1. We first have

$$
\begin{aligned}
\nabla_l \nabla_k A^i \equiv A^i_{;k;l} &= A^i_{;k,l} + \Gamma^i_{ml} A^m_{;k} - \Gamma^m_{kl} A^i_{;m} \\
&= \left(A^i_{,k} + \Gamma^i_{mk} A^m \right)_{,l} + \Gamma^i_{ml} \left(A^m_{,k} + \Gamma^m_{nk} A^n \right) - \Gamma^m_{kl} A^i_{;m} \\
&= \underbrace{A^i_{,k,l}}_{\text{symmetric about } k \text{ and } l} + \Gamma^i_{mk,l} A^m \underbrace{+ \Gamma^i_{mk} A^m_{,l} + \Gamma^i_{ml} A^m_{,k}}_{\text{symmetric about } k \text{ and } l} + \Gamma^i_{ml} \Gamma^m_{nk} A^n - \underbrace{\Gamma^m_{kl} A^i_{;m}}_{\text{symmetric about } k \text{ and } l}.
\end{aligned}
$$

Then

Similarly,

$$
\begin{aligned}
\nabla_l \nabla_k A_i \equiv A_{i;k;l} &= A_{i;k,l} - \Gamma^m_{il} A_{m;k} - \Gamma^m_{kl} A_{i;m} \\
&= \left(A_{i,k} + \Gamma^m_{ik} A_m \right)_{,l} + \Gamma^m_{il} \left(A_{m,k} + \Gamma^n_{mk} A_n \right) - \Gamma^m_{kl} A_{i;m} \\
&= \underbrace{A_{i,k,l}}_{\text{symmetric about } k \text{ and } l} + \Gamma^m_{ik,l} A_m \underbrace{+ \Gamma^m_{ik} A_{m,l} + \Gamma^m_{il} A_{m,k}}_{\text{symmetric about } k \text{ and } l} + \Gamma^m_{il} \Gamma^n_{mk} A_n - \underbrace{\Gamma^m_{kl} A_{i;m}}_{\text{symmetric about } k \text{ and } l}.
\end{aligned}
$$

Then

$$
\begin{aligned}
& A^i_{;k;l} - A^i_{;l;k} \\
&= A_{i,k,l} - A_{i,l,k} + \left(\Gamma^m_{ik,l} - \Gamma^m_{il,k} \right) A_m + \left[\left(\Gamma^m_{ik} A_{m,l} + \Gamma^m_{il} A_{m,k} \right) - \left(\Gamma^m_{il} A_{m,k} + \Gamma^m_{ik} A_{m,l} \right) \right] + \left(\Gamma^m_{il} \Gamma^n_{mk} - \Gamma^m_{ik} \Gamma^n_{ml} \right) A_n - \left(\Gamma^m_{kl} - \Gamma^m_{lk} \right) A^i_{;m} \\
&= \left(\Gamma^m_{ik,l} - \Gamma^m_{il,k} \right) A_m + \left(\Gamma^m_{il} \Gamma^n_{mk} - \Gamma^m_{ik} \Gamma^n_{ml} \right) A_n = \left(\Gamma^m_{ik,l} - \Gamma^m_{il,k} + \Gamma^n_{il} \Gamma^m_{nk} - \Gamma^n_{ik} \Gamma^m_{nl} \right) A_m \\
&= R^m_{ilk} A_m = R^m_{ikl} A_m .
\end{aligned}
$$

2. In a locally geodesic frame, $\Gamma^i_{kl} = 0$. Then

$$
R^n_{ikl;m} = \left(\Gamma^n_{il,k} - \Gamma^n_{ik,l} + \Gamma^n_{jk} \Gamma^j_{il} - \Gamma^n_{jl} \Gamma^j_{ik} \right)_{;m} = \Gamma^n_{il,k,m} - \Gamma^n_{ik,l,m} .
$$

Likewise, $R^n_{imk;l} = \Gamma^n_{ik,m,l} - \Gamma^n_{im,k,l}$, $R^n_{ilm;k} = \Gamma^n_{im,l,k} - \Gamma^n_{il,m,k}$. Then

$$
R^n_{ikl;m} + R^n_{imk;l} + R^n_{ilm;k} = \left(\Gamma^n_{il,k,m} - \Gamma^n_{il,m,k} \right) + \left(\Gamma^n_{ik,m,l} - \Gamma^n_{ik,l,m} \right) + \left(\Gamma^n_{im,l,k} - \Gamma^n_{im,k,l} \right) = 0 .
$$

3. For $ds^2 = g_{ij} dx^i dx^j$, the Christoffel symbol corresponding to the metric g_{ij} is

$$
\Gamma^i_{kl} = \frac{g^{im}}{2} \left(g_{mk,l} + g_{ml,k} - g_{kl,m} \right) = \frac{1}{2 g_{im}} \left(g_{mk,l} + g_{ml,k} - g_{kl,m} \right),
$$

noticing that $g_{ij} = g_{ji}, g^{ij} = \dfrac{1}{g_{ij}}$.

For $d\bar{s}^2 = \bar{g}_{ij} dx^i dx^j = e^{\sigma} g_{ij} dx^i dx^j$,

$$
\begin{aligned}
\bar{\Gamma}^i_{kl} &= \frac{1}{2 \bar{g}_{im}} \left(\bar{g}_{mk,l} + \bar{g}_{ml,k} - \bar{g}_{kl,m} \right) \\
&= \frac{1}{2 e^{\sigma} g_{im}} \left[e^{\sigma} \left(g_{mk,l} + g_{ml,k} - g_{kl,m} \right) + e^{\sigma} \left(\sigma_{,l} g_{mk} + \sigma_{,k} g_{ml} - \sigma_{,m} g_{kl} \right) \right] \\
&= \Gamma^i_{kl} + \frac{1}{2 g_{im}} \left(\sigma_{,l} g_{mk} + \sigma_{,k} g_{ml} - \sigma_{,m} g_{kl} \right).
\end{aligned}
$$