

Wigner's Theorem

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A proof based on *The quantum theory of fields, vol. 1: Foundations* by Weinberg, S. (1995).

QM Assumptions

- (1) Physical states are vectors/rays \in **Hilbert space** \mathcal{H} —some complex vector space with a norm s.t. (i) $(\Phi, \Psi) = (\Psi, \Phi)^*$
 Φ, Ψ , etc. $\xi\Phi + \eta\Psi \in \mathcal{H}, \forall \xi, \eta \in \mathbb{C}$ $(\Phi, \Psi) \in \mathbb{C}$
- (ii) $(\xi_1\Phi_1 + \xi_2\Phi_2, \eta\Psi) = \eta[\xi_1^*(\Phi_1, \Psi) + \xi_2^*(\Phi_2, \Psi)]$ (iii) $(\Psi \neq 0, \Psi) > (0, 0) = 0$.
- (2) Observables are Hermitian operators s.t. $A(\xi\Phi + \eta\Psi) = \xi A\Phi + \eta A\Psi$.
 $A^\dagger = A$ an automorphism $\text{Aut}(\mathcal{H})$

(3) A ray $\mathcal{R}|_{\ni\Psi}$ has a definite eigenvalue $\alpha \forall$ operator A .¹ Testing a system in \mathcal{R} brings itself into one of the orthogonal rays $\mathcal{R}_{n=1,2,\dots}|_{\ni\Psi_n}$ with transition probability $P(\mathcal{R} \rightarrow \mathcal{R}_n) := |(\Psi, \Psi_n)|^2 = P(\mathcal{R}_n \rightarrow \mathcal{R})$, s.t. $\sum_n P(\mathcal{R} \rightarrow \mathcal{R}_n) = 1$ for a complete set $\{\Psi_n\}$.

NB (i) Pick a normalised vector Ψ as (a representative of) a ray $\in \mathcal{H}$ (ii) define the **adjoint** A^\dagger of a linear operator A as $(\Phi, A^\dagger\Psi) := (A\Phi, \Psi)$, or $(\Phi, A^\dagger\Psi) := (A\Phi, \Psi)^*$ for an antilinear operator A .
 $(\Psi, A\Phi)^*$ $(\Psi, A\Phi)$

Wigner's Theorem An invertible transition-probability-preserving ray-transformation T s.t.
symmetry

$$\underbrace{P\left(\mathcal{R}'\right)\Big|_{\ni\Psi'=U\Psi}}_{|(\Psi', \Psi'_n)|^2} = \underbrace{TR \leftrightarrow \mathcal{R}'_n\Big|_{\ni\Psi'_n=U\Psi_n}}_{= TR_n} = \underbrace{P(\mathcal{R} \leftrightarrow \mathcal{R}_n)}_{|(\Psi, \Psi_n)|^2} \quad (1)$$

\Rightarrow a(n) (anti)unitary & (anti)linear operator U on \mathcal{H} s.t.

$$\begin{cases} (U\Phi, U\Psi) = (\Phi, \Psi) \Leftrightarrow U^\dagger U = 1 \\ U(\xi\Phi + \eta\Psi) = \xi U\Phi + \eta U\Psi \end{cases} \quad \text{or} \quad \begin{cases} (U\Phi, U\Psi) = (\Phi, \Psi)^* \Leftrightarrow U^\dagger U = 1 \\ U(\xi\Phi + \eta\Psi) = \xi^* U\Phi + \eta^* U\Psi \end{cases} \quad (2)$$

Pf. A complete orthogonal set $\{\Psi_n\}$ $\xrightarrow[\text{the expansions}]{\begin{matrix} (\Psi_{n_1}, \Psi_{n_2}) = \delta_{n_1, n_2} \\ |(\Psi'_{n_1}, \Psi'_{n_2})|^2 \stackrel{(i)}{=} \delta_{n_1, n_2} \stackrel{(\Psi'_{n_1}, \Psi'_{n_2}) \geq 0}{\Rightarrow} (\Psi'_{n_1}, \Psi'_{n_2}) = \delta_{n_1, n_2} \end{matrix}}} \text{another complete orthogonal set } \{U\Psi_n\}^2 \Rightarrow$

$$\Psi\Big|_{\in\mathcal{R}} = \sum_{n=1}^N C_n \Psi_n \quad \& \quad (U\Psi)\Big|_{\in T\mathcal{R}} = \sum_{n=1}^N C'_n U\Psi_n \quad (3)$$

for a given Ψ & its counterpart $U\Psi$. To carry on, we must decide the *relative phases* for the transformed basis $\{U\Psi_n\}$:

$$\left|\left(\sum_{i=1}^{I=1,\dots,N} \Psi_{n_i}, \Psi\right)\right|^2 \stackrel{(i)}{=} \left|\left(U \sum_{i=1}^{I=1,\dots,N} \Psi_{n_i}, U\Psi\right)\right|^2 \xrightarrow[\text{(3)}]{\Psi = \Psi_{n_i=1,\dots,I}} \left|\sum_{i=1}^{I=1,\dots,N} \Psi_{n_i}\right|^2 = \left|\sum_{i=1}^{I=1,\dots,N} e^{i\theta_i} U\Psi_{n_i}\right|^2 \quad (4)$$

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¹ An order- N Hermitian matrix has N orthogonal eigenvectors with distinct *real* eigenvalues.

² Given a non-zero $\Psi' \notin \{\Psi_n\}$ but \perp all Ψ_n , the 1-1 inverse map will take it back to some non-zero $\Psi'' \notin \{\Psi_n\}$ s.t. $|(\Psi_n, \Psi'')|^2 = |(\Psi'_n, \Psi')|^2 = 0 \Rightarrow$ impossible, as $\{\Psi_n\}$ is already complete.

& our *convention* is $\theta_{i=1,\dots,I} \equiv 0 \forall I \in \mathbb{N}$. Now we can investigate the relation between the two sets of expansion coefficients, $\{C_n\}$ & $\{C'_n\}$. First, $|(\Psi_n, \Psi)|^2 \stackrel{(1)}{=} |(U\Psi_n, U\Psi)|^2 \stackrel{(3)}{\Rightarrow}$

$$\left| \frac{C'_n}{C_n} \right| = 1. \quad (5)$$

$$\begin{aligned} \text{Then } |(\sum_{i=1}^2 \Psi_{n_i}, \Psi)|^2 &\stackrel{(1)}{=} \left| \left(U \sum_{i=1}^2 \Psi_{n_i} \xrightarrow{\text{phase convention}} \sum_{i=1}^2 U\Psi_{n_i}, U\Psi \right) \right|^2 \stackrel{(3)}{\Rightarrow} \\ &\left| \frac{\sum_{i=1}^2 C'_{n_i}}{\sum_{i=1}^2 C_{n_i}} \right| = 1 \stackrel{(5)}{\Rightarrow} \frac{|1 + C'_{n_2}/C'_{n_1}|^2 - 1 - |C'_{n_2}/C'_{n_1}|^2}{|1 + C_{n_2}/C_{n_1}|^2 - 1 - |C_{n_2}/C_{n_1}|^2} = 1 \xrightarrow{(5)} \frac{|C'_{n_2}/C'_{n_1}| = |C_{n_2}/C_{n_1}|}{\text{Re}(C'_{n_2}/C'_{n_1})/\text{Re}(C_{n_2}/C_{n_1})} \frac{C_{n_2}}{C_{n_1}} = \frac{C'_{n_2}}{C'_{n_1}} \text{ or } \left(\frac{C'_{n_2}}{C'_{n_1}} \right)^*. \end{aligned} \quad (6)$$

Next, think of $\left| \left(\sum_{i=1}^{N \geq 3} \Psi_{n_i}, \Psi \right) \right|^2 \stackrel{(1)}{=} \left| \left(U \sum_{i=1}^{N \geq 3} \Psi_{n_i} \xrightarrow{\text{phase convention}} \sum_{i=1}^{N \geq 3} U\Psi_{n_i}, U\Psi \right) \right|^2$, with $\{C_n\}$ & $\{C'_n\}$ satisfying eq (6) in the way that $C_{n_2}/C_{n_1} \equiv C'_{n_2}/C'_{n_1}$ or $\equiv (C'_{n_2}/C'_{n_1})^*$. To prove this ' \equiv ', let us single out some C_{n_1} , & assume that $C_{n_i}/C_{n_1} = C'_{n_i}/C'_{n_1} \forall i \in \{2, \dots, M < N\}$ & $C_{n_i}/C_{n_1} = (C'_{n_i}/C'_{n_1})^* \forall i \in \{M+1, \dots, N\}$. Then, from eqs (5) & (6),

$$\begin{aligned} 0 &= \left| 1 + \sum_{i=2}^N \frac{C'_{n_i}}{C'_{n_1}} \right|^2 - \left| 1 + \sum_{i=2}^N \frac{C_{n_i}}{C_{n_1}} \right|^2 = \sum_{i,j=2}^N \left(\frac{C'_{n_i} C'^*_{n_j} - C_{n_i} C^*_{n_j}}{|C_{n_1}|^2} + i \leftrightarrow j \right) \\ &= \sum_{i=2}^M \sum_{j=M+1}^N \left[-4 \left(\text{Im} \frac{C_{n_i}}{C_{n_1}} \right) \text{Im} \frac{C_{n_j}}{C_{n_1}} \right] = (-4)^{(M-1)(N-M)} \underbrace{\left(\sum_{i=2}^M \text{Im} \frac{C_{n_i}}{C_{n_1}} \right)}_x \underbrace{\left(\sum_{j=M+1}^N \text{Im} \frac{C_{n_j}}{C_{n_1}} \right)}_y \end{aligned} \quad (7)$$

\Rightarrow either x or y must $\in \mathbb{R}$, which is an unreasonable stronger statement (transcendental coef-constraint). In brief, \forall symmetric transformation T , the corresponding U satisfies

$$U \sum_n C_n \Psi_n = \sum_n C_n U \Psi_n \quad \text{or} \quad \sum_n C_n^* U \Psi_n. \quad (8)$$

Our final step is to prove that the '=' in eq (8) is actually ' \equiv ', which leads directly to the property (2). In fact, the mixed unitary-antiunitary case with a stronger constraint

$$\left\{ \begin{array}{l} \underbrace{U \sum_n A_n \Psi_n}_{\Psi} = \underbrace{\sum_n A_n U \Psi_n}_{\Psi'} \\ \underbrace{U \sum_n B_n \Psi_n}_{\Phi} = \underbrace{\sum_n B_n^* U \Psi_n}_{\Phi'} \end{array} \right. \xrightarrow[(1)]{|(\Psi, \Phi)|^2 = |(\Psi', \Phi')|^2} \underbrace{\sum_{n_1, n_2} (\text{Im} A_{n_1}^* A_{n_2}) \text{Im} B_{n_1}^* B_{n_2}}_{(|\sum_n A_n B_n^*|^2 - |\sum_n A_n B_n|^2)/\prod_{n_1, n_2} (-4)} = 0 \quad (9)$$

is again unreasonable, as the probability preservation (1) is automatically satisfied with the overall (anti)unitary condition ' \equiv '. \square