1. From frame $\mathcal{O}$ to frame $\mathcal{O}'$, the matrix of moduli is $a_{ij} = \begin{bmatrix} 4 & -13 & 4 & -15 \\ -13 & 14 & -3 & 12 \\ 1 & -4 & 11 & -5 \\ 8 & 14 & -3 & 17 \end{bmatrix}$.

In frame $\mathcal{O}$, $A_{kl} = \begin{bmatrix} 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \end{bmatrix}$. Then in frame $\mathcal{O}'$,

$$A'_{ij} = a_{ik}a_{jl}A_{kl} = aAa^T = \begin{bmatrix} 4 & -13 & 4 & -15 \\ -13 & 14 & -3 & 12 \\ 1 & -4 & 11 & -5 \\ 8 & 14 & -3 & 17 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & -13 & 1 & 8 \\ -13 & 14 & -4 & 14 \\ 4 & -3 & 11 & -3 \\ -15 & 12 & -5 & 17 \end{bmatrix} = \begin{bmatrix} 598 & -788 & 147 & -347 \\ -788 & 882 & -126 & 641 \\ 147 & -126 & 38 & -177 \\ -347 & 614 & -177 & -20 \end{bmatrix}.$$

2. We have known that $dg = \frac{\partial g}{\partial g_{ik}} dg_{ik}$, where the determinant g is made up from the components of the tensor g_{ik}, and the coefficient $\frac{\partial g}{\partial g_{ik}}$ is the corresponding minor of g.

Form $\begin{cases} \frac{1}{4} g_{ik} \frac{\partial g}{\partial g_{ik}} = g, \\ g_{ik}g^{ik} = g_{ik}g^{ki} = \delta_i^i = 4, \end{cases}$ we can get that $g^{ik} = \frac{1}{g}\frac{\partial g}{\partial g_{ik}} \cdots (1)$. Thus, $dg = \frac{\partial g}{\partial g_{ik}} dg_{ik} = g g^{im} dg_{im} \cdots (2)$.

For $g_{ik} = \begin{bmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ 0 & 0 & 0 & g_{33} \end{bmatrix}$, we have $g^{ik} = \frac{1}{g}\frac{\partial g}{\partial g_{ik}} = \frac{1}{\prod_{p=0}^{4} g_{pp}} \frac{\partial \prod_{p=0}^{4} g_{pp}}{\partial g_{ik}}$. That is,

$$g^{ik} = \begin{cases} \frac{1}{g_{ik}}, i = k \\ 0, i \neq k \end{cases} = \begin{bmatrix} 1/g_{00} & 0 & 0 & 0 \\ 0 & 1/g_{11} & 0 & 0 \\ 0 & 0 & 1/g_{22} & 0 \\ 0 & 0 & 0 & 1/g_{33} \end{bmatrix}.$$

Similarly, for $g_{ik} = \begin{bmatrix} g_{00} & g_{01} & 0 & 0 \\ g_{10} & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ 0 & 0 & 0 & g_{33} \end{bmatrix}$, $g = g_{22}g_{33}\left(g_{00}g_{11} - g_{01}^{\ 2}\right)$. Using (1), we have

$$g^{ik} = \begin{bmatrix} g_{11}/\left(g_{00}g_{11} - g_{01}^{\ 2}\right) & -2g_{01}/\left(g_{00}g_{11} - g_{01}^{\ 2}\right) & 0 & 0 \\ -2g_{01}/\left(g_{00}g_{11} - g_{01}^{\ 2}\right) & g_{00}/\left(g_{00}g_{11} - g_{01}^{\ 2}\right) & 0 & 0 \\ 0 & 0 & 1/g_{22} & 0 \\ 0 & 0 & 0 & 1/g_{33} \end{bmatrix}.$$

3. The covariant derivative $A_{\mu;\nu} \equiv A_{\mu,\nu} - \Gamma^{\lambda}_{\mu\nu} A_{\lambda}$ is itself a tensor. Therefore, $A'_{\mu;\nu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} A_{\alpha;\beta}$. On the other hand ,the normal derivative of A'_{μ} is

$$A'_{\mu,\nu} \equiv \frac{\partial A'_{\mu}}{\partial x'^{\nu}} = \left(\frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} A_{\alpha,\beta} + \frac{\partial^2 x^{\alpha}}{\partial x'^{\nu} \partial x'^{\mu}} A_{\alpha} \right).$$

Then

$$\begin{aligned}
& A'_{\mu,\nu} - \Gamma'^{\lambda}_{\mu\nu} A'_{\lambda} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \left(A_{\alpha,\beta} - \Gamma^{\gamma}_{\alpha\beta} A_{\gamma} \right) \\
& \Rightarrow \left(\frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} A_{\alpha,\beta} + \frac{\partial^2 x^{\alpha}}{\partial x'^{\nu} \partial x'^{\mu}} A_{\alpha} \right) - \Gamma'^{\lambda}_{\mu\nu} A'_{\lambda} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} A_{\alpha,\beta} - \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \Gamma^{\gamma}_{\alpha\beta} A_{\gamma} \\
& \Rightarrow \Gamma'^{\lambda}_{\mu\nu} A'_{\lambda} = \frac{\partial^2 x^{\alpha}}{\partial x'^{\nu} \partial x'^{\mu}} A_{\alpha} + \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \Gamma^{\gamma}_{\alpha\beta} A_{\gamma} \\
& \qquad = \frac{\partial^2 x^{\alpha}}{\partial x'^{\nu} \partial x'^{\mu}} \frac{\partial x'^{\lambda}}{\partial x^{\alpha}} A'_{\lambda} + \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \Gamma^{\gamma}_{\alpha\beta} \frac{\partial x'^{\lambda}}{\partial x^{\gamma}} A'_{\lambda}.
\end{aligned}$$

That is

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial^2 x^{\alpha}}{\partial x'^{\nu} \partial x'^{\mu}} \frac{\partial x'^{\lambda}}{\partial x^{\alpha}} + \frac{\partial x'^{\lambda}}{\partial x^{\gamma}} \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \Gamma^{\gamma}_{\alpha\beta}.$$

This is the law of transformation of the Christoffel symbols.

4. We have known that $\begin{cases} \Gamma^{i}_{kl} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^{l}} + \frac{\partial g_{ml}}{\partial x^{k}} - \frac{\partial g_{kl}}{\partial x^{m}} \right), \\ dg = g g^{im} dg_{im} \cdots (2). \end{cases}$ Then

$$\Gamma^{i}_{ki} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^{i}} + \frac{\partial g_{mi}}{\partial x^{k}} - \frac{\partial g_{ki}}{\partial x^{m}} \right).$$

Changing the positions of the indices m and i in the third and first terms in parentheses, we see that these two terms cancel each other, so that

$$\Gamma^{i}_{ik} = \Gamma^{i}_{ki} = \frac{1}{2} g^{im} \frac{\partial g_{mi}}{\partial x^{k}} = \frac{1}{2g} \frac{\partial g}{\partial x^{k}} = \frac{\partial \ln \sqrt{-g}}{\partial x^{k}}.$$