Deriving the Lagrange Equation from the Principle of the Least Action

The **action**, denoted by S, of a physical system, is a **functional** of $x(t)$:

$S=\int_{t_1}^{t_2} L(x,\dot{x})dt.$

Mathematically, the **Principle of the Least Action** is $\delta S=0$, where δ means a *infinitesimal* change of S.

We now look into a possible path near the actual path $x(t)$:

$$\tilde{x}(t)=x(t)+\delta x(t).$$

The two paths above take the same positions at t_1 and t_2. Hence,

$$\tilde{S}-S=\int_{t_1}^{t_2}(L(\tilde{x},\dot{\tilde{x}};t)-L(x,\dot{x};t))dt, \qquad (1)$$

$$\delta x(t_1)=\delta x(t_2)=0. \qquad (2)$$

As the difference δx between $\tilde{x}(t)$ and $x(t)$ is infinitesimal, we can do the **Taylor expansion** as follows:

$$\begin{aligned} L(\tilde{x},\dot{\tilde{x}};t) &= L(x,\dot{x};t)+(\frac{\partial L}{\partial x}\delta x+\frac{\partial L}{\partial \dot{x}}\delta \dot{x})+O(\delta^2) \\ &= L(x,\dot{x};t)+\left[\frac{\partial L}{\partial x}\delta x+\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}\delta x)-\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}})\delta x\right]+O(\delta^2). \end{aligned} \qquad (3)$$

With Eq. (1), (2) and (3), we could have:

$$\begin{aligned} \tilde{S}-S=\delta S &= \int_{t_1}^{t_2}\left[\frac{\partial L}{\partial x}-\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}})\right]\delta x\,dt+(\frac{\partial L}{\partial \dot{x}}\delta x)\Big|_{t_1}^{t_2}+O(\delta^2) \\ &= \int_{t_1}^{t_2}\left[\frac{\partial L}{\partial x}-\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}})\right]\delta x\,dt. \end{aligned}$$

According to the Principle of the Least Action, $\delta S=0$, then there must be $\frac{\partial L}{\partial x}-\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}})=0$, which is so-called the **Lagrange equation**.