IX

1. According to the virial theorem, for a stable system consisting of N particles, bound by potential forces, with that of the total potential energy, $\langle \Phi_{TOT} \rangle$, the average over time of the total kinetic energy, $\langle T \rangle = -\frac{1}{2}\sum_{k=1}^{N}\langle \mathbf{F}_k \bullet \mathbf{r}_k \rangle$, where angle brackets represent the average over time of the enclosed quantity; $\langle \mathbf{F}_k \rangle$ represents the force on the *k*th particle, which is located at position $\langle \mathbf{r}_k \rangle$.

If the force between any two particles of the system results from a potential energy $\Phi(r) = \alpha r^n$, where r is the inter-particle distance, the virial theorem takes the simple form $2\langle T \rangle = n\langle \Phi_{TOT} \rangle$.

Now we have $\Phi = -\frac{3GM^2}{5R}$, then

$$\begin{cases} \Phi + 2T = 0, \\ T = \dfrac{3GM^2}{10R} = \dfrac{M}{2}V^2 \ ...(1.1), \end{cases}$$

where we have use the fact that $T = \frac{1}{2}\sum_{k=1}^{N} m{v_k}^2 = \frac{m}{2}\sum_{k=1}^{N}{v_k}^2 = \frac{M}{2N}\sum_{k=1}^{N}{v_k}^2$, and the definition that $V^2 = \frac{1}{N}\sum_{k=1}^{N}{v_k}^2$. As the question is 3-dimensional, then for isotropy, the dispersion of the radial velocity of the stars must satisfies that:

$$\sigma_r^{\ 2} = \frac{V^2}{3} \ ...(1.2).$$

Using (1.1) and (1.2), we get $\sigma_r^{\ 2} = \frac{GM}{5R}$.

2.
$$\begin{cases} \sigma_r^{\ 2} = \dfrac{GM}{5R}, \\ \dfrac{L}{L_*} \approx (a\sigma)^4 \\ L_* = 1.0\times 10^{10} h^{-2} L_{\odot}, a = \dfrac{km^{-1}\bullet s}{220} \end{cases} \Rightarrow L \approx \frac{a^4 L_* G^2 M^2}{25R^2}.$$

Besides, we have known that $M = \frac{4\pi R^3 \bar{\rho}}{3}, R = \left(\frac{3M}{4\pi\bar{\rho}}\right)^{1/3}$. then

$$L \approx \frac{a^4 L_* G^2 M^2}{25R^2} = \frac{a^4 L_* G^2}{25}\left(\frac{4\pi\bar{\rho} M^2}{3}\right)^{2/3} = 0.44\bar{\rho}^{2/3} M^{4/3} L_{\odot}.$$

3. The speed of sound is $v_s = \sqrt{\frac{\gamma P}{\rho}}$, where $\gamma = \frac{C_p}{C_V}$ and ρ are constants; $\frac{dP}{dr} =$
$-\frac{GM(r)\rho}{r^2} = -\frac{4\pi G r^3 \rho^2}{3r^2} = -\frac{4\pi G r \rho^2}{3}$.

Thus,

$$\int_{P_c}^{P(r)} dp = P(r) - P_c = \int_0^r -\frac{4\pi G r \rho^2}{3} dr = -\frac{2\pi G r^2 \rho^2}{3},$$

where $P_c = P(0)$. If $P(R) = P_c - \frac{2\pi G R^2 \rho^2}{3} = 0$, then

$$\begin{cases} P(r) = P_c - \frac{2\pi G r^2 \rho^2}{3} = \frac{2\pi G\left(R^2 - r^2\right)\rho^2}{3}, \\ v_s(r) = \sqrt{\frac{\gamma P(r)}{\rho}} = \sqrt{\frac{2\pi\gamma G\left(R^2 - r^2\right)\rho}{3}}. \end{cases}$$

Let's take $k = 1/\sqrt{\frac{2\pi\gamma G\rho}{3}} = \text{constant}$. Then

$$\begin{cases} v_s(r) = \frac{dr}{dt} = \frac{1}{k}\sqrt{\left(R^2 - r^2\right)}, \\ dt = \frac{dr}{v_s}, \\ \Pi = \int dt = \left|\int_0^R \frac{dr}{v_s}\right| + \left|\int_R^0 \frac{dr}{v_s}\right| = 2\int_0^R \frac{dr}{v_s} = 2k\int_0^R \frac{dr}{\sqrt{\left(R^2 - r^2\right)}} = 2k\left(\arcsin\frac{r}{R}\right)\Big|_0^R, \end{cases}$$

$$\Pi = 2k \arcsin 1 = k\pi = \sqrt{\frac{3\pi}{2\gamma G\rho}}.$$

X

1. As $z = \lambda/\lambda_0 - 1 = \sqrt{1 + \frac{v}{c}} / \sqrt{1 - \frac{v}{c}} - 1$, then

a) $z = \sqrt{1 + \frac{v}{c}} / \sqrt{1 - \frac{v}{c}} - 1 \Rightarrow (z+1)^2 = \frac{1 + \frac{v}{c}}{1 - \frac{v}{c}} \Rightarrow v = \frac{z(z+2)c}{z^2 + 2z + z}$.

b) $v = \frac{z(z+2)c}{z^2 + 2z + 2} = \frac{c}{1 + \frac{2}{z(z+2)}} \le c, z \ge 0$.

c) When $z \to 0$, $v = \frac{z(z+2)c}{z^2+2z+2} = \left(1 - \frac{1}{\frac{z^2+2z}{2}+1}\right)c \approx \left[1 - \left(1 - \frac{z^2+2z}{2}\right)\right]c = \frac{z^2+2z}{2}c \approx zc.$

2.

a) From the figure, we get $\lambda = 5700 \times 10^{-10} m = 5.7 \times 10^{-7} m$. From the equation that:

$$\frac{1}{\lambda_0} = R\left(\frac{1}{2^2} - \frac{1}{n^2}\right), n = 4,$$

we get $\lambda_0 = \frac{16}{3R} = \frac{16}{3 \times 1.097} \times 10^{-7} m$. Then the redshift $z = \lambda / \lambda_0 - 1 = 0.172$.

b) As $\lambda / \lambda_0 = \sqrt{1 + \frac{v}{c}} / \sqrt{1 - \frac{v}{c}} = 1.1724 \Rightarrow v = 0.158c$.

c) According to Hubble's law, the distance to 3C 273 is

$$d = \frac{v}{H_0} = \frac{0.158c}{100h\ km \bullet s^{-1} \bullet Mpc^{-1}},$$

where h is a dimensionless parameter falling somewhere between 0.5 and 1. If we take $h = 0.7$, then $d = 2.2$ bilion light-year.