QFT HW 02

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Contents

1 Quantum Scattering

This section is just a reminder of some Basic concepts of the quantum scattering. In this passage, we will not derive everything from the bottom. Instead, we will employ the known rules to calculate some simple cases of the quantum scattering.

1.1 Dyson Series

Consider a Hamiltonian

$$H = H_0 + V(t), \tag{1}$$

where $V(t)$ is the **perturbative** time-dependent potential, H_0 does not contain time explicitly and its energy eigenkets and eigenvalues are

$$H_0 |n\rangle = E_n |n\rangle . \tag{2}$$

In the **interaction picture** [1],

$$|\psi_I(t)\rangle \equiv e^{\mathrm{i}H_0 t} |\psi_S(t)\rangle , \tag{3}$$

$$A_I \equiv e^{\mathrm{i}H_0 t} A_S e^{-\mathrm{i}H_0 t}, \tag{4}$$

$$|\psi_I(t)\rangle = U_I(t) |\psi_I(0)\rangle , \quad t > 0. \tag{5}$$

$$\mathrm{i}\frac{\partial}{\partial t} |\psi_I(t)\rangle = V_I(t) |\psi_I(t)\rangle , \tag{6}$$

$$\mathrm{i}\frac{dA_I}{dt} = [A_I, H_0] . \tag{7}$$

where

$$V_I(t) = e^{\mathrm{i}H_0 t} V e^{-\mathrm{i}H_0 t}. \tag{8}$$

From above, we get

$$\mathrm{i}\frac{\partial U_I(t)}{\partial t} = V_I(t) U_I(t). \tag{9}$$

This differential equation along with the initial condition, $U_I(0) = 1$, is equivalent to

$$U_I(t) = 1 - \mathrm{i} \int_0^t V_I(t_1) U_I(t_1) dt_1. \tag{10}$$

We can obtain an approximation solution to the equation by iteration,

$$\begin{aligned} U_I(t) &= 1 + \sum_{n=1}^{\infty} (-\mathrm{i})^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n V_I(t_1) \cdots V_I(t_n) \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-\mathrm{i})^n}{n!} \int_0^t dt_1 \cdots \int_0^t dt_n T \left\{ V_I(t_1) \cdots V_I(t_n) \right\} \\ &\equiv \mathcal{T} \left\{ \exp \left[-\mathrm{i} \int_0^t V_I(\tau) d\tau \right] \right\} . \end{aligned} \tag{11}$$

[1] "I" and "S" (usually omitted) refer to the interaction and the Schrödinger picture, respectively.

This perturbation expansion is known as the **Dyson series**. For the last step in Eq. (11), we have introduced the identical "τ" to denote "$t_1, t_2, \cdots, t_n$" as they are mathematically equivalent; the time-ordered exponential is just a compact way of writing and remembering the correct expression ($\mathcal{T}$ means the **time ordering**).

1.2 Transition Probability

In the interaction picture we continue using $|n\rangle$ as our base kets. If the initial state at $t = 0$ is one of the energy eigenstates of H_0, then

$$|i(t)\rangle_I = U_I(t)\,|i\rangle = \sum_n |n\rangle\,\langle n|\,U_I(t)\,|i\rangle \equiv \sum_n c_n(t)\,|n\rangle\,. \tag{12}$$

Note that $|\alpha(t)\rangle_I = e^{\mathrm{i}H_0 t}\,|\alpha(t)\rangle_S = e^{\mathrm{i}H_0 t}U(t)\,|\alpha(0)\rangle_S = e^{\mathrm{i}H_0 t}U(t)e^{-iH_0 t}\,|\alpha(0)\rangle_I$, thus

$$U_I(t) = e^{\mathrm{i}H_0 t}U(t)e^{-iH_0 t}. \tag{13}$$

Let us now look at the matrix element of $U_I(t)$ between energy eigenstates of H_0:

$$\langle n|\,U_I(t)\,|i\rangle = e^{\mathrm{i}(E_n - E_i)t}\,\langle n|\,U(t)\,|i\rangle = e^{\mathrm{i}\omega_{ni}t}\,\langle n|\,U(t)\,|i\rangle\,, \quad \omega_{ni} \equiv E_n - E_i = -\omega_{in}; \tag{14}$$

$$|\langle n|\,U_I(t)\,|i\rangle|^2 = |\langle n|\,U(t)\,|i\rangle|^2\,, \tag{15}$$

where $\langle n|\,U(t)\,|i\rangle$ has been defined to be the **transition amplitude** and $|\langle n|\,U_I(t)\,|i\rangle|^2$ is now defined to be the **transition probability** [2]. Premultiplying both sides of Eq. (6) by $\langle n|$, we can write the differential equation for $c_n(t)$:

$$\mathrm{i}\frac{d}{dt}c_n(t) = \mathrm{i}\frac{\partial}{\partial t}\,\langle n|\alpha(t)\rangle_I = \sum_m \langle n|V_I(t)|m\rangle\,\langle m|\alpha(t)\rangle_I = \sum_m V_{nm}(t)e^{\mathrm{i}\omega_{nm}t}c_m(t). \tag{17}$$

Exact solutions to Eq.(17) for $c_n(t)$ are usually not available.We must be content with approximate solutions obtained by **perturbation expansion**:

$$c_n(t) = \langle n|\,U_I(t)\,|i\rangle = c_n^{(0)}(t) + c_n^{(1)}(t) + c_n^{(2)}(t) + \cdots\,, \tag{18}$$

where $c_n^{(0)}(t) = \delta_{ni}$; $c_n^{(1)}(t), c_n^{(2)}(t), \ldots$ signify amplitudes of first order, second order, and so on in the strength parameter of the **time-dependent potential**. Using Eq.s (8) and (11),

$$c_n^{(1)}(t) = -\mathrm{i}\int_0^t \langle n|\,V_I(\tau)\,|i\rangle\,d\tau = -\mathrm{i}\int_0^t e^{\mathrm{i}\omega_{ni}\tau}V_{ni}(\tau)d\tau, \tag{19}$$

$$c_n^{(2)}(t) = -\mathrm{i}\int_0^t\int_0^t d\tau d\tau' e^{\mathrm{i}(\omega_{nm}\tau + \omega_{mi}\tau')}\mathcal{T}\left\{V_{nm}(\tau)V_{mi}(\tau')\right\}. \tag{20}$$

The **transition probability** for $|i\rangle \to |n\rangle$ with $n \neq i$ is

$$P(i \to n) = \left|c_n^{(1)}(t) + c_n^{(2)}(t) + \cdots\right|^2. \tag{21}$$

[2] Parenthetically, if the matrix elements of $U_I(t)$ are taken between initial and final states that are not energy eigenstates—for example, between $|a'\rangle$ and $|b'\rangle$ (eigenkets of A and B, respectively), where $[H_0, A] \neq 0$ and/or $[H_0, B] \neq 0$—we have, in general,

$$|\langle a'|\,U_I(t)\,|b'\rangle|^2 \neq |\langle a'|\,U(t)\,|b'\rangle|^2\,. \tag{16}$$

1.3 Cross Section and Scattering Matrix

We define the **cross section** σ as the effective area of each target $\mathcal{A}$ particle to be seen by an incoming beam ($\mathcal{B}$ particles). Imagine a thin target with $N_{\mathcal{A}}$ particles in it, each $\mathcal{A}$ with effective area σ. If we aim a beam of $N_{\mathcal{B}}$ particles at the target with area A, then the chance of hitting one of these particles is $N_{\mathcal{A}}\sigma/A$, and the number of scattering events is $N = N_{\mathcal{B}}N_{\mathcal{A}}\sigma/A$. Thus [3]

$$\sigma \equiv \frac{N}{N_{\mathcal{B}}N_{\mathcal{A}}/A} = \frac{N/N_{\mathcal{A}}t}{N_{\mathcal{B}}/At} = \frac{\text{decay rate } \Gamma_{\mathcal{A}}}{\text{incident flux } J_{\mathcal{B}}}. \tag{22}$$

We introduce the (unitary) **scattering matrix** $\boldsymbol{S}$ to describe a scattering event that takes us from an initial state $|i\rangle$ consisting of a collection of free, asymptotic states at $t \to -\infty$ to a final state $|f\rangle$ at $t \to \infty$. If we take the states in the **scattering amplitude** $S_{fi} = \langle f|\, S\, |i\rangle$ to be the usual plane wave states, then

$$\langle f|\, S\, |i\rangle \;\; \equiv \langle f|\, \mathbf{1} + \mathrm{i}T\, |i\rangle = \delta_{fi} + \mathrm{i}T_{fi} \equiv \delta_{fi} + \mathrm{i}(2\pi)^4\delta^4\left(\sum_i q_i - \sum_f p_f\right)\mathcal{M}_{fi}, \tag{23}$$

where δ_{fi} symbolically represents the particles not interacting at all and the **transition matrix** T describes non-trivial scattering; $S^\dagger S = \mathbf{1}$ or $\sum_j S^*_{jf}S_{ji} = \delta_{fi} \Rightarrow T^\dagger T = \mathrm{Im}T$; the factor $\delta^4\left(\sum_i q_i - \sum_f p_f\right)$, associated with the **invariant matrix element** $\mathcal{M}_{fi} \equiv \left\langle \mathbf{p}_f|\mathcal{M}|\mathbf{q}_i\right\rangle$, reflects the **four-momentum conservation**. The transition probability, $P(i \to f) = \sum_{f,i}|S_{fi}|^2$, of the scattering contains the square of $\delta^4\left(\sum_i q_i - \sum_f p_f\right)$, which is senseless. What went wrong is that the plane wave states are not normalisable. They extend throughout all of space and two plane waves never get far apart no matter how long you wait, thus the scattering occurs at every point in space and goes on for all time. One possible approach is to replace one $\delta^4\left(\sum_i q_i - \sum_f p_f\right)$ by an infinite constant—the total volume of spacetime $VT = \lim_{L,t\to\infty}\int_{-L/2}^{L/2}\int_{-L/2}^{L/2}\int_{-L/2}^{L/2}\int_{-t}^{t} d^4x\; e^{\mathrm{i}0\cdot x} = (2\pi)^4\,\delta^4(0)$. A more feasible way to salvage the situation is to build wave packets, which are normalisable, and do get far apart in the far past/future. A **wave packet** [4] representing the initial or final particle state can schematically be expressed as

$$|\phi\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{\phi(\mathbf{p})}{\sqrt{2E_{\mathbf{p}}}}\,|\mathbf{p}\rangle\,, \tag{24}$$

where $\phi(\mathbf{p})$ is the Fourier transformation of the spatial wave function, $|\mathbf{p}\rangle = \sqrt{2E_{\mathbf{p}}}a^\dagger_{\mathbf{p}}\,|0\rangle$ is a one-particle state of momentum $\mathbf{p}$ in the interacting theory and the normalisation is

$$\langle\phi|\phi\rangle = 1 \quad \text{or} \quad \int \frac{d\mathbf{p}}{(2\pi)^3}\,|\phi(\mathbf{p})|^2 = 1. \tag{25}$$

[3] Using classical wave function techniques, a plane wave $e^{\mathrm{i}kz}$ scattering off a stationary, hard target becomes $e^{\mathrm{i}kz} + \frac{f(\theta)}{r}e^{\mathrm{i}kr}$, where the term with $e^{\mathrm{i}kr}$ represents the scattered wave. Thus the **differential cross section** $\frac{d\sigma}{d\Omega} = |f(\theta)|^2 \propto$ the probability that a particle scatters into an angle $\theta \in [0,\pi]$ (**colatitude**), where the **solid angle** $\Omega = \int d\Omega = \int_0^{2\pi}\int_{\cos\theta}^{1} d\left(\cos\theta'\right)d\phi = 2\pi\left(1-\cos\theta\right)$ and ϕ is the **longitude**. The **total cross section** $\sigma = \int d\Omega\frac{d\sigma}{d\Omega} = \int d\Omega|f(\theta)|^2$. However, this formulation is not suitable because it is inherently non-relativistic.

[4] A superposition of waves with one predominant wave number k, but with several other wave numbers near k.

Consider a scattering of one incident $\mathcal{B}$ particle off one target $\mathcal{A}$ particle with **impact parameter b**, with Eq. (25), the probability we wish to compute is then

$$\begin{aligned}
&P\left(i \to f\right) = \left|{}_{\text{out}}\left\langle \{\phi_f\} \mid \{\phi_i\}\right\rangle_{\text{in}}\right|^2 \Big|_{i=\mathcal{A},\mathcal{B}} \\
=& \left| \left[\prod_f \int \frac{d\mathbf{p}_f}{(2\pi)^3} \frac{\phi_f\left(\mathbf{p}_f\right)}{\sqrt{2E\left(\mathbf{p}_f\right)}}\right] \left[\prod_i \int \frac{d\mathbf{q}_i}{(2\pi)^3} \frac{\phi_i\left(\mathbf{q}_i\right)}{\sqrt{2E\left(\mathbf{q}_i\right)}}\right] e^{-i\mathbf{b}\cdot\mathbf{q}_{\mathcal{B}}} \cdot \left({}_{\text{out}}\left\langle \{\mathbf{p}_f\} \mid \{\mathbf{q}_i\}\right\rangle_{\text{in}}\right)\right|^2 \\
=& \left[\prod_f \frac{d\mathbf{p}_f}{(2\pi)^3 2E\left(\mathbf{p}_f\right)}\right] \left[\prod_f \int \frac{d\mathbf{p}_f}{(2\pi)^3} \left|\phi_f(\mathbf{p}_f)\right|^2\right] \left[\prod_i \int \frac{d\mathbf{q}_i d\bar{\mathbf{q}}_i}{(2\pi)^6} \frac{\phi_i\left(\mathbf{q}_i\right)\phi_i^*\left(\bar{\mathbf{q}}_i\right)}{2\sqrt{E\left(\mathbf{q}_i\right)E\left(\bar{\mathbf{q}}_i\right)}}\right] \\
&\times e^{i\mathbf{b}\cdot\left(\bar{\mathbf{q}}_{\mathcal{B}}-\mathbf{q}_{\mathcal{B}}\right)} \left({}_{\text{out}}\left\langle \{\mathbf{p}_f\} \mid \{\mathbf{q}_i\}\right\rangle_{\text{in}}\right) \left({}_{\text{out}}\left\langle \{\mathbf{p}_f\} \mid \{\bar{\mathbf{q}}_i\}\right\rangle_{\text{in}}\right)^* \\
=& \left[\prod_f \frac{d\mathbf{p}_f}{(2\pi)^3 2E\left(\mathbf{p}_f\right)}\right] \left[\prod_i \int \frac{d\mathbf{q}_i d\bar{\mathbf{q}}_i}{(2\pi)^6} \frac{\phi_i\left(\mathbf{q}_i\right)\phi_i^*\left(\bar{\mathbf{q}}_i\right)}{2\sqrt{E\left(\mathbf{q}_i\right)E\left(\bar{\mathbf{q}}_i\right)}}\right] e^{i\mathbf{b}\cdot\left(\bar{\mathbf{q}}_{\mathcal{B}}-\mathbf{q}_{\mathcal{B}}\right)} \\
&\times \left({}_{\text{out}}\left\langle \{\mathbf{p}_f\} \mid \{\mathbf{q}_i\}\right\rangle_{\text{in}}\right) \left({}_{\text{out}}\left\langle \{\mathbf{p}_f\} \mid \{\bar{\mathbf{q}}_i\}\right\rangle_{\text{in}}\right)^*.
\end{aligned} \tag{26}$$

where the factor $e^{-i\mathbf{b}\cdot\mathbf{q}_{\mathcal{B}}}$ associated with $\phi_{\mathcal{B}}(\mathbf{q}_{\mathcal{B}})$ accounts for the spatial translation. Following Peskin's QFT, the **differential cross section**

$$\begin{aligned}
&d\sigma_{fi} = \int d^2b \; P(i \to f) \\
=& \frac{1}{4E\left(\mathbf{q}_{\mathcal{A}}\right)E\left(\mathbf{q}_{\mathcal{B}}\right)\left|\mathbf{v}_{\mathcal{A}}-\mathbf{v}_{\mathcal{B}}\right|} \left[\prod_f \frac{d\mathbf{p}_f}{(2\pi)^3 2E\left(\mathbf{p}_f\right)}\right] (2\pi)^4 \delta^4\left(\sum_i q_i - \sum_f p_f\right) \left|\mathcal{M}_{fi}\right|^2 \\
\equiv& \frac{d\Pi_n \left|\mathcal{M}_{fi}\right|^2}{4E\left(\mathbf{q}_{\mathcal{A}}\right)E\left(\mathbf{q}_{\mathcal{B}}\right)\left|\mathbf{v}_{\mathcal{A}}-\mathbf{v}_{\mathcal{B}}\right|}.
\end{aligned} \tag{27}$$

where $\left|\mathbf{v}_{\mathcal{A}}-\mathbf{v}_{\mathcal{B}}\right| \equiv \left|\frac{\mathbf{q}_{\mathcal{A}}}{E(\mathbf{q}_{\mathcal{A}})} - \frac{\mathbf{q}_{\mathcal{B}}}{E(\mathbf{q}_{\mathcal{B}})}\right|$ is the **relative velocity**, $d\Pi_n$ is a product of the **final-state momenta**, $E\left(\mathbf{q}_{\mathcal{A}}\right) = \sqrt{\left|\mathbf{q}_{\mathcal{A}}\right|^2 + m_{\mathcal{A}}{}^2}$ and $E\left(\mathbf{q}_{\mathcal{B}}\right) = \sqrt{\left|\mathbf{q}_{\mathcal{B}}\right|^2 + m_{\mathcal{B}}{}^2}$. The integral $\int d\Pi_n$ is known as the **relativistic invariant n-body phase space**.

In the **centre-of-mass frame** where $\mathbf{q}_{\mathcal{B}} = -\mathbf{q}_{\mathcal{A}}$,

$$\begin{aligned}
&\left[E\left(\mathbf{q}_{\mathcal{A}}\right)E\left(\mathbf{q}_{\mathcal{B}}\right)\left|\mathbf{v}_{\mathcal{A}}-\mathbf{v}_{\mathcal{B}}\right|\right]^2 \\
=& \left|\mathbf{q}_{\mathcal{A}}\right|^2 \left[E\left(\mathbf{q}_{\mathcal{A}}\right)+E\left(\mathbf{q}_{\mathcal{B}}\right)\right]^2 \\
=& \left|\mathbf{q}_{\mathcal{A}}\right|^2 \left[E\left(\mathbf{q}_{\mathcal{A}}\right)^2+E\left(\mathbf{q}_{\mathcal{B}}\right)^2\right] + \left[E\left(\mathbf{q}_{\mathcal{A}}\right)E\left(\mathbf{q}_{\mathcal{B}}\right)-\mathbf{q}_{\mathcal{A}}\cdot\mathbf{q}_{\mathcal{B}}\right]^2 - E^2\left(\mathbf{q}_{\mathcal{A}}\right)E^2\left(\mathbf{q}_{\mathcal{B}}\right) - \left|\mathbf{q}_{\mathcal{A}}\right|^4 \\
=& \left[E\left(\mathbf{q}_{\mathcal{A}}\right)E\left(\mathbf{q}_{\mathcal{B}}\right)-\mathbf{q}_{\mathcal{A}}\cdot\mathbf{q}_{\mathcal{B}}\right]^2 - m_{\mathcal{A}}{}^2 m_{\mathcal{B}}{}^2 \\
=& \left(q_{\mathcal{A}}\cdot q_{\mathcal{B}}\right)^2 - m_{\mathcal{A}}{}^2 m_{\mathcal{B}}{}^2.
\end{aligned} \tag{28}$$

Thus

$$d\sigma_{fi} = \frac{d\Pi_n \left|\mathcal{M}_{fi}\right|^2}{4\sqrt{\left(q_{\mathcal{A}}\cdot q_{\mathcal{B}}\right)^2 - m_{\mathcal{A}}{}^2 m_{\mathcal{B}}{}^2}}. \tag{29}$$

A two-body phase space in the CM frame ($\mathbf{q}_\mathcal{B} = -\mathbf{q}_\mathcal{A}$, $\mathbf{p}_2 = -\mathbf{p}_1$) is [5]

$$\begin{aligned}
\int d\Pi_2 &= \left[\prod_f \int \frac{d\mathbf{p}_f}{(2\pi)^3 2E\left(\mathbf{p}_f\right)}\right](2\pi)^4\,\delta^4\left(\sum_i q_i - \sum_f p_f\right) \\
&= \int \frac{d\mathbf{p}_1 d\mathbf{p}_2}{16\pi^2 E\left(\mathbf{p}_1\right)E\left(\mathbf{p}_2\right)}\delta^3(0)\delta\left[E_\text{CM} - E\left(\mathbf{p}_1\right) - E\left(\mathbf{p}_2\right)\right] \\
&= \int \frac{d\mathbf{p}_1}{16\pi^2 E\left(\mathbf{p}_1\right)E\left(\mathbf{p}_2\right)}\delta\left[E_\text{CM} - E\left(\mathbf{p}_1\right) - E\left(\mathbf{p}_2\right)\right] \\
&= \int \frac{d\left|\mathbf{p}_1\right| d\Omega\,\left|\mathbf{p}_1\right|^2}{16\pi^2 E\left(\mathbf{p}_1\right)E\left(\mathbf{p}_2\right)}\delta\left[E_\text{CM} - E\left(\mathbf{p}_1\right) - E\left(\mathbf{p}_2\right)\right] \\
&= \int \frac{d\Omega\,\,\left|\mathbf{p}_1\right|_0}{16\pi^2 E_\text{CM}},
\end{aligned} \tag{30}$$

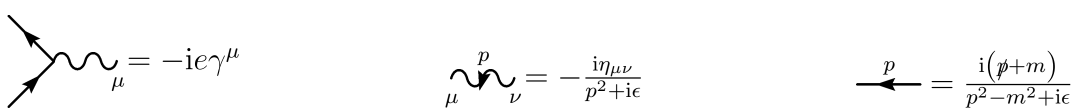
where $E_\text{CM} \equiv E\left(\mathbf{q}_\mathcal{A}\right) + E\left(\mathbf{q}_\mathcal{B}\right)$ is the **total initial energy**, $E\left(\mathbf{p}_1\right) = \sqrt{\left|\mathbf{p}_1\right|^2 + {m_1}^2}$, $E\left(\mathbf{p}_2\right) = \sqrt{\left|\mathbf{p}_1\right|^2 + {m_2}^2}$ and $\left|\mathbf{p}_1\right|_0 \equiv \frac{1}{2}\sqrt{{E_\text{CM}}^2 - 2\left({m_1}^2 + {m_2}^2\right) + \left(\frac{{m_1}^2 - {m_2}^2}{E_\text{CM}}\right)^2}$. In that case,

$$\left(\frac{d\sigma}{d\Omega}\right)_\text{CM} = \frac{\left(d\Pi_2/d\Omega\right)\left|\mathcal{M}\right|^2}{4\sqrt{\left(q_\mathcal{A}\cdot q_\mathcal{B}\right)^2 - {m_\mathcal{A}}^2{m_\mathcal{B}}^2}} = \frac{\left|\mathbf{p}_1\right|_0\left|\mathcal{M}\right|^2}{64\pi^2 E_\text{CM}\sqrt{\left(q_\mathcal{A}\cdot q_\mathcal{B}\right)^2 - {m_\mathcal{A}}^2{m_\mathcal{B}}^2}}. \tag{31}$$

2 Scattering Samples

2.1 Feynman Rules for QED

$$\mathcal{L} = \bar{\psi}\left(\mathrm{i}\not{D} - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad D_\mu \equiv \partial_\mu + \mathrm{i}eA_\mu. \tag{32}$$



(a) **Vetex** (b) **Photon propagator** (c) **Fermion propagator**

Figure 1: Vertex and Propagators

- **Vertex** See Fig. 1a. We impose momentum conservation at each vertex.

[5] Let $f\left(\left|\mathbf{p}_1\right|\right) = E_\text{CM} - E\left(\mathbf{p}_1\right) - E\left(\mathbf{p}_2\right)$. $f\left(\left|\mathbf{p}_1\right|_0\right) = 0 \Rightarrow \left|\mathbf{p}_1\right|_0 = \frac{1}{2}\sqrt{{E_\text{CM}}^2 - 2\left({m_1}^2 + {m_2}^2\right) + \left(\frac{{m_1}^2 - {m_2}^2}{E_\text{CM}}\right)^2}$ $\Rightarrow \int_{-\infty}^{\infty}\frac{\left|\mathbf{p}_1\right|^2 d\left|\mathbf{p}_1\right|}{E(\mathbf{p}_1)E(\mathbf{p}_2)}\delta\left[f\left(\left|\mathbf{p}_1\right|\right)\right] = \int_{-\infty}^{\infty}\frac{\left|\mathbf{p}_1\right|^2 d\left|\mathbf{p}_1\right|}{E(\mathbf{p}_1)E(\mathbf{p}_2)}\frac{\delta\left(\left|\mathbf{p}_1\right| - \left|\mathbf{p}_1\right|_0\right)}{\left|df\left(\left|\mathbf{p}_1\right|_0\right)/d\left|\mathbf{p}_1\right|\right|} = \left.\frac{\left|\mathbf{p}_1\right|}{E(\mathbf{p}_1)+E(\mathbf{p}_2)}\right|_{\left|\mathbf{p}_1\right| = \left|\mathbf{p}_1\right|_0} = \frac{\left|\mathbf{p}_1\right|_0}{E_\text{CM}}$.

- **Photon propagator** See Fig. 1b.
- **Fermion propagator** See Fig. 1c.
- An initial/final external **photons** line appears with a **polarisation vector** $\epsilon_\mu/\epsilon^*_\mu$. In the Coulomb gauge, $\epsilon^0 = 0$ and $\epsilon_\mu \cdot \mathbf{p} = 0$.
- An initial/final external **fermion** line appears with a spinor $u^s(p)/\bar{u}^s(p)$; for an incoming/outgoing **anti-fermion**, we attach a spinor $\bar{v}^{\dot{s}}(p)/v^{\dot{s}}(p)$.
- Figure out the **overall sign** of the diagram. For instance, an open e^+ line going through the diagram from the initial to the final state brings about an extra minus sign, as it requires an odd number of transpositions of **fermionic operators**.

2.2 Identities of Gamma Matrices

2.2.1 The Miscellaneous

$$\begin{aligned}
\gamma^\mu \gamma_\mu &= \gamma^\mu \eta_{\mu\nu} \gamma^\nu = \eta_{\mu\nu} \gamma^\mu \gamma^\nu \\
&= \frac{1}{2} \left(\eta_{\mu\nu} + \eta_{\nu\mu} \right) \gamma^\mu \gamma^\nu = \frac{1}{2} \left(\eta_{\mu\nu} \gamma^\mu \gamma^\nu + \eta_{\nu\mu} \gamma^\mu \gamma^\nu \right) \\
&= \frac{1}{2} \left(\eta_{\mu\nu} \gamma^\mu \gamma^\nu + \eta_{\mu\nu} \gamma^\nu \gamma^\mu \right) = \frac{\eta_{\mu\nu}}{2} \left\{ \gamma^\mu, \gamma^\nu \right\} \\
&= \eta_{\mu\nu} \eta^{\mu\nu} I_4 = 4 I_4.
\end{aligned} \tag{33}$$

$$\gamma^\mu \gamma^\nu \gamma_\mu = 2\eta^{\mu\nu} I_4 \gamma_\mu - \gamma^\nu \gamma^\mu \gamma_\mu = 2\gamma^\nu - 4\gamma^\nu = -2\gamma^\nu. \tag{34}$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 2\eta^{\mu\nu} I_4 \gamma^\rho \gamma_\mu - \gamma^\nu \left(\gamma^\mu \gamma^\rho \gamma_\mu \right) = 2\gamma^\rho \gamma^\nu - \gamma^\nu \left(-2\gamma^\nu \right) = 4\eta^{\rho\nu} I_4. \tag{35}$$

$$\begin{aligned}
\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu &= 2\eta^{\mu\nu} I_4 \gamma^\rho \gamma^\sigma \gamma_\mu - \gamma^\nu \left(\gamma^\mu \gamma^\sigma \gamma^\rho \gamma_\mu \right) = 2\gamma^\rho \gamma^\sigma \gamma^\nu - \gamma^\nu \left(4\eta^{\rho\sigma} I_4 \right) \\
&= 2\gamma^\rho \gamma^\sigma \gamma^\nu - 4\eta^{\rho\sigma} \gamma^\nu = 2 \left(2\eta^{\rho\sigma} - \gamma^\sigma \gamma^\rho \right) \gamma^\nu - 4\eta^{\rho\sigma} \gamma^\nu \\
&= -2\gamma^\sigma \gamma^\rho \gamma^\nu.
\end{aligned} \tag{36}$$

$$\gamma^\mu \gamma^\nu \gamma^\rho = \eta^{\mu\nu} \gamma^\rho + \eta^{\nu\rho} \gamma^\mu - \eta^{\mu\rho} \gamma^\nu - \mathrm{i} \epsilon^{\sigma\mu\nu\rho} \gamma_\sigma \gamma^5, \tag{37}$$

where $\delta^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma} = \epsilon^{\alpha\beta\gamma\delta} \epsilon_{\mu\nu\rho\sigma}$ is the **generalised Kronecker delta**, $\epsilon^{\mu\nu\rho\sigma} = \delta^{\mu\nu\rho\sigma}_{0123}$ and $\epsilon_{\mu\nu\rho\sigma} = \delta^{0123}_{\mu\nu\rho\sigma}$ are the **Levi-Civita symbols**.

2.2.2 Trace Identities

$$\mathrm{tr} \left(A + B \right) = \mathrm{tr} \left(A \right) + \mathrm{tr} \left(B \right); \tag{38}$$

$$\mathrm{tr} \left(aA \right) = a \mathrm{tr} \left(A \right), \quad a \text{ is a constant number;} \tag{39}$$

$$\mathrm{tr} \left(ABC \right) = \mathrm{tr} \left(BCA \right) = \mathrm{tr} \left(CAB \right). \tag{40}$$

We can prove the following identities with the above properties of the trace operator:

$$
\begin{aligned}
\mathrm{tr}\gamma^\mu &= \mathrm{tr}\left(\gamma^5\gamma^5\gamma^\mu\right) = -\mathrm{tr}\left(\gamma^5\gamma^\mu\gamma^5\right) = -\mathrm{tr}\left(\gamma^5\gamma^5\gamma^\mu\right) = -\mathrm{tr}\gamma^\mu \\
&= \mathrm{tr}\gamma^5 = \mathrm{tr}\left(\gamma^0\gamma^0\gamma^5\right) = -\mathrm{tr}\left(\gamma^0\gamma^5\gamma^0\right) = -\mathrm{tr}\left(\gamma^0\gamma^0\gamma^5\right) = -\mathrm{tr}\gamma^5 \\
&= \mathrm{tr}\left(\text{any odd number of }\gamma\text{s}\right) \\
&= \mathrm{tr}\left(\gamma^5 \times \text{any odd number of }\gamma\text{s}\right) \\
&= 0;
\end{aligned} \tag{41}
$$

$$\mathrm{tr}\left(\gamma^\mu\gamma^\nu\right) = \mathrm{tr}\left(2\eta^{\mu\nu}I_4\right) - \mathrm{tr}\left(\gamma^\nu\gamma^\mu\right) = 8\eta^{\mu\nu} - \mathrm{tr}\left(\gamma^\mu\gamma^\nu\right) = 4\eta^{\mu\nu}; \tag{42}$$

$$\mathrm{tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) = 4\left(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right); \tag{43}$$

$$\mathrm{tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5\right) = -4\mathrm{i}\epsilon^{\mu\nu\rho\sigma}; \tag{44}$$

$$
\begin{aligned}
\mathrm{tr}\left(\prod_{i=1}^{n}\gamma^{\mu_i}\right) &= (-1)^n\,\mathrm{tr}\left[\prod_{i=1}^{n} C\left(\gamma^{\mu_i}\right)^{\mathrm{T}} C^{-1}\right] \\
&= (-1)^n\,\mathrm{tr}\left(C\gamma^{\mu_n}CC^{-1}\cdots CC^{-1}\gamma^{\mu_2}CC^{-1}\gamma^{\mu_1}C^{-1}\right) \\
&= -\mathrm{tr}\left(C\gamma^{\mu_n}\cdots\gamma^{\mu_2}\gamma^{\mu_1}C^{-1}\right) \\
&= -\mathrm{tr}\left(C^{-1}C\gamma^{\mu_n}\cdots\gamma^{\mu_2}\gamma^{\mu_1}\right) \\
&= \mathrm{tr}\left(\gamma^{\mu_n}\cdots\gamma^{\mu_2}\gamma^{\mu_1}\right),
\end{aligned} \tag{45}
$$

where the **charge conjugation matrix**

$$C \equiv \mathrm{i}\gamma^2\gamma^0 = -C^{-1} = -C^{\mathrm{T}} = -C^\dagger \quad \text{and} \quad C\left(\gamma^\mu\right)^{\mathrm{T}} C^{-1} = -\gamma^\mu. \tag{46}$$

2.3 Mandelstam Variables

For a $2 \to 2$ scattering ($p_1, p_2 \to p_3, q_4$) that satisfies the four-momentum conservation $p_1 + p_2 = p_3 + q_4 = k$ and the **on-shell condition** ${p_i}^2 = {m_i}^2$ ($i = 1, 2, 3, 4$), we can introduce variables

$$s \equiv \left(p_1 + p_2\right)^2 = \left(p_3 + p_4\right)^2 = k^2, \tag{47}$$

$$t \equiv \left(p_1 - p_3\right)^2 = \left(p_4 - p_2\right)^2 \tag{48}$$

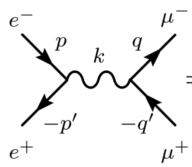
$$\text{and} \quad u \equiv \left(p_1 - p_4\right)^2 = \left(p_3 - p_2\right)^2 \tag{49}$$

such that $s + t + u = \sum_{i=1}^{4} {m_i}^2$.

2.4 Positron Electron Scattering

2.4.1 Unpolarised Cross Section

The leading-order **Feynman diagram** for $e^-e^+ \to \mu^-\mu^+$ is where the amplitude



$$= \mathrm{i}\mathcal{M}\left[e^-(p)e^+(p') \to \mu^-(q)\mu^+(q')\right],$$

$$\begin{aligned}\mathcal{M} &= \frac{1}{\mathrm{i}}\bar{v}^{\dot{s}}(p')\left(-\mathrm{i}e\gamma^\alpha\right)u^s(p)\left(\frac{-\mathrm{i}\eta_{\alpha\beta}}{k^2}\right)\bar{u}^r(q)\left(-\mathrm{i}e\gamma^\beta\right)v^{\dot{r}}(q') \\ &= \frac{e^2}{k^2}\bar{v}^{\dot{s}}(p')\gamma^\alpha u^s(p)\bar{u}^r(q)\gamma_\alpha v^{\dot{r}}(q').\end{aligned} \tag{50}$$

A bi-spinor product such as $\bar{v}\gamma^\mu u$ can be complex-conjugated as follow:

$$\left(\bar{v}\gamma^\alpha u\right)^* = \left(\bar{v}\gamma^\alpha u\right)^\dagger = u^\dagger\left(\gamma^\alpha\right)^\dagger\left(\gamma^0\right)^\dagger v = u^\dagger\gamma^0\gamma^\alpha v = \bar{u}\gamma^\alpha v. \tag{51}$$

Using Eq.s (43) and (51) and the **completeness relations**

$$\sum_s u^s(p)\bar{u}^s(p) = \not{p} + m \quad \text{and} \quad \sum_{\dot{s}} v^{\dot{s}}(p)\bar{v}^{\dot{s}}(p) = \not{p} - m, \tag{52}$$

we can compute that [6]

$$\begin{aligned}&\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 \\ &= \frac{e^4}{k^4}\left[\frac{1}{4}\sum_{s,\dot{s}}\bar{v}^{\dot{s}}(p')\gamma^\alpha u^s(p)\bar{u}^s(p)\gamma^\beta v^{\dot{s}}(p')\right]\left[\sum_{r,\dot{r}}\bar{u}^r(q)\gamma_\alpha v^{\dot{r}}(q')\bar{v}^{\dot{r}}(q')\gamma_\beta u^r(q)\right] \\ &= \frac{e^4}{4k^4}\left[\sum_{s,\dot{s}}\bar{v}^{\dot{s}}_a(p')\gamma^\alpha_{ab}u^s_b(p)\bar{u}^s_c(p)\gamma^\beta_{cd}v^{\dot{s}}_d(p')\right]\left[\sum_{r,\dot{r}}\bar{u}^r_a(q)\gamma_{\alpha,ab}v^{\dot{r}}_b(q')\bar{v}^{\dot{r}}_c(q')\gamma_{\beta,cd}u^r_d(q)\right] \\ &= \frac{e^4}{4k^4}\left[\left(\not{p}' - m_e\right)_{da}\gamma^\alpha_{ab}\left(\not{p} + m_e\right)_{bc}\gamma^\beta_{cd}\right]\left[\left(\not{q} + m_\mu\right)_{da}\gamma_{\alpha,ab}\left(\not{q}' - m_\mu\right)_{bc}\gamma_{\beta,cd}\right] \\ &= \frac{e^4}{4k^4}\mathrm{tr}\left[\left(\not{p}' - m_e\right)\gamma^\alpha\left(\not{p} + m_e\right)\gamma^\beta\right]\mathrm{tr}\left[\left(\not{q} + m_\mu\right)\gamma_\alpha\left(\not{q}' - m_\mu\right)\gamma_\beta\right] \\ &= \frac{e^4}{4k^4}\cdot 4\left[p'^\alpha p^\beta + p'^\beta p^\alpha - \eta^{\alpha\beta}\left(p\cdot p' + {m_e}^2\right)\right]\cdot 4\left[q'_\alpha q_\beta + q'_\beta q_\alpha - \eta_{\alpha\beta}\left(q\cdot q' + {m_\mu}^2\right)\right] \\ &= \frac{8e^4}{k^4}\left[\left(p\cdot q\right)\left(p'\cdot q'\right) + \left(p\cdot q'\right)\left(p'\cdot q\right) + {m_\mu}^2\left(p\cdot p'\right) + {m_e}^2\left(q\cdot q'\right) + 2{m_e}^2{m_\mu}^2\right].\end{aligned} \tag{53}$$

[6] In practice, electron and positron beams are often unpolarised, and muon detectors are normally blind to the muon polarisation. Hence in this section, we wish to first suppress the electron and positron spin orientations, and sum the cross section over muon spin orientations.

Taking the limit $m_e/m_\mu \ll 1$, then in the CM frame,

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{8e^4}{k^4}[(p\cdot q)\,(p'\cdot q') + (p\cdot q')\,(p'\cdot q) + {m_\mu}^2\,(p\cdot p')]; \tag{54}$$

$$p = (E, E\mathbf{z})\,, \quad p' = (E, -E\mathbf{z})\,, \quad q = (E, \mathbf{q})\,, \quad q' = (E, -\mathbf{q})\,; \tag{55}$$

$${E_{\text{CM}}}^2 = 2E^2 = E\cdot E - E\mathbf{z}\cdot(-E\mathbf{z}) = p\cdot p'; \tag{56}$$

$$|\mathbf{p}_1|_0 \equiv \frac{1}{2}\sqrt{{E_{\text{CM}}}^2 - 2\left({m_\mu}^2 + {m_\mu}^2\right) + \left(\frac{{m_\mu}^2 - {m_\mu}^2}{E_{\text{CM}}}\right)^2} = \frac{1}{2}\sqrt{2E^2 - 4{m_\mu}^2}; \tag{57}$$

$$\sqrt{(p\cdot p')^2 - {m_e}^2{m_e}^2} = \sqrt{(p\cdot p')^2} = p\cdot p', \tag{58}$$

where $E \equiv |\mathbf{p}| = -|\mathbf{p}'|$, $|\mathbf{z}| = 1$, $\mathbf{q}\cdot\mathbf{z} = |\mathbf{q}|\cos\theta = \sqrt{E^2 - {m_\mu}^2}\cos\theta$, θ is the **scattering angle** and E_{CM} is the total initial energy. Plugging Eq.s (54)–(58) into Eq. (31),

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} =& \frac{|\mathbf{p}_1|_0\cdot\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2}{64\pi^2 E_{\text{CM}}\sqrt{(p\cdot p')^2 - {m_e}^2{m_e}^2}} \\ =& \,\alpha^2\sqrt{1 - \frac{m_\mu^2}{E^2}}\cdot\frac{(p\cdot q)\,(p'\cdot q') + (p\cdot q')\,(p'\cdot q) + {m_\mu}^2\,(p\cdot p')}{k^4\,(p\cdot p')},\ \alpha = \frac{e^2}{4\pi}.\end{aligned} \tag{59}$$

If we go on with the Mandelstam variables (along with Eq. (56)), then

$$2p\cdot p' = s - \left(p^2 + {p'}^2\right) = s - 2{m_e}^2 = s = k^2 = 4E^4; \tag{60}$$

$$\begin{aligned}-2p\cdot q \ &= \ -2p\cdot q' = t - \left({m_e}^2 + {m_\mu}^2\right) = t - {m_\mu}^2 \\ &= \ -2E^2\left(1 - \sqrt{1 - \tfrac{{m_\mu}^2}{E^2}}\cos\theta\right) = -\tfrac{s}{2}\left(1 - \sqrt{1 - \tfrac{4{m_\mu}^2}{s}}\cos\theta\right);\end{aligned} \tag{61}$$

$$-2p\cdot q' = -2p'\cdot q = u - {m_\mu}^2 = -\frac{s}{2}\left(1 + \sqrt{1 - \frac{4{m_\mu}^2}{s}}\cos\theta\right). \tag{62}$$

From Eq.s (60)–(62),

$$\begin{aligned}&\frac{(p\cdot q)\,(p'\cdot q') + (p\cdot q')\,(p'\cdot q) + {m_\mu}^2\,(p\cdot p')}{k^4\,(p\cdot p')} \\ =& \ \frac{\left(t - {m_\mu}^2\right)^2 + \left(u - {m_\mu}^2\right)^2 + 2{m_\mu}^2 s}{2s^3} \\ =& \ \left[\frac{1}{4s} + \frac{1}{4s}\left(1 - \frac{4{m_\mu}^2}{s}\right)\cos^2\theta + \frac{{m_\mu}^2}{s^2}\right] \\ =& \ \frac{1}{4s}\left[\left(1 + \frac{4{m_\mu}^2}{s}\right) + \left(1 - \frac{4{m_\mu}^2}{s}\right)\cos^2\theta\right].\end{aligned} \tag{63}$$

Substituting Eq.s (60) and (63) into Eq. (59),

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{\alpha^2}{4s}\sqrt{1 - \frac{4m_\mu^2}{s}}\left[\left(1 + \frac{4{m_\mu}^2}{s}\right) + \left(1 - \frac{4{m_\mu}^2}{s}\right)\cos^2\theta\right],\ \alpha = \frac{e^2}{4\pi}. \tag{64}$$

2.4.2 Dependence on the Spins of the Initial Particles

In the Weyl basis, the spinor field

$$\psi(x) = \psi_L + \psi_R \equiv \left(\frac{1-\gamma^5}{2}\right)\psi + \left(\frac{1+\gamma^5}{2}\right)\psi = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}\psi + \begin{pmatrix} 0 & 0 \\ 0 & I_2 \end{pmatrix}\psi. \tag{65}$$

With the **Van der Waerden notation**, we can define the left and right handed **Weyl spinors**

$$\psi_L \equiv \begin{pmatrix} \phi_a \\ 0 \end{pmatrix} \quad \text{and} \quad \psi_R \equiv \begin{pmatrix} 0 \\ \bar{\chi}^{\dot{a}} \end{pmatrix}. \tag{66}$$

The product $\bar{\psi}_1\gamma^\mu\psi_2$ $(1,2 = L, R)$ would vanish when both $\bar{\psi}_1$ and ψ_2 are left or right handed [7]. For instance,

$$\left(\bar{\psi}\right)_R \gamma^\mu \psi_R = \bar{\psi}_L\gamma^\mu\psi_R = \begin{pmatrix} 0 & \bar{\phi}_{\dot{a}} \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \bar{\chi}^{\dot{a}} \end{pmatrix} = 0. \tag{68}$$

Thus $\bar{\psi}_1\gamma^\mu\psi_2 \neq 0$ only if $\bar{\psi}_1$ and ψ_2 are of different **chiralities**. When considering the chirality of $\bar{\psi}\gamma\psi$, we only need to be concerned about

$$\bar{\psi}_L\gamma^\mu\psi_L = \bar{\psi}\gamma^\mu\psi_L = \bar{\psi}\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)\psi \quad \text{and} \quad \bar{\psi}_R\gamma^\mu\psi_R = \bar{\psi}\gamma^\mu\psi_R = \bar{\psi}\gamma^\mu\left(\frac{1+\gamma^5}{2}\right)\psi. \tag{69}$$

Recall the amplitude $\mathcal{M}$ in Eq. (50) and think about $|\mathcal{M}|^2$'s dependence on spins [8] of the initial particles. As $\bar{v}(p')$, $u(p) \sim \psi(x)$, we know the recipe, according to the discussion herein above, could be inserting $\frac{1+\gamma^5}{2}$ or $\frac{1-\gamma^5}{2}$ into $\bar{v}^{\dot{s}}(p')\gamma^\alpha u^s(p)$:

$$\bar{v}(p')\gamma^\alpha u(p) \to \bar{v}(p')\gamma^\alpha\left(\frac{1\pm\gamma^5}{2}\right)u(p). \tag{70}$$

After the insertion, we are free to sum over the electron and positron spins in $|\mathcal{M}|^2$ since only one of the four terms in the sum is nonzero. That is, we compute $\sum_{s,\dot{s}}\left|\bar{v}^{\dot{s}}(p')\gamma^\alpha\left(\frac{1+\gamma^5}{2}\right)u^s(p)\right|^2$ for part of $|\mathcal{M}|^2$ and $\sum_{s,\dot{s}}\left|\bar{v}^{\dot{s}}(p')\gamma^\alpha\left(\frac{1-\gamma^5}{2}\right)u^s(p)\right|^2$ for the remaining part. Similar to Eq.s (51) and

[7] Note that the left/right handed $\bar{\psi}$ correspond to the right/left handed ψ:

$$\left(\bar{\psi}\right)_L \equiv \bar{\psi}\left(\frac{1-\gamma^5}{2}\right) = \bar{\psi}_R = \psi_R^\dagger\gamma^0 = \begin{pmatrix} \chi^a & 0 \end{pmatrix}, \quad \left(\bar{\psi}\right)_R \equiv \bar{\psi}\left(\frac{1+\gamma^5}{2}\right) = \bar{\psi}_L = \psi_L^\dagger\gamma^0 = \begin{pmatrix} 0 & \bar{\phi}_{\dot{a}} \end{pmatrix}. \tag{67}$$

[8] Here for the massless electron and positron, **helicity** $\sim$ chirality.

(53), we have [9]

$$\left(\bar{v}\gamma^\alpha\gamma^5 u\right)^* = \left(\bar{v}\gamma^\alpha\gamma^5 u\right)^\dagger = u^\dagger\gamma^5\gamma^0\gamma^\alpha v = u^\dagger\gamma^0\gamma^\alpha\gamma^5 v = \bar{u}\gamma^\alpha\gamma^5 v; \tag{72}$$

$$\begin{aligned}
&\sum_{s,\dot{s}} \left|\bar{v}^{\dot{s}}(p')\gamma^\alpha\left(\frac{1\pm\gamma^5}{2}\right)u^s(p)\right|^2 \\
=& \sum_{s,\dot{s}} \bar{v}^{\dot{s}}(p')\gamma^\alpha\left(\frac{1\pm\gamma^5}{2}\right)u^s(p)\bar{u}^s(p)\gamma^\beta\left(\frac{1\pm\gamma^5}{2}\right)v^{\dot{s}}(p') \\
=& \operatorname{tr}\left[\left(\not{p}'-m_e\right)\gamma^\alpha\left(\frac{1\pm\gamma^5}{2}\right)\left(\not{p}+m_e\right)\gamma^\beta\left(\frac{1\pm\gamma^5}{2}\right)\right] \\
=& \operatorname{tr}\left[\not{p}'\gamma^\alpha\left(\frac{1\pm\gamma^5}{2}\right)\not{p}\gamma^\beta\left(\frac{1\pm\gamma^5}{2}\right)\right] \\
=& \frac{1}{2}\left[\operatorname{tr}\left(\not{p}'\gamma^\alpha\not{p}\gamma^\beta\right)\pm\operatorname{tr}\left(\not{p}'\gamma^\alpha\not{p}\gamma^\beta\gamma^5\right)\right] \\
=& \frac{1}{2}\operatorname{tr}\left(\not{p}'\gamma^\alpha\not{p}\gamma^\beta\right)\mp 2\mathrm{i}\epsilon^{\mu\alpha\nu\beta}p'_\mu p_\nu.
\end{aligned} \tag{73}$$

Consequently,

$$\begin{aligned}
&\sum_{s,\dot{s}}\left|\bar{v}^{\dot{s}}(p')\gamma^\alpha u^s(p)\right|^2 = \sum_{s,\dot{s}}\bar{v}^{\dot{s}}(p')\gamma^\alpha u^s(p)\bar{u}^s(p)\gamma^\beta v^{\dot{s}}(p') \\
=& \sum_{s,\dot{s}}\Bigg[\bar{v}^{\dot{s}}(p')\gamma^\alpha\left(\frac{1+\gamma^5}{2}\right)u^s(p)\bar{u}^s(p)\gamma^\beta\left(\frac{1+\gamma^5}{2}\right)v^{\dot{s}}(p') \\
&+\bar{v}^{\dot{s}}(p')\gamma^\alpha\left(\frac{1-\gamma^5}{2}\right)u^s(p)\bar{u}^s(p)\gamma^\beta\left(\frac{1-\gamma^5}{2}\right)v^{\dot{s}}(p')\Bigg] \\
=& \sum_{s,\dot{s}}\left[\left|\bar{v}^{\dot{s}}(p')\gamma^\alpha\left(\frac{1+\gamma^5}{2}\right)u^s(p)\right|^2+\left|\bar{v}^{\dot{s}}(p')\gamma^\alpha\left(\frac{1-\gamma^5}{2}\right)u^s(p)\right|^2\right] \\
=& \frac{1}{2}\operatorname{tr}\left(\not{p}'\gamma^\alpha\not{p}\gamma^\beta\right)+2\mathrm{i}\epsilon^{\mu\alpha\nu\beta}p'_\mu p_\nu+\frac{1}{2}\operatorname{tr}\left(\not{p}'\gamma^\alpha\not{p}\gamma^\beta\right)-2\mathrm{i}\epsilon^{\mu\alpha\nu\beta}p'_\mu p_\nu \\
=& \operatorname{tr}\left(\not{p}'\gamma^\alpha\not{p}\gamma^\beta\right).
\end{aligned} \tag{74}$$

That is, $|\mathcal{M}|^2$'s dependence on spins of the initial particles is reflected by the term $2\mathrm{i}\epsilon^{\mu\alpha\nu\beta}p'_\mu p_\nu$.

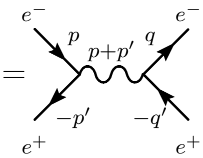
---

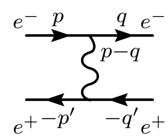
[9]

$$\begin{aligned}
\gamma^\mu\gamma^\alpha\left(\frac{1\pm\gamma^5}{2}\right)\gamma^\nu\gamma^\beta\left(\frac{1\pm\gamma^5}{2}\right) &= \frac{1}{4}\left(\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\pm\gamma^\mu\gamma^\alpha\gamma^5\gamma^\nu\gamma^\beta\pm\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^5+\gamma^\mu\gamma^\alpha\gamma^5\gamma^\nu\gamma^\beta\gamma^5\right) \\
&= \frac{1}{4}\left(\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\pm\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^5\pm\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^5+\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^5\gamma^5\right) \\
&= \frac{1}{4}\left(\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\pm\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^5\pm\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^5+\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\right) \\
&= \frac{1}{2}\left(\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\pm\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^5\right).
\end{aligned} \tag{71}$$

2.5 Bhabha Scattering

The scattering $e^-e^+ \to e^-e^+$ has two leading-order Feynman diagrams:

$\mathrm{i}\mathcal{M} \;=$ 

(Annihilation) $+$ 

(Scattering)

$$= \mathrm{i}\left(\mathcal{M}_A + \mathcal{M}_S\right) = \mathrm{i}e^2 \left[\frac{\bar{v}^{\dot{s}}(p')\gamma^\alpha u^s(p)\bar{u}^r(q)\gamma_\alpha v^{\dot{r}}(q')}{(p+p')^2} - \frac{\bar{v}^{\dot{s}}(p')\gamma^\beta v^{\dot{r}}(q')\bar{u}^r(q)\gamma_\beta u^s(p)}{(p-q)^2} \right].$$

At a **high-energy** (**relativistic**) **limit** where $E \equiv |\mathbf{p}| = -|\mathbf{p}'| \gg m_e$, we suppose that

$$p = (E, E\mathbf{z}), \quad p' = (E, -E\mathbf{z}), \quad q = (E, \mathbf{q}), \quad q' = (E, -\mathbf{q}), \tag{75}$$

where $|\mathbf{z}| = 1$, $\mathbf{q}\cdot\mathbf{z} = |\mathbf{q}|\cos\theta = \sqrt{E^2 - {m_e}^2}\cos\theta = E\cos\theta$ and θ is the scattering angle. Due to the on-shell condition $p^2 = p'^2 = q^2 = q'^2 = {m_e}^2 \to 0$, the Mandelstam variables

$$\begin{aligned} s &= (p+p')^2 = (q+q')^2 = 2p\cdot p' = 2q\cdot q' = 4E^2 = {E_{\mathrm{CM}}}^2, \\ t &= (p-q)^2 = (q'-p')^2 = -2p\cdot q = -2q'\cdot p' = -2E^2\left(1-\cos\theta\right), \\ u &= (p-q')^2 = (q-p')^2 = -2p\cdot q' = -2q\cdot p' = -2E^2\left(1+\cos\theta\right), \end{aligned} \tag{76}$$

where E_{CM} is the total initial energy. To compute the unpolarised cross section, we shall calculate

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{1}{4}\sum_{\text{spins}} \left(|\mathcal{M}_A|^2 + \mathcal{M}_A\mathcal{M}_S^* + \mathcal{M}_S\mathcal{M}_A^* + |\mathcal{M}_S|^2 \right), \tag{77}$$

where we have known from Eq.s (53) ($m_\mu \to m_e \to 0$) and (76) that

$$\begin{aligned} &\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_A|^2 \\ =& \frac{e^4}{4\left(p+p'\right)^4} \sum_{\text{spins}} \left[\bar{v}^{\dot{s}}(p')\gamma^\alpha u^s(p)\bar{u}^s(p)\gamma^\beta v^{\dot{s}}(p') \cdot \bar{u}^r(q)\gamma_\alpha v^{\dot{r}}(q')\bar{v}^{\dot{r}}(q')\gamma_\beta u^r(q) \right] \\ =& \frac{e^4}{4\left(p+p'\right)^4} \mathrm{tr}\left(\not{p}'\gamma^\alpha \not{p}\gamma^\beta \right) \mathrm{tr}\left(\not{q}\gamma_\alpha \not{q}'\gamma_\beta \right) = 2e^4 \left(\frac{t^2+u^2}{s^2} \right). \end{aligned} \tag{78}$$

Comparable to eq (53), [10]

$$
\begin{aligned}
\frac{1}{4}\sum_{\text{spins}}\mathcal{M}_A\mathcal{M}_S^* &= \frac{-e^4}{4\left(p+p'\right)^2\left(p-q\right)^2}\sum_{\text{spins}}\big[\bar{v}^{\dot{s}}(p')\gamma^\alpha u^s(p)\cdot\bar{u}^s(p)\gamma^\beta u^r(q)\\
&\quad\times\bar{u}^r(q)\gamma_\alpha v^{\dot{r}}(q')\cdot\bar{v}^{\dot{r}}(q')\gamma_\beta v^{\dot{s}}(p')\big]\\
&= \frac{-e^4}{4\left(p+p'\right)^2\left(p-q\right)^2}\mathrm{tr}\Bigg[\sum_{\text{spins}} v^{\dot{s}}(p')\bar{v}^{\dot{s}}(p')\gamma^\alpha u^s(p)\bar{u}^s(p)\gamma^\beta u^r(q)\bar{u}^r(q)\gamma_\alpha\\
&\quad\times v^{\dot{r}}(q')\bar{v}^{\dot{r}}(q')\gamma_\beta\Bigg]\\
&= \frac{-e^4}{4\left(p+p'\right)^2\left(p-q\right)^2}\mathrm{tr}\left(\not{p}'\gamma^\alpha\not{p}\gamma^\beta\not{q}\gamma_\alpha\not{q}'\gamma_\beta\right)\\
&= \frac{-e^4}{4\left(p+p'\right)^2\left(p-q\right)^2}\mathrm{tr}\left(p'_\nu p_\mu q_\rho q'_\sigma\gamma^\nu\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\rho\gamma_\alpha\gamma^\sigma\gamma_\beta\right)\\
&= \frac{-e^4}{4\left(p+p'\right)^2\left(p-q\right)^2}\mathrm{tr}\left(-8p'_\nu p_\mu q_\rho q'_\sigma\gamma^\nu\gamma^\rho\eta^{\sigma\mu}\right)\\
&= \frac{-e^4}{\left(p+p'\right)^2\left(p-q\right)^2}\mathrm{tr}\left[\left(-2p\cdot q'\right)p'_\nu q_\rho\gamma^\nu\gamma^\rho\right] = \frac{-e^4}{st}\mathrm{tr}\left(u\cdot p'_\nu q_\rho\gamma^\nu\gamma^\rho\right);
\end{aligned}
\tag{80}
$$

$$
\begin{aligned}
\frac{1}{4}\sum_{\text{spins}}\mathcal{M}_S\mathcal{M}_A^* &= \frac{-e^4}{4\left(p-q\right)^2\left(p+p'\right)^2}\sum_{\text{spins}}\big[\bar{v}^{\dot{s}}(p')\gamma^\alpha v^{\dot{r}}(q')\cdot\bar{v}^{\dot{r}}(q')\gamma^\beta u^r(q)\\
&\quad\times\bar{u}^r(q)\gamma_\alpha u^s(p)\cdot\bar{u}^s(p)\gamma_\beta v^{\dot{s}}(p')\big]\\
&= \frac{-e^4}{4\left(p-q\right)^2\left(p+p'\right)^2}\mathrm{tr}\left(\not{p}'\gamma^\alpha\not{q}'\gamma^\beta\not{q}\gamma_\alpha\not{p}\gamma_\beta\right) = \frac{-e^4}{st}\mathrm{tr}\left(u\cdot p'_\nu q_\rho\gamma^\rho\gamma^\nu\right);
\end{aligned}
\tag{81}
$$

$$
\begin{aligned}
\frac{1}{4}\sum_{\text{spins}}\left(\mathcal{M}_A\mathcal{M}_S^*+\mathcal{M}_S\mathcal{M}_A^*\right) &= \frac{-e^4}{st}\mathrm{tr}\left[u\cdot p'_\nu q_\rho\left(\gamma^\nu\gamma^\rho+\gamma^\rho\gamma^\nu\right)\right]\\
&= \frac{e^4}{st}\mathrm{tr}\left[u\cdot\left(-2p'\cdot q\right)I_4\right] = \frac{e^4}{st}\mathrm{tr}\left(u^2 I_4\right) = 4e^4\left(\frac{u^2}{st}\right);
\end{aligned}
\tag{82}
$$

$$
\begin{aligned}
\frac{1}{4}\sum_{\text{spins}}\left|\mathcal{M}_S\right|^2 &= \frac{e^4}{4\left(p+p'\right)^4}\sum_{\text{spins}}\big[\bar{v}^{\dot{s}}(p')\gamma^\alpha v^{\dot{r}}(q')\bar{v}^{\dot{r}}(q')\gamma^\beta v^{\dot{s}}(p')\\
&\quad\times\bar{u}^r(q)\gamma_\alpha u^s(p)\bar{u}^s(p)\gamma_\beta u^r(q)\big]\\
&= \frac{e^4}{4\left(p+p'\right)^4}\mathrm{tr}\left(\not{p}'\gamma^\alpha\not{q}'\gamma^\beta\right)\mathrm{tr}\left(\not{q}\gamma_\alpha\not{p}\gamma_\beta\right) = 2e^4\left(\frac{s^2+u^2}{t^2}\right).
\end{aligned}
\tag{83}
$$

[10] From Eq.s (35) and (36),

$$
\begin{aligned}
\gamma^\nu\left(\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\rho\gamma_\alpha\right)\gamma^\sigma\gamma_\beta &= \gamma^\nu\left(-2\gamma^\rho\gamma^\beta\gamma^\mu\right)\gamma^\sigma\gamma_\beta = -2\gamma^\nu\gamma^\rho\left(\gamma^\beta\gamma^\mu\gamma^\sigma\gamma_\beta\right)\\
&= -2\gamma^\nu\gamma^\rho\left(4\eta^{\sigma\mu}I_4\right) = -8\gamma^\nu\gamma^\rho\eta^{\sigma\mu}.
\end{aligned}
\tag{79}
$$

From Eq.s (77)–(83),

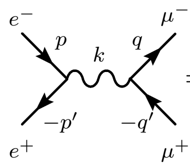
$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = 2e^4\left(\frac{t^2+u^2}{s^2}+\frac{2u^2}{st}+\frac{s^2+u^2}{t^2}\right) = 2e^4\left[\frac{t^2}{s^2}+\frac{s^2}{t^2}+u^2\left(\frac{1}{s}+\frac{1}{t}\right)^2\right]. \tag{84}$$

Substituting Eq.s (86), (76) and (84) into (31),

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{\alpha^2}{2s}\left[\frac{t^2}{s^2}+\frac{s^2}{t^2}+u^2\left(\frac{1}{s}+\frac{1}{t}\right)^2\right],\ \alpha=\frac{e^2}{4\pi},\ u=-(s+t)\,. \tag{85}$$

2.6 Møller Scattering

The scattering $e^-e^- \to e^-e^-$ has two leading-order Feynman diagrams:



$$= \mathrm{i}\mathcal{M}\left[e^-(p)e^+(p') \to \mu^-(q)\mu^+(q')\right],$$

At a high-energy limit where $E \equiv |\mathbf{p}| = -|\mathbf{p}'| \gg m_e$, we suppose that

$$p=(E,E\mathbf{z})\,,\quad p'=(E,-E\mathbf{z})\,,\quad q=(E,\mathbf{q})\,,\quad q'=(E,-\mathbf{q})\,, \tag{86}$$

where $|\mathbf{z}|=1$, $\mathbf{q}\cdot\mathbf{z}=|\mathbf{q}|\cos\theta=E\cos\theta$ and θ is the scattering angle. To compute the unpolarised cross section, we shall calculate

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{1}{4}\sum_{\text{spins}}\left(|\mathcal{M}_t|^2+\mathcal{M}_t\mathcal{M}_u^*+\mathcal{M}_u\mathcal{M}_t^*+|\mathcal{M}_u|^2\right), \tag{87}$$

where

$$\begin{aligned}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}_u|^2 &= \frac{e^4}{4\,(p-q')^4}\sum_{\text{spins}}\big[\bar{u}^{\dot{r}}(q')\gamma^\alpha u^s(p)\bar{u}^s(p)\gamma^\beta u^{\dot{r}}(q') \\ &\qquad \times \bar{u}^r(q)\gamma_\alpha u^{\dot{s}}(p')\bar{u}^{\dot{s}}(p')\gamma_\beta u^r(q)\big] \\ &= \frac{e^4}{4\,(p-q')^4}\mathrm{tr}\left(\not{q}'\gamma^\alpha\not{p}\gamma^\beta\right)\mathrm{tr}\left(\not{q}\gamma_\alpha\not{p}'\gamma_\beta\right);\end{aligned} \tag{88}$$

$$\begin{aligned}\frac{1}{4}\sum_{\text{spins}}\mathcal{M}_u\mathcal{M}_t^* &= \frac{-e^4}{4\,(p-q')^2\,(p-q)^2}\sum_{\text{spins}}\big[\bar{u}^{\dot{r}}(q')\gamma^\alpha u^s(p)\cdot\bar{u}^s(p)\gamma^\beta u^r(q) \\ &\qquad \times \bar{u}^r(q)\gamma_\alpha u^{\dot{s}}(p')\cdot\bar{u}^{\dot{s}}(p')\gamma_\beta u^{\dot{r}}(q')\big] \\ &= \frac{-e^4}{4\,(p-q')^2\,(p-q)^2}\mathrm{tr}\left(\not{q}'\gamma^\alpha\not{p}\gamma^\beta\not{q}\gamma_\alpha\not{p}'\gamma_\beta\right);\end{aligned} \tag{89}$$

$$
\begin{aligned}
\frac{1}{4}\sum_{\text{spins}}\mathcal{M}_t\mathcal{M}_u^* = {} & \frac{-e^4}{4\left(p-q\right)^2\left(p-q'\right)^2}\sum_{\text{spins}}\big[\bar{u}^{\dot{r}}(q')\gamma^\alpha u^{\dot{s}}(p')\cdot\bar{u}^{\dot{s}}(p')\gamma^\beta u^r(q) \\
& \times\bar{u}^r(q)\gamma_\alpha u^s(p)\cdot\bar{u}^s(p)\gamma_\beta u^{\dot{r}}(q')\big] \\
= {} & \frac{-e^4}{4\left(p-q\right)^2\left(p-q'\right)^2}\operatorname{tr}\left(\not{q}'\gamma^\alpha\not{p}'\gamma^\beta\not{q}\gamma_\alpha\not{p}\gamma_\beta\right);
\end{aligned}
\tag{90}
$$

$$
\begin{aligned}
\frac{1}{4}\sum_{\text{spins}}\left|\mathcal{M}_t\right|^2 = {} & \frac{e^4}{4\left(p-q\right)^4}\sum_{\text{spins}}\big[\bar{u}^{\dot{r}}(q')\gamma^\alpha u^{\dot{s}}(p')\bar{u}^{\dot{s}}(p')\gamma^\beta u^{\dot{r}}(q') \\
& \times\bar{u}^r(q)\gamma_\alpha u^s(p)\bar{u}^s(p)\gamma_\beta u^r(q)\big] \\
= {} & \frac{e^4}{4\left(p-q\right)^4}\operatorname{tr}\left(\not{q}'\gamma^\alpha\not{p}'\gamma^\beta\right)\operatorname{tr}\left(\not{q}\gamma_\alpha\not{p}\gamma_\beta\right).
\end{aligned}
\tag{91}
$$

Comparing Eq.s (88)–(91) with Eq.s (78)–(85), we can immediately write

$$
\begin{aligned}
\frac{1}{4}\sum_{\text{spins}}\left|\mathcal{M}\right|^2 &= \frac{1}{4}\sum_{\text{spins}}\left[\left|\mathcal{M}_t\right|^2+\left(\mathcal{M}_t\mathcal{M}_u^*+\mathcal{M}_u\mathcal{M}_t^*\right)+\left|\mathcal{M}_u\right|^2\right] \\
&= 2e^4\left(\frac{s^2+u^2}{t^2}+\frac{2s^2}{tu}+\frac{s^2+t^2}{u^2}\right) = 2e^4\left[\frac{s^2\left(t+u\right)^2+t^4+u^4}{t^2u^2}\right] \\
&= 2e^4\left[\frac{s^4+\left(t^2+u^2\right)^2-2t^2u^2}{t^2u^2}\right] = 2e^4\left[\frac{s^4+\left(s^2-2tu\right)^2-2t^2u^2}{t^2u^2}\right] \\
&= 4e^4\left(\frac{s^4}{t^2u^2}-\frac{2s^2}{tu}+1\right),
\end{aligned}
\tag{92}
$$

$$
\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{\alpha^2}{s}\left(\frac{s^4}{t^2u^2}-\frac{2s^2}{tu}+1\right).
\tag{93}
$$

2.7 Phase Space Calculations

According to Eq. (27), the relativistic invariant n-body phase space

$$
R_n \equiv \int d\Pi_n \equiv \left[\prod_{f=1}^{n}\int\frac{d\mathbf{p}_f}{(2\pi)^3 2E\left(\mathbf{p}_f\right)}\right](2\pi)^4\,\delta^4\left(\sum_i q_i-\sum_{f=1}^{n}p_f\right),
\tag{94}
$$

where $E\left(\mathbf{p}_f\right) = \sqrt{\left|\mathbf{p}_f\right|^2+m_f{}^2}$. We have seen in Eq. (30), Sec. 1.3, that a two-body phase space in the CM frame is

$$
R_2 = \int d\Pi_2 = \int\frac{d\Omega\ \left|\mathbf{p}_1\right|_0}{16\pi^2 E_{\text{CM}}},
\tag{95}
$$

where E_{CM} is the total initial energy, $|\mathbf{p}_1|_0 \equiv \frac{1}{2}\sqrt{{E_{\text{CM}}}^2 - 2\left({m_1}^2 + {m_2}^2\right) + \left(\frac{{m_1}^2 - {m_2}^2}{E_{\text{CM}}}\right)^2}$ and Ω the solid angle in the **p**-space. With the help of the **recursion formula**

$$
\begin{aligned}
& R_n(s_n) \sim R_n\left({E_{\text{CM}}}^2\right) = R_n\left(E_{\text{CM}}\right) \\
&= \left[\prod_{f=1}^{n}\int \frac{d\mathbf{p}_f}{(2\pi)^3 2E\left(\mathbf{p}_f\right)}\right](2\pi)^4\,\delta^4\left(\sum_i q_i - \sum_{f=1}^{n} p_f\right) \\
&= \int \frac{d\mathbf{p}_n}{(2\pi)^3 2E\left(\mathbf{p}_n\right)}\left[\prod_{f=1}^{n-1}\int \frac{d\mathbf{p}_f}{(2\pi)^3 2E\left(\mathbf{p}_f\right)}\right](2\pi)^4\,\delta^3(0)\delta\left[E_{\text{CM}} - \sum_{f=1}^{n} E\left(\mathbf{p}_f\right)\right] \\
&= \int \frac{d\mathbf{p}_n}{(2\pi)^3 2E\left(\mathbf{p}_n\right)}\left[\prod_{f=1}^{n-1}\int \frac{d\mathbf{p}_f}{(2\pi)^3 2E\left(\mathbf{p}_f\right)}\right](2\pi)^4\,\delta^3(0)\delta\Big\{\left[E_{\text{CM}} - E(\mathbf{p}_n)\right] - \sum_{f=1}^{n-1} E\left(\mathbf{p}_f\right)\Big\} \\
&= \int \frac{d\mathbf{p}_n}{(2\pi)^3 2E\left(\mathbf{p}_n\right)}\left\{\left[\prod_{f=1}^{n-1}\int \frac{d\mathbf{p}_f}{(2\pi)^3 2E\left(\mathbf{p}_f\right)}\right](2\pi)^4\,\delta^3(0)\delta\left[E'_{\text{CM}} - \sum_{f=1}^{n-1} E\left(\mathbf{p}_f\right)\right]\right\} \\
&= \int \frac{d\mathbf{p}_n}{(2\pi)^3 2E\left(\mathbf{p}_n\right)}\left\{\left[\prod_{f=1}^{n-1}\int \frac{d\mathbf{p}_f}{(2\pi)^3 2E\left(\mathbf{p}_f\right)}\right](2\pi)^4\,\delta^4\left(\sum_i q_i - \sum_{f=1}^{n-1} p_f\right)\right\} \\
&= \int \frac{d\mathbf{p}_n}{(2\pi)^3 2E\left(\mathbf{p}_n\right)} R_{n-1}\left(E'_{\text{CM}}\right) = \int \frac{d\mathbf{p}_n}{(2\pi)^3 2E\left(\mathbf{p}_n\right)} R_{n-1}\left({E'_{\text{CM}}}^2\right) \\
&\sim \int \frac{d\mathbf{p}_n}{(2\pi)^3 2E\left(\mathbf{p}_n\right)} R_{n-1}(s_{n-1}),
\end{aligned}
\tag{96}
$$

where s_n is the square of the total energy in the n-particle CM system, one can derive from R_2 the expressions of R_3, R_4, etc..

2.8 Charged Pion Decay Rate

The leading-order Feynman diagram for the π^+ decay processes is

Setting $m_{\nu_l} = 0$ and using the Dirac equation, we can write $\mathcal{M}_S$ and $\mathcal{M}_V$ respectively for the **scalar** and **vector coupling** :

$$
\mathcal{M}_S = -\mathrm{i} g_\pi \bar{u}^{\dot{s}}_{\nu_l}(p')\left(1 + \gamma^5\right) v^s_l(p), \tag{97}
$$

$$
\mathcal{M}_V = \frac{f_\pi}{m_\pi}\bar{u}^{\dot{s}}_{\nu_l}(p')\left(1 + \gamma^5\right)\left(\not{p} + \not{p}'\right) v^s_l(p) = -f_\pi \frac{m_l}{m_\pi}\bar{u}^{\dot{s}}_{\nu_l}(p')\left(1 + \gamma^5\right) v^s_l(p), \tag{98}
$$

where g_π and f_π are two coupling constants. Similar to Eq.s (51) and (53), [11]

$$\left[\bar{u}\left(1+\gamma^5\right)v\right]^* = \bar{v}\left(1-\gamma^5\right)u, \tag{101}$$

$$\begin{aligned}\sum_{s,\dot{s}}|\mathcal{M}_S|^2 &= {g_\pi}^2\sum_{s,\dot{s}}\bar{u}^{\dot{s}}(p')\left(1+\gamma^5\right)v^s(p)\bar{v}^s(p)\left(1-\gamma^5\right)u^{\dot{s}}(p')\\ &= {g_\pi}^2\mathrm{tr}\left[\not{p}'\left(1+\gamma^5\right)\left(\not{p}-m_l\right)\left(1-\gamma^5\right)\right]\\ &= 2{g_\pi}^2\mathrm{tr}\left[p'_\alpha p_\beta\gamma^\alpha\gamma^\beta\left(1-\gamma^5\right)\right] = 2{g_\pi}^2\mathrm{tr}\left[p'_\alpha p_\beta\eta^{\alpha\beta}\left(1-\gamma^5\right)\right]\\ &= 2{g_\pi}^2\left(p\cdot p'\right)\mathrm{tr}\left(1-\gamma^5\right) = 8{g_\pi}^2\left(p\cdot p'\right)\\ &= 4{g_\pi}^2\left({m_\pi}^2-{m_l}^2\right),\end{aligned} \tag{102}$$

$$\sum_{s,\dot{s}}|\mathcal{M}_V|^2 = -8{f_\pi}^2\frac{{m_l}^2}{{m_\pi}^2}\left(p\cdot p'\right) = -4{f_\pi}^2\frac{{m_l}^2}{{m_\pi}^2}\left({m_\pi}^2-{m_l}^2\right), \tag{103}$$

where

$$p\cdot p' = \frac{\left(p+p'\right)^2-\left(p^2+p'^2\right)}{2} = \frac{q^2-\left(p^2+p'^2\right)}{2} = \frac{{m_\pi}^2-{m_l}^2}{2}. \tag{104}$$

Recall Eq.s (27) and (30), the decay rate for the spinless π^+ is

$$\begin{aligned}\Gamma &= \frac{(2\pi)^4}{2E(q)}\int\frac{d\mathbf{p}}{(2\pi)^3 2E\left(\mathbf{p}\right)}\int\frac{d\mathbf{p}'}{(2\pi)^3 2E\left(\mathbf{p}'\right)}\delta^4\left(q-p-p'\right)\sum_{s,\dot{s}}|\mathcal{M}|^2\\ &\propto \frac{\int d\Pi_2}{E(q)}\sum_{s,\dot{s}}|\mathcal{M}|^2 \propto \frac{|\mathbf{p}_1|_0}{E^2(q)}\sum_{s,\dot{s}}|\mathcal{M}|^2\\ &= \frac{\sqrt{E^2(q)-2{m_l}^2+\left[\frac{{m_l}^2}{E(q)}\right]^2}}{2E^2(q)}\sum_{s,\dot{s}}|\mathcal{M}|^2 = \frac{\left|E^2(q)-{m_l}^2\right|}{2E^3(q)}\sum_{s,\dot{s}}|\mathcal{M}|^2\\ &= \frac{{m_\pi}^2-{m_l}^2}{2{m_\pi}^3}\sum_{s,\dot{s}}|\mathcal{M}|^2.\end{aligned} \tag{105}$$

Combining Eq. (105) with Eq.s (102) and (103),

$$\left(\frac{\Gamma_e}{\Gamma_\mu}\right)_S = \frac{\left({m_\pi}^2-{m_e}^2\right)^2}{\left({m_\pi}^2-{m_\mu}^2\right)^2}, \quad \left(\frac{\Gamma_e}{\Gamma_\mu}\right)_V = \frac{{m_e}^2\left({m_\pi}^2-{m_e}^2\right)^2}{{m_\mu}^2\left({m_\pi}^2-{m_\mu}^2\right)^2}. \tag{106}$$

[11]

$$\begin{aligned}\gamma^\alpha\left(1+\gamma^5\right)\gamma^\beta\left(1-\gamma^5\right) &= \gamma^\alpha\gamma^\beta+\gamma^\alpha\gamma^5\gamma^\beta-\gamma^\alpha\gamma^\beta\gamma^5-\gamma^\alpha\gamma^5\gamma^\beta\gamma^5\\ &= \gamma^\alpha\gamma^\beta-\gamma^\alpha\gamma^\beta\gamma^5-\gamma^\alpha\gamma^\beta\gamma^5+\gamma^\alpha\gamma^\beta\gamma^5\gamma^5\\ &= \gamma^\alpha\gamma^\beta-\gamma^\alpha\gamma^\beta\gamma^5-\gamma^\alpha\gamma^\beta\gamma^5+\gamma^\alpha\gamma^\beta\\ &= 2\gamma^\alpha\gamma^\beta\left(1-\gamma^5\right);\end{aligned} \tag{99}$$

$$\gamma^\alpha\left(1+\gamma^5\right)m_l\left(1-\gamma^5\right) = \gamma^\alpha m_l\left[1-\left(\gamma^5\right)^2\right] = \gamma^\alpha m_l\left(1-1\right) = 0. \tag{100}$$

2.9 Gauge Field

2.9.1 A Gauge Invariant Lagrangian

In the Weyl basis, we construct [12]

$$
\begin{aligned}
&\mathcal{L} = \mathcal{L}_{\text{M}} + (\mathcal{L}_{\text{D}} + \mathcal{L}_{\text{D-M}}) + \mathcal{L}_{\text{D-KG}} + (\mathcal{L}_{\text{KG}} + \mathcal{L}_{\text{KG-M}} + \mathcal{L}_{\text{KG-KG}}) \\
= &-\frac{1}{4}(F_{\mu\nu})^2 + \left[\bar{\psi}_L(\mathrm{i}\not{D}_1 - m)\psi_L + \bar{\psi}_R(\mathrm{i}\not{D}_2 - m)\psi_R\right] + g\bar{\psi}\psi(\phi+\phi^*) + \left[|D_\mu\phi|^2 - \mathcal{V}\right] \\
= &-\frac{1}{4}(F_{\mu\nu})^2 + \mathrm{i}\left(\bar{\psi}_L\not{D}_1\psi_L + \bar{\psi}_R\not{D}_2\psi_R\right) - \left(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L\right)[m - g(\phi+\phi^*)] + |D_\mu\phi|^2 - \mathcal{V},
\end{aligned}
\tag{107}
$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\psi = \psi_L + \psi_R \equiv \frac{1}{2}(1-\gamma^5)\psi + \frac{1}{2}(1+\gamma^5)\psi$; $(D_1)_\mu \equiv \partial_\mu + \mathrm{i}eA_\mu$, $(D_2)_\mu \equiv \partial_\mu - \mathrm{i}eA_\mu$, $D_\mu \equiv \partial_\mu - 2\mathrm{i}eA_\mu$; $\mathcal{V}(\phi) \equiv m^2(\phi\phi^*) + \lambda(\phi\phi^*)^2$; g is a coupling constant; "M", "D" and "KG" stand for free Maxwell, Dirac and KG field, respectively. Under the gauge transformations

$$
\psi_L \to e^{-\mathrm{i}e\theta}\psi_L, \quad \psi_R \to e^{\mathrm{i}e\theta}\psi_R, \quad \phi \to e^{2\mathrm{i}e\theta}\phi \quad \text{and} \quad A_\mu \to A_\mu + \partial_\mu\theta, \tag{108}
$$

we have

1.

$$
\begin{aligned}
F'_{\mu\nu} &= \partial_\mu A'_\nu - \partial_\nu A'_\mu \\
&= \partial_\mu(A_\nu + \partial_\nu\theta) - \partial_\nu(A_\mu + \partial_\mu\theta) \\
&= (\partial_\mu A_\nu - \partial_\nu A_\mu) + (\partial_{\mu\nu} - \partial_{\nu\mu})\theta \\
&= \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu},
\end{aligned}
\tag{109}
$$

$$
-\frac{1}{4}\left(F'_{\mu\nu}\right)^2 = -\frac{1}{4}(F_{\mu\nu})^2 \quad \text{(invariant)}; \tag{110}
$$

2.

$$
\begin{aligned}
D'_\mu\phi' &= [\partial_\mu - 2\mathrm{i}e(A_\mu + \partial_\mu\theta)]\left(e^{2\mathrm{i}e\theta}\phi\right) \\
&= e^{2\mathrm{i}e\theta}[(\partial_\mu + 2\mathrm{i}e\partial_\mu\theta) - 2\mathrm{i}e(A_\mu + \partial_\mu\theta)]\phi \\
&= e^{2\mathrm{i}e\theta}(\partial_\mu - 2\mathrm{i}eA_\mu)\phi = D_\mu\phi,
\end{aligned}
\tag{111}
$$

$$
\begin{aligned}
\left|D'_\mu\phi'\right|^2 &= \eta^{\mu\nu}\left[e^{2\mathrm{i}e\theta}(\partial_\mu - 2\mathrm{i}eA_\mu)\phi\right]\left[e^{-2\mathrm{i}e\theta}(\partial_\nu + 2\mathrm{i}eA_\nu)\phi^*\right] \\
&= \eta^{\mu\nu}[(\partial_\mu - 2\mathrm{i}eA_\mu)\phi][(\partial_\nu + 2\mathrm{i}eA_\nu)\phi^*] \\
&= |D_\mu\phi|^2 \quad \text{(invariant)};
\end{aligned}
\tag{112}
$$

[12] Much the same as we did in Sec. 2.4.2 (eqs (66)–(69)), we know that $\bar{\psi}\not{D}\psi = \bar{\psi}\not{D}\psi_L + \bar{\psi}\not{D}\psi_R = \bar{\psi}_L\not{D}\psi_L + \bar{\psi}_R\not{D}\psi_R$ and $\bar{\psi}\psi = \bar{\psi}\psi_R + \bar{\psi}\psi_L = \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L$.

3.

$$m^2\left(\phi'\phi'^*\right)=m^2\left(e^{2\mathrm{i}e\theta}\phi\right)\left(e^{-2\mathrm{i}e\theta}\phi^*\right)=m^2\left(\phi\phi^*\right)\quad\text{(invariant)},\tag{113}$$

$$\lambda\left(\phi'\phi'^*\right)^2=\lambda\left(\phi\phi^*\right)^2\quad\text{(invariant)}.\tag{114}$$

4.

$$\bar{\psi}'_L=\left(e^{-\mathrm{i}e\theta}\psi_L\right)^\dagger\gamma^0=e^{\mathrm{i}e\theta}\psi_L^\dagger\gamma^0=e^{\mathrm{i}e\theta}\bar{\psi}_L,\quad \bar{\psi}'_R=e^{-\mathrm{i}e\theta}\bar{\psi}_R;\tag{115}$$

$$\begin{aligned}\bar{\psi}'_L\not{D}_1\psi'_L&=\gamma^\mu\cdot e^{\mathrm{i}e\theta}\bar{\psi}_L\cdot\left[\partial_\mu+\mathrm{i}e\left(A_\mu+\partial_\mu\theta\right)\right]\cdot e^{-\mathrm{i}e\theta}\psi_L\\&=\gamma^\mu\cdot e^{\mathrm{i}e\theta}\bar{\psi}_L\cdot e^{-\mathrm{i}e\theta}\left[\partial_\mu-\mathrm{i}e\partial_\mu\theta+\mathrm{i}e\left(A_\mu+\partial_\mu\theta\right)\right]\cdot\psi_L\\&=\gamma^\mu\bar{\psi}_L\left(\partial_\mu+\mathrm{i}eA_\mu\right)\psi_L=\bar{\psi}_L\not{D}_1\psi_L\quad\text{(invariant)},\end{aligned}\tag{116}$$

$$\bar{\psi}'_R\not{D}_2\psi'_R=\bar{\psi}_R\not{D}_2\psi_R\quad\text{(invariant)};\tag{117}$$

5.

$$m\bar{\psi}'_L\psi'_R=m\left(e^{\mathrm{i}e\theta}\bar{\psi}_L\right)\left(e^{\mathrm{i}e\theta}\psi_R\right)=me^{2\mathrm{i}e\theta}\bar{\psi}_L\psi_R\quad\text{(non-invariant)},\tag{118}$$

$$m\bar{\psi}'_R\psi'_L=me^{-2\mathrm{i}e\theta}\bar{\psi}_R\psi_L\quad\text{(non-invariant)};\tag{119}$$

6.

$$g\bar{\psi}'_L\psi'_R\phi'=ge^{2\mathrm{i}e\theta}\bar{\psi}_L\psi_R\cdot e^{2\mathrm{i}e\theta}\phi=ge^{4\mathrm{i}e\theta}\bar{\psi}_L\psi_R\phi\quad\text{(non-invariant)},\tag{120}$$

$$g\bar{\psi}'_L\psi'_R\left(\phi^*\right)'=ge^{2\mathrm{i}e\theta}\bar{\psi}_L\psi_R\cdot e^{-2\mathrm{i}e\theta}\phi^*=g\bar{\psi}_L\psi_R\phi^*\quad\text{(invariant)},\tag{121}$$

$$g\bar{\psi}'_R\psi'_L\phi'=ge^{2\mathrm{i}e\theta}\bar{\psi}_R\psi_L\cdot e^{-2\mathrm{i}e\theta}\phi=g\bar{\psi}_R\psi_L\phi\quad\text{(invariant)},\tag{122}$$

$$g\bar{\psi}'_R\psi'_L\left(\phi^*\right)'=ge^{-4\mathrm{i}e\theta}\bar{\psi}_R\psi_L\phi^*\quad\text{(non-invariant)}.\tag{123}$$

Getting rid of the non-invariant terms, Eq. (107) becomes gauge invariant:

$$\begin{aligned}\mathcal{L}=&-\frac{1}{4}\left(F_{\mu\nu}\right)^2+\left|D_\mu\phi\right|^2-\mathcal{V}+\mathrm{i}\left(\bar{\psi}_L\not{D}_1\psi_L+\bar{\psi}_R\not{D}_2\psi_R\right)+g\left(\bar{\psi}_L\psi_R\phi^*+\bar{\psi}_R\psi_L\phi\right)\\=&-\frac{1}{4}\left(F_{\mu\nu}\right)^2+\left|D_\mu\phi\right|^2-\mathcal{V}+\mathrm{i}\left(\bar{\psi}_L\not{D}_1\psi_L+\bar{\psi}_R\not{D}_2\psi_R\right)+g\bar{\psi}\psi\left(\frac{\phi^*+\phi}{2}\right)\\&+g\bar{\psi}\gamma^5\psi\times\left(\frac{\phi^*-\phi}{2}\right),\end{aligned}\tag{124}$$

where the fermion field $\mathrm{i}\left(\bar{\psi}_L\not{D}_1\psi_L+\bar{\psi}_R\not{D}_2\psi_R\right)$ is massless and the gauge field couples to the axial fermionic current $g\bar{\psi}\gamma^5\psi\left(\frac{\phi^*-\phi}{2}\right)$.