Conformal Transformations in Riemannian Geometry

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Contents

1 Conformal Metric and Connections

$$\tilde{g}_{\mu\nu} \equiv g_{\mu\nu} e^{2w}, \quad \tilde{g}^{\mu\nu} = g^{\mu\nu} e^{-2w}, \tag{1}$$

where w is a scalar and $\tilde{g}_{\mu\nu}\tilde{g}^{\mu\nu} = 4$. The new Christoffel symbols, Riemann tensor, Ricci tensor, Ricci scalar, Einstein tensor and Weyl tensor can be defined at once by replacing all the $g^{\mu\nu}$s with $\tilde{g}^{\mu\nu}$s. However, we would like to do it slightly different here. We first define $\tilde{R}^{\rho}_{\mu\sigma\nu}$, $\tilde{R}_{\mu\nu}$ and $\tilde{\nabla}_\mu$ by replacing the original

$$\Gamma^{\rho}_{\mu\nu} = \frac{g^{\rho\sigma}}{2}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right) \tag{2}$$

in them with a new connection $\tilde{\Gamma}^{\rho}_{\mu\nu}$. Assume that

$$\tilde{\nabla}_\rho \tilde{g}_{\mu\nu} = 0 \quad \text{or} \quad \partial_\rho \tilde{g}_{\mu\nu} = \tilde{\Gamma}^{\lambda}_{\nu\rho}\tilde{g}_{\mu\lambda} + \tilde{\Gamma}^{\lambda}_{\mu\rho}\tilde{g}_{\nu\lambda} \tag{3}$$

still holds. After writing down the similar expressions for $\partial_\mu \tilde{g}_{\nu\sigma}$, $\partial_\nu \tilde{g}_{\sigma\mu}$ and $\partial_\sigma \tilde{g}_{\mu\nu}$, we have

$$\tilde{\Gamma}^{\rho}_{\mu\nu} = \frac{\tilde{g}^{\rho\sigma}}{2}\left(\partial_\mu \tilde{g}_{\nu\sigma} + \partial_\nu \tilde{g}_{\sigma\mu} - \partial_\sigma \tilde{g}_{\mu\nu}\right) = \Gamma^{\rho}_{\mu\nu} + \left(\delta^{\rho}_{\nu}\partial_\mu + \delta^{\rho}_{\mu}\partial_\nu - g_{\mu\nu}\partial^\rho\right) w. \tag{4}$$

Thus $\Delta\tilde{\Gamma}^{\rho}_{\mu\nu} \equiv \tilde{\Gamma}^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\mu\nu} = \left(\delta^{\rho}_{\nu}\nabla_\mu + \delta^{\rho}_{\mu}\nabla_\nu - g_{\mu\nu}\nabla^\rho\right) w$. Under a coordinate transformation,

$$\Delta\tilde{\Gamma}'^{\rho}_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\frac{\partial x'^\rho}{\partial x^\gamma}\Delta\tilde{\Gamma}^{\gamma}_{\alpha\beta}. \tag{5}$$

2 Riemann Tensor, Ricci Tensor and Ricci Scalar

$$
\begin{aligned}
&\Delta\tilde{R}^{\rho}{}_{\sigma\mu\nu} \equiv \tilde{R}^{\rho}{}_{\sigma\mu\nu} - R^{\rho}{}_{\sigma\mu\nu} \\
&= \partial_\mu \tilde{\Gamma}^{\rho}_{\nu\sigma} + \tilde{\Gamma}^{\rho}_{\mu\lambda}\tilde{\Gamma}^{\lambda}_{\nu\sigma} - \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \mu \leftrightarrow \nu \\
&= \partial_\mu \Delta\tilde{\Gamma}^{\rho}_{\nu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Delta\tilde{\Gamma}^{\lambda}_{\nu\sigma} + \Delta\tilde{\Gamma}^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} + \Delta\tilde{\Gamma}^{\rho}_{\mu\lambda}\Delta\tilde{\Gamma}^{\lambda}_{\nu\sigma} - \mu \leftrightarrow \nu \\
&= \partial_\mu \Delta\tilde{\Gamma}^{\rho}_{\nu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Delta\tilde{\Gamma}^{\lambda}_{\nu\sigma} - \Gamma^{\lambda}_{\mu\sigma}\Delta\tilde{\Gamma}^{\rho}_{\nu\lambda} + \Delta\tilde{\Gamma}^{\rho}_{\mu\lambda}\Delta\tilde{\Gamma}^{\lambda}_{\nu\sigma} - \mu \leftrightarrow \nu \\
&= \nabla_\mu \Delta\tilde{\Gamma}^{\rho}_{\nu\sigma} + \Gamma^{\lambda}_{\mu\nu}\Delta\tilde{\Gamma}^{\rho}_{\lambda\sigma} + \Delta\tilde{\Gamma}^{\rho}_{\mu\lambda}\Delta\tilde{\Gamma}^{\lambda}_{\nu\sigma} - \mu \leftrightarrow \nu \\
&= \nabla_\mu \Delta\tilde{\Gamma}^{\rho}_{\nu\sigma} + \Delta\tilde{\Gamma}^{\rho}_{\mu\lambda}\Delta\tilde{\Gamma}^{\lambda}_{\nu\sigma} - \mu \leftrightarrow \nu,
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
\nabla_\mu \Delta\tilde{\Gamma}^{\rho}_{\nu\sigma} &= \nabla_\mu \left(\delta^\rho_\nu \nabla_\sigma + \delta^\rho_\sigma \nabla_\nu - g_{\sigma\nu}\nabla^\rho\right) w \\
&= \left[\delta^\rho_\nu \nabla_\mu\nabla_\sigma + \delta^\rho_\sigma \nabla_\mu\nabla_\nu - g_{\sigma\nu}\nabla_\mu\nabla^\rho - \left(\nabla_\mu g_{\sigma\nu}\right)\nabla^\rho\right] w \\
&= \left(\delta^\rho_\nu \nabla_\mu\nabla_\sigma + \delta^\rho_\sigma \nabla_\mu\nabla_\nu - g_{\sigma\nu}\nabla_\mu\nabla^\rho\right) w,
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
&\Delta\tilde{\Gamma}^{\rho}_{\mu\lambda}\Delta\tilde{\Gamma}^{\lambda}_{\nu\sigma} = \left(\delta^\rho_\lambda \nabla_\mu + \delta^\rho_\mu \nabla_\lambda - g_{\mu\lambda}\nabla^\rho\right) w \cdot \left(\delta^\lambda_\nu \nabla_\sigma + \delta^\lambda_\sigma \nabla_\nu - g_{\sigma\nu}\nabla^\lambda\right) w \\
&= \delta^\rho_\lambda \left(\nabla_\mu w\right)\delta^\lambda_\nu \left(\nabla_\sigma w\right) + \delta^\rho_\lambda \left(\nabla_\mu w\right)\delta^\lambda_\sigma \left(\nabla_\nu w\right) - \delta^\rho_\lambda \left(\nabla_\mu w\right) g_{\sigma\nu}\left(\nabla^\lambda w\right) \\
&\quad + \delta^\rho_\mu \left(\nabla_\lambda w\right)\delta^\lambda_\nu \left(\nabla_\sigma w\right) + \delta^\rho_\mu \left(\nabla_\lambda w\right)\delta^\lambda_\sigma \left(\nabla_\nu w\right) - \delta^\rho_\mu \left(\nabla_\lambda w\right) g_{\sigma\nu}\left(\nabla^\lambda w\right) \\
&\quad - g_{\mu\lambda}\left(\nabla^\rho w\right)\delta^\lambda_\nu \left(\nabla_\sigma w\right) - g_{\mu\lambda}\left(\nabla^\rho w\right)\delta^\lambda_\sigma \left(\nabla_\nu w\right) + g_{\mu\lambda}\left(\nabla^\rho w\right) g_{\sigma\nu}\left(\nabla^\lambda w\right) \\
&= \delta^\rho_\nu \nabla_\mu w \nabla_\sigma w + \delta^\rho_\sigma \nabla_\mu w \nabla_\nu w - g_{\sigma\nu}\nabla_\mu w \nabla^\rho w + \delta^\rho_\mu \nabla_\nu w \nabla_\sigma w + \delta^\rho_\mu \nabla_\sigma w \nabla_\nu w - \delta^\rho_\mu g_{\sigma\nu}\left(\nabla w\right)^2 \\
&\quad - g_{\mu\nu}\nabla^\rho w \nabla_\sigma w - g_{\mu\sigma}\nabla^\rho w \nabla_\nu w + g_{\sigma\nu}\nabla^\rho w \nabla_\mu w \\
&= \delta^\rho_\nu \nabla_\mu w \nabla_\sigma w + \delta^\rho_\sigma \nabla_\mu w \nabla_\nu w + 2\delta^\rho_\mu \nabla_\nu w \nabla_\sigma w - \delta^\rho_\mu g_{\sigma\nu}\left(\nabla w\right)^2 - g_{\mu\nu}\nabla^\rho w \nabla_\sigma w - g_{\mu\sigma}\nabla^\rho w \nabla_\nu w;
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
\Delta\tilde{R}^{\rho}{}_{\sigma\mu\nu} &= \left(\delta^\rho_\nu \nabla_\mu\nabla_\sigma - \delta^\rho_\mu \nabla_\nu\nabla_\sigma - g_{\sigma\nu}\nabla_\mu\nabla^\rho + g_{\sigma\mu}\nabla_\nu\nabla^\rho\right) w + \delta^\rho_\mu \nabla_\nu w \nabla_\sigma w - \delta^\rho_\nu \nabla_\mu w \nabla_\sigma w \\
&\quad - \left(\delta^\rho_\mu g_{\sigma\nu} - \delta^\rho_\nu g_{\sigma\mu}\right)\left(\nabla w\right)^2 - g_{\mu\sigma}\nabla_\nu w \nabla^\rho w + g_{\nu\sigma}\nabla_\mu w \nabla^\rho w.
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
&\Delta\tilde{R}_{\rho\sigma\mu\nu} = \tilde{g}_{\rho\lambda}\Delta\tilde{R}^{\lambda}{}_{\sigma\mu\nu} \\
&= \Big[\left(g_{\rho\nu}\nabla_\mu\nabla_\sigma - g_{\rho\mu}\nabla_\nu\nabla_\sigma - g_{\sigma\nu}\nabla_\mu\nabla_\rho + g_{\sigma\mu}\nabla_\nu\nabla_\rho\right) w + g_{\rho\mu}\nabla_\nu w \nabla_\sigma w - g_{\rho\nu}\nabla_\mu w \nabla_\sigma w \\
&\quad - \left(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}\right)\left(\nabla w\right)^2 - g_{\mu\sigma}\nabla_\nu w \nabla_\rho w + g_{\nu\sigma}\nabla_\mu w \nabla_\rho w\Big] e^{2w} \\
&= \Big[-\left(g_{\rho\mu}\nabla_\nu\nabla_\sigma + g_{\sigma\nu}\nabla_\mu\nabla_\rho\right) w + g_{\rho\mu}\nabla_\nu w \nabla_\sigma w + g_{\nu\sigma}\nabla_\mu w \nabla_\rho w - g_{\rho\mu}g_{\sigma\nu}\left(\nabla w\right)^2 - \mu \leftrightarrow \nu\Big] e^{2w}.
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
&\Delta\tilde{R}_{\mu\nu} = \Delta\tilde{R}^{\rho}{}_{\mu\rho\nu} = \delta^\sigma_\rho \Delta\tilde{R}^{\rho}{}_{\mu\sigma\nu} \\
&= \delta^\sigma_\rho \Big[\left(\delta^\rho_\nu \nabla_\sigma\nabla_\mu - \delta^\rho_\sigma \nabla_\nu\nabla_\mu - g_{\mu\nu}\nabla_\sigma\nabla^\rho + g_{\mu\sigma}\nabla_\nu\nabla^\rho\right) w + \delta^\rho_\sigma \nabla_\nu w \nabla_\mu w - \delta^\rho_\nu \nabla_\sigma w \nabla_\mu w \\
&\quad - \left(\delta^\rho_\sigma g_{\mu\nu} - \delta^\rho_\nu g_{\mu\sigma}\right)\left(\nabla w\right)^2 - g_{\sigma\mu}\nabla_\nu w \nabla^\rho w + g_{\nu\mu}\nabla_\sigma w \nabla^\rho w\Big] \\
&= \left(\nabla_\nu\nabla_\mu - 4\nabla_\nu\nabla_\mu - g_{\mu\nu}\Box + \nabla_\nu\nabla_\mu\right) w + 4\nabla_\nu w \nabla_\mu w - \nabla_\nu w \nabla_\mu w - \left(4g_{\mu\nu} - g_{\mu\nu}\right)\left(\nabla w\right)^2 \\
&\quad - \nabla_\nu w \nabla_\mu w + g_{\nu\mu}\left(\nabla w\right)^2 \\
&= -\left(2\nabla_\nu\nabla_\mu + g_{\mu\nu}\Box\right) w + 2\nabla_\mu w \nabla_\nu w - 2g_{\mu\nu}\left(\nabla w\right)^2 \\
&= -\left(2\nabla_\mu\nabla_\nu + g_{\mu\nu}\Box\right) w + 2\nabla_\mu w \nabla_\nu w - 2g_{\mu\nu}\left(\nabla w\right)^2.
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
\Delta\tilde{R} \equiv \tilde{R} - R &= \tilde{g}^{\mu\nu}\Delta\tilde{R}_{\mu\nu} + (\tilde{g}^{\mu\nu} - g^{\mu\nu})\, R_{\mu\nu} \\
&= \left[-(2\Box + 4\Box)\, w + 2\,(\nabla w)^2 - 8\,(\nabla w)^2\right] e^{-2w} + \left(e^{-2w} - 1\right) R \\
&= -6\left[\Box w + (\nabla w)^2\right] e^{-2w} + \left(e^{-2w} - 1\right) R.
\end{aligned} \tag{12}
$$

3 Einstein Tensor

$$
\begin{aligned}
\Delta\tilde{G}_{\mu\nu} \equiv \tilde{G}_{\mu\nu} - G_{\mu\nu} &= -\left(2\nabla_\mu\nabla_\nu + g_{\mu\nu}\Box\right) w + 2\nabla_\mu w\nabla_\nu w - 2g_{\mu\nu}\,(\nabla w)^2 + 3g_{\mu\nu}\left[\Box w + (\nabla w)^2\right] \\
&= 2\left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right) w + 2\nabla_\mu w\nabla_\nu w + g_{\mu\nu}\,(\nabla w)^2 .
\end{aligned} \tag{13}
$$

When $w \to 0$, $\Delta\tilde{G}_{\mu\nu} = 2\left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right) w$.

4 Weyl Tensor

For

$$
C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{2}\left(g_{\mu\rho}R_{\nu\sigma} + R_{\mu\rho}g_{\nu\sigma} - \frac{g_{\mu\rho}g_{\nu\sigma}R}{3} - \mu \leftrightarrow \nu\right), \tag{14}
$$

one can easily check that $\Delta\tilde{C}^{\mu}_{\ \nu\rho\sigma} \equiv \tilde{C}^{\mu}_{\ \nu\rho\sigma} - C^{\mu}_{\ \nu\rho\sigma} = 0$ or

$$
\Delta\tilde{C}_{\mu\nu\rho\sigma} = \tilde{C}_{\mu\nu\rho\sigma} - C_{\mu\nu\rho\sigma}e^{2w} = \Delta\tilde{R}_{\mu\nu\rho\sigma} - \frac{1}{2}\left(\tilde{g}_{\mu\rho}\Delta\tilde{R}_{\nu\sigma} + \Delta\tilde{R}_{\mu\rho}\tilde{g}_{\nu\sigma} - \frac{\tilde{g}_{\mu\rho}\tilde{g}_{\nu\sigma}\Delta\tilde{R}}{3} - \mu \leftrightarrow \nu\right) = 0, \tag{15}
$$

with all properties above.