\omega t}+a^\dagger e^{\mathrm{i}\omega t}, ae^{-\mathrm{i}\omega t'}+a^\dagger e^{\mathrm{i}\omega t'}\right] \\ &= f(t)f(t')\left[e^{-\mathrm{i}\omega(t-t')}-e^{\mathrm{i}\omega(t-t')}\right] \\ &= -2\mathrm{i}f(t)f(t')\sin\omega(t-t'). \end{aligned} \tag{58}$$

Then

$$\left[V_I(t''), \left[V_I(t), V_I(t')\right]\right] = 0. \tag{59}$$

[1] $\epsilon \sim d\tau \to \mathcal{T}\left\{e^{V_n}\right\} \sim \mathcal{T}\left\{\exp\left[\frac{V_I(\tau)d\tau}{\mathrm{i}\hbar}\right]\right\} = \mathcal{T}\left\{\sum_{n=0}^{\infty}\frac{(V_I(\tau)d\tau/\mathrm{i}\hbar)^n}{n!}\right\} = \sum_{n=0}^{\infty}\frac{(V_I(\tau)d\tau/\mathrm{i}\hbar)^n}{n!} = e^{V_n}$. (I don't think it's rigorous enough here, but I can't think of a better way to prove that for now.)

As for V_l, V_m and V_n defined by Eq. (56), one can say that the above relationship still holds, taking the form of

$$[V_l, [V_m, V_n]] = 0. \tag{60}$$

The reason is that, from Eq. (59) to Eq. (60), we should do the integration three times over three non-overlapping domains:

$$\begin{aligned}
& [V_I(t''), [V_I(t), V_I(t')]] = 0,\ \forall t'', t', t \\
\Longrightarrow\ & [V_l, [V_I(t), V_I(t')]] = \tfrac{1}{\mathrm{i}\hbar} \int_{(l-1)\epsilon}^{l\epsilon} [V_I(\tau), [V_I(t), V_I(t')]]\, d\tau = 0, \forall t, t' \notin [l\epsilon, (l-1)\epsilon] \\
& [V_I(t), [V_l, V_I(t')]] = \tfrac{1}{\mathrm{i}\hbar} \int_{(l-1)\epsilon}^{l\epsilon} [V_I(t), [V_I(\tau), V_I(t')]]\, d\tau = 0, \forall t, t' \notin [l\epsilon, (l-1)\epsilon] \\
& \ldots \\
\Longrightarrow\ & [V_l, [V_m, V_n]] = 0.
\end{aligned}$$

Combining Eq.s (43), (56)-(58) and (60), as well as the Baker-Hausdorff lemma, we have

$$\begin{aligned}
U_I(t) & \xlongequal{\text{Eq. 57, Eq. 60, BH lemma}} \lim_{N\to\infty} \exp \sum_{n=1}^{N} \left(V_n + \frac{1}{2} \left[V_n, \sum_{k=1}^{n} V_k \right] \right) \\
& \xlongequal{\text{Eq. 56}} \exp \left[\frac{1}{\mathrm{i}\hbar} \int_0^t V_I(\tau) d\tau - \frac{1}{2\hbar^2} \int_0^t d\tau \int_0^{t'} d\tau' \left[V_I(\tau), V_I(\tau') \right] \right], \\
& \xlongequal{\text{Eq. 43, Eq. 58}} \exp \left[z(t) a^\dagger - z^*(t) a + \mathrm{i}\beta(t) \right] \\
& = e^{\mathrm{i}\beta(t)} \exp \left[z(t) a^\dagger - z^*(t) a \right] \\
& = e^{\mathrm{i}\beta(t)} D \left[z(t) \right].
\end{aligned} \tag{61}$$

where $z(t) \equiv \frac{1}{\mathrm{i}\hbar} \int_0^t f(\tau) e^{i\omega t} d\tau$, $\beta(t) \equiv \frac{1}{\hbar^2} \int_0^t d\tau \int_0^{t'} d\tau' \sin \omega (\tau - \tau')$ and the **translation operator** $D\left[z(t)\right] \equiv \exp \left[z(t) a^\dagger - z^*(t) a \right]$. Therefore, $U_I(t)$, up to a phase $e^{\mathrm{i}\beta(t)}$, equals $D\left[z(t)\right]$.

(3) From $|0\rangle$ to $|\psi(t)\rangle$

$$\begin{aligned}
& |\psi_I(t)\rangle = U_I(t) |0\rangle = e^{\mathrm{i}\beta(t)} D\left[z(t)\right] |0\rangle \\
= & e^{\mathrm{i}\beta(t)} \sum_n \frac{\left[z(t) a^\dagger - z^*(t) a \right]^n}{n!} |0\rangle = e^{\mathrm{i}\beta(t)} \left\{ 1 + \sum_{n=1} \frac{\left[z(t) a^\dagger \right]^n}{n!} \right\} |0\rangle \\
= & e^{\mathrm{i}\beta(t)} e^{z(t) a^\dagger} |0\rangle .
\end{aligned} \tag{62}$$

1.3 $|z\rangle$ for a Free EM Field

We have seen the coherent states $|z\rangle$ for a SHO in Sec. 1.1.1:

$$a |z\rangle = z |z\rangle ,$$

$$\begin{aligned}
|z\rangle = & \exp \left(-\frac{|z|^2}{2} \right) \sum_n \frac{z^n}{\sqrt{n!}} |n\rangle = \exp \left(-\frac{|z|^2}{2} \right) \exp \left(z a^\dagger \right) |0\rangle \\
= & \exp \left(z a^\dagger - z^* a \right) |0\rangle = D(z) |0\rangle ,
\end{aligned}$$

where "a" and "$a^\dagger$" are the ladder operators. If a free EM field has a SHO-form Hamiltonian, then surely we can define the "$|z\rangle$" for it. The key to the practical solution is a belief that such a field can ultimately be broken down into some " **quanta**" or "particles"—**photons**.

1.3.1 EM Field in Vacuum

$\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B} = \mathbf{B}(\mathbf{r}, t)$, represented by the **electric scalar potential** $\phi(\mathbf{r}, t)$ and the **magnetic vector potential** $\mathbf{A}(\mathbf{r}, t)$:

$$\begin{cases} \mathbf{E} &= -\nabla\phi - \partial_t \mathbf{A}, \\ \mathbf{B} &= \nabla \times \mathbf{A}, \end{cases} \tag{63}$$

are invariant under the **gauge transformation**

$$\begin{cases} \mathbf{A} &\to \mathbf{A} + \nabla\psi, \\ \phi &\to \phi - \partial_t \psi. \end{cases} \tag{64}$$

Choosing the **Coulomb gauge** in particular, we make $\mathbf{A}$ a transverse field for $\nabla \cdot \mathbf{A} = 0$. Plugging this into to the source free **Maxwell's equations**, we get $\nabla^2\phi = 0$ and the **wave equation**,

$$\Box \mathbf{A} = 0, \quad \Box \equiv \frac{\partial_{tt}}{c^2} - \nabla^2, \tag{65}$$

where $\Box$ is the **d'Alembert operator**. Note that, due to the gauge transformation, it is alright to set $\phi = 0$ here. As a result,

$$\begin{cases} \mathbf{E} &= -\partial_t \mathbf{A}, \\ \mathbf{B} &= \nabla \times \mathbf{A}. \end{cases} \tag{66}$$

1.3.2 $\mathbf{A}(\mathbf{r}, t)$ in a Box

Consider a free EM field in a cubical volume of side L [2] [3] [4], then all quantities characterizing the field can be expanded in this box in Fourier series [5]. Take $\mathbf{A}(\mathbf{r}, t)$ as an example:

$$\mathbf{A}(\mathbf{r}, t) = \sum_{\mathbf{k}} \sum_{\mu=\pm 1} \mathbf{A}_{\mathbf{k}}^{(\mu)}(t) e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}, \tag{67}$$

where $\mathbf{A}_{\mathbf{k}}^{(\mu)}(t)$ are coefficients of the Fourier components (**polarised** monochromatic waves) of $\mathbf{A}$; $\mu = \pm 1$ the **polarisation** is the eigenvalue of the z-component of the photon spin and the wave vector $\mathbf{k}$ gives the **propagation direction** of the corresponding $\mathbf{A}_{\mathbf{k}}^{(\mu)}(t)$. Ignore μ for the moment, then

$$\mathbf{A} = \sum_{\mathbf{k}} \mathbf{A}_{\mathbf{k}} e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}, \tag{68}$$

[2] Landau, L. D. (Ed.). (2013). *The classical theory of fields* (Vol. 2). Elsevier.

[3] Quantization of the electromagnetic field. (2017, 2 July). Retrieved from https:\\en.wikipedia.org\wiki\Quantization_of_the_electromagnetic_field

[4] Walls, D. F., & Milburn, G. J. (2007). *Quantum optics*. Springer Science & Business Media.

[5] This is because we wish to introduce here the **periodic boundary conditions**, corresponding to travelling-wave modes or conditions appropriate to reflecting walls which lead to standing waves.

with $\mathbf{A}_{-\mathbf{k}} = \mathbf{A}_{\mathbf{k}}^*$ since $\mathbf{A}$ is real ("*" denotes the **complex conjugate** here). Now we separate $\mathbf{A}$ into two parts,

$$\mathbf{A} = \mathbf{A}^{(+)} + \mathbf{A}^{(-)} = \sum_{\mathbf{k}} \left(\mathbf{A}_{\mathbf{k}} e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} + \mathbf{A}_{\mathbf{k}}^* e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \right). \tag{69}$$

In this summation, $\mathbf{k}$ run over one side, positive or negative. Thus the components of $\mathbf{k}$ are

$$k_x = \frac{2\pi n_x}{L}, k_y = \frac{2\pi n_y}{L}, k_z = \frac{2\pi n_z}{L}, \quad n_x, n_y, n_z = 0, 1, 2, \cdots. \tag{70}$$

From $\nabla \cdot \mathbf{A} = 0$, it follows that for each $\mathbf{k}$:

$$\mathbf{k} \cdot \mathbf{A}_{\mathbf{k}}(t) = 0, \tag{71}$$

i.e., $\mathbf{A}_{\mathbf{k}}$ are perpendicular to the corresponding $\mathbf{k}$; from the wave equation (65), we know that $\mathbf{A}_{\mathbf{k}}$ satisfy the equation

$$\left(\partial_{tt} + c^2 k^2 \right) \mathbf{A}_{\mathbf{k}} = 0. \tag{72}$$

1.3.3 Photons as the Quanta of the Field

If $\mathbf{k}$ are wave vectors of photons, then their length satisfy the **dispersion relation** of light:

$$|\mathbf{k}| = \frac{\omega(\mathbf{k})}{c} > 0. \tag{73}$$

Combing Eq. (72), we know that

$$\mathbf{A}_{\mathbf{k}}(t) \sim e^{-\mathrm{i}\omega t}, \tag{74}$$

or equivalently,

$$\mathbf{A}_{\mathbf{k}}^{(\mu)}(t) = \mathbf{e}_{\mathbf{k}}^{(\mu)} A_{\mathbf{k}}^{(\mu)} e^{-\mathrm{i}\omega t}, \tag{75}$$

where we have recalled the polarisation μ; and the two **polarisation vectors**, $\mathbf{e}_{\mathbf{k}}^{(\mu)}$, are conventional unit vectors for left and right hand circular polarised (LCP and RCP) EM waves and perpendicular to their corresponding $\mathbf{k}$ (Eq. (71)):

$$\mathbf{e}_{\mathbf{k}}^{(\mu)} \equiv \frac{\mp 1}{\sqrt{2}} \left(\mathbf{e}_x^{(\mu)}(\mathbf{k}) \pm \mathrm{i}\mathbf{e}_y^{(\mu)}(\mathbf{k}) \right) \quad \text{with } \mathbf{e}_x^{(\mu)}(\mathbf{k}) \cdot \mathbf{k} = 0 \text{ and } \mathbf{e}_y^{(\mu)}(\mathbf{k}) \cdot \mathbf{k} = 0. \tag{76}$$

Using Eq.s (66), (69) and (75), we get [6]

$$\begin{cases} \mathbf{E}(\mathbf{r}, t) & = \mathrm{i} \sum_{\mathbf{k}} \sum_{\mu=\pm 1} \omega(\mathbf{k}) \left(\mathbf{A}_{\mathbf{k}}^{(\mu)} e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} - \mathbf{A}_{\mathbf{k}}^{(\mu)*} e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \right), \\ \mathbf{B}(\mathbf{r}, t) & = \mathrm{i} \sum_{\mathbf{k}} \sum_{\mu=\pm 1} \left[\left(\mathbf{k} \times \mathbf{A}_{\mathbf{k}}^{(\mu)} \right) e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} - \left(\mathbf{k} \times \mathbf{A}_{\mathbf{k}}^{(\mu)*} \right) e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \right]. \end{cases} \tag{77}$$

Hence the classical Hamiltonian of the field is [7]

$$H = \mathcal{E} = \frac{\epsilon_0}{2} \iiint_V \left(|\mathbf{E}|^2 + c^2 |\mathbf{B}|^2 \right) d\mathbf{r} = V\epsilon_0 \sum_{\mathbf{k}} \sum_{\mu=\pm 1} \omega^2(\mathbf{k}) \left(\mathbf{A}_{\mathbf{k}}^{(\mu)} \mathbf{A}_{\mathbf{k}}^{(\mu)*} + \mathbf{A}_{\mathbf{k}}^{(\mu)*} \mathbf{A}_{\mathbf{k}}^{(\mu)} \right), \tag{78}$$

where the right-hand-side can be easily obtained by using the equation

$$\int_V e^{i(\mathbf{k}-\mathbf{k}')\mathbf{r}} d\mathbf{r} = V\delta_{kk'}. \tag{79}$$

[6] $\nabla \times e^{\mathbf{kr}} = \mathbf{k} \times e^{\mathbf{kr}}$, $\forall$ vectors $\mathbf{k}$.

[7] $\mathbf{k} \cdot \mathbf{A}_{\mathbf{k}} = 0 \Rightarrow (\mathbf{k} \times \mathbf{A}_{\mathbf{k}}) \cdot (\mathbf{k} \times \mathbf{A}_{\mathbf{k}}^*) = k^2 \mathbf{A}_{\mathbf{k}} \cdot \mathbf{A}_{\mathbf{k}}^*$.

1.3.4 Canonical Variables and Fock State

From Eq. (75),

$$\partial_t \mathbf{A}_{\mathbf{k}}^{(\mu)} = -\mathrm{i}\omega(\mathbf{k})\mathbf{A}_{\mathbf{k}}^{(\mu)}. \tag{80}$$

We can introduce the "canonical" $P_{\mathbf{k}}$ and $Q_{\mathbf{k}}$ ("μ" have been omitted here for simplicity):

$$\begin{cases} Q_{\mathbf{k}} & = -\frac{\sqrt{2V\epsilon_0}}{\omega(k)}\left(\mathbf{A}_{\mathbf{k}} + \mathbf{A}_{\mathbf{k}}^*\right), \\ P_{\mathbf{k}} & = \mathrm{i}\sqrt{2V\epsilon_0}\left(\mathbf{A}_{\mathbf{k}} - \mathbf{A}_{\mathbf{k}}^*\right) = \partial_t Q_{\mathbf{k}}. \end{cases} \tag{81}$$

Substituting this into Eq. (78),

$$H = \frac{1}{2}\sum_{\mathbf{k}}\left[P_{\mathbf{k}}^2 + \omega^2(\mathbf{k})Q_{\mathbf{k}}^2\right], \tag{82}$$

where $\omega(\mathbf{k}) = c\,|\mathbf{k}| > 0$ as we have already set the components of **k** positive. At the moment, we have proven an EM field to have a SHO-form Hamiltonian (using the Coulomb gauge). All we have to do now is follow the SOP in Sec. 1.1 and define

$$\begin{cases} a_{\mathbf{k}} & \equiv \sqrt{\frac{\omega}{2\hbar}}\left(Q_{\mathbf{k}} + \frac{\mathrm{i}P_{\mathbf{k}}}{\omega}\right) = -2\sqrt{\frac{V\epsilon_0}{\hbar\omega}}\mathbf{A}_{\mathbf{k}}, \\ a_{\mathbf{k}}^\dagger & \equiv \sqrt{\frac{\omega}{2\hbar}}\left(Q_{\mathbf{k}} - \frac{\mathrm{i}P_{\mathbf{k}}}{\omega}\right) = -2\sqrt{\frac{V\epsilon_0}{\hbar\omega}}\mathbf{A}_{\mathbf{k}}^* \\ N_{\mathbf{k}} & = a_{\mathbf{k}}^\dagger a_{\mathbf{k}} = \frac{4V\epsilon_0}{\hbar\omega}\mathbf{A}_{\mathbf{k}}^*\mathbf{A}_{\mathbf{k}}. \end{cases} \tag{83}$$

The Hamiltonian can then be written as

$$H = \sum_{\mathbf{k}} H_{\mathbf{k}} = \sum_{|\mathbf{k}|>0} \hbar\omega(\mathbf{k})\left(N_{\mathbf{k}} + \frac{1}{2}\right). \tag{84}$$

The energy eigenstate $|n_{\mathbf{k}}\rangle$ of $H_{\mathbf{k}}$ is defined in a manner similar to the single-mode field via the energy eigenvalue equation

$$H_{\mathbf{k}}\,|n_{\mathbf{k}}\rangle = \hbar\omega(\mathbf{k})\left(N_{\mathbf{k}} + \frac{1}{2}\right)|n_{\mathbf{k}}\rangle \tag{85}$$

The general eigenstate of H can therefore have $n_{\mathbf{k}l}$ photons in the l-th state, and can be written as

$$|n_{\mathbf{k}1}, n_{\mathbf{k}2}, \cdots n_{\mathbf{k}l}, \cdots\rangle \equiv |\{n_{\mathbf{k}}\}\rangle. \tag{86}$$

This is so-called the **Fock state**. Note that the annihilation and creation operators $a_{\mathbf{k}l}$ and $a_{\mathbf{k}l}^\dagger$ lower and raise the l-th entry alone, i.e.,

$$\begin{aligned} a_{\mathbf{k}l}\,|\{n_{\mathbf{k}}\}\rangle &= \sqrt{n_{\mathbf{k}l}}\,|n_{\mathbf{k}1}, n_{\mathbf{k}2}, \cdots n_{\mathbf{k}i-1}, \cdots\rangle, \\ a_{\mathbf{k}l}^\dagger\,|\{n_{\mathbf{k}}\}\rangle &= \sqrt{n_{\mathbf{k}i+1}}\,|n_{\mathbf{k}1}, n_{\mathbf{k}2}, \cdots n_{\mathbf{k}i+1}, \cdots\rangle. \end{aligned} \tag{87}$$

Thus the number operator for the l-th state, $N_{\mathbf{k}l}$, acts on the Fock state in the following way:

$$N_{\mathbf{k}l}\,|\{n_{\mathbf{k}}\}\rangle = n_{\mathbf{k}l}\,|\{n_{\mathbf{k}}\}\rangle. \tag{88}$$

Hence the Fock state is an eigenstate of the number operator with eigenvalue $n_{\mathbf{k}l}$.

1.3.5 Coherent States $|z\rangle$

We know from section 1.1.5 that $|z_{\mathbf{k}}\rangle = D(z_{\mathbf{k}})\,|0_{\mathbf{k}}\rangle$, where the transition operator $D(z_{\mathbf{k}}) \equiv \exp\left(z_{\mathbf{k}} a^{\dagger}_{\mathbf{k}} - z^{*}_{\mathbf{k}} a_{\mathbf{k}}\right)$. Since $a_{\mathbf{k}l}$ and $a^{\dagger}_{\mathbf{k}l}$ act only on the l-th entry, then the coherent state $|z\rangle$ can be defined as

$$|z\rangle = \left[\prod_{l=1}^{\infty} D\left(z_{\mathbf{k}l}\right)\right] |\{0_{\mathbf{k}}\}\rangle = \exp\left[\sum_{l=1}^{\infty}\left(z_{\mathbf{k}l} a^{\dagger}_{\mathbf{k}l} - z_{\mathbf{k}l} * a_{\mathbf{k}l}\right)\right] |\{0_{\mathbf{k}}\}\rangle \,. \tag{89}$$

1.3.6 $|z\rangle$ at a Later Time

Similarly, from section 1.2.1 we know that (using Eq. (34)) if $|\psi(0)\rangle = |z\rangle$ is the coherent state describing the system at $t = 0$, then $|\psi(t)\rangle$ at later time t should be

$$\begin{aligned} |\psi(t)\rangle &= \left\{\prod_{l=1}^{\infty} \exp\left(\frac{\omega_{\mathbf{k}l} t}{2\mathrm{i}} - \frac{|z_{\mathbf{k}l}|^2}{2}\right) \prod_{l=1}^{\infty} \exp\left[\exp(\frac{\omega_{\mathbf{k}l} t}{\mathrm{i}\hbar}) z_{\mathbf{k}l} a^{\dagger}\right]\right\} |\{0_{\mathbf{k}}\}\rangle \\ &= \exp\left(\frac{t}{2\mathrm{i}} \sum_{l=1}^{\infty} \omega_{\mathbf{k}l} - \frac{\sum_{l=1}^{\infty} |z_{\mathbf{k}l}|^2}{2}\right) \exp\left\{\sum_{l=1}^{\infty}\left[\exp(\frac{\omega_{\mathbf{k}l} t}{\mathrm{i}\hbar}) z_{\mathbf{k}l} a^{\dagger}\right]\right\} |\{0_{\mathbf{k}}\}\rangle \,. \end{aligned} \tag{90}$$

2 Green's Functions for the KG Operator

2.1 Definition of a Green's Function

Given a **linear differential operator** $L = L_x$ acting on the collection of **distributions** over a subset Ω of some Euclidean space $\mathbb{R}^n$, a Green's function $G = G(x, s)$ at the point $s \in \Omega$ corresponding to L_x is any solution of

$$L_x G(x, s) = \delta(x - s), \tag{91}$$

where δ is the δ-function. By multiplying the above identity by a function $f(s)$ and integrating with respect to s yields

$$L_x \int G(x, s) f(s) ds = \int L_x G(x, s) f(s) ds = \int \delta(x - s) f(s) ds = f(x). \tag{92}$$

That is, the solution to the **differential equation**,

$$L_x u(x) = f(x), \tag{93}$$

(where $f(x)$ can be considered as some **source term**) is

$$u(x) = \int G(x, s) f(s) ds. \tag{94}$$

This process relies upon the linearity of L_x.

A Loose Prove of $G(x,y) = G(x-y)$

First we define the **translation operator** T_h over a **function space** $(x, y, h \in \mathbb{R}^n)$:

$$T_h[f(x)] = f(x+h). \tag{95}$$

An operator A on functions is said to be **translationally invariant** with respect to T_h, if the order of them applying on the argument function does not matter. It is easy to see that L_x must be sunch an "A". Now consider an equation

$$L_x G_y(x) = \delta_y(x) \equiv \delta(x-y). \tag{96}$$

Since

$$L_x G_0(x) = \delta_0(x) = \delta(x), \tag{97}$$

after multiplying both side by T_{-y} on the left, we have

$$T_{-y}[L_x G_0(x)] = L_x[T_{-y} G_0(x)] = L_x G_0(x-y) = T_{-y}[\delta(x)] = \delta(x-y). \tag{98}$$

Comparing $L_x G_y(x) = \delta(x-y)$ and $L_x G_0(x-y) = \delta(x-y)$, we know that

$$G(x,y) \equiv G_y(x) = G_0(x-y) = G(x-y,0). \tag{99}$$

That is, $G(x,y) = G(x-y,0)$, $G(x,y)$ depends merely on the difference between x and y. If we simply denote $G(x-y,0)$ as $G(x-y)$, then

$$G(x,y) = G(x-y). \tag{100}$$

2.2 Real KG Field

We try a recipe called **canonical quantisation** to quantise a classical scalar field governed by the Lagrangian [8]

$$\mathcal{L}(x) = \frac{1}{2}\left[\partial_\mu \phi(x)\right]^2 - \frac{1}{2} m^2 \phi^2(x). \tag{101}$$

In the **Schrödinger picture**, we promote ϕ and $\pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$ to operators:

$$\left\{ \begin{array}{lcl} \phi(\mathbf{x}) & = & \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}} + a^\dagger_{\mathbf{p}} e^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}} \right) \big|_{p^0 = E_{\mathbf{p}}}, \\ \pi(\mathbf{x}) & = & -\,\mathrm{i} \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_{\mathbf{p}}}{2}} \left(a_{\mathbf{p}} e^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}} - a^\dagger_{\mathbf{p}} e^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}} \right) \big|_{p^0 = E_{\mathbf{p}}}, \end{array} \right. \tag{102}$$

where $\omega_{\mathbf{p}} = E_{\mathbf{p}} \equiv \sqrt{|\mathbf{p}|^2 + m^2}$. [9] Remember that the commutation relations for ϕ and π here are equivalent to the following ones for $a_{\mathbf{p}}$ and $a^\dagger_{\mathbf{p}}$:

$$\left\{ \begin{array}{ll} [\phi(\mathbf{x}), \phi(\mathbf{y})] & = [\pi(\mathbf{x}), \pi(\mathbf{y})] = 0, \\ [\phi(\mathbf{x}), \pi(\mathbf{y})] & = \mathrm{i}\delta^3(\mathbf{x}-\mathbf{y}), \end{array} \right. \Leftrightarrow \left\{ \begin{array}{ll} \left[a_{\mathbf{p}}, a_{\mathbf{q}}\right] & = \left[a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}\right] = 0, \\ \left[a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}\right] & = (2\pi)^3 \delta^3(\mathbf{p}-\mathbf{q}). \end{array} \right. \tag{103}$$

[8] We will work with $\hbar = c = 1$, $x = (t, \mathbf{x})$ and $p = (E, \mathbf{p})$, $p^2 = p^\mu p_\mu = E^2 - |\mathbf{p}|^2 = m^2$ (for a massive particle) henceforth.

[9] Note that the energy $E_{\mathbf{p}}$ here is always positive.

The Hamiltonian operator is

$$\begin{aligned}
H &= \int d^3x\mathcal{H} = \frac{1}{2}\int d^3x\left[\pi^2 + (\nabla\phi)^2 + m\phi^2\right] \\
&= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}E_{\mathbf{p}}\left(a_{\mathbf{p}}a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}\right) \\
&= \int \frac{d^3p}{(2\pi)^3}E_{\mathbf{p}}\left[a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + \frac{(2\pi)^3}{2}\delta^3(0)\right] \\
&\sim \int \frac{d^3p}{(2\pi)^3}E_{\mathbf{p}}a_{\mathbf{p}}^\dagger a_{\mathbf{p}}.
\end{aligned} \tag{104}$$

The state $|0\rangle$ satisfying $a_{\mathbf{p}}|0\rangle = 0$ for all $\mathbf{p}$ is the **ground state** or **vacuum**, and has $E = 0$ after we have dropped the infinite constant $\int d^3p\frac{\omega_{\mathbf{p}}}{2}\delta^3(0)$ in H. In QFT, a product of quantum fields, or equivalently their ladder operators, is said to be **normal ordered** when all creation operators are to the left of all annihilation operators in the product.

In the **Heisenberg picture**, the time dependence is assigned to the operators $\mathcal{O}$, that is,

$$\mathcal{O}_H = e^{\mathrm{i}Ht}\mathcal{O}_S e^{-\mathrm{i}Ht} \quad \text{and} \quad \mathrm{i}\partial_t\mathcal{O}_H = [\mathcal{O}_H, H]. \tag{105}$$

Now we make ϕ and π time-dependent in the usual way:

$$\begin{cases} \phi(x) &= \phi(\mathbf{x}, t) = e^{\mathrm{i}Ht}\phi(\mathbf{x})e^{-\mathrm{i}Ht}, \\ \pi(x) &= \pi(\mathbf{x}, t) = e^{\mathrm{i}Ht}\pi(\mathbf{x})e^{-\mathrm{i}Ht}. \end{cases}$$

The dire expression for ϕ and π is then [10]:

$$\begin{cases} \phi(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{\mathrm{i}p\cdot x} + a_{\mathbf{p}}^\dagger e^{-\mathrm{i}p\cdot x}\right)\big|_{p_0=E_{\mathbf{p}}}, \\ \pi(x) &= \partial_t\phi(x). \end{cases} \tag{106}$$

Note that the **Kliein-Gordon equation** holds for both $\phi(x)$ and $\phi(\mathbf{x})$:

$$(\Box + m^2)\phi = 0, \quad \Box = \partial_{tt} - \nabla^2. \tag{107}$$

2.3 Final Results

2.3.1 Inhomogeneous KG Field

Consider a KG equation in the presence of a classical source:

$$(\Box + m^2)\phi(x) = \rho(x). \tag{108}$$

One standard way to do this is finding the Green's function $G(x, y)$ with the property

$$(\Box + m^2)G(x, y) = \delta^4(x - y). \tag{109}$$

[10] We will always regard $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^\dagger$ as the time-independent, Schrödinger-picture ladder operators.

Given $G(x, y)$, we can write a *formal* solution to the inhomogeneous equation:

$$\phi(x) = \int d^4 y G(x, y) \rho(y). \tag{110}$$

Plugging the Fourier transformation of $G(x, y)$,

$$G(x, y) = \int \frac{d^4 p}{(2\pi)^4} e^{-\mathrm{i}p\cdot x} g(p, y) \tag{111}$$

into Eq. (109), then

$$\begin{aligned} (\Box + m^2) G(x, y) = & \int \frac{d^4 p}{(2\pi)^4} e^{-\mathrm{i}p\cdot x} \left(-E^2 + |\mathbf{p}|^2 + m^2\right) g(p, y) \\ & = \delta^4(x - y) = \int \frac{d^4 p}{(2\pi)^4} e^{-\mathrm{i}p\cdot(x-y)}. \end{aligned} \tag{112}$$

That is,

$$\left(p^2 - m^2\right) g(k, y) = -e^{\mathrm{i}p\cdot y}, \quad \text{or} \quad g(p, y) = \frac{-e^{\mathrm{i}p\cdot y}}{p^2 - m^2}. \tag{113}$$

Note that the second expression above is **singular** when $p^2 = m^2$. Using this expression anyway and leave the singularities to be handled afterwards, immediately we get

$$G(x, y) = -\int \frac{d^4 p}{(2\pi)^4} \frac{e^{-\mathrm{i}p\cdot(x-y)}}{p^2 - m^2}. \tag{114}$$

We will start with the integration over $p^0 = E$ (suppose that $x - y = (t, \mathbf{x})$),

$$G(x, y) \quad = \int \frac{d^3 p}{(2\pi)^4} e^{\mathbf{i}\mathbf{p}\cdot\mathbf{x}} \int dE \frac{e^{-\mathrm{i}Et}}{E_{\mathbf{p}}{}^2 - E^2} = \int \frac{d^3 p}{(2\pi)^4 2E_{\mathbf{p}}} e^{\mathbf{i}\mathbf{p}\cdot\mathbf{x}} \int dE \left(\frac{e^{-\mathrm{i}Et}}{E + E_{\mathbf{p}}} - \frac{e^{-\mathrm{i}Et}}{E - E_{\mathbf{p}}}\right), \tag{115}$$

that is, we have to first evaluate

$$\int_{-\infty}^{\infty} \left(\frac{e^{-\mathrm{i}Et}}{E + E_{\mathbf{p}}} - \frac{e^{-\mathrm{i}Et}}{E - E_{\mathbf{p}}}\right) dE. \tag{116}$$

2.3.2 Jordan's Lemma

Consider a complex-valued, conqinuous function f, defined on a semicircular contour

$$C_R = \left\{Re^{\mathrm{i}\theta} | \theta \in [0, \pi]\right\} \tag{117}$$

of positive radius R lying in the **upper half-plane** (**UHP**), centred at the origin. If f is of the form

$$f(z) = e^{\mathrm{i}az} g(z), \quad z \in C_R, \tag{118}$$

with a positive parameter a, then **Jordan's lemma** states the upper bound for the **line integral** hereinafter:

$$\left|\int_{C_R} f(z) dz\right| \leq \frac{\pi}{a} M_R, \quad \text{where} \quad M_R \equiv \max_{\theta\in[0,\pi]} \left|g\left(Re^{\mathrm{i}\theta}\right)\right|. \tag{119}$$

where equal sign is when g vanishes everywhere. An analogous statement for a semicircular contour in the **lower half-plane** (**LHP**, $C_R = \left\{Re^{\mathrm{i}\theta} | \theta \in [-\pi, 0]\right\}$) holds when $a < 0$.

2.3.3 Proof of Jordan's Lemma

$$I_R \equiv \int_{C_R} f(z)dz = R\int_0^\pi g\left(Re^{\mathrm{i}\theta}\right) e^{\mathrm{i}aP(\cos\theta+\mathrm{i}\sin\theta)}\mathrm{i}Re^{\mathrm{i}\theta}d\theta. \tag{120}$$

Now the inequality

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|\,dx \tag{121}$$

yields

$$I_R \leq R\int_0^\pi \left|g\left(Re^{\mathrm{i}\theta}\right)\right| e^{-aR\sin\theta}d\theta \leq RM_R\int_0^\pi e^{-aR\sin\theta}d\theta = 2RM_R\int_0^{\pi/2} e^{-aR\sin\theta}d\theta. \tag{122}$$

Furthermore, by plotting $\sin\theta$ and $2\theta/\pi$ on the same graph, one can easily see that $\sin\theta \geq 2\theta/\pi$ for $0 \leq \theta \leq \pi/2$. Thus,

$$I_R \leq 2RM_R\int_0^{\pi/2} e^{-aR\sin\theta}d\theta \leq 2RM_R\int_0^{\pi/2} e^{-2aR\theta/\pi}d\theta = \frac{\pi}{a}\left(1-e^{-aR}\right)M_R \leq \frac{\pi}{a}M_R. \tag{123}$$

2.3.4 Contour Integral

In order to deal with the **singularity** of the intrgral (116), we will treat E as a complex number treat the integration as a **contour integral**.

First, we have to avoid the singularity at $E = 0$ by pushing it infinitesimally away from the real axis into the UHP or LHP. Secondly, we will convert the open-contour in the E plane into a close contour by adding a semicircle at infinity that *contributes zero to the integral* (using Jordan's lemma). Whether we do this in the UHP or the LHP is dictated by the sign of t in the exponential $e^{-\mathrm{i}Et}$: **close the contour in the LHP when $t > 0$ and in the UHP when $t < 0$.**

Generally, consider the integral

$$\int_{-\infty}^{\infty} \frac{f(x)}{x-x_0}dx, \tag{124}$$

where $x_0 \in \mathbb{R}$ and f is analytic at x_0. We bypass x_0 as shown in Fig. 1 (note that the **large semicircle** C_∞ at infinity is not plotted here),

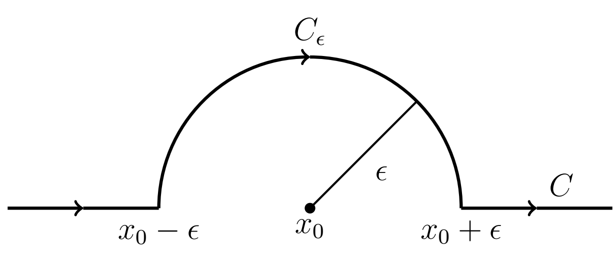

Figure 1: The Contour C avoids x_0

The **principal value** of the integral (in the limit $\epsilon \to 0^+$) is [11] [12]

$$
\begin{aligned}
\mathcal{P}\int_{-\infty}^{\infty}\frac{f(x)}{x-x_0}dx \equiv & \int_{-\infty}^{x_0-\epsilon}\frac{f(x)}{x-x_0}dx+\int_{x_0+\epsilon}^{\infty}\frac{f(x)}{x-x_0}dx \\
= & \pm\mathrm{i}\pi f(x_0)+2\pi\mathrm{i}\sum_i \mathrm{Res}\left[\frac{f(z_i)}{z_i-x_0}\right] \\
= & \pm\mathrm{i}\pi f(x_0)+\int_{-\infty}^{\infty}\frac{f(x)}{x-x_0\pm\mathrm{i}\epsilon}dx,
\end{aligned}
\tag{127}
$$

where $\{z_i\}$ are poles of $f(z)$ in the contour C, the plus (minus) sign refers to the infinitesimal semicircle C_ϵ in the UHP (LHP), as shown in Fig. 1. By saying the evaluation of the integral (116), we are actually talking about that of its principal value plus the integration along C_ϵ, i.e.,

$$
\mathcal{P}\int_{-\infty}^{\infty}\frac{f(x)}{x-x_0}dx+\int_{C_\epsilon}\frac{f(z)dz}{z-x_0}=\mathcal{P}\int_{-\infty}^{\infty}\frac{f(x)}{x-x_0}dx\mp\mathrm{i}\pi f(x_0)=\int_{-\infty}^{\infty}\frac{f(x)}{x-x_0\pm\mathrm{i}\epsilon}. \tag{128}
$$

2.3.5 Retarded, Advanced and Feynman GFs

There are three ways to push the singularities of t he integral (116) infinitesimally away from the real axis:

(i) both singularities are in the LHP;

(ii) both singularities are in the UHP;

(iii) one singularity is in the LHP and the other is in the UHP (here we specifically push $E = -E_{\mathbf{p}}$ in the UHP).

Remembering the correct rule to close the large semicircle C_∞ and using Eq. (128), we can get three different GFs respectively for the above three conditions, after substituting the integral (116)

---

[11] $z - x_0 = \epsilon e^{\mathrm{i}\theta}$ and $dz = \mathrm{i}\epsilon e^{\mathrm{i}\theta}d\theta \Rightarrow \int_{C_\epsilon}\frac{f(z)dz}{z-x_0} = \mathrm{i}\int_0^\pi f\left(x_0+\epsilon e^{\mathrm{i}\theta}\right)d\theta = -\mathrm{i}\pi f(x_0), \epsilon\to 0^+$, for C_ϵ in the UHP.

[12] Two useful lemmas: (1) $f(z)$ is continuous in the (deleted) neighbourhood of $z = a$. If $(z-a)f(z)$ uniformly converges k when $\theta_1 \le \arg(z-a) \le \theta_2$ and $|z-a| \to 0$, then

$$
\lim_{\epsilon\to 0^+}\int_{C_\epsilon} f(z)dz = \mathrm{i}k(\theta_2-\theta_1), \tag{125}
$$

where C_ϵ is an arc of radius ϵ centred at $z = a$, having a centre angle $(\theta_2-\theta_1)$ and an anticlockwise direction.

(2) $f(z)$ is continuous in the neighbourhood of $z = \infty$. If $zf(z)$ uniformly converges K when $\theta_1 \le \arg(z-a) \le \theta_2$ and $z \to \infty$, then

$$
\lim_{R\to\infty}\int_{C_R} f(z)dz = \mathrm{i}K(\theta_2-\theta_1), \tag{126}
$$

where C_R is an arc of radius R centred at the origin, having a centre angle $(\theta_2-\theta_1)$ and an anticlockwise direction.

of different values into Eq. (115):

$$\begin{aligned} G_{\text{ret}}(x) &= \theta(t)\int \frac{d^3p}{(2\pi)^3 E_{\mathbf{p}}} e^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}} \sin E_{\mathbf{p}} t, \\ G_{\text{adv}}(x) &= -\theta(-t)\int \frac{d^3p}{(2\pi)^3 E_{\mathbf{p}}} e^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}} \sin E_{\mathbf{p}} t, \\ G_{\text{F}}(x) &= \int \frac{d^3p}{(2\pi)^4} \frac{\mathrm{i}e^{-\mathrm{i}p\cdot x}}{{E_{\mathbf{p}}}^2 - E^2 - \mathrm{i}\epsilon} = \mathrm{i}\int \frac{d^3p}{(2\pi)^4 2E_{\mathbf{p}}} e^{\mathrm{i}(\mathbf{p}\cdot\mathbf{x} - E_{\mathbf{p}}|t|)}. \end{aligned} \tag{129}$$

where the Heaviside step function is defined as $\theta(t) = 0$ for $t < 0$, $\theta(t) = 1$ for $t > 0$ and $\theta(t) = 1/2$ for $t = 0$.

2.3.6 GFs Expressed by $D(x - y)$

The amplitude for a particle (in the KG field) to propagate from y to x is

$$D(x-y) = \langle 0|\, \phi(x)\phi(y)\, |0\rangle = \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} e^{-\mathrm{i}p\cdot(x-y)}\Big|_{p^0 = E_{\mathbf{p}}}. \tag{130}$$

It can be proved that

$$[\phi(x), \phi(y)] = \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} \left[e^{-\mathrm{i}p\cdot(x-y)} - e^{\mathrm{i}p\cdot(x-y)}\right] = D(x-y) - D(y-x). \tag{131}$$

Since $[\phi(x), \phi(y)]$ is not an operator (a **c-number** rather than a **q-number**), then

$$\langle 0|\, [\phi(x), \phi(y)]\, |0\rangle = [\phi(x), \phi(y)]\, \langle 0|0\rangle = [\phi(x), \phi(y)]\,. \tag{132}$$

If we redefine the retarded, advanced and Feynman GFs as

$$\begin{aligned} D_{\text{ret}}(x-y) &\equiv \theta(x^0 - y^0)\, \langle 0|\, [\phi(x), \phi(y)]\, |0\rangle\,, \\ D_{\text{adv}}(x-y) &\equiv \theta(y^0 - x^0)\, \langle 0|\, [\phi(y), \phi(x)]\, |0\rangle\,, \\ D_{\text{F}}(x-y) &\equiv \langle 0|\, \mathcal{T}\left\{\phi(x)\phi(y)\right\} |0\rangle \\ &= \theta(x^0 - y^0)\, \langle 0|\, \phi(x)\phi(y)\, |0\rangle + \theta(y^0 - x^0)\, \langle 0|\, \phi(y)\phi(x)\, |0\rangle \\ &= \begin{cases} D(x-y), & \text{for } x^0 > y^0, \\ D(y-x), & \text{for } y^0 < x^0. \end{cases} \end{aligned} \tag{133}$$

then we can demonstrate that

$$\begin{aligned} D_{\text{ret}}(x) &= -\mathrm{i}G_{\text{ret}}(x) = \theta(t)\left[D(x) - D(-x)\right], \\ D_{\text{adv}}(x) &= -\mathrm{i}G_{\text{adv}}(x) = \theta(-t)\left[D(-x) - D(x)\right], \\ D_{\text{F}}(x) &= -\mathrm{i}G_{\text{F}}(x) = \theta(t)D(x) + \theta(-t)D(-x). \end{aligned} \tag{134}$$

3 Dirac Field Bi-linears

3.1 Three Sets of γ Matrices

3.1.1 Dirac Algebra

γ **matrices** are a set of four matrices that generate a matrix representation of the **Dirac algebra**:

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} I_4, \quad \mu = 0, 1, 2, 3, \tag{135}$$

where $\eta^{\mu\nu}$ are components of the **Minkowski metric** with signature (+ - - -).

3.1.2 Feynman Slash Notation

For a **covariant vector** A, the **Feynman slash notation** is defined by

$$\not{A} \equiv \gamma^\mu A_\mu, \tag{136}$$

where there is an implied summation over μ. If A^μ and A^ν "commute", then

$$\not{A}^2 = \gamma^\mu\gamma^\nu A_\mu A_\nu = \frac{1}{2}\left(\gamma^\mu\gamma^\nu A_\mu A_\nu + \gamma^\nu\gamma^\mu A_\nu A_\mu\right) = \frac{\{\gamma^\mu, \gamma^\nu\}}{2} A_\mu A_\nu = A^2. \tag{137}$$

3.1.3 Standard (Dirac) Representation

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{138}$$

where σ^is are the **Pauli matrices**

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{139}$$

We now introduce the fifth γ matrix

$$\gamma^5 \equiv \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \tag{140}$$

satisfying

$$\left(\gamma^5\right)^\dagger = \gamma^5, \quad \left(\gamma^5\right)^2 = I_4 \quad \text{and} \quad \left\{\gamma^5, \gamma^\mu\right\} = 0. \tag{141}$$

All the four γ matrices anti-commute, so

$$\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^{[0}\gamma^1\gamma^2\gamma^{3]} = \frac{1}{4!}\delta^{0123}_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = \frac{1}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma. \tag{142}$$

where $\delta^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma} = \epsilon^{\alpha\beta\gamma\delta}\epsilon_{\mu\nu\rho\sigma}$ is the **generalised Kronecker delta**, $\epsilon^{\mu\nu\rho\sigma} = \delta^{\mu\nu\rho\sigma}_{0123}$ and $\epsilon_{\mu\nu\rho\sigma} = \delta^{0123}_{\mu\nu\rho\sigma}$ are the **Levi-Civita symbols**. Hence

$$\gamma^5 = \frac{\mathrm{i}}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma. \tag{143}$$

3.1.4 Weyl (Chiral) Representation and Van der Waerden Notation

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}, \tag{144}$$

where

$$\sigma^\mu \equiv (I_2, \boldsymbol{\sigma}), \quad \bar{\sigma}^\mu \equiv (I_2, -\boldsymbol{\sigma}), \ \boldsymbol{\sigma} \equiv \left(\sigma^1, \sigma^2, \sigma^3\right). \tag{145}$$

In the Weyl basis, the chiral decomposition of a **Dirac field** $\psi(x)$ take a simple form,

$$\psi = \psi_L + \psi_R \equiv \frac{1}{2}\left(1-\gamma^5\right)\psi + \frac{1}{2}\left(1+\gamma^5\right)\psi = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}\psi + \begin{pmatrix} 0 & 0 \\ 0 & I_2 \end{pmatrix}\psi. \tag{146}$$

where ψ_L and ψ_R are the left and right handed **Weyl spinors**. Both ψ_L and ψ_R have only two nonzero components and can be defined through **Van der Waerden's notation** [13] :

$$\psi_L \equiv \begin{pmatrix} \phi_a \\ 0 \end{pmatrix} \quad \text{and} \quad \psi_R \equiv \begin{pmatrix} 0 \\ \bar{\chi}^{\dot{a}} \end{pmatrix}, \tag{147}$$

where ϕ_a and $\bar{\chi}_{\dot{a}}$ are both two-component spinors; the **undotted** (**chiral**) **indices** $a = 1, 2$ and the **dotted** (**anti-chiral**) **indices** $\dot{a} = \dot{1}, \dot{2}$ are perfectly independent; the over-bar is just a reminder of $\chi^{\dot{a}}$ belonging to the right handed ψ_R, when there is no a and $\dot{a}$.

We define the **Dirac adjoint** for a Dirac spinor ψ:

$$\bar{\psi}(x) \equiv \psi^\dagger(x)\gamma^0, \tag{148}$$

where $\psi^\dagger$ is the **hermitian adjoint** of ψ. From Eq. (147),

$$\psi = \begin{pmatrix} \phi_a \\ \bar{\chi}^{\dot{a}} \end{pmatrix} \quad \text{and} \quad \bar{\psi} = \begin{pmatrix} \phi_a^\dagger & \bar{\chi}^{\dot{a}\dagger} \end{pmatrix}\gamma^0 = \begin{pmatrix} \bar{\chi}^{\dot{a}\dagger} & \phi_a^\dagger \end{pmatrix}. \tag{149}$$

However, both $\dot{a}$ and the bar (not the one on ψ) indicate that $\bar{\chi}^{\dot{a}\dagger}$ should be "right handed". If we want to define the right handed (hermitian conjugate) spinor is always to the right of the left handed one in $\bar{\psi}$, then Eq. (149) is obviously not the case here. Thus we have to take off the dot and bar from $\bar{\chi}^{\dot{a}\dagger}$, and likewise, put on the dot and bar to $\phi_a^\dagger$:

$$\chi^a \equiv \bar{\chi}^{\dot{a}\dagger}, \quad \bar{\phi}_{\dot{a}} \equiv \phi_a^\dagger. \tag{150}$$

Therefore

$$\bar{\chi}^{\dot{a}} = \chi^{a\dagger}, \quad \phi_a = \bar{\phi}_{\dot{a}}^\dagger, \tag{151}$$

and

$$\bar{\psi} = \begin{pmatrix} \chi^a & \bar{\phi}_{\dot{a}} \end{pmatrix}. \tag{152}$$

Let us sum up a bit now. Eq.s (149) and (152) define two independent sets of "covariant" and "contravariant" variables akin to the theory of relativity. They live in two chiral spaces and are distinguished by a and $\dot{a}$ (coupled with an over-bar). Eq.s (150) and (151) tell us how the covariant

[13] This notation works only for the Weyl basis.

(contravariant) ones switch from one space to the other. But we still need the rules to raise and lower the indices a and $\dot{a}$. This can be done with the Levi-Civita symbols (the "metrics"):

$$\epsilon_{ab} = \epsilon_{\dot{a}\dot{b}} = -\epsilon^{ab} = -\epsilon^{\dot{a}\dot{b}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mathrm{i}\sigma^2, \tag{153}$$

such that $\epsilon^{ac}\epsilon_{cb} = \delta^a_{\ b}$ and $\epsilon^{\dot{a}\dot{c}}\epsilon_{\dot{c}\dot{b}} = \delta^{\dot{a}}_{\ \dot{b}}$; and the proper rules are

$$\phi_a = \epsilon_{ab}\phi^b, \quad \phi^a = \epsilon^{ab}\phi_b \quad \text{and} \quad \bar{\chi}_{\dot{a}} = \epsilon_{\dot{a}\dot{b}}\bar{\chi}^{\dot{b}}, \quad \bar{\chi}^{\dot{a}} = \epsilon^{\dot{a}\dot{b}}\bar{\chi}_{\dot{b}}, \tag{154}$$

where there is an implied summation for b and $\dot{b}$ ("Einstein's summation"). Finally, we can associate the σ^μ and $\bar{\sigma}^\mu$ in Eq. (145) with indices a and $\dot{a}$ ("covariant and contravariant tensors"):

$$\sigma^\mu_{a\dot{a}} = \sigma^\mu = (I_2, \boldsymbol{\sigma}), \quad \bar{\sigma}^{\mu\dot{a}a} = \bar{\sigma}^\mu = (I_2, -\boldsymbol{\sigma}), \tag{155}$$

such that

$$\bar{\sigma}^{\mu\dot{a}a} = \epsilon^{\dot{a}\dot{b}}\epsilon^{ab}\sigma^\mu_{b\dot{b}}. \tag{156}$$

Mind that the row indices are always to the left of the column ones here.

3.1.5 Majorana Representation

$$\begin{aligned} &\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} \mathrm{i}\sigma^3 & 0 \\ 0 & \mathrm{i}\sigma^3 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} -\mathrm{i}\sigma^1 & 0 \\ 0 & -\mathrm{i}\sigma^1 \end{pmatrix}, \\ &\gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix}, \end{aligned} \tag{157}$$

where $(\gamma^\mu)^* = -\gamma^\mu$ and $(\gamma^5)^* = -\gamma^5$.

3.2 Dirac Lagrangian and Dirac Equation

In the **Dirac basis**, the free **Dirac equation** for a Dirac field can be obtained by applying the **Euler-Lagrangian equation** to the **Lagrangian density**

$$\mathcal{L} = \bar{\psi}\left(\mathrm{i}\partial\!\!\!/ - m\right)\psi = \mathrm{i}\gamma^\mu\bar{\psi}\left(\partial_\mu\psi\right) - m\bar{\psi}\psi. \tag{158}$$

Explicitly, that is

$$\begin{cases} \frac{\partial\mathcal{L}}{\partial\bar{\psi}} - \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\bar{\psi}\right)}\right] & = \left(\mathrm{i}\gamma^\mu\partial_\mu - m\right)\psi = \left(\mathrm{i}\partial\!\!\!/ - m\right)\psi = 0, \quad \textbf{Dirac equation}, \\ \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right] - \frac{\partial\mathcal{L}}{\partial\psi} & = \bar{\psi}\left(\mathrm{i}\partial\!\!\!/ + m\right) = 0, \quad \textbf{adjoint Dirac equation}. \end{cases} \tag{159}$$

3.3 Generators of Λ_S

We wish Dirac's equation to be **covariant** under the **Lorentz transformation** ($x' = \Lambda x$):

$$(\mathrm{i}\gamma^\nu \partial'_\nu - m)\,\psi'(x') = 0. \tag{160}$$

Suppose that there exists an invertible matrix Λ_S [14] such that

$$\psi'^\nu(x) = (\Lambda_S)^\nu{}_\mu \psi^\mu(\Lambda^{-1}x) \quad \text{or} \quad \psi'(x') = \Lambda_S\psi(x),\ x'^\nu = \Lambda^\nu{}_\mu x^\mu. \tag{161}$$

Plugging it into Eq. (160):

$$\left(\mathrm{i}\gamma^\nu \frac{\partial x^\mu}{\partial x'^\nu}\partial_\nu - m\right)\Lambda_S\psi(x) = \left[\mathrm{i}\gamma^\nu(\Lambda^{-1})^\mu{}_\nu\partial_\nu - m\right]\Lambda_S\psi(x) = 0. \tag{162}$$

Plugging the above equation by Λ_S^{-1}, then

$$\left[\mathrm{i}\Lambda_S^{-1}\gamma^\nu\Lambda_S(\Lambda^{-1})^\mu{}_\nu\partial_\nu - m\right]\psi(x) = 0. \tag{163}$$

Compare it with Eq. (159), we know that there must be

$$\Lambda_S^{-1}\gamma^\nu\Lambda_S(\Lambda^{-1})^\mu_\nu = \gamma^\mu \quad \text{or} \quad \Lambda_S^{-1}\gamma^\mu\Lambda_S = \Lambda^\mu{}_\nu\gamma^\nu. \tag{164}$$

Consider an **infinitesimal Lorentz transformation**

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \delta\omega^\mu_\nu, \tag{165}$$

where the **transformation parameters** ($\delta\omega^\mu_\nu$) are six tensors (antisymmetric with both indices up or down [15]; $\delta\omega^\mu_\nu = g^{\mu\rho}\delta\omega_{\rho\nu} = -g^{\mu\rho}\delta\omega_{\nu\rho} = -\delta\omega_\nu{}^\mu$) that tell us what transformation (boosts or rotations) we are doing. Assume that transformation parameters for Λ_S are the same as those of Λ's, then the corresponding

$$\Lambda_S = \exp\left(-\frac{\mathrm{i}}{4}\delta\omega_{\mu\nu}\sigma^{\mu\nu}\right), \tag{166}$$

where $\sigma^{\mu\nu}$ are 4×4 antisymmetric matrices that can be interpreted as **generators** of Λ_S. Substituting Eq.s (165) and (166) into Eq. (164), we get

$$\begin{aligned}
&\Lambda_S^{-1}\gamma^\mu\Lambda_S - (\delta^\mu_\nu + \delta\omega^\mu_\nu)\,\gamma^\nu \\
=& \left(1 + \frac{\mathrm{i}}{4}\delta\omega_{\rho\sigma}\sigma^{\rho\sigma}\right)\gamma^\mu\left(1 - \frac{\mathrm{i}}{4}\delta\omega_{\rho\sigma}\sigma^{\rho\sigma}\right) + O(\delta\omega_{\rho\sigma}) - \gamma^\mu - \delta\omega^\nu{}_\nu\gamma^\nu \\
=& \gamma^\mu + \frac{\mathrm{i}}{4}\delta\omega_{\rho\sigma}\left[\sigma^{\rho\sigma},\gamma^\mu\right] + O(\delta\omega_{\rho\sigma}) - \gamma^\mu - \frac{1}{2}\left(\delta\omega^\rho{}_\nu\gamma^\rho + \delta\omega^\sigma{}_\nu\gamma^\sigma\right) \\
=& \frac{\mathrm{i}}{4}\delta\omega_{\rho\sigma}\left[\sigma^{\rho\sigma},\gamma^\mu\right] - \frac{1}{2}\delta\omega_{\rho\sigma}\left(\gamma^\rho\eta^{\sigma\mu} - \gamma^\sigma\eta^{\mu\rho}\right).
\end{aligned} \tag{167}$$

[14]Note that Λ_S acts specifically on the spinor field $\psi^\mu(x)$, but not x or the ordinary Lorentz 4-vectors.

[15]It can be proved by using the property of a Lorentz transformation: $\eta^\sigma_\rho = \Lambda^\mu_\rho\Lambda^\nu_\sigma\eta^\mu_\nu$.

That is,

$$[\sigma^{\mu\nu}, \gamma^\rho] = 2\mathrm{i}\left(\gamma^\mu \eta^{\nu\rho} - \gamma^\nu \eta^{\rho\mu}\right). \tag{168}$$

It can be proved that

$$\sigma^{\mu\nu} = \frac{\mathrm{i}}{2}\left[\gamma^\mu, \gamma^\nu\right] = \mathrm{i}\left(\gamma^\mu\gamma^\nu - \eta^{\mu\nu}\right), \tag{169}$$

with the property

$$[\sigma^{\mu\nu}, \sigma^{\rho\sigma}] = -2\left(\gamma^\mu\gamma^\sigma\eta^{\nu\rho} - \gamma^\nu\gamma^\sigma\eta^{\rho\mu} + \gamma^\rho\gamma^\mu\eta^{\nu\sigma} - \gamma^\rho\gamma^\nu\eta^{\sigma\mu}\right). \tag{170}$$

In the **Dirac basis**, the boost and rotation generators are

$$\sigma^{0i} = \frac{\mathrm{i}}{2}\left[\gamma^0, \gamma^i\right] = \mathrm{i}\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \tag{171}$$

and

$$\sigma^{ij} = \frac{\mathrm{i}}{2}\left[\gamma^i, \gamma^j\right] = -\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix},\ i \neq j. \tag{172}$$

A **finite** Lorentz transformation can be regarded as the product of infinite number of infinitesimal transformations. With the finite transformation parameters $\omega_{\mu\nu}$, the correspondent Λ_S will be [16]

$$\Lambda_S = \exp\left(-\frac{\mathrm{i}}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right). \tag{176}$$

3.4 Transformation Properties of Bi-linears

3.4.1 Lorentz Scalar

$$\begin{aligned} &(\gamma^0)^2 = -(\gamma^i)^2 = 1 \quad \text{and} \quad (\gamma^0)^\dagger = \gamma^0,\ (\gamma^i)^\dagger = -\gamma^i, \\ \Longrightarrow &\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger \quad \text{and} \quad (\sigma^{\mu\nu})^\dagger = \frac{\mathrm{i}}{2}\left[(\gamma^\nu)^\dagger, (\gamma^\mu)^\dagger\right] = -\gamma^0\sigma^{\mu\nu}\gamma^0. \end{aligned} \tag{177}$$

[16] For the ordinary Lorentz transformation,

$$\Lambda = \exp\left(-\frac{\mathrm{i}}{2}\omega_{\mu\nu}L^{\mu\nu}\right), \tag{173}$$

where

$$\left(L^{\mu\nu}\right)^\alpha_{\ \beta} = \mathrm{i}\left(\eta^{\mu\alpha}\delta^\nu_{\ \beta} - \eta^{\nu\alpha}\delta^\mu_{\ \beta}\right). \tag{174}$$

These Lorentz generators, $L^{\mu\nu}$, obey the Lie algebra relationship

$$[L^{\mu\nu}, L^{\rho\sigma}] = \mathrm{i}\left(\eta^{\nu\rho}L^{\mu\sigma} - \eta^{\rho\mu}L^{\nu\sigma} - \eta^{\nu\sigma}L^{\rho\mu} + \eta^{\sigma\mu}L^{\rho\nu}\right). \tag{175}$$

As a consequence, $\bar{\psi}\psi$ transforms as a scalar under a Lorentz transformation:

$$\begin{aligned}
\bar{\psi}'(x')\psi'(x') &= \left[\psi^\dagger(x')\right]'\gamma^0\Lambda_S\psi(x) \\
&= \psi^\dagger(x)\Lambda_S^\dagger\gamma^0\Lambda_S\psi(x) \\
&= \psi^\dagger(x)\left(1-\frac{\mathrm{i}}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right)^\dagger\gamma^0\left(1-\frac{\mathrm{i}}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right)\psi(x) \\
&= \psi^\dagger(x)\gamma^0\psi(x) \\
&= \bar{\psi}(x)\psi(x).
\end{aligned} \tag{178}$$

3.4.2 Lorentz Tensor

Using Eq. (164) and the identity

$$\left(\gamma^0\right)^{-1}\Lambda_S^\dagger\gamma^0 = \Lambda_S^{-1}, \tag{179}$$

we have

$$\begin{aligned}
\bar{\psi}'(x')\gamma^\mu\psi'(x') &= \left[\psi^\dagger(x')\right]'\gamma^\mu\Lambda_S\psi(x) \\
&= \left[\psi^\dagger(x)\cdot\gamma^0\left(\gamma^0\right)^{-1}\cdot\Lambda_S^\dagger\right]\gamma^0\cdot\gamma^\mu\Lambda_S\psi(x) \\
&= \left[\psi^\dagger(x)\gamma^0\right]\left[\left(\gamma^0\right)^{-1}\Lambda_S^\dagger\gamma^0\right]\gamma^\mu\Lambda_S\psi(x) \\
&= \bar{\psi}(x)\Lambda_S^{-1}\gamma^\mu\Lambda_S\psi(x) \\
&= \bar{\psi}(x)\Lambda^\mu_{\ \nu}\gamma^\nu\psi(x) \\
&= \Lambda^\mu_{\ \nu}\bar{\psi}(x)\gamma^\nu\psi(x).
\end{aligned} \tag{180}$$

3.4.3 Lorentz Antisymmetric Tensor

Similar to Eq. (180), we have for $\sigma^{\mu\nu} = \frac{\mathrm{i}}{2}\left[\gamma^\mu,\gamma^\nu\right]$:

$$\bar{\psi}'(x')\sigma^{\mu\nu}\psi'(x') = \bar{\psi}(x)\Lambda_S^{-1}\sigma^{\mu\nu}\Lambda_S\psi(x). \tag{181}$$

Form Eq. (164),

$$\begin{aligned}
-2\mathrm{i}\Lambda_S^{-1}\sigma^{\mu\nu}\Lambda_S &= \Lambda_S^{-1}\left[\gamma^\mu,\gamma^\nu\right]\Lambda_S \\
&= \Lambda_S^{-1}\gamma^\mu\gamma^\nu\Lambda_S - \Lambda_S^{-1}\gamma^\nu\gamma^\mu\Lambda_S \\
&= \left(\Lambda_S^{-1}\gamma^\mu\Lambda_S\right)\left(\Lambda_S^{-1}\gamma^\nu\Lambda_S\right) - \left(\Lambda_S^{-1}\gamma^\nu\Lambda_S\right)\left(\Lambda_S^{-1}\gamma^\mu\Lambda_S\right) \\
&= \Lambda^\mu_{\ \rho}\gamma^\rho\Lambda^\nu_{\ \sigma}\gamma^\sigma - \Lambda^\nu_{\ \sigma}\gamma^\sigma\Lambda^\mu_{\ \rho}\gamma^\rho \\
&= \Lambda^\mu_{\ \rho}\Lambda^\nu_{\ \sigma}\left[\gamma^\rho,\gamma^\sigma\right] \\
&= -2\mathrm{i}\Lambda^\mu_{\ \rho}\Lambda^\nu_{\ \sigma}\sigma^{\rho\sigma}.
\end{aligned} \tag{182}$$

Hence

$$\bar{\psi}'(x')\sigma^{\mu\nu}\psi'(x') = \bar{\psi}(x)\Lambda_S^{-1}\sigma^{\mu\nu}\Lambda_S\psi(x) = \Lambda^\mu_{\ \rho}\Lambda^\nu_{\ \sigma}\bar{\psi}(x)\sigma^{\rho\sigma}\psi(x). \tag{183}$$

3.4.4 Lorentz Pseudo-vector

Form Eq. (164),

$$
\begin{aligned}
\Lambda_S^{-1}\gamma^5\Lambda_S &= \frac{\mathrm{i}}{4!}\epsilon_{\mu\nu\rho\sigma}\Lambda_S^{-1}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\Lambda_S \\
&= \frac{\mathrm{i}}{4!}\epsilon_{\mu\nu\rho\sigma}\left(\Lambda_S^{-1}\gamma^\mu\Lambda_S\right)\left(\Lambda_S^{-1}\gamma^\nu\Lambda_S\right)\left(\Lambda_S^{-1}\gamma^\rho\Lambda_S\right)\left(\Lambda_S^{-1}\gamma^\sigma\Lambda_S\right) \\
&= \frac{\mathrm{i}}{4!}\epsilon_{\mu\nu\rho\sigma}\Lambda^\mu_{\ \alpha}\gamma^\alpha\Lambda^\nu_{\ \beta}\gamma^\beta\Lambda^\rho_{\ \gamma}\gamma^\gamma\Lambda^\sigma_{\ \delta}\gamma^\delta \\
&= \frac{\mathrm{i}}{4!}\epsilon_{\mu\nu\rho\sigma}\Lambda^\mu_{\ \alpha}\Lambda^\nu_{\ \beta}\Lambda^\rho_{\ \gamma}\Lambda^\sigma_{\ \delta}\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta \\
&= \frac{\mathrm{i}}{4!}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\gamma\delta}\Lambda^\mu_{\ \alpha}\Lambda^\nu_{\ \beta}\Lambda^\rho_{\ \gamma}\Lambda^\sigma_{\ \delta}\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta \\
&= \left(\epsilon_{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma\delta}\Lambda^\mu_{\ \alpha}\Lambda^\nu_{\ \beta}\Lambda^\rho_{\ \gamma}\Lambda^\sigma_{\ \delta}\right)\left(\frac{\mathrm{i}}{4!}\epsilon_{\alpha\beta\gamma\delta}\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\right) \\
&= \det\left(\Lambda\right)\gamma^5.
\end{aligned}
\tag{184}
$$

Then similar to Eq. (180),

$$
\begin{aligned}
\bar{\psi}'(x')\gamma^5\gamma^\mu\psi'(x') &= \bar{\psi}(x)\Lambda_S^{-1}\gamma^5\Lambda_S\Lambda_S^{-1}\gamma^\mu\Lambda_S\psi(x) \\
&= \bar{\psi}(x)\det\left(\Lambda\right)\gamma^5\Lambda^\mu_{\ \nu}\gamma^\nu\psi(x) \\
&= \det\left(\Lambda\right)\Lambda^\mu_{\ \nu}\bar{\psi}(x)\gamma^5\gamma^\nu\psi(x).
\end{aligned}
\tag{185}
$$

3.4.5 Lorentz Pseudo-scalar

Much the same as Eq. (185),

$$
\bar{\psi}'(x')\gamma^5\psi'(x') = \bar{\psi}(x)\Lambda_S^{-1}\gamma^5\Lambda_S\psi(x) = \det\left(\Lambda\right)\bar{\psi}(x)\gamma^5\psi(x). \tag{186}
$$

4 A Cute Trick

To prove that

$$
\left(\not{p}\pm m\right)\gamma^0\left(\not{p}\pm m\right) = 2p^0\left(\not{p}\pm m\right) \tag{187}
$$

for $p^2 = E^2 - \mathbf{p}^2 = m^2$, we shall always remember that $\left(\gamma^0\right)^2 = -\left(\gamma^i\right)^2 = 1$ and $\gamma^0\gamma^i = -\gamma^i\gamma^0$:

$$\begin{aligned}
&\left[\left(\not{p} \pm m\right)\gamma^0 - 2p^0\right]\left(\not{p} \pm m\right) \\
=& \left(\eta_{\mu\rho} p^\mu \gamma^\rho \gamma^0 \pm m\gamma^0 - 2E\right)\left(\eta_{\nu\sigma} p^\nu \gamma^\sigma \pm m\right) \\
=& \left(\eta_{ik} p^i \gamma^k \gamma^0 \pm m\gamma^0 - E\right)\left(\eta_{jl} p^j \gamma^l + E\gamma^0 \pm m\right) \\
=& \eta_{ik}\eta_{jl} p^i p^j \gamma^k \gamma^0 \gamma^l \pm m\left(\eta_{ik} p^i \gamma^k \gamma^0 + \eta_{jl} p^j \gamma^0 \gamma^l\right) + E\left[\eta_{ik} p^i \gamma^k \left(\gamma^0\right)^2 - \eta_{jl} p^j \gamma^l\right] \\
&\pm mE\left[\left(\gamma^0\right)^2 - 1\right] + \left(m^2 - E^2\right)\gamma^0 \\
=& -\eta_{ik}\eta_{jl} p^i p^j \gamma^k \gamma^l \gamma^0 \pm m\left(\eta_{ik} p^i \gamma^k \gamma^0 - \eta_{ik} p^i \gamma^k \gamma^0\right) + E\left[\eta_{ik} p^i \gamma^k - \eta_{ik} p^i \gamma^k\right] - \mathbf{p}^2\gamma^0 \\
=& \left(-\mathbf{p}^2 - \eta_{ik}\eta_{jl} p^i p^j \gamma^k \gamma^l\right)\gamma^0 \\
=& \left(-\mathbf{p}^2 - \sum_{i=1}^{3}\sum_{j=1}^{3} p^i p^j \gamma^i \gamma^j\right)\gamma^0 \\
=& \left(-\mathbf{p}^2 + \mathbf{p}^2 - \sum_{1\le i,j\le 3, i\neq j} p^i p^j \gamma^i \gamma^j\right)\gamma^0 \\
=& -\left(\sum_{1\le i,j\le 3, i\neq j} p^i p^j \gamma^i \gamma^j\right)\gamma^0 \\
=& 0.
\end{aligned} \tag{188}$$

The reason for the last step is that $p^i p^j = p^j p^i$ and $\gamma^i\gamma^j = -\gamma^j\gamma^i$, for any two different i and j.

5 C and T for the Dirac Field

5.1 Plane Wave Solutions

5.1.1 Positive and Negative Solutions

Premultiplying Dirac's equation by $\left(-\mathrm{i}\not{\partial} - m\right)$, we see that the $\psi(x)$ obeys the KG equation:

$$\left(-\mathrm{i}\not{\partial} - m\right)\left(\mathrm{i}\not{\partial} - m\right)\psi = \left(\not{\partial}^2 + m^2\right)\psi = \left(\frac{\{\gamma^\mu, \gamma^\nu\}}{2}\partial_\mu\partial_\nu + m^2\right)\psi = (\Box + m^2)\phi = 0. \tag{189}$$

In consequence, we can write the (positive energy) plane wave solution to Dirac's equation as

$$\psi(x) = u(p)e^{-\mathrm{i}p\cdot x}, \quad p^2 = m^2,\ p^0 > 0. \tag{190}$$

Plugging it into Dirac's equation, we get (in the **Weyl basis**)

$$\left(\not{p} - m\right)u(p) = \begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\bar{\sigma} & -m \end{pmatrix} u(p) = 0, \tag{191}$$

where $\sigma^\mu = (1, \boldsymbol{\sigma})$, $\quad \bar{\sigma}^\mu = (1, -\boldsymbol{\sigma})$. Note that $(p \cdot \sigma)(p \cdot \bar{\sigma}) = p_0{}^2 - p_i p_j \sigma^i \sigma^j = p_0{}^2 - p_i p_j \delta^{ij} = p^2 = m^2$, the solution to Eq. (191) is then (up to a constant)

$$u(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\xi \\ \sqrt{p \cdot \bar{\sigma}}\xi \end{pmatrix}. \tag{192}$$

We saw above that $\bar{\psi}\psi$ is a Lorentz scalar. That is,

$$\bar{u}u = \left(\xi^\dagger \sqrt{p \cdot \sigma}, \xi^\dagger \sqrt{p \cdot \bar{\sigma}}\right) \gamma^0 \begin{pmatrix} \sqrt{p \cdot \sigma}\xi \\ \sqrt{p \cdot \bar{\sigma}}\xi \end{pmatrix} = 2m\xi^\dagger \xi \tag{193}$$

is also supposed to be a Lorentz scalar. In fact, it is convenient to introduce an orthonormal basis

$$\xi^{r\dagger}\xi^s = \delta^{rs}, \quad r, s = 1, 2, \tag{194}$$

such as the "up" and "down" spinors

$$\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{195}$$

In that case, $u(p)$ can be written as

$$u^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\xi^s \\ \sqrt{p \cdot \bar{\sigma}}\xi^s \end{pmatrix}, \quad s = 1, 2. \tag{196}$$

Similarly, the (negative energy) plane wave solution to Dirac's equation are (we use $\dot{s}$ here to distinguish it from s)

$$\psi^{\dot{s}}(x) = v^{\dot{s}}(p)e^{\mathrm{i}p\cdot x}, \quad v^{\dot{s}}(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\eta^{\dot{s}} \\ -\sqrt{p \cdot \bar{\sigma}}\eta^{\dot{s}} \end{pmatrix}, \; \dot{s} = \dot{1}, \dot{2}. \tag{197}$$

where we still have $p^2 = m^2$, $p^0 > 0$ and

$$\eta^{\dot{r}\dagger}\eta^{\dot{s}} = \delta^{\dot{r}\dot{s}}, \quad \dot{r}, \dot{s} = \dot{1}, \dot{2}. \tag{198}$$

5.1.2 Properties of the Solutions

$$\begin{aligned} u^{r\dagger}(p)u^s(p) &= \left(\xi^{r\dagger}\sqrt{p \cdot \sigma}, \xi^{r\dagger}\sqrt{p \cdot \bar{\sigma}}\right) \begin{pmatrix} \sqrt{p \cdot \sigma}\xi^s \\ \sqrt{p \cdot \bar{\sigma}}\xi^s \end{pmatrix} = 2p_0\delta^{rs}, \\ \bar{u}^r(p)u^s(p) &= \left(\xi^{r\dagger}\sqrt{p \cdot \sigma}, \xi^{r\dagger}\sqrt{p \cdot \bar{\sigma}}\right) \gamma^0 \begin{pmatrix} \sqrt{p \cdot \sigma}\xi^s \\ \sqrt{p \cdot \bar{\sigma}}\xi^s \end{pmatrix} = 2m\delta^{rs}, \end{aligned} \tag{199}$$

$$v^{\dot{r}\dagger}(p)v^{\dot{s}}(p) = 2p_0\delta^{\dot{r}\dot{s}}, \quad \bar{v}^{\dot{r}}(p)v^{\dot{s}}(p) = -2m\delta^{\dot{r}\dot{s}}; \tag{200}$$

$$\bar{u}^r(p)v^{\dot{s}}(p) = \bar{v}^{\dot{r}}(p)u^s(p) = u^{r\dagger}(p)v^{\dot{s}}(p') = v^{\dot{r}\dagger}(p)u^s(p'), \quad p' \equiv (p_0, -\mathbf{p}); \tag{201}$$

$$\sum_s u^s(p)\bar{u}^s(p) = \not{p} + m, \quad \sum_{\dot{s}} v^{\dot{s}}(p)\bar{v}^{\dot{s}}(p) = \not{p} - m. \tag{202}$$

Note that $u^\dagger v$ and $v^\dagger u$ are slightly different, for $(p \cdot \sigma)(p' \cdot \sigma) = (p \cdot \bar{\sigma})(p' \cdot \bar{\sigma}) = p^2 = m^2$.

5.2 Fermionic Quantisation

Once again, we use the canonical quantisation to quantise the spinor field governed by the Lagrangian

$$\mathcal{L}(x) = \bar{\psi}(x)\left(\mathrm{i}\partial\!\!\!/ - m\right)\psi(x) \tag{203}$$

In Schrödinger's picture, we promote ψ and $\pi \equiv \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = i\bar{\psi}\gamma^0 = i\psi^\dagger$ to operators [17]:

$$\left\{\begin{array}{rcl} \psi(\mathbf{x}) & = & \sum_s \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\left[a^s_\mathbf{p} u^s(\mathbf{p})e^{\mathbf{ip\cdot x}} + b^{\dot{s}\dagger}_\mathbf{p} v^{\dot{s}}(\mathbf{p})e^{-\mathbf{ip\cdot x}}\right]\big|_{p^0=E_\mathbf{p}}, \\ \psi^\dagger(\mathbf{x}) & = & \sum_s \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\left[a^{s\dagger}_\mathbf{p} u^{s\dagger}(\mathbf{p})e^{\mathbf{ip\cdot x}} + b^{\dot{s}}_\mathbf{p} v^{\dot{s}\dagger}(\mathbf{p})e^{-\mathbf{ip\cdot x}}\right]\big|_{p^0=E_\mathbf{p}}, \end{array}\right. \tag{205}$$

where $E_\mathbf{p} \equiv \sqrt{|\mathbf{p}|^2+m^2}$. For the spin $^1/_2$ particles, we should consider the fermionic anti-commutation relations rather than the bosonic commutation ones, otherwise H will have no bound below. Suppose that the equal-time anti-commutation relations are

$$\{\psi_\alpha(\mathbf{x}),\psi_\beta(\mathbf{y})\} = \left\{\psi^\dagger_\alpha(\mathbf{x}),\psi^\dagger_\beta(\mathbf{y})\right\} = 0 \quad \text{and} \quad \left\{\psi_\alpha(\mathbf{x}),\psi^\dagger_\beta(\mathbf{y})\right\} = \mathrm{i}\delta_{\alpha\beta}\delta^3(\mathbf{x}-\mathbf{y}), \tag{206}$$

or equivalently,

$$\left\{a^r_\mathbf{p}, a^{s\dagger}_\mathbf{q}\right\} = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})\delta^{rs}, \quad \left\{b^{\dot{r}}_\mathbf{p}, b^{\dot{s}\dagger}_\mathbf{q}\right\} = -(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})\delta^{\dot{r}\dot{s}}, \tag{207}$$

with all the other anti-commutators vanishing. Note that Eq. (191) yields

$$\left(-\gamma^i\partial_i + m\right)u^s(p) = \gamma^0 p_0 u^s(p); \tag{208}$$

and in like manner,

$$\left(\gamma^i\partial_i + m\right)v^{\dot{s}}(p) = -\gamma^0 p_0 v^{\dot{s}}(p). \tag{209}$$

Then from Eq.s (199)–(209), we can compute the Hamiltonian operator [18]

$$\begin{aligned} H &= \int d^3x\mathcal{H} = \int d^3x\left(\pi\dot{\psi} - \mathcal{L}\right) = \int d^3x\bar{\psi}\left(-\mathrm{i}\gamma^i\partial_i + m\right)\psi \\ &= \sum_s\int\frac{d^3p}{(2\pi)^3}E_\mathbf{p}\left(a^{s\dagger}_\mathbf{p}a^s_\mathbf{p} - b^{\dot{s}}_\mathbf{p}b^{\dot{s}\dagger}_\mathbf{p}\right) \\ &= \sum_s\int\frac{d^3p}{(2\pi)^3}E_\mathbf{p}\left[a^{s\dagger}_\mathbf{p}a^s_\mathbf{p} + b^{\dot{s}\dagger}_\mathbf{p}b^{\dot{s}}_\mathbf{p} - (2\pi)^3\delta^3(0)\right] \\ &\sim \sum_s\int\frac{d^3p}{(2\pi)^3}E_\mathbf{p}\left(a^{s\dagger}_\mathbf{p}a^s_\mathbf{p} + b^{\dot{s}\dagger}_\mathbf{p}b^{\dot{s}}_\mathbf{p}\right) \end{aligned} \tag{210}$$

[17] We can make ψ and $\psi^\dagger$ time-dependent in Heisenberg's picture:

$$\left\{\begin{array}{rcl} \psi(x) & = & e^{\mathrm{i}Ht}\psi(\mathbf{x})e^{-\mathrm{i}Ht} = \sum_s\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\left[a^s_\mathbf{p}u^s(p)e^{-\mathrm{i}p\cdot x} + b^{\dot{s}\dagger}_\mathbf{p}v^{\dot{s}}(p)e^{\mathrm{i}p\cdot x}\right]\big|_{p^0=E_\mathbf{p}}, \\ \psi^\dagger(x) & = & \sum_s\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\left[a^{s\dagger}_\mathbf{p}u^{s\dagger}(p)e^{-\mathrm{i}p\cdot x} + b^{\dot{s}}_\mathbf{p}v^{\dot{s}\dagger}(p)e^{\mathrm{i}p\cdot x}\right]\big|_{p^0=E_\mathbf{p}}. \end{array}\right. \tag{204}$$

[18] $H = \sum_s\int\frac{d^3p}{(2\pi)^3}E_\mathbf{p}\left(a^{s\dagger}_\mathbf{p}a^s_\mathbf{p} - b^{\dot{s}}_\mathbf{p}b^{\dot{s}\dagger}_\mathbf{p}\right) = \sum_s\int\frac{d^3p}{(2\pi)^3}E_\mathbf{p}\left[a^{s\dagger}_\mathbf{p}a^s_\mathbf{p} - b^{\dot{s}\dagger}_\mathbf{p}b^{\dot{s}}_\mathbf{p} + (2\pi)^3\delta^3(0)\right]$ in the bosonic quantisation. $\delta^3(0)$ can be dealt with by **normal ordering**, but $-b^{\dot{s}\dagger}_\mathbf{p}b^{\dot{s}}_\mathbf{p}$ makes H with no bound below, which is a disaster!

and the momentum operator

$$\mathbf{P} = \int d^3x \psi^\dagger \left(-\mathrm{i}\nabla\right) \psi = \sum_s \int \frac{d^3p}{(2\pi)^3} \mathbf{p} \left(a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s + b_{\mathbf{p}}^{\dot{s}\dagger} b_{\mathbf{p}}^{\dot{s}} \right). \tag{211}$$

The vacuum $|0\rangle$ is define to be the state such that

$$a_{\mathbf{p}}^s |0\rangle = b_{\mathbf{p}}^{\dot{s}} |0\rangle = 0. \tag{212}$$

Thus both $a_{\mathbf{p}}^{s\dagger}$ and $b_{\mathbf{p}}^{\dot{s}\dagger}$ create particles with energy $+E_{\mathbf{p}}$ and momentum $\mathbf{p}$. We will refer to the particles created by $a_{\mathbf{p}}^{s\dagger}$ as **fermions** and to those created by $b_{\mathbf{p}}^{\dot{s}\dagger}$ as **anti-fermions**.

5.3 Charge Conjugation

The purpose of this section is to find the charge conjugation operator C that turns a particle into its antiparticle, reversing its charge and all other "charge-like" numbers. In addition, we will compute the transformation properties of the five bi-linears in Sec. 3.4 under C.

5.3.1 Coupling with the EM Filed

Following Sec. 3.4, it is all right to couple a scalar field ϕ to the ψ by adding $g\phi\bar{\psi}\psi$ (with g some coupling constant) to $\mathcal{L} = \bar{\psi}\left(\mathrm{i}\partial\!\!\!/ - m\right)\psi$ (and of course also adding the Lagrangian for ϕ). Similarly, we can couple a vector field A_μ by adding $-eA_\mu\bar{\psi}\gamma^\mu\psi$. Introducing the **covariant derivative** $D_\mu = \partial_\mu + \mathrm{i}eA_\mu$ and write $\mathcal{L} = \bar{\psi}\left(\mathrm{i}\partial\!\!\!/ - m\right)\psi - eA_\mu\bar{\psi}\gamma^\mu\psi = \bar{\psi}\left(\mathrm{i}D\!\!\!\!/ - m\right)\psi$. Thus, the Lagrangian for a Dirac field interacting with A_μ reads

$$\mathcal{L} = \bar{\psi}\left(\mathrm{i}D\!\!\!\!/ - m\right)\psi + j^\mu A_\mu - \frac{1}{4\mu_0} F_{\mu\nu}F^{\mu\nu}, \tag{213}$$

where $j^\mu = \frac{1}{\mu_0}\partial_\nu F^{\mu\nu}$ is the **four-current**, μ_0 the **free space permeability** and $F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ the **electromagnetic tensor**. Varying with respect to $\bar{\psi}$ or ψ, we obtain the Dirac or adjoint Dirac equation in the presence of an EM field:

$$\left(\mathrm{i}\partial\!\!\!/ - eA\!\!\!/ - m\right)\psi = 0 \quad \text{or} \quad \bar{\psi}\left(\mathrm{i}\overleftarrow{\partial\!\!\!/} + eA\!\!\!/ + m\right) = 0. \tag{214}$$

The conjugate of the adjoint Dirac equation is

$$\left[\gamma^{\mu\,\mathrm{T}}\left(\mathrm{i}\partial_\mu + eA_\mu\right) + m\right]\bar{\psi}^{\,\mathrm{T}} = 0. \tag{215}$$

Suppose that there exists an invertible matrix C, such that

$$C\gamma^{\mu\,\mathrm{T}}C^{-1} = -\gamma^\mu, \tag{216}$$

then after plugging C into Eq. (215),

$$\begin{aligned} C\left[\gamma^{\mu\,\mathrm{T}}\left(\mathrm{i}\partial_\mu + eA_\mu\right) + m\right]C^{-1}C\bar{\psi}^{\,\mathrm{T}} = & \left[\left(C\gamma^{\mu\,\mathrm{T}}C^{-1}\right)\left(\mathrm{i}\partial_\mu + eA_\mu\right) + m\right]C\bar{\psi}^{\,\mathrm{T}} \\ = & -\left(\mathrm{i}\partial\!\!\!/ + eA\!\!\!/ - m\right)C\bar{\psi}^{\,\mathrm{T}} = 0. \end{aligned} \tag{217}$$

If ψ is the field of the electron, then we could define a field ψ^c of the positron (with a charge opposite to that of the electron but with the same mass). From the Dirac equation, we could write

$$\left(\mathrm{i}\not{\partial} + e\not{A} - m\right)\psi^c = 0. \tag{218}$$

Comparing Eq. (218) with Eq. (217), we could therefore define

$$\psi^c \equiv C\bar{\psi}^{\,\mathrm{T}}. \tag{219}$$

and refer to C as the **charge conjugation matrix**.

5.3.2 Flipping the Spins

To verify that C can also flip the spin of ψ, we consider C of a specific form (in Weyl's basis)

$$C = \mathrm{i}\gamma^2\gamma^0 = \begin{pmatrix} \mathrm{i}\sigma^2 & 0 \\ 0 & -\mathrm{i}\sigma^2 \end{pmatrix} = -C^{-1} = -C^{\mathrm{T}} = -C^{\dagger}. \tag{220}$$

Using Van der Waerden's notation in Sec. 3.1.4, we suppose that $\psi = \begin{pmatrix} \phi_a \\ \bar{\chi}^{\dot{a}} \end{pmatrix}$, then

$$\bar{\psi} = \psi^{\dagger}\gamma^0 = \begin{pmatrix} \bar{\phi}_{\dot{a}} & \chi^a \end{pmatrix}\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} = \begin{pmatrix} \chi^a & \bar{\phi}_{\dot{a}} \end{pmatrix}, \tag{221}$$

$$\psi^c = C\bar{\psi}^{\,\mathrm{T}} = \begin{pmatrix} \epsilon_{\dot{b}\dot{a}} & 0 \\ 0 & \epsilon^{ba} \end{pmatrix}\begin{pmatrix} \bar{\chi}^{\dot{a}} \\ \phi_a \end{pmatrix}^{*} = \begin{pmatrix} \bar{\chi}_{\dot{b}} \\ \phi^b \end{pmatrix}^{*}. \tag{222}$$

However, the asterisks look redundant here. To get rid of them, we refer to the two-component spinors, ϕ and χ, as single elements such that $\phi^{\mathrm{T}} \sim \phi$ and $\chi^{\mathrm{T}} \sim \chi$. Hence the function of the asterisks in Eq. (222) is just taking off or putting on dots and bars (the same as daggers):

$$\psi^c = \begin{pmatrix} \bar{\chi}_{\dot{b}} \\ \phi^b \end{pmatrix}^{*} = \begin{pmatrix} \chi_b \\ \bar{\phi}^{\dot{b}} \end{pmatrix}. \tag{223}$$

In conclusion, $\psi \to \psi^c$ is just a "$\phi \leftrightarrow \chi$" process which can be interpreted as the flipping of spins:

$$\begin{pmatrix} \phi \\ \bar{\chi} \end{pmatrix} \to \begin{pmatrix} \chi \\ \bar{\phi} \end{pmatrix}. \tag{224}$$

5.3.3 C or $\mathcal{C}$

Consider a different definition of the charge conjugation:

$$\psi^c(x) \equiv \mathcal{C}\psi(x)\mathcal{C}^{\dagger}, \tag{225}$$

where $\mathcal{C}$ is a unitary operator called the **charge conjugation operator**, $\mathcal{C}^{-1} = \mathcal{C}^{\dagger}$. At the level of the full QFT, we will require the vacuum state $|0\rangle$ in Eq. (212) to be invariant under $\mathcal{C}$:

$$\mathcal{C}\,|0\rangle = |0\rangle\,. \tag{226}$$

We have seen in Sec. 5.2 that both $a_{\mathbf{p}}^{s\dagger}$ and $b_{\mathbf{p}}^{\dot{s}\dagger}$ create particles with energy $+E_{\mathbf{p}}$ and momentum $\mathbf{p}$. To fully determine the one-particle states $a_{\mathbf{p}}^{s\dagger}|0\rangle$ and $b_{\mathbf{p}}^{\dot{s}\dagger}|0\rangle$, we need to talk about the spins. We demand that particle and anti-particle states to be exchanged under $\mathcal{C}$:

$$\begin{aligned}\mathcal{C}a_{\mathbf{p}}^{s\dagger}|0\rangle &= \mathcal{C}a_{\mathbf{p}}^{s\dagger}\mathcal{C}^{-1}\mathcal{C}|0\rangle = \mathcal{C}a_{\mathbf{p}}^{s\dagger}\mathcal{C}^{-1}|0\rangle = b_{\mathbf{p}}^{\dot{s}\dagger}|0\rangle\,,\\ \mathcal{C}b_{\mathbf{p}}^{\dot{s}\dagger}|0\rangle &= \mathcal{C}b_{\mathbf{p}}^{\dot{s}\dagger}\mathcal{C}^{-1}|0\rangle = a_{\mathbf{p}}^{s\dagger}|0\rangle\,.\end{aligned} \tag{227}$$

Hence, for the one-particle states to satisfy these rules it is sufficient to require that

$$\mathcal{C}a_{\mathbf{p}}^{s\dagger}\mathcal{C}^{-1} = b_{\mathbf{p}}^{\dot{s}\dagger} \quad \text{and} \quad \mathcal{C}b_{\mathbf{p}}^{\dot{s}\dagger}\mathcal{C}^{-1} = a_{\mathbf{p}}^{s\dagger}, \tag{228}$$

or equivalently,

$$\mathcal{C}a_{\mathbf{p}}^{s}\mathcal{C}^{\dagger} = b_{\mathbf{p}}^{\dot{s}} \quad \text{and} \quad \mathcal{C}b_{\mathbf{p}}^{\dot{s}}\mathcal{C}^{\dagger} = a_{\mathbf{p}}^{s}. \tag{229}$$

The explicit form of one-particle states could be, for instance,

$$|\mathbf{p}, s\rangle \equiv \sqrt{2E_{\mathbf{p}}}\,a_{\mathbf{p}}^{s\dagger}|0\rangle \tag{230}$$

so that the inner product (according to Eq. (207))

$$\langle\mathbf{p}, r|\mathbf{q}, s\rangle = 2E_{\mathbf{p}}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})\delta^{rs}, \quad r, s = 1, 2, \tag{231}$$

is Lorentz invariant. With the knowledge of the canonical quantisation (in Sec. 5.2), the technique to flip the spins and inspired by the methods in Peskin's QFT, we can as well compute that

$$\psi^c(x) = \mathcal{C}\psi(x)\mathcal{C}^\dagger = \mathrm{i}\gamma^2\psi^* = C\bar{\psi}^{\,\mathrm{T}} \quad \text{(in Weyl's basis).} \tag{232}$$

5.3.4 Bi-linears

Follow the methods in Sec. 5.3.2, we can work out the transformation properties of the bi-linears in Sec. 3.4 under the charge conjugation,

$$\psi(x) \to \psi^c(x) = \mathcal{C}\psi(x)\mathcal{C}^\dagger = \mathcal{C}\begin{pmatrix}\phi_a\\ \bar{\chi}^{\dot{a}}\end{pmatrix}\mathcal{C}^\dagger = C\bar{\psi}^{\mathrm{T}} = \begin{pmatrix}\chi_b\\ \bar{\phi}^{\dot{b}}\end{pmatrix}, \tag{233}$$

where

$$C \equiv \mathrm{i}\gamma^2\gamma^0 = -C^{-1} = -C^{\mathrm{T}} = -C^\dagger \tag{234}$$

$$C^{-1}\gamma^\mu C = -\gamma^{\mu\,\mathrm{T}}, \quad C^{-1}\gamma^5 C = \gamma^5 = \begin{pmatrix}-I_2 & 0\\ 0 & I_2\end{pmatrix}. \tag{235}$$

Note that order of charge conjugation and Hermitian conjugation does not matter:

$$\left(\psi^\dagger\right)^c = \mathcal{C}\psi^\dagger\mathcal{C}^\dagger = \left(\mathcal{C}^\dagger\right)^\dagger\psi^\dagger\mathcal{C}^\dagger = \left(\mathcal{C}\psi\mathcal{C}^\dagger\right)^\dagger = \left(\psi^c\right)^\dagger. \tag{236}$$

Thus

$$\mathcal{C}\bar{\psi}\mathcal{C}^\dagger = \mathcal{C}\psi^\dagger\mathcal{C}^\dagger\gamma^0 = \left(\psi^c\right)^\dagger\gamma^0 = \begin{pmatrix}\phi^b & \bar{\chi}_{\dot{b}}\end{pmatrix} = \mathcal{C}\begin{pmatrix}\chi^a & \bar{\phi}_{\dot{a}}\end{pmatrix}\mathcal{C}^\dagger = \mathcal{C}\bar{\psi}\mathcal{C}^\dagger = \psi^{\mathrm{T}}C^{-1}. \tag{237}$$

The bilinear $\bar{\psi}A\psi$ therefore transforms as

$$\mathcal{C}\bar{\psi}A\psi\mathcal{C}^\dagger = \psi^{\mathrm{T}}C^{-1}AC\bar{\psi}^{\mathrm{T}} = \psi^{\mathrm{T}}C^{\mathrm{T}}\left(A^{\mathrm{T}}\right)^{\mathrm{T}}\left(C^{-1}\right)^{\mathrm{T}}\bar{\psi}^{\mathrm{T}} = \left(\bar{\psi}C^{-1}A^{\mathrm{T}}C\psi\right)^{\mathrm{T}}. \tag{238}$$

When $A = 1$,

$$\bar{\psi}A\psi = \begin{pmatrix} \chi^a & \bar{\phi}_{\dot{a}} \end{pmatrix}\begin{pmatrix} \phi_a \\ \bar{\chi}^{\dot{a}} \end{pmatrix} = \chi^a\phi_a + \bar{\phi}_{\dot{a}}\bar{\chi}^{\dot{a}} \quad \text{(scalar)}, \tag{239}$$

$$\mathcal{C}\bar{\psi}\psi\mathcal{C}^\dagger = \left(\bar{\psi}C^{-1}C\psi\right)^{\mathrm{T}} = \left(\bar{\psi}\psi\right)^{\mathrm{T}} = \bar{\psi}\psi. \tag{240}$$

In the same way, for $A = \gamma^\mu$,

$$\bar{\psi}\gamma^\mu\psi = \begin{pmatrix} \chi^a & \bar{\phi}_{\dot{a}} \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu_{a\dot{a}} \\ \bar{\sigma}^{\mu\dot{a}a} & 0 \end{pmatrix}\begin{pmatrix} \phi_a \\ \bar{\chi}^{\dot{a}} \end{pmatrix} = \chi^a\sigma^\mu_{a\dot{a}}\bar{\chi}^{\dot{a}} + \bar{\phi}_{\dot{a}}\bar{\sigma}^{\mu\dot{a}a}\phi_a, \tag{241}$$

$$\mathcal{C}\bar{\psi}A\psi\mathcal{C}^\dagger = \left(\bar{\psi}C^{-1}\gamma^{\mu\,\mathrm{T}}C\psi\right)^{\mathrm{T}} = -\left(\bar{\psi}\gamma^\mu\psi\right)^{\mathrm{T}} = -\left(\bar{\psi}\gamma^\mu\psi\right); \tag{242}$$

for $A = \gamma^5$, $A^{\mathrm{T}} = \gamma^5$,

$$\mathcal{C}\bar{\psi}A\psi\mathcal{C}^\dagger = \left(\bar{\psi}C^{-1}\gamma^5C\psi\right)^{\mathrm{T}} = \left(\bar{\psi}\gamma^5\psi\right)^{\mathrm{T}} = \left(\bar{\psi}\gamma^5\psi\right); \tag{243}$$

for $A = \gamma^5\gamma^\mu$, $A^{\mathrm{T}} = \gamma^{\mu\,\mathrm{T}}\gamma^5$,

$$C^{-1}A^{\mathrm{T}}C = C^{-1}\gamma^{\mu\,\mathrm{T}}\gamma^5C = C^{-1}\gamma^{\mu\,\mathrm{T}}CC^{-1}\gamma^5C = -\gamma^\mu\gamma^5 = \gamma^5\gamma^\mu, \tag{244}$$

$$\mathcal{C}\bar{\psi}A\psi\mathcal{C}^\dagger = \left(\bar{\psi}\gamma^5\gamma^\mu\psi\right)^{\mathrm{T}} = \bar{\psi}\gamma^5\gamma^\mu\psi. \tag{245}$$

Finally, for $A = \sigma^{\mu\nu} = \frac{\mathrm{i}}{2}\left[\gamma^\mu, \gamma^\nu\right]$, $A^{\mathrm{T}} = \frac{\mathrm{i}}{2}\left[\gamma^{\nu\,\mathrm{T}}, \gamma^{\mu\,\mathrm{T}}\right]$,

$$C^{-1}A^{\mathrm{T}}C = \frac{\mathrm{i}}{2}\left[C^{-1}\gamma^{\nu\,\mathrm{T}}C, C^{-1}\gamma^{\mu\,\mathrm{T}}C\right] = \frac{\mathrm{i}}{2}\left[-\gamma^\nu, -\gamma^\mu\right] = -\sigma^{\mu\nu}, \tag{246}$$

$$\mathcal{C}\bar{\psi}A\psi\mathcal{C}^\dagger = -\left(\bar{\psi}\sigma^{\mu\nu}\psi\right)^{\mathrm{T}} = -\left(\bar{\psi}\sigma^{\mu\nu}\psi\right). \tag{247}$$

Since the essence of the charge conjugation is just "$\phi \leftrightarrow \chi$" for ψ, it might be more straightforward to simply flip the spins (Eq.s (233) and (237)) in the expressions of As in Van der Waerden's notation; compare the final results with the original ones, and we can see the transformation properties. Take $A = \gamma^\mu$ as an example. Switch ϕ and χ in Eq. (241), then

$$\begin{aligned}
\mathcal{C}\bar{\psi}\gamma^\mu\psi\mathcal{C}^\dagger &= \bar{\chi}_{\dot{a}}\bar{\sigma}^{\mu\dot{a}a}\chi_a + \phi^a\sigma^\mu_{a\dot{a}}\bar{\phi}^{\dot{a}} \\
&= \epsilon^{\dot{a}\dot{b}}\epsilon^{ab}\bar{\chi}_{\dot{a}}\sigma^\mu_{b\dot{b}}\chi_a + \epsilon_{\dot{a}\dot{b}}\epsilon_{ab}\phi^a\bar{\sigma}^{\mu\dot{b}b}\bar{\phi}^{\dot{a}} \\
&= \epsilon^{\dot{b}\dot{a}}\epsilon^{ba}\bar{\chi}_{\dot{a}}\sigma^\mu_{b\dot{b}}\chi_a + \epsilon_{\dot{b}\dot{a}}\epsilon_{ba}\phi^a\bar{\sigma}^{\mu\dot{b}b}\bar{\phi}^{\dot{a}} \\
&= \bar{\chi}^{\dot{b}}\sigma^\mu_{b\dot{b}}\chi^b + \phi_b\bar{\sigma}^{\mu\dot{b}b}\bar{\phi}_{\dot{b}}.
\end{aligned} \tag{248}$$

Remember that we are dealing with fermions and anti-fermions here, thus the anti-commutation relation yields an extra minus sign for switching ψ and $\psi^\dagger$ (recall that $\dot{a}$s and bars are just "$\dagger$"s):

$$\mathcal{C}\bar{\psi}\gamma^\mu\psi\mathcal{C}^\dagger = \bar{\chi}^{\dot{b}}\sigma^\mu_{b\dot{b}}\chi^b + \phi_b\bar{\sigma}^{\mu\dot{b}b}\bar{\phi}_{\dot{b}} = -\left(\chi^b\sigma^\mu_{b\dot{b}}\bar{\chi}^{\dot{b}} + \bar{\phi}_{\dot{b}}\bar{\sigma}^{\mu\dot{b}b}\phi_b\right) = -\bar{\psi}\gamma^\mu\psi. \tag{249}$$

5.4 Time Reversal

Define the time reversal of the Dirac field [19]

$$\mathcal{T}\psi(x)\mathcal{T}^\dagger = C\gamma^5\psi(I_t x), \tag{250}$$

where C is the charge conjugation matrix in Sec. 5.3.2; I_t is the time reversal for $x = (t, \mathbf{x})$,

$$I_t x = (-t, \mathbf{x}), \quad (I_t)^\mu_{\ \nu} = \left(I_t^{-1}\right)^\mu_{\ \nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \tag{251}$$

and $\mathcal{T}$ is the **time-reversal operator** which is **antiunitary**, meaning that it acts on c-numbers as well as operators, as follows,

$$\mathcal{T}\mathrm{i}\mathcal{T}^\dagger = -\mathrm{i} \quad (\text{i is the imaginary unit})\,. \tag{252}$$

From Eq. 250,

$$\begin{aligned} \mathcal{T}\bar{\psi}\mathcal{T}^\dagger &= \mathcal{T}\psi^\dagger\mathcal{T}^\dagger\gamma^0 = \left(\mathcal{T}\psi\mathcal{T}^\dagger\right)^\dagger\gamma^0 \\ &= \left(C\gamma^5\psi\right)^\dagger\gamma^0 = \psi^\dagger\gamma^{5\,\dagger}C^\dagger\gamma^0 \\ &= \psi^\dagger\gamma^5C^{-1}\gamma^0 = \psi^\dagger\gamma^0\gamma^5C^{-1} \\ &= \bar{\psi}\gamma^5C^{-1}, \end{aligned} \tag{253}$$

where we have suppressed the spacetime arguments (which transform in the obvious way). Thus for all bi-linears (do not forget Eq. (252)),

$$\mathcal{T}\bar{\psi}A\psi\mathcal{T}^\dagger = \bar{\psi}\gamma^5C^{-1}A^*C\gamma^5\psi. \tag{254}$$

It is very easy to find

$$\begin{aligned} \gamma^5C^{-1}1^*C\gamma^5 &= 1, \\ \gamma^5C^{-1}\left(\gamma^5\right)^*C\gamma^5 &= \gamma^5, \\ \gamma^5C^{-1}\left(\gamma^0\right)^*C\gamma^5 &= \gamma^0, \\ \gamma^5C^{-1}\left(\gamma^i\right)^*C\gamma^5 &= -\gamma^i, \\ \gamma^5C^{-1}\left(\gamma^5\gamma^0\right)^*C\gamma^5 &= \gamma^5\gamma^0, \\ \gamma^5C^{-1}\left(\gamma^5\gamma^i\right)^*C\gamma^5 &= -\gamma^5\gamma^i. \end{aligned} \tag{255}$$

[19] This section is chiefly based on the *Quantum field theory* from Srednicki, M. (2007), Cambridge University Press. Although the choice of Van der Waerden's notation might be slightly different here, the final results ought to be identical.

Therefore,

$$
\begin{aligned}
\mathcal{T}\bar{\psi}\psi\mathcal{T}^{\dagger} &= \bar{\psi}\psi, \\
\mathcal{T}\bar{\psi}\gamma^{5}\psi\mathcal{T}^{\dagger} &= \bar{\psi}\gamma^{5}\psi, \\
\mathcal{T}\bar{\psi}\gamma^{\mu}\psi\mathcal{T}^{\dagger} &= -\bar{\psi}\left(I_t\right)^{\mu}_{\ \nu}\gamma^{\nu}\psi, \\
\mathcal{T}\bar{\psi}\gamma^{5}\gamma^{\mu}\psi\mathcal{T}^{\dagger} &= -\bar{\psi}\left(I_t\right)^{\mu}_{\ \nu}\gamma^{5}\gamma^{\nu}\psi, \\
\mathcal{T}\bar{\psi}\sigma^{\mu\nu}\psi\mathcal{T}^{\dagger} &= -\left(I_t\right)^{\mu}_{\ \alpha}\left(I_t\right)^{\nu}_{\ \beta}\bar{\psi}\sigma^{\alpha\beta}\psi.
\end{aligned}
\tag{256}
$$