1. If the space is homogeneity and isotropy, we can take the Robertson-Walker metric, whose spatial component is time-dependent:

$$ds^2 = c^2dt^2 - a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right\},$$

where all of the time dependence is in the scalar factor $a(t)$ and k is a constant representing the curvature of the space.

There are two independent Friedmann equations for modelling a homogeneous, isotropic universe. The first is:

$$\frac{\dot{a}^2 + kc^2}{a^2} = \frac{8\pi G\rho + \Lambda c^2}{3}, \ldots\ldots(1.1)$$

which is derived from the 00 component of Einstein's field equation,

$$R_i^k - \frac{1}{2}\delta_i^k R = \frac{8\pi G}{c^4}T_i^k + \Lambda\delta_i^k.$$

The second is:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3}, \ldots\ldots(1.2)$$

which is derived from the first together with the trace of Einstein's field equations. $H \equiv \frac{\dot{a}}{a}$ is the Hubble parameter. G is Newton's gravitational constant and Λ is the cosmological constant.

The density parameter Ω is defined as the ratio of the actual (or observed) density ρ to the critical density ρ_c of the Friedmann universe. An expression for ρ_c is found by assuming $\Lambda = 0$ (as it is for all basic Friedmann universes) and setting the normalised spatial curvature $k = 0$. When the substitutions are applied to the first of the Friedmann equations we find:

$$\begin{cases} \rho_c = \dfrac{3H^2}{8\pi G}, \\ \Omega \equiv \dfrac{\rho}{\rho_c} = \dfrac{8\pi G\rho}{3H^2}. \end{cases}$$

In terms of the present values of the density parameters, the first Friedmann equation can be written as:

$$\frac{H^2}{H_0^{\ 2}} = \Omega_R a^{-4} + \Omega_M a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda, \ldots\ldots(1.3)$$

where Ω_R is the radiation density today (i.e. when $a = 1$), Ω_M is the matter (dark plus baryonic) density today, $\Omega_k = 1 - \Omega$ is the "spatial curvature density" today, and Ω_Λ is the cosmological constant or vacuum density today.

If the universe contains only radiation but no matter, we can derive from Eq. (1.1) and (1.3) that (setting $c = 1$):

$$a^2\dot{a}^2 = H_0^{\ 2}\Omega_R - ka^2 + \frac{\Lambda a^4}{3}.\ldots\ldots(1.4)$$

For $k = 1$ and $\Lambda = \frac{3}{4H_0^{\ 2}\Omega_R}$, Eq. (1.4) becomes

$$a^2\dot{a}^2 = H_0{}^2\Omega_R - a^2 + \frac{a^4}{4H_0{}^2\Omega_R}.$$

that is,

$$\beta^2\left[\frac{d}{dt}\left(\frac{a^2}{2\beta}-\beta\right)\right]^2 = \left(\frac{a^2}{2\beta}-\beta\right)^2, \beta \equiv H_0\sqrt{\Omega_R}.$$

$$\frac{a^2}{2\beta}-\beta = Ce^{\pm\beta t}.$$

Considering the initial condition that $a|_{t=0}=0$, then $C=-\beta, \frac{a^2}{2\beta}-\beta=-\beta e^{\pm\beta t}$, and

$$t = \pm\frac{1}{\beta}\ln\left(1-\frac{a^2}{2\beta^2}\right), \beta \equiv H_0\sqrt{\Omega_R}.$$

2. Stetting $k=0, c=1$ and assuming that the universe contains only matter and dark energy, then we can get from Eq. (1.1) and (1.3) that:

$$\left(\frac{\dot{a}}{a}\right)^2 = \beta^2 a^{-3} + \frac{\Lambda}{3}, \beta \equiv H_0\sqrt{\Omega_R}.$$

Considering the initial condition $a|_{t=0}=0$, we can solve this equation and obtain:

$$a = \left(\frac{\beta}{2}\sqrt{\frac{3}{\Lambda}}\left(e^{\frac{t\sqrt{3\Lambda}}{2}} - e^{-\frac{t\sqrt{3\Lambda}}{2}}\right)\right)^{2/3},$$

$$t = \frac{2\ln\left(A\Lambda+\sqrt{A^2\Lambda^2+1}\right)}{\sqrt{3\Lambda}}, A = \frac{a^{3/2}}{\sqrt{3}\beta}.$$

Here we consider a as positive. In the FLRW spacetime (expanding universe), if at the present time, t_0, we receive light from a distant object with red shift of z (>0; for Doppler effect blue shift, $z<0$), then the scale factor at the time the object originally emitted that light is $a(t)=\frac{a_0}{1+z}=\frac{1}{1+z}$.

As Λ is infinitesimal, we can have the Taylor series for a and t about the point $\Lambda=0$ to order Λ^2:

$$a = \left(\frac{3\beta t}{2}\right)^{2/3}\left[1+\frac{t^2\Lambda}{12}+\frac{1}{5}\left(\frac{t^2\Lambda}{12}\right)^2\right], \beta \equiv H_0\sqrt{\Omega_R},$$

$$t = \frac{2A}{\sqrt{3}}\left[1-\frac{A}{2}\Lambda^{1/2}+\frac{A^2}{3}\Lambda+\left(\frac{A}{2}-\frac{A^3}{4}\right)\Lambda^{3/2}+\left(-\frac{A^2}{2}+\frac{A^4}{5}\right)\Lambda^{3/2}\right], A = \frac{a^{3/2}}{\sqrt{3}\beta}.$$

From above, we could tell roughly that the matter-dominated relationship $a(t)\propto t^{2/3}$ still holds.

3. $\dot{a}^2 = \frac{\Lambda a^2}{3} \Rightarrow a = Ce^{\pm t\sqrt{\frac{\Lambda}{3}}}$, where C is a constant. Considering the initial condition that $a|_{t=0}=0$, the constant C must be extremely small. (We should check the result in the second Friedmann equation)

4. According to the Birkhoff's theorem and Hubble's law:

$$\begin{cases} \ddot{a}(t) = -\frac{4}{3}\pi G(\rho + 3p)a(t) = -\frac{4}{3}\pi G(\rho + 3p)Ce^{\pm t\sqrt{\frac{\Lambda}{3}}}, \\ \dot{d}(t) = \frac{d(t)\dot{a}(t)}{a(t)} = Hd(t), \end{cases}$$

the acceleration is

$$\ddot{d} = \frac{\dot{d}\dot{a}}{a} + \frac{d\ddot{a}}{a} - \frac{\dot{d}\dot{a}^2}{a^2} = \left(\frac{\dot{a}^2}{a^2} - \frac{4}{3}\pi G(\rho + 3p) - \frac{\dot{a}^3}{a^3}\right)d = \left[\frac{\Lambda}{3} \mp \left(\frac{\Lambda}{3}\right)^{3/2} - \frac{4}{3}\pi G(\rho + 3p)\right]d.$$