1. For a static spherically symmetric star in equilibrium,

$$ds^2 = e^{\nu(r)}c^2dt^2 - e^{\lambda(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

Denoting by x^0, x^1, x^2, x^3, respectively, the coordinates ct, r, θ, φ, we have for the nonzero Christoffel symbols the expressions:

$$\Gamma^0_{10} = \frac{\nu'}{2}, \Gamma^1_{11} = \frac{\lambda'}{2}, \Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r}, \Gamma^3_{23} = \cot\theta,$$
$$\Gamma^1_{00} = \frac{\nu'}{2}e^{\nu-\lambda}, \Gamma^1_{22} = -re^{-\lambda}, \Gamma^1_{33} = -r\sin^2\theta e^{-\lambda}, \quad ...(1.1)$$
$$\Gamma^2_{33} = -\sin\theta\cos\theta,$$

By the perfect fluid assumption, the energy-momentum tensor is diagonal:

$$T_i^k = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix}, ...(1.2)$$

Substituting (1.1) and (1.2) in $T^k_{i;k} = 0$, we have

$$\begin{aligned} T^k_{i;k} &= T^k_{i,k} - \Gamma^m_{ik}T^k_m + \Gamma^k_{mk}T^m_i \\ &= T^1_{1,1} - \left[T^0_0\Gamma^0_{10} + T^1_1\Gamma^1_{11} + T^2_2\Gamma^2_{12} + T^3_3\Gamma^3_{13}\right] + T^1_1\left(\Gamma^0_{10} + \Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13}\right) \\ &= T^1_{1,1} - T^0_0\Gamma^0_{10} + p\left[\Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13} - \left(\Gamma^0_{10} + \Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13}\right)\right] \\ &= T^1_{1,1} - T^0_0\Gamma^0_{10} + p\left(-\Gamma^0_{10}\right) \\ &= -p' - \left(\rho c^2 + p\right)\frac{\nu'}{2} = 0. \end{aligned}$$

That is, $\nu' = -\dfrac{2p'}{\rho c^2 + p}$ ....(1.3)

2. Following our assumption in question 1, we get:

$$\frac{8\pi G}{c^4}T^0_0 = \frac{e^{-\lambda}\lambda'}{r} + \frac{1-e^{-\lambda}}{r^2} = \frac{8\pi G\rho(r)}{c^2} \text{....(2.1)}$$
$$\frac{8\pi G}{c^4}T^1_1 = -\frac{e^{-\lambda}\nu'}{r} + \frac{1-e^{-\lambda}}{r^2} = -\frac{8\pi Gp(r)}{c^4} \text{....(2.2)}$$
$$\frac{8\pi G}{c^4}T^2_2 = \frac{8\pi G}{c^4}T^3_3 = -\frac{e^{-\lambda}}{2}\left(\nu'' + \frac{\nu'^2}{2} + \frac{\nu' - \lambda'}{r} - \frac{\nu'\lambda'}{2}\right) = -\frac{8\pi Gp(r)}{c^4} \text{....(2.3)}$$

(1) From $1 - \frac{1}{r}\int_0^r (2.1) r^2\, dr$ we get:

$$e^{-\lambda} = 1 - \frac{1}{r}\int_0^r \rho r^2\, dr = 1 - \frac{2GM(r)}{c^2 r} > 0 \text{....(2.4)}$$

(2) Substituting (2.4) in (2.2):

$$\nu' = \frac{2G}{c^2r^2}\left(1-\frac{2GM}{c^2r}\right)^{-1}\left(M+\frac{4\pi r^3 p}{c^2}\right)....(2.5)$$

(3) Eliminating ν' in (1.3) and (2.5), we obtain:

$$p' = -\frac{G}{r^2}\left(\rho+\frac{p}{c^2}\right)\left(1-\frac{2GM}{c^2r}\right)^{-1}\left(M+\frac{4\pi r^3 p}{c^2}\right)....(2.6)$$

That is so-called the TOV equation.

3. We take $r = a\eta,\ \rho = \rho_c\varepsilon(\eta),\ p = \rho_c c^2 P(\eta), M = M^* m(\eta),$ where a and M^* are constants, then

$$M = M^* m = 4\pi\int_0^r \rho r^2\, dr = 4\pi\int_0^{a\eta}\rho_c\varepsilon\bullet(a\eta)^2\, a\, d\eta = 4\pi\rho_c a^3\int_0^{a\eta}\eta^2\varepsilon\, d\eta.$$

Comparing both sides of the equation $M^* m(\eta) = 4\pi\rho_c a^3\int_0^{a\eta}\eta^2\varepsilon(\eta)d\eta,$ we could know that $M^* = 4\pi\rho_c a^3$. Substituting these in (2.6), we get:

$$\frac{\rho_c c^2}{a}\frac{dP}{d\eta} = -\frac{G}{a^2\eta^2}(\rho_c\varepsilon+\rho_c P)\left(1-\frac{2GM^* m}{c^2 a\eta}\right)^{-1}\left[M^* m + 4\pi(a\eta)^3\rho_c P\right],$$

that is,

$$2\eta\frac{dP}{d\eta}\left(m-\frac{c^2\eta}{8\pi G\rho_c a^2}\right) = (\varepsilon+P)\left(m+\eta^3 P\right)....(3.1).$$

Comparing both sides of (3.1), we could find that:

$$\begin{cases} 2\eta\dfrac{dP}{d\eta} = (\varepsilon+P), \\ -\dfrac{c^2\eta}{8\pi G\rho_c a^2} = \eta^3 P, \end{cases}$$

and $a = \dfrac{c}{2\eta\sqrt{-2\pi G\rho_c P}}$.

4. Assuming that $\rho = \dfrac{\rho_0}{r^2},\ p = kc^2\rho,$ where ρ_0 and $k\left(\in R^+\right)$ are constants, we then have $M(r) = 4\pi\int_0^r \rho r^2\, dr = 4\pi\int_0^r \dfrac{\rho_0}{r^2}\bullet r^2\, dr = 4\pi\rho_0 r.$ Substituting these in (2.6), we get:

$$kc^2\rho' = -\frac{G}{r^2}(1+k)\rho\left(1-\frac{2GM}{c^2r}\right)^{-1}\left(M+4\pi r^3 k\rho\right),$$

$$-\frac{2kc^2\rho_0}{r^3} = -\frac{G\rho_0}{r^4}(1+k)\left(1-\frac{8\pi G\rho_0 r}{c^2r}\right)^{-1}\left(4\pi\rho_0 r+\frac{4\pi r^3 k\rho_0}{r^2}\right),$$

$$k^2+\left(6-\frac{c^2}{2\pi G\rho_0}\right)k+1=0,$$

$$k=\frac{c^2}{4\pi G\rho_0}-3\pm\sqrt{\left(\frac{c^2}{4\pi G\rho_0}-2\right)\left(\frac{c^2}{4\pi G\rho_0}-4\right)}.$$