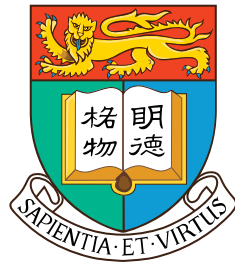


Formalized Higher-Ranked Polymorphic Type Inference Algorithms

by

Jinxu Zhao
(赵锦煦)



A thesis submitted in partial fulfillment of the requirements for
the degree of Doctor of Philosophy
at The University of Hong Kong

February 2021

Abstract of thesis entitled
“Formalized Higher-Ranked Polymorphic Type Inference Algorithms”

Submitted by
Jinxu Zhao

for the degree of Doctor of Philosophy
at The University of Hong Kong
in February 2021

DECLARATION

I declare that this thesis represents my own work, except where due acknowledgment is made, and that it has not been previously included in a thesis, dissertation or report submitted to this University or to any other institution for a degree, diploma or other qualifications.

.....

Jinxu Zhao

February 2021

ACKNOWLEDGMENTS

CONTENTS

DECLARATION	I
ACKNOWLEDGMENTS	II
LIST OF FIGURES	VII
LIST OF TABLES	IX
I PROLOGUE	1
1 INTRODUCTION	2
1.1 Type Systems and Type Inference Algorithms	2
1.1.1 Functional Programming and System F	3
1.1.2 Hindley-Milner Type System	4
1.1.3 Higher-Ranked Polymorphism	5
1.1.4 Bidirectional Typing	5
1.1.5 Subtyping	7
1.2 Mechanical Formalizations and Theorem Provers	7
1.3 Contributions and Outline	8
2 BACKGROUND	11
2.1 Hindley-Milner Type System	11
2.1.1 Declarative System	11
2.1.2 Algorithmic System and Principality	13
2.2 Odersky-Läufer Type System	14
2.2.1 Higher-Ranked Types	14
2.2.2 Declarative System	14
2.2.3 Relating to HM	17
2.3 Dunfield's Bidirectional Type System	17
2.3.1 Declarative System	17

2.4	MLsub	20
2.4.1	Types and Polar Types	20
2.4.2	Biunification	21
3	HIGHER-RANK POLYMORPHISM SUBTYPING ALGORITHM	23
3.1	Overview: Polymorphic Subtyping	23
3.1.1	Declarative Polymorphic Subtyping	23
3.1.2	Finding Solutions for Variable Instantiation	24
3.1.3	The Worklist Approach	26
3.2	A Worklist Algorithm for Polymorphic Subtyping	27
3.2.1	Syntax and Well-Formedness of the Algorithmic System	27
3.2.2	Algorithmic Subtyping	28
3.3	Metatheory	31
3.3.1	Transfer to the Declarative System	31
3.3.2	Soundness	32
3.3.3	Completeness	32
3.3.4	Decidability	33
3.4	The Choice of Abella	34
3.4.1	Statistics and Discussion	36
4	HIGHER-RANK POLYMORPHISM WORKLIST ALGORITHM	37
4.1	Overview	38
4.1.1	DK's Declarative System	38
4.1.2	DK's Algorithm	41
4.1.3	Judgment Lists	43
4.1.4	Single-Context Worklist Algorithm for Subtyping	44
4.1.5	Algorithmic Type Inference for Higher-Ranked Types: Key Ideas	44
4.2	Algorithmic System	46
4.2.1	Syntax and Well-Formedness	46
4.2.2	Algorithmic System	48
4.3	Metatheory	54
4.3.1	Declarative Worklist and Transfer	54
4.3.2	Non-Overlapping Declarative System	56
4.3.3	Soundness	58
4.3.4	Completeness	59
4.3.5	Decidability	60
4.3.6	Abella and Proof Statistics	63

Contents

4.4	Discussion	63
4.4.1	Contrasting Our Scoping Mechanisms with DK's	63
4.4.2	Elaboration	67
4.4.3	Lexically-Scoped Type Variables	68
4.5	Technical Discussions on DK's Manual Proofs	69
4.5.1	Broken Lemmas in DK's Papers	69
4.5.2	A Fix to DK's Subsumption Lemma	70
5	HIGHER-RANK POLYMORPHISM WITH OBJECT-ORIENTED SUBTYPING	72
5.1	Overview	72
5.1.1	Type Inference in Presence of Subtyping	72
5.1.2	Judgment List and Eager Substitution	74
5.1.3	Our Solution: Backtracking Algorithm	74
5.2	Declarative System	75
5.3	Backtracking Algorithm	77
5.3.1	Syntax	77
5.3.2	Algorithmic Subtyping	78
5.3.3	Algorithmic Typing	81
5.4	Metatheory	82
5.4.1	Declarative Properties	83
5.4.2	Transfer	85
5.4.3	Soundness	85
5.4.4	Partial Completeness of Subtyping: Rank-1 Restriction	85
5.4.5	Algorithmic Rank-1 Restriction (Partial Completeness)	86
5.4.6	Termination	87
5.4.7	Formalization in the Abella Proof Assistant	88
5.5	Discussion	88
5.5.1	A Complete Algorithm Under Monotype Guessing Restrictions	88
5.5.2	Lazy Substitution and Non-terminating Loops	89
II	RELATED AND FUTURE WORK	91
6	RELATED WORK	92
6.1	Higher-Ranked Polymorphic Type Inference Algorithms	92
6.1.1	Predicative Algorithms	92
6.1.2	Impredicative Algorithms	93
6.2	Type Inference Algorithms with Subtyping	94

Contents

6.3	Techniques Used in Type Inference Algorithms	94
6.3.1	Ordered Contexts in Type Inference	94
6.3.2	The Essence of ML Type Inference	94
6.3.3	Lists of Judgments in Unification	94
6.4	Mechanical Formalization of Polymorphic Type Systems	95
III	EPILOGUE	96
7	CONCLUSION AND FUTURE WORK	97
7.1	Conclusion	97
7.2	Future Work	97
	BIBLIOGRAPHY	98

LIST OF FIGURES

2.1	HM Syntax	12
2.2	HM Type System	12
2.3	Syntax of Odersky-Läufer System	14
2.4	Well-formedness of types in the Odersky-Läufer System	15
2.5	Subtyping of the Odersky-Läufer System	15
2.6	Typing of the Odersky-Läufer System	16
2.7	Syntax of Declarative System	17
2.8	Declarative Well-formedness and Subtyping	18
2.9	Declarative Typing	19
2.10	Types of MLsub	20
3.1	Syntax of Declarative System	24
3.2	Well-formedness of Declarative Types and Declarative Subtyping	24
3.3	Syntax and Well-Formedness Judgement for the Algorithmic System.	28
3.4	Algorithmic Subtyping	29
3.5	Successful and Failing Derivations for the Algorithmic Subtyping Relation	30
3.6	Transfer Rules	32
3.7	Statistics for the proof scripts	36
4.1	Syntax of Declarative System	39
4.2	Declarative Well-formedness and Subtyping	39
4.3	Declarative Typing	40
4.4	Extended Syntax and Well-Formedness for the Algorithmic System	47
4.5	Algorithmic Typing	49
4.6	A Sample Derivation for Algorithmic Typing	54
4.7	Declarative Worklists and Instantiation	55
4.8	Declarative Transfer	56
4.9	Context Subtyping	58
4.10	Worklist measures	61
4.11	Worklist Update	62

List of Figures

5.1	Declarative Syntax	75
5.2	Declarative Subtyping	76
5.3	Declarative Typing	77
5.4	Algorithmic Syntax	78
5.5	Algorithmic Garbage Collection and Subtyping	79
5.6	Algorithmic Typing	81
5.7	Declarative Worklists and Instantiation	85
5.8	Declarative Transfer	86

LIST OF TABLES

4.1	Statistics for the proof scripts	64
4.2	Translation Table for the Proof Scripts	65
5.1	Statistics for the proof scripts	89

PART I

PROLOGUE

1 INTRODUCTION

1.1 TYPE SYSTEMS AND TYPE INFERENCE ALGORITHMS

Statically typed programming languages are widely used nowadays. Programs are categorized by various types before it is compiled and executed. Type errors caught before execution usually indicate potential bugs, letting the programmers to realize and correct in advance.

In early stages, programming languages like C is built on a simple explicit type system, where only primitive types, function types and records are supported. People soon realize the need to generalize similar programs that have different types when used. For example, a typical way to define a swap function on integers is

```
void swap(int *x, int *y) {  
    int t = *x; *x = *y; *y = *t;  
}
```

Similarly, the swap function for float values can be defined as

```
void swap(float *x, float *y) {  
    float t = *x; *x = *y; *y = *t;  
}
```

which mostly shares a the same body with the above function, except that the types changes from `int` to `float`. Therefore, a type-safe approach to get a generic function for free is by text generation (pre-processing) macros:

```
#define SWAP(X, Y, T) { T _tmp = X; X = Y; Y = _tmp; }
```

Macros provides a generic way to program while maintaining type-safety, but the drawback of macro programming is also obvious: the type information for the abstraction is neither declared nor verified by the type system. In fact, users are responsible for the correctness of an expression after substitution.

In comparison, C++ provides a better way to define such a generic function:

```
template<typename T> void swap(T& t1, T& t2) {  
    T tmp(t1); t1 = t2; t2 = tmp;  
}
```

where T denotes the generic type that programmers may arbitrarily pick. The C++ `swap` function utilizes the feature of type system, template, to improve modularity. In other words, introduction of good features of type systems accepts more meaningful programs.

On the other hand, being able to accept all syntactically correct programs, as dynamically-typed programming languages do, are not desirable as well. Ill-typed programs are easy to write by mistake, like `"3" / 3`; or even well-typed programs with ambiguous/unexpected meaning like `"3" + 4 + 5` (for someone who is not familiar with the conventions, she might think this evaluates to "39").

1.1.1 FUNCTIONAL PROGRAMMING AND SYSTEM F

Nowadays, more and more programming languages adopt the functional programming paradigm, where functions are first-class citizens, and programs are mostly constructed with function applications. Functional programming originates from lambda calculus, and the first lambda calculus that has a type system is the simply-typed lambda calculus, which has a simple static type checking algorithm, preventing ill-typed program before actual execution. However, the system does not allow polymorphic functions and thus is quite tedious to express higher-level logics.

In order to improve the expressiveness of functional programming languages, System F introduces polymorphism via the universal quantifier \forall for type system and the Λ binder (to introduce type-level functions) for expressions. For example, the identity function that can range over any type of form $A \rightarrow A$ can be encoded in System F:

$$\text{id} = \Lambda a. \lambda x : a. x : \forall a. a \rightarrow a$$

To enjoy the polymorphism of such a function, one needs to first supply a type argument like `id @Int` (we use the `@` sign to denote a type-level application), so that the polymorphic function is *instantiated* with a concrete type.

IMPLICIT PARAMETRIC POLYMORPHISM Although being much more expressive than simply-typed systems, plain System F is still tedious to use. It feels a bit silly to write `id @Int 3` compared with `id 3`, because the missing type is quite easy to be figured out, in presence of the argument, which is of type `Int` and should represent a for the function application at the same time. With the feature called “implicit parametric polymorphism” [Reynolds 1983], type arguments like `@Int` is not written by the programmer explicitly, in contrast, it is the type inference algorithm’s responsibility to guess them. Unfortunately, there does not exist such a perfect algorithm that can automatically guess missing type applications for every possible program [Jimmy: cite](#).

For example, the following expression is ambiguous when the implicit type argument is not given:

$$f = (\text{choose} : \forall a. a \rightarrow a \rightarrow a) (\text{id} : \forall b. b \rightarrow b)$$

It is unclear how the type variable is instantiated during the polymorphic application: one possibility is that a is chosen to be the type of id , or $\forall b. b \rightarrow b$, resulting in the type $(\forall b. b \rightarrow b) \rightarrow (\forall b. b \rightarrow b)$. However, another type that is also valid is $\forall b. (b \rightarrow b) \rightarrow (b \rightarrow b)$, by first instantiating id with a fresh type variable b , and generalize the type after calculating the type of the application. Furthermore, between the two possible types, neither one is better: there exist programs that type checks under one of them and fails under the other.

The fact that implicit parametric algorithm for full System F is impossible motivates people to discover restrictions on the type system under which type inference algorithms are capable of guessing the best types.

1.1.2 HINDLEY-MILNER TYPE SYSTEM

The Hindley-Milner (HM) type system [Damas and Milner 1982; Hindley 1969; Milner 1978] restricts System F types to *type schemes*, or *first-order polymorphism*, where polymorphic types can only have universal quantifiers in the top level. For example, $\forall a b. (a \rightarrow b) \rightarrow a \rightarrow b$ is allowed, but not $(\forall b. b \rightarrow b) \rightarrow (\forall b. b \rightarrow b)$. An important property of the HM type inference algorithm is *principality*, where any unannotated program can always be inferred to a most general type within its type system. which supports full type-inference without any type annotations.

For example, the following function

$$g = \lambda x. \lambda y. x$$

can be assigned to types $\text{Int} \rightarrow \text{Bool} \rightarrow \text{Int}$, $\text{Int} \rightarrow \text{Int} \rightarrow \text{Int}$ or infinitely many others. The HM inference algorithm will infer a more general type $\forall a. \forall b. a \rightarrow b \rightarrow a$. In order to use the function as the types mentioned above, a more-general-than notion \leq is used to describe that the polymorphic type can be instantiated to more concrete types:

$$\begin{aligned} \forall a. \forall b. a \rightarrow b \rightarrow a &\leq \text{Int} \rightarrow \text{Bool} \rightarrow \text{Int} \\ \forall a. \forall b. a \rightarrow b \rightarrow a &\leq \text{Int} \rightarrow \text{Int} \rightarrow \text{Int} \\ \forall a. \forall b. a \rightarrow b \rightarrow a &\leq \forall a. a \rightarrow a \rightarrow a \end{aligned}$$

PREDICATIVITY In the HM system, \forall quantifiers can appear only on the top level, type instantiations will always be *monotypes*, i.e. types without the \forall quantifier. We refer to such

system as *predicative*. In contrast, System F does not restrict the types to instantiate, thus being a impredicative system. An important challenge is that full type inference for impredicative polymorphism is known to be undecidable [Wells 1999]. There are works that focus on practical inference of impredicative systems [Jimmy: cite](#). However, throughout this work, we study predicative type systems only.

1.1.3 HIGHER-RANKED POLYMORPHISM

As functional languages evolved the need for more expressive power has motivated language designers to look beyond HM, where there is still one obvious weakness that prevents some useful programs to type check: HM only have types of rank-1, since all the \forall 's appear on the top level. Thus one expected feature is to allow *higher-ranked polymorphism* where polymorphic types can occur anywhere in a type signature. This enables more code reuse and more expressions to type-check, and has numerous applications [Gill et al. 1993; Jones 1995; Lämmel and Jones 2003; Launchbury and Peyton Jones 1995].

One of the interesting examples is the ST monad of Haskell, where the `runST` function is only possible to express in a rank-2 type system:

$$\text{runST} :: \forall a. (\forall s. \text{ST } s \ a) \rightarrow a$$

The type is rank-2 because of the inner \forall quantifier in the argument position of the type. Such a type encapsulates the state and makes sure that program states from different computation threads do not escape their scopes, otherwise the type checker should reject in advance.

In order to support higher-ranked types, we need to extend the type system of HM, but not taking a too big step since type inference for full System F would be impossible. A simple polymorphic subtyping relation proposed by Odersky and Läufer [1996] extends the HM system by allowing higher-ranked types, but instantiations are still limited to monotypes, thus the system remains predicative.

1.1.4 BIDIRECTIONAL TYPING

In order to improve the expressiveness for higher-ranked systems, some type annotations are necessary to guide type inference. In response to this challenge several decidable type systems requiring some annotations have been proposed [Dunfield and Krishnaswami 2013; Le Botlan and Rémy 2003; Leijen 2008; Peyton Jones et al. 2007; Serrano et al. 2018; Vytiniotis et al. 2008].

As an example,

$$\text{hpoly} = \lambda(f : \forall a. a \rightarrow a). (f \ 1, f \ 'c')$$

the type of `hpoly` is $(\forall a. a \rightarrow a) \rightarrow (\text{Int}, \text{Char})$, which is a rank-2 type and is not typeable in HM. Notably (and unlike Hindley-Milner) the lambda argument f requires a *polymorphic* type annotation. This annotation is needed because the single universal quantifier does not appear at the top-level. Instead it is used to quantify a type variable a used in the first argument of the function.

One of the key feature that improves type inference algorithms with type annotations is bidirectional typing [Pierce and Turner 2000], a technique that combines two modes of typing: type checking, which checks an expression against a given type, and type synthesis (inference), which infers a type from an expression. Bidirectional typing is quite useful when the language supports type annotations, because those “hints” are handled with the checking mode. For the typing judgment $\Gamma \vdash e : A$, the input of checking mode ($e \Leftarrow A$) both e and A , while synthesis mode ($e \Rightarrow A$) needs to calculate the output A from the input expression e . It is clear that, with less information, synthesis mode is more difficult than checking mode. Therefore, bidirectional type checking with type annotations provides a way for the programmer to guide the type inference algorithm when the expression is tricky to analyse. In addition, bidirectional typing algorithms in practice improve the quality of error messages, due to the fact that they report errors in a relatively local range, compared with global unification algorithms.

Two closely related type systems that support *predicative* higher-ranked type inference were proposed by Peyton Jones et al. [Peyton Jones et al. 2007] and Dunfield and Krishnaswami [Dunfield and Krishnaswami 2013] (henceforth denoted as DK). These type systems are popular among language designers and their ideas have been adopted by several modern functional languages, including Haskell, PureScript [Freeman 2017] and Unison [Chiusano and Bjarnason 2015] among others.

DK developed a higher-ranked global bidirectional type system based on the declarative system by Odersky and Läufer [Odersky and Läufer 1996]. Beyond the existing works, they introduce a third form of judgment, the application inference $A \bullet e \Rightarrow C$, where the function type A and argument expression e are input, and the type C is the output return type of the function application. One can tell the procedure for type checking application expressions from the shape of the judgment — firstly, the type of the function is inferred to A ; then the application inference judgment calls the checking mode, to verify if the argument e checks against the argument part of A ; finally, output the return part of A .

$$\frac{\Psi \vdash \tau \quad \Psi \vdash [\tau/a]A \bullet e \Rightarrow C}{\Psi \vdash \forall a. A \bullet e \Rightarrow C} \text{Decl}\forall\text{App}$$

The use of application inference judgment avoids implicit instantiations of types like HM, instead, when the function type A is a polymorphic type, it is explicitly instantiated by the application inference until it becomes a function type:

$$\frac{\Psi \vdash e_1 \Rightarrow A \quad \Psi \vdash A \bullet e_2 \Rightarrow C}{\Psi \vdash e_1 e_2 \Rightarrow C} \text{Decl} \rightarrow \text{E} \qquad \frac{\Psi \vdash e \Leftarrow A}{\Psi \vdash A \rightarrow C \bullet e \Rightarrow C} \text{Decl} \rightarrow \text{App}$$

As a result, DK is in a more syntax-directed system compared with HM-like systems.

DK also provided an elegant formalization of their sound and complete algorithm, which has also inspired implementations of type inference in some polymorphic programming languages (such as PureScript [Freeman 2017] or DDC [Disciple Development Team 2017]).

The focus of this thesis is also on predicative implicit higher-ranked bidirectional type inference algorithms.

1.1.5 SUBTYPING

The term “subtyping” is used as two slightly different concepts in this thesis. One of them refers to the polymorphic subtyping relation used in polymorphic type systems, which compares the degree of polymorphism between types, i.e. the more-general-than relation. Chapters 3 and 4 only focus on this type of subtyping relation. The other one is the subtyping usually seen in object-oriented programming, where there are some built-in or user-defined type conversion rules. We introduce object-oriented subtyping as a feature in Chapter 5. We will then introduce the top and bottom types as the minimal support.

1.2 MECHANICAL FORMALIZATIONS AND THEOREM PROVERS

While many aspects of programming languages and type systems have been mechanically formalized in theorem provers, there is little work on formalizing algorithms related to type-inference. The main exceptions to the rule are mechanical formalizations of algorithm \mathcal{W} and other aspects of traditional Hindler-Milner type-inference [Dubois 2000; Dubois and Menissier-Morain 1999; Garrigue 2015; Naraschewski and Nipkow 1999; Urban and Nipkow 2008]. However, as far as we know, there is no mechanisation of algorithms used by modern functional languages like Haskell, and polymorphic subtyping included is no exception. This is a shame because recently there has been a lot of effort in promoting the use of theorem provers to check the meta-theory of programming languages, e.g., through well-known examples like the POPLMARK challenge [Aydemir et al. 2005] and the CompCert project [Leroy et al. 2012]. Mechanical formalizations are especially valuable for proving the

correctness of the semantics and type systems of programming languages. Type-inference algorithms are arguably among the most non-trivial aspects of the implementations of programming languages. In particular the information discovery process required by many algorithms (through unification-like or constraint-based approaches), is quite subtle and tricky to get right. Moreover, extending type-inference algorithms with new programming language features is often quite delicate. Studying the meta-theory for such extensions would be greatly aided by the existence of a mechanical formalization of the base language, which could then be extended by the language designer.

Handling variable binding is particularly challenging in type inference, because the algorithms typically do not rely simply on local environments, but instead propagate information across judgements. Yet, there is little work on how to deal with these complex forms of binding in theorem provers. We believe that this is the primary reason why theorem provers have still not been widely adopted for formalizing type-inference algorithms.

ABELLA The Abella theorem prover [Gacek 2008] is one that specifically ease formalization on binders. Its λ -tree syntax, or HOAS, and features including the ∇ quantifier and higher-order unification, have better experiences than using Coq libraries. Two common treatments of binding are to use the De Bruijn index [de Bruijn 1972] and the nominal logic framework of Pitts [Pitts 2003]. Abella is based on another approach that uses the abstraction operator in a typed λ -calculus to encode binding. In practise, Abella uses the ∇ quantifier and nominal constants to help quantify a “fresh” variable during formalization. For example, the common type checking rule

$$\frac{e \Leftarrow A \quad a \text{ fresh}}{e \Leftarrow \forall a. A}$$

is encoded as $\text{check E}(\text{all } A) := \text{nabla } a, \text{check E}(A \ a)$ in Abella, where the ∇ quantifier introduces a fresh type variable a and later use it to “open” the body of $\forall a. A$.

Throughout the thesis, all the type systems and declared properties are mechanically formalized in Abella.

1.3 CONTRIBUTIONS AND OUTLINE

CONTRIBUTIONS In summary, the main contributions of this thesis are:

- **Chapter 3** presents a predicative polymorphic subtyping judging algorithm for OL’s higher-ranked subtyping specification. Central to type inference, subtyping is the most difficult component. Similar to DK’s algorithm, we employ an ordered context to collect type variables and existential variables (used as placeholders for guessing

monotypes). However, our unification process is novel. DK’s algorithm solves variables on-the-fly and pass the partial solutions through an output context. In contrast, our algorithm collects a list of judgments and propagate partial solutions across them via eager substitutions. Such technique eliminates the use of output contexts, and thus simplifies the metatheory and make mechanical formalizations easier. Besides, using only a single context keeps the definition of well-formedness simple, resulting in an easy and elegant algorithm.

We showed that our algorithm is *sound*, *complete* and *decidable* in the Abella theorem prover. And we are the first to formalize the meta-theoretical results of a polymorphic higher-ranked subtyping algorithm.

- Chapter 4 proposes *worklist judgments*, a new technique that unifies ordered contexts and judgments. This enables precise scope tracking of variables in judgments, and avoids the duplication of context information across judgments in worklists. Similar to the previous algorithm, partial solutions are propagated across judgments in a single list consist of both variable bindings and judgments. Nevertheless, the unification of worklists and contexts exploits the fact that judgments are usually sharing a large part of common information. And one can easily tell when a variable is no longer referred. Furthermore, we support inference judgment so that bidirectional typing can be encoded as worklist judgments. The idea is to use a continuation passing style to enable the transfer of inferred information across judgments.

Utilizing worklist judgments, we present a new bidirectional higher-ranked typing inference algorithm based on DK’s declarative system. We are the first to present a full mechanical formalization for type inference of higher-ranked type system. The *soundness*, *completeness* and *decidability* are shown in the Abella theorem prover.

- Chapter 5 further extends the higher-ranked system with object-oriented subtyping.
 - We propose a bidirectional declarative system extended with the top and bottom types and relevant subtyping and typing rules. Several desirable properties are satisfied and mechanically proven.
 - A new backtracking-based worklist algorithm is presented and proven to be *sound* with respect to our declarative specification. Since the top or bottom type often satisfies a subtyping constraint easily with a simple solution, which should be tried in parallel with other possibilities. Therefore, the backtracking technique is well-suited for the non-deterministic trial without missing any of them.
 - We formalize the rank-1 restriction of subtyping relation, and proved that our algorithmic subtyping is *complete* under such restriction.

PRIOR PUBLICATIONS. This thesis is partially based on the publications by the author [Zhao et al. 2018, 2019], as indicated below.

Chapter 3: Jinxu Zhao, Bruno C. d. S. Oliveira, and Tom Schrijvers. 2018. “Formalization of a Polymorphic Subtyping Algorithm”. In *International Conference on Interactive Theorem Proving (ITP)*.

Chapter 4: Jinxu Zhao, Bruno C. d. S. Oliveira, and Tom Schrijvers. 2019. “A Mechanical Formalization of Higher-Ranked Polymorphic Type Inference”. In *International Conference on Functional Programming (ICFP)*.

MECHANIZED PROOFS. All proofs in this thesis is mechanically formalized in the Abella theorem prover and are available online:

Chapter 3: <https://github.com/JimmyZJX/Abella-subtyping-algorithm>

Chapter 4: <https://github.com/JimmyZJX/TypingFormalization>

Chapter 5: <https://github.com/JimmyZJX/TODO> [Jimmy: TODO URL](#)

2 BACKGROUND

2.1 HINDLEY-MILNER TYPE SYSTEM

2.1.1 DECLARATIVE SYSTEM

SYNTAX The declarative syntax is shown in Figure 2.1. The HM types are consist of polymorphic types (or type schemes) and monomorphic types. A polymorphic type contains zero or more universal quantifiers only at the top level. When no universal quantifier occurs, the type belongs to a mono-type. Mono-types are constructed by a unit type 1, a type variable a , or a function type $\tau_1 \rightarrow \tau_2$.

Expressions e includes variables x , literals $()$, lambda abstractions $\lambda x. e$, applications $e_1 e_2$ and the let expression **let** $x = e_1$ **in** e_2 . A context Ψ is a collection of type bindings for variables.

TYPE INSTANTIATION The relations between types are described via type instantiations. The rule shown to the top of Figure 2.2 checks if $\forall \bar{a}. \tau$ is a *generic instance* of $\forall \bar{b}. \tau'$. This relation is valid when $\tau' = [\bar{\tau}/\bar{a}]\tau$ for a series of mono-types $\bar{\tau}$ and each variable in \bar{b} is not free in $\forall \bar{a}. \tau$.

For example,

$$\forall a. a \rightarrow a \sqsubseteq 1 \rightarrow 1$$

is obtained by the substitution $[1/a]$, and

$$\forall a. a \rightarrow a \sqsubseteq \forall b. (b \rightarrow b) \rightarrow (b \rightarrow b)$$

substitutes a by $b \rightarrow b$, and generalizes b after the substitution.

TYPING The typing relation $\Psi \vdash_{HM} e : \sigma$ synthesizes a type σ for an expression e under the context Ψ . Rule HM-Var looks up the binding of a variable x in the context. Rule HM-Unit always give the unit type 1 to the unit expression $()$. For a lambda abstraction $\lambda x. e$, rule HM-Abs guesses its input type (τ_1) and compute the type of its body (τ_2) as the return type. Rule HM-App eliminates a function type by an application $e_1 e_2$, where the argument type

2 Background

Type variables	a, b
Types	$\sigma ::= \tau \mid \forall a. \sigma$
Monotypes	$\tau ::= 1 \mid a \mid \tau_1 \rightarrow \tau_2$
Expressions	$e ::= x \mid () \mid \lambda x. e \mid e_1 e_2 \mid \mathbf{let} \ x = e_1 \ \mathbf{in} \ e_2$
Contexts	$\Psi ::= \cdot \mid \Psi, x : \sigma$

Figure 2.1: HM Syntax

$\sigma_1 \sqsubseteq \sigma_2$ HM Type Instantiation

$$\frac{\tau' = [\bar{\tau}/\bar{a}]\tau \quad \bar{b} \notin \text{FV}(\forall \bar{a}. \tau)}{\forall \bar{a}. \tau \sqsubseteq \forall \bar{b}. \tau'} \text{HM-TInst}$$

$\Psi \vdash_{HM} e : \sigma$ HM Typing

$$\begin{array}{c}
\frac{(x : \sigma) \in \Psi}{\Psi \vdash_{HM} x : \sigma} \text{HM-Var} \quad \frac{}{\Psi \vdash_{HM} () : 1} \text{HM-Unit} \quad \frac{\Psi, x : \tau_1 \vdash_{HM} e : \tau_2}{\Psi \vdash_{HM} \lambda x. e : \tau_1 \rightarrow \tau_2} \text{HM-Abs} \\
\\
\frac{\Psi \vdash_{HM} e_1 : \tau_1 \rightarrow \tau_2 \quad \Psi \vdash_{HM} e_2 : \tau_1}{\Psi \vdash_{HM} e_1 e_2 : \tau_2} \text{HM-App} \\
\\
\frac{\Psi \vdash_{HM} e_1 : \sigma \quad \Psi, x : \sigma \vdash_{HM} e_2 : \tau}{\Psi \vdash_{HM} \mathbf{let} \ x = e_1 \ \mathbf{in} \ e_2 : \tau} \text{HM-Let} \\
\\
\frac{\Psi \vdash_{HM} e : \sigma \quad \bar{a} \notin \text{FV}(\Psi)}{\Psi \vdash_{HM} e : \forall \bar{a}. \sigma} \text{HM-Gen} \quad \frac{\Psi \vdash_{HM} e : \sigma_1 \quad \sigma_1 \sqsubseteq \sigma_2}{\Psi \vdash_{HM} e : \sigma_2} \text{HM-Inst}
\end{array}$$

Figure 2.2: HM Type System

must be the same as the input type of the function, and the type of the whole application is τ_2 .

Rule HM-Let is also referred as let-polymorphism. In (untyped) lambda calculus, $\mathbf{let} \ x = e_1 \ \mathbf{in} \ e_2$ behaves the same as $(\lambda x. e_2) e_1$. However, the HM let rule derives the type of e_1 first, and binds the polymorphic type into the context before e_2 . This enables polymorphic expressions to be reused multiple times in different instantiated types.

Rules HM-Gen and HM-Inst changes the type of an expression at any time during the derivation. Rule HM-Gen generalizes over fresh type variables \bar{a} . Rule HM-Inst, as opposed to generalization, specializes a type according to the type instantiation relation.

The type system of HM supports *implicit instantiation* through rule HM-Inst. This means that any expression (function) that has a polymorphic type can be automatically instantiated

with a proper monotype for any reasonable application. The fact that only monotypes are guessed indicates that the system is *predicative*. In contrast, an *impredicative* system might guess polymorphic types. Unfortunately, type inference on impredicative systems is undecidable [Wells 1999]. In this thesis, we focus on predicative systems only.

2.1.2 ALGORITHMIC SYSTEM AND PRINCIPALITY

SYNTAX-DIRECTED SYSTEM The declarative system is not syntax-directed due to rules HM-Gen and HM-Inst, which can be applied on any expression. A syntax-directed system can be obtained by replacing rules HM-Var and HM-Let by the following rules:

$$\frac{(x : \sigma) \in \Psi \quad \sigma \sqsubseteq \tau}{\Psi \vdash_{HM}^S x : \tau} \text{ HM-Var-Inst} \qquad \frac{\Psi \vdash_{HM} e_1 : \sigma \quad \bar{a} = \text{FV}(\sigma) - \text{FV}(\Psi) \quad \Psi, x : \forall \bar{a}. \sigma \vdash_{HM} e_2 : \tau}{\Psi \vdash_{HM}^S \text{let } x = e_1 \text{ in } e_2 : \tau} \text{ HM-Let-Gen}$$

A generalization on σ , the synthesized type of e_1 , is added to rule HM-Let, since it is the source place where a polymorphic type is generated. However, a too generalized type might reject applications due to its shape, therefore, an instantiation procedure is added to eliminate all the universal quantifiers on rule HM-Var. We omit rules HM-Unit, HM-Abs, and HM-App for the syntax-directed system $\Psi \vdash_{HM}^S$. The following property shows that the new system is (almost) equivalent to the original declarative system.

Theorem 2.1 (Equivalence of Syntax-Directed System).

1. If $\Psi \vdash_{HM}^S e : \sigma$ then $\Psi \vdash_{HM} e : \sigma$
2. If $\Psi \vdash_{HM} e : \sigma$ then $\Psi \vdash_{HM} e : \tau$, and $\forall \bar{a}. \tau \sqsubseteq \sigma$, where $\bar{a} = \text{FV}(\tau) - \text{FV}(\Psi)$.

TYPE INFERENCE ALGORITHM Although being syntax-directed solves some problems, the rules still requires some guessings, including rule HM-Abs and HM-Var-Inst. Algorithm W [Milner 1978], based on unification, is proven to be sound and complete w.r.t the declarative specifications.

Theorem 2.2 (Algorithmic Completeness (Principality)). If $\Psi \vdash_{HM} e : \sigma$, then W computes a principal type scheme σ_p , i.e.

1. $\Psi \vdash_{HM} e : \sigma_p$
2. $\sigma_p \sqsubseteq \sigma$.

2 Background

Type variables	a, b
Types	$\sigma ::= 1 \mid a \mid \sigma_1 \rightarrow \sigma_2 \mid \forall a. \sigma$
Monotypes	$\tau ::= 1 \mid a \mid \tau_1 \rightarrow \tau_2$
Expressions	$e ::= x \mid () \mid \lambda x : \sigma. e \mid e : \sigma \mid \lambda x. e \mid e_1 e_2 \mid \mathbf{let} \ x = e_1 \ \mathbf{in} \ e_2$
Contexts	$\Psi ::= \cdot \mid \Psi, x : \sigma \mid \Psi, a$

Figure 2.3: Syntax of Odersky-Läufer System

2.2 ODERSKY-LÄUFER TYPE SYSTEM

2.2.1 HIGHER-RANKED TYPES

The *rank* of a type represents how deep a universal quantifier appear at the contravariant position [Kfoury and Tiuryn 1992]. Formally speaking,

$$\begin{array}{ll}
 \text{Rank 0 / Monotypes} & \tau, \sigma^0 ::= 1 \mid a \mid \tau_1 \rightarrow \tau_2 \\
 \text{Rank } k (k \geq 1), \text{ Polytypes} & \sigma^k ::= \sigma^{k-1} \mid \sigma^{k-1} \rightarrow \sigma^k \mid \forall a. \sigma^k
 \end{array}$$

The following example illustrates what rank a type belongs to:

$$\begin{array}{ll}
 1 \rightarrow 1 & \text{Rank 0} \\
 \forall a. a \rightarrow a & \text{Rank 1} \\
 1 \rightarrow \forall a. a \rightarrow a & \text{Rank 1} \\
 (\forall a. a \rightarrow a) \rightarrow (\forall a. a \rightarrow a) & \text{Rank 2}
 \end{array}$$

According to the definition, monotypes are types that does not contain any universal quantifier. In the HM type system, all polymorphic types have rank 1.

[Jimmy: TODO Background about System F incompleteness, examples about higher-rank types](#)

2.2.2 DECLARATIVE SYSTEM

The syntax of Odersky-Läufer system is shown in Figure 2.3. There are several differences compared to the HM system.

First, polymorphic types can be of arbitrary rank, i.e. the forall quantifier may occur at any part of a type. Yet, mono-type remains the same definition as HM's.

2 Background

$$\begin{array}{c}
\frac{}{\Psi \vdash_{OL} 1} \text{OL-WF-Unit} \qquad \frac{a \in \Psi}{\Psi \vdash_{OL} a} \text{OL-WF-TVar} \\
\frac{\Psi \vdash_{OL} \sigma_1 \quad \Psi \vdash_{OL} \sigma_2}{\Psi \vdash_{OL} \sigma_1 \rightarrow \sigma_2} \text{OL-WF-Arr} \qquad \frac{\Psi, a \vdash_{OL} \sigma}{\Psi \vdash_{OL} \forall a. \sigma} \text{OL-WF-Forall}
\end{array}$$

Figure 2.4: Well-formedness of types in the Odersky-Läufer System

$$\begin{array}{c}
\frac{}{\Psi \vdash_{OL} 1 \leq 1} \text{OL-SUB-Unit} \qquad \frac{a \in \Psi}{\Psi \vdash_{OL} a \leq a} \text{OL-SUB-Var} \\
\frac{\Psi \vdash_{OL} \sigma'_1 \leq \sigma_1 \quad \Psi \vdash_{OL} \sigma_2 \leq \sigma'_2}{\Psi \vdash_{OL} \sigma_1 \rightarrow \sigma_2 \leq \sigma'_1 \rightarrow \sigma'_2} \text{OL-SUB-Arr} \\
\frac{\Psi \vdash_{OL} \tau \quad \Psi \vdash_{OL} [\tau/a]\sigma \leq \sigma'}{\Psi \vdash_{OL} \forall a. \sigma \leq \sigma'} \text{OL-SUB-}\forall\text{L} \qquad \frac{\Psi, a \vdash_{OL} \sigma \leq \sigma'}{\Psi \vdash_{OL} \sigma \leq \forall a. \sigma'} \text{OL-SUB-}\forall\text{R}
\end{array}$$

Figure 2.5: Subtyping of the Odersky-Läufer System

Second, expressions now allows annotations $e : \sigma$ and (argument) annotated lambda functions $\lambda x : \sigma. e$. Annotations on expressions help guide the type system properly, acting as a machine-checked document by the programmers. By annotating the argument of a lambda function with a polymorphic type σ , one may encode a function of higher rank in this system compared to HM's.

Finally, contexts consist of not only variable bindings, but also type variable declarations. Here we adopt a slightly different approach than the original work [Odersky and Läufer 1996], which does not track type variables explicitly in a context. Such explicit declarations reduce formalization difficulties especially when dealing with freshness conditions or variable name encodings. This also enables us to formally define well-formedness of types, shown in Figure 2.4.

SUBTYPING The subtyping relation, defined in Figure 2.5, is more powerful than that (type instantiation) of HM. In contrast to HM's subtyping, higher-ranked types can be compared thanks to rule OL-SUB-Arr. Functions are contravariant on argument types and covariant on return types. Rule OL-SUB- $\forall\text{L}$ instantiates a polymorphic type by a monotype τ . Rule OL-SUB- $\forall\text{R}$ picks a fresh type variable for the right-hand-side polymorphic type, which is made possible by renaming according to alpha-equivalence. Given that such implicit freshness condition is widely adopt, we omit that throughout the thesis.

$$\begin{array}{c}
 \frac{(x : \sigma) \in \Psi}{\Psi \vdash_{OL} x : \sigma} \text{OL-Var} \quad \frac{}{\Psi \vdash_{OL} () : 1} \text{OL-Unit} \quad \frac{\Psi \vdash_{OL} e : \sigma}{\Psi \vdash_{OL} (e : \sigma) : \sigma} \text{OL-Anno} \\
 \frac{\Psi \vdash_{OL} \tau \quad \Psi, x : \tau \vdash_{OL} e : \sigma}{\Psi \vdash_{OL} \lambda x. e : \tau \rightarrow \sigma} \text{OL-Lam} \quad \frac{\Psi, x : \sigma_1 \vdash_{OL} e : \sigma_2}{\Psi \vdash_{OL} \lambda x : \sigma_1. e : \sigma_1 \rightarrow \sigma_2} \text{OL-LamAnno} \\
 \frac{\Psi \vdash_{OL} e_1 : \sigma_1 \rightarrow \sigma_2 \quad \Psi \vdash_{OL} e_2 : \sigma_1}{\Psi \vdash_{OL} e_1 e_2 : \sigma_2} \text{OL-App} \quad \frac{\Psi, a \vdash_{OL} e : \sigma}{\Psi \vdash_{OL} e : \forall a. \sigma} \text{OL-Gen} \\
 \frac{\Psi \vdash_{OL} e_1 : \sigma_1 \quad \Psi, x : \sigma_1 \vdash_{OL} e_2 : \sigma_2}{\Psi \vdash_{OL} \text{let } x = e_1 \text{ in } e_2 : \sigma_2} \text{OL-Let} \quad \frac{\Psi \vdash_{OL} e : \sigma_1 \quad \Psi \vdash_{OL} \sigma_1 \leq \sigma_2}{\Psi \vdash_{OL} e : \sigma_2} \text{OL-Sub}
 \end{array}$$

Figure 2.6: Typing of the Odersky-Läufer System

TYPING The type system of Odersky-Läufer, shown in Figure 2.6, extends HM's type system in the following aspects.

Rule OL-Lam now accepts polymorphic return type, because such type is well-formed. The guess on parameter types is still limited to monotypes like HM's. However, if a parameter type is specified in advance, the type system accepts polymorphic argument type with rule OL-LamAnno. Functions of arbitrary rank can be encoded through proper annotations. The application and let-generalization rules also accept polymorphic return types.

Rule OL-Gen encodes the generalization rule of HM in a different way under explicit type variable declarations. A fresh type variable is introduced into the context before the type of expression e is calculated. Then we conclude that e has a polymorphic type by generalizing the type variable. For example, the type of the identity function is derived as follows

$$\begin{array}{c}
 \frac{(x : a) \in (\cdot, a, x : a)}{\cdot, a \vdash_{OL} a \quad \cdot, a, x : a \vdash_{OL} x : a} \text{OL-Var} \\
 \frac{}{\cdot, a \vdash_{OL} \lambda x. x : a \rightarrow a} \text{OL-Lam} \\
 \frac{}{\cdot \vdash_{OL} \lambda x. x : \forall a. a \rightarrow a} \text{OL-Gen}
 \end{array}$$

The subsumption rule OL-Sub converts the type of an expression with the help of the subtyping relation.

2 Background

Type variables	a, b		
Types	A, B, C	$::=$	$1 \mid a \mid \forall a. A \mid A \rightarrow B$
Monotypes	τ, σ	$::=$	$1 \mid a \mid \tau \rightarrow \sigma$
Expressions	e	$::=$	$x \mid () \mid \lambda x. e \mid e_1 e_2 \mid (e : A)$
Contexts	Ψ	$::=$	$\cdot \mid \Psi, a \mid \Psi, x : A$

Figure 2.7: Syntax of Declarative System

2.2.3 RELATING TO HM

2.3 DUNFIELD’S BIDIRECTIONAL TYPE SYSTEM

2.3.1 DECLARATIVE SYSTEM

SYNTAX. The syntax of DK’s declarative system [Dunfield and Krishnaswami 2013] is shown in Figure 3.1. A declarative type A is either the unit type 1 , a type variable a , a universal quantification $\forall a. A$ or a function type $A \rightarrow B$. Nested universal quantifiers are allowed for types, but monotypes τ do not have any universal quantifier. Terms include a unit term $()$, variables x , lambda-functions $\lambda x. e$, applications $e_1 e_2$ and annotations $(e : A)$. Contexts Ψ are sequences of type variable declarations and term variables with their types declared $x : A$.

WELL-FORMEDNESS Well-formedness of types and terms is shown at the top of Figure 2.8. The rules are standard and simply ensure that variables in types and terms are well-scoped.

DECLARATIVE SUBTYPING The bottom of Figure 2.8 shows DK’s declarative subtyping judgment $\Psi \vdash A \leq B$, which was adopted from Odersky and Läufer [1996]. It compares the degree of polymorphism between A and B in DK’s implicit polymorphic type system. Essentially, if A can always be instantiated to match any instantiation of B , then A is “at least as polymorphic as” B . We also say that A is “more polymorphic than” B and write $A \leq B$.

Subtyping rules \leq_{Var} , \leq_{Unit} and \leq_{\rightarrow} handle simple cases that do not involve universal quantifiers. The subtyping rule for function types \leq_{\rightarrow} is standard, being covariant on the return type and contravariant on the argument type. Rule $\leq_{\forall R}$ states that if A is a subtype of B in the context Ψ, a , where a is fresh in A , then $A \leq \forall a. B$. Intuitively, if A is more general than $\forall a. B$ (where the universal quantifier already indicates that $\forall a. B$ is a general type), then A must instantiate to $[\tau/a]B$ for every τ .

The most interesting rule is $\leq_{\forall L}$. If some instantiation of $\forall a. A$, $[\tau/a]A$, is a subtype of B , then $\forall a. A \leq B$. The monotype τ we used to instantiate a is *guessed* in this declarative

2 Background

$$\begin{array}{c}
\boxed{\Psi \vdash A} \text{ Well-formed declarative type} \\
\\
\frac{}{\Psi \vdash 1} \text{wf}_{\text{dunit}} \quad \frac{a \in \Psi}{\Psi \vdash a} \text{wf}_{\text{dvar}} \quad \frac{\Psi \vdash A \quad \Psi \vdash B}{\Psi \vdash A \rightarrow B} \text{wf}_{\text{d}\rightarrow} \quad \frac{\Psi, a \vdash A}{\Psi \vdash \forall a. A} \text{wf}_{\text{d}\forall} \\
\\
\boxed{\Psi \vdash e} \text{ Well-formed declarative expression} \\
\\
\frac{x : A \in \Psi}{\Psi \vdash x} \text{wf}_{\text{d}\text{tmvar}} \quad \frac{}{\Psi \vdash ()} \text{wf}_{\text{d}\text{tmunit}} \quad \frac{\Psi, x : A \vdash e}{\Psi \vdash \lambda x. e} \text{wf}_{\text{d}\text{abs}} \\
\\
\frac{\Psi \vdash e_1 \quad \Psi \vdash e_2}{\Psi \vdash e_1 e_2} \text{wf}_{\text{d}\text{app}} \quad \frac{\Psi \vdash A \quad \Psi \vdash e}{\Psi \vdash (e : A)} \text{wf}_{\text{d}\text{anno}} \\
\\
\boxed{\Psi \vdash A \leq B} \text{ Declarative subtyping} \\
\\
\frac{a \in \Psi}{\Psi \vdash a \leq a} \leq \text{Var} \quad \frac{}{\Psi \vdash 1 \leq 1} \leq \text{Unit} \quad \frac{\Psi \vdash B_1 \leq A_1 \quad \Psi \vdash A_2 \leq B_2}{\Psi \vdash A_1 \rightarrow A_2 \leq B_1 \rightarrow B_2} \leq \rightarrow \\
\\
\frac{\Psi \vdash \tau \quad \Psi \vdash [\tau/a]A \leq B}{\Psi \vdash \forall a. A \leq B} \leq \forall \text{L} \quad \frac{\Psi, b \vdash A \leq B}{\Psi \vdash A \leq \forall b. B} \leq \forall \text{R}
\end{array}$$

Figure 2.8: Declarative Well-formedness and Subtyping

rule, but the algorithmic system does not guess and defers the instantiation until it can determine the monotype deterministically. The fact that τ is a monotype rules out the possibility of polymorphic (or impredicative) instantiation. However this restriction ensures that the subtyping relation remains decidable. Allowing an arbitrary type (rather than a monotype) in rule $\leq \forall \text{L}$ is known to give rise to an undecidable subtyping relation [Tiuryn and Urzyczyn 1996]. Peyton Jones et al. [2007] also impose the restriction of predicative instantiation in their type system. Both systems are adopted by several practical programming languages.

Note that when we introduce a new binder in the premise, we implicitly pick a fresh one. This applies to rules such as $\text{wf}_{\text{d}\forall}$, $\text{wf}_{\text{d}\text{abs}}$, $\leq \forall \text{R}$, throughout the whole text.

DECLARATIVE TYPING The bidirectional type system, shown in Figure 2.9, has three judgments. The checking judgment $\Psi \vdash e \Leftarrow A$ checks expression e against the type A in the context Ψ . The synthesis judgment $\Psi \vdash e \Rightarrow A$ synthesizes the type A of expression e in the context Ψ . The application judgment $\Psi \vdash A \bullet e \Rightarrow C$ synthesizes the type C of the application of a function of type A (which could be polymorphic) to the argument e .

Many rules are standard. Rule `DeclVar` looks up term variables in the context. Rules `Decl1I` and `Decl1I \Rightarrow` respectively check and synthesize the unit type. Rule `DeclAnno` syn-

2 Background

$\boxed{\Psi \vdash e \Leftarrow A}$	e checks against input type A .
$\boxed{\Psi \vdash e \Rightarrow A}$	e synthesizes output type A .
$\boxed{\Psi \vdash A \bullet e \Rightarrow C}$	Applying a function of type A to e synthesizes type C .

$\frac{(x : A) \in \Psi}{\Psi \vdash x \Rightarrow A} \text{DeclVar}$	$\frac{\Psi \vdash e \Rightarrow A \quad \Psi \vdash A \leq B}{\Psi \vdash e \Leftarrow B} \text{DeclSub}$
$\frac{\Psi \vdash A \quad \Psi \vdash e \Leftarrow A}{\Psi \vdash (e : A) \Rightarrow A} \text{DeclAnno}$	$\frac{}{\Psi \vdash () \Leftarrow 1} \text{Decl1I} \quad \frac{}{\Psi \vdash () \Rightarrow 1} \text{Decl1I}\Rightarrow$
$\frac{\Psi, a \vdash e \Leftarrow A}{\Psi \vdash e \Leftarrow \forall a. A} \text{Decl}\forall\text{I}$	$\frac{\Psi \vdash \tau \quad \Psi \vdash [\tau/a]A \bullet e \Rightarrow C}{\Psi \vdash \forall a. A \bullet e \Rightarrow C} \text{Decl}\forall\text{App}$
$\frac{\Psi, x : A \vdash e \Leftarrow B}{\Psi \vdash \lambda x. e \Leftarrow A \rightarrow B} \text{Decl}\rightarrow\text{I}$	$\frac{\Psi \vdash \sigma \rightarrow \tau \quad \Psi, x : \sigma \vdash e \Leftarrow \tau}{\Psi \vdash \lambda x. e \Rightarrow \sigma \rightarrow \tau} \text{Decl}\rightarrow\text{I}\Rightarrow$
$\frac{\Psi \vdash e_1 \Rightarrow A \quad \Psi \vdash A \bullet e_2 \Rightarrow C}{\Psi \vdash e_1 e_2 \Rightarrow C} \text{Decl}\rightarrow\text{E}$	$\frac{\Psi \vdash e \Leftarrow A}{\Psi \vdash A \rightarrow C \bullet e \Rightarrow C} \text{Decl}\rightarrow\text{App}$

Figure 2.9: Declarative Typing

thesizes the annotated type A of the annotated expression $(e : A)$ and checks that e has type A . Checking an expression e against a polymorphic type $\forall a. A$ in the context Ψ succeeds if e checks against A in the extended context (Ψ, a) . The subsumption rule DeclSub depends on the subtyping relation, and changes mode from checking to synthesis: if e synthesizes type A and $A \leq B$ (A is more polymorphic than B), then e checks against B . If a checking problem does not match any other rules, this rule can be applied to synthesize a type instead and then check whether the synthesized type entails the checked type. Lambda abstractions are the hardest construct of the bidirectional type system to deal with. Checking $\lambda x. e$ against function type $A \rightarrow B$ is easy: we check the body e against B in the context extended with $x : A$. However, synthesizing a lambda-function is a lot harder, and this type system only synthesizes monotypes $\sigma \rightarrow \tau$.

Application $e_1 e_2$ is handled by Rule $\text{Decl}\rightarrow\text{E}$, which first synthesizes the type A of the function e_1 . If A is a function type $B \rightarrow C$, Rule $\text{Decl}\rightarrow\text{App}$ is applied; it checks the argument e_2 against B and returns type C . The synthesized type of function e_1 can also be polymorphic, of the form $\forall a. A$. In that case, we instantiate A to $[\tau/a]A$ with a monotype τ using according to Rule $\text{Decl}\rightarrow\text{I}\Rightarrow$. If $[\tau/a]A$ is a function type, Rule $\text{Decl}\rightarrow\text{App}$ proceeds; if $[\tau/a]A$ is another universal quantified type, Rule $\text{Decl}\rightarrow\text{I}\Rightarrow$ is recursively applied.

2 Background

Types	τ	$::=$	$1 \mid a \mid \top \mid \perp \mid \tau_1 \rightarrow \tau_2 \mid \tau_1 \sqcup \tau_2 \mid \tau_1 \sqcap \tau_2$
Positive Types	τ^+	$::=$	$1 \mid a \mid \perp \mid \tau_1^- \rightarrow \tau_2^+ \mid \tau_1^+ \sqcup \tau_2^+$
Negative Types	τ^-	$::=$	$1 \mid a \mid \top \mid \tau_1^+ \rightarrow \tau_2^- \mid \tau_1^- \sqcap \tau_2^-$

Figure 2.10: Types of MLsub

2.4 MLSUB

MLsub [Dolan and Mycroft 2017] extends the HM type system with subtyping. In presence of subtyping, type inference does not simply handle equality during unification. Therefore, types are extended with lattice to express bounds properly. Furthermore, polar types are introduced to help separate input and output types, which simplifies the type inference algorithm. Like HM's type inference, MLsub always infers a principal type.

2.4.1 TYPES AND POLAR TYPES

In comparison to the type system of HM, types (Figure 2.10) now include \top and \perp , as minimal components to support subtyping. Besides, the least-upper-bound (\sqcup) and greatest-lower-bound (\sqcap) lattice operations are used to represent a bound expressed by two types. For finite types, a distributive lattice can be defined via a set of equivalence classes of \equiv [Dolan and Mycroft 2017]. The most interesting equations are the distributivity rule and rules for function types:

$$\begin{aligned}
\tau_1 \sqcup (\tau_2 \sqcap \tau_3) &\equiv (\tau_1 \sqcup \tau_2) \sqcap (\tau_1 \sqcup \tau_3) \\
\tau_1 \sqcap (\tau_2 \sqcup \tau_3) &\equiv (\tau_1 \sqcap \tau_2) \sqcup (\tau_1 \sqcap \tau_3) \\
(\tau_1 \rightarrow \tau_2) \sqcup (\tau_1' \rightarrow \tau_2') &\equiv (\tau_1 \sqcap \tau_1') \rightarrow (\tau_2 \sqcup \tau_2') \\
(\tau_1 \rightarrow \tau_2) \sqcap (\tau_1' \rightarrow \tau_2') &\equiv (\tau_1 \sqcup \tau_1') \rightarrow (\tau_2 \sqcap \tau_2')
\end{aligned}$$

The partial order $\tau_1 \leq \tau_2$ is defined as $\tau_1 \sqcup \tau_2 \equiv \tau_2$ or $\tau_1 \sqcap \tau_2 \equiv \tau_1$. \top and \perp are the least and greatest types. The above rules on function types imply the usual subtyping rule for function types, considering the definition of partial order:

$$\frac{\tau_1' \leq \tau_1 \quad \tau_2 \leq \tau_2'}{\tau_1 \rightarrow \tau_2 \leq \tau_1' \rightarrow \tau_2'}$$

Type schemes are not defined as the usual σ . Instead, a monotype τ already represent a type scheme by omitting the \forall quantifiers—all the free type variables are implicitly generalized.

Recursive types play an important role regarding the principality of type inference, but we omit them for simplicity.

POLAR TYPES Polar types are restrictions on the lattice operations; they should not occur arbitrarily in any position. Specifically, function outputs consist of types (τ_1, τ_2) from different branches, resulting in $\tau_1 \sqcup \tau_2$; a function input might be used in various ways (under different constraints), thus $\tau_1 \sqcap \tau_2$ is more suitable. In summary, \sqcup only arises in return types, while \sqcap only arises in argument types. Figure 2.10 formally defines the restriction, where positive types τ^+ describe return types, and negative types τ^- describe argument types.

An important consequence is that all the constraints are of the form $\tau^+ \leq \tau^-$, which represents the subtyping relation when using an output expression in a function application as argument. The following subtyping rules involving the lattice operations reflects their basic properties:

$$\frac{\tau_1^+ \leq \tau^- \quad \tau_2^+ \leq \tau^-}{\tau_1^+ \sqcup \tau_2^+ \leq \tau^-} \quad \frac{\tau^+ \leq \tau_1^- \quad \tau^+ \leq \tau_2^-}{\tau^- \leq \tau_1^- \sqcap \tau_2^-}.$$

Interestingly, the polar subtyping judgments avoids difficult judgments like $\tau_1 \sqcap \tau_2 \leq \tau$ or $\tau \leq \tau_1 \sqcup \tau_2$ through its syntactic restriction.

2.4.2 BIUNIFICATION

Type inference for MLsub, similar to that for ML, is mainly a unification algorithm. However, in presence of subtyping, equality-based unification loses information about subtyping constraints.

For the atomic constraint $\hat{\alpha} = \tau$ where $\hat{\alpha} \notin \text{FV}(\tau)$, the ML unification algorithm produces the substitution $[\tau/\hat{\alpha}]$. In contrast, an MLsub atomic constraint might be $\hat{\alpha} \leq \tau$, and the substitution $[\tau/\hat{\alpha}]$ treat the subtyping constraint as a equality constraint, which eliminates a whole set of possibilities.

Luckily, lattices in MLsub helps express subtyping constraints on types directly. The constraint $\hat{\alpha} \leq \tau$ ($\hat{\alpha} \notin \text{FV}(\tau)$) may produce the substitution $[\hat{\alpha} \sqcap \tau/\hat{\alpha}]$, since $\hat{\alpha} \sqcap \tau \leq \tau$. In the meantime, $\hat{\alpha} \sqcap \tau$ does not lose any expressiveness: for any τ_0 s.t. $\tau_0 \leq \tau$, picking $\hat{\alpha} = \tau_0$ gives $\hat{\alpha} \sqcap \tau = \tau_0$, and the substitution $[\hat{\alpha} \sqcap \tau/\hat{\alpha}]$ is equivalent to $[\tau_0/\hat{\alpha}]$.

In presence of polar types, the biunification algorithm of MLsub produces a *bisubstitution* $[\hat{\alpha} \sqcap \tau^-/\hat{\alpha}^-]$ against the constraint $\hat{\alpha} \leq \tau^-$, where only negative occurrences are substituted, keeping polar types properly “polarized”. For example, a positive type $\hat{\alpha} \rightarrow \hat{\alpha}$ becomes $(\hat{\alpha} \sqcap \tau^-) \rightarrow \hat{\alpha}$ under such substitution and remains a positive type. A more important fact

2 Background

is that this type is equivalent to the original type with the constraint $\hat{\alpha} \leq \tau^-$. Similarly, a constraint like $\tau^+ \leq \hat{\alpha}$ is reduced to a substitution $[\hat{\alpha} \sqcup \tau^+ / \hat{\alpha}^+]$.

For example, the *choose* function is typed $\forall a. a \rightarrow a \rightarrow a$ in ML. However, MLsub might also infer an equivalent type $\forall a b. a \rightarrow b \rightarrow a \sqcup b$. One can easily read the MLsub type in a form where constraints are explicitly stated

$$\forall a b c. a \rightarrow b \rightarrow c \text{ where } a \leq c, b \leq c.$$

Therefore, MLsub encodes the constraints directly onto types with the help of the lattice operations. Furthermore, a simplification step is taken after the type inference algorithm, reducing the size and improving readability of the type inferred.

As a result, biunification for MLsub extends unification for ML, accepting subtyping in addition to type schemes, while maintaining principality.

3 HIGHER-RANK POLYMORPHISM

SUBTYPING ALGORITHM

In this chapter, we present a new algorithm for polymorphic subtyping with mechanical formalizations in the Abella theorem prover. In particular, we show that the problem we have identified from existing systems can be overcome by means of *worklist judgments*. These are a form of judgement that turns the complicated global propagation of unifications into a simple local substitution. Moreover, we exploit several ideas in the recent inductive formulation of a type-inference algorithm by Dunfield and Krishnaswami [2013], which turn out to be useful for mechanization in a theorem prover.

Building on these ideas we develop a complete formalization of polymorphic subtyping in the Abella theorem prover. Moreover, we show that the algorithm is *sound*, *complete* and *decidable* with respect to the well-known declarative formulation of polymorphic subtyping by Odersky and Läufer [1996]. While these meta-theoretical results are not new, as far as we know our work is the first to mechanically formalize them.

3.1 OVERVIEW: POLYMORPHIC SUBTYPING

This section discusses Odersky and Läufer declarative subtyping rules further in depth, and identifies the challenges in formalizing a corresponding algorithmic version. Then the key ideas of our approach that address those challenges are introduced.

3.1.1 DECLARATIVE POLYMORPHIC SUBTYPING

In implicitly polymorphic type systems, the subtyping relation compares the degree of polymorphism of types. In short, if a polymorphic type A can always be instantiated to any instantiation of B , then A is “at least as polymorphic as” B , or we just say that A is “more polymorphic than” B , or $A \leq B$.

There is a very simple declarative formulation of polymorphic subtyping due to Odersky and Läufer [1996]. The syntax of this declarative system is shown in Figure 3.1. Types, represented by A, B, C , are the unit type 1 , type variables a, b , universal quantification $\forall a. A$

Type variables	a, b
Types	$A, B, C ::= 1 \mid a \mid \forall a. A \mid A \rightarrow B$
Monotypes	$\tau ::= 1 \mid a \mid \tau_1 \rightarrow \tau_2$
Contexts	$\Psi ::= \cdot \mid \Psi, a$

Figure 3.1: Syntax of Declarative System

$$\begin{array}{c}
 \boxed{\Psi \vdash A} \\
 \\
 \frac{}{\Psi \vdash 1} \text{wf}_{\text{dunit}} \quad \frac{a \in \Psi}{\Psi \vdash a} \text{wf}_{\text{dvar}} \quad \frac{\Psi \vdash A \quad \Psi \vdash B}{\Psi \vdash A \rightarrow B} \text{wf}_{\text{d}\rightarrow} \quad \frac{\Psi, a \vdash A}{\Psi \vdash \forall a. A} \text{wf}_{\text{d}\forall} \\
 \\
 \boxed{\Psi \vdash A \leq B} \\
 \\
 \frac{a \in \Psi}{\Psi \vdash a \leq a} \leq_{\text{Var}} \quad \frac{}{\Psi \vdash 1 \leq 1} \leq_{\text{Unit}} \quad \frac{\Psi \vdash B_1 \leq A_1 \quad \Psi \vdash A_2 \leq B_2}{\Psi \vdash A_1 \rightarrow A_2 \leq B_1 \rightarrow B_2} \leq_{\rightarrow} \\
 \\
 \frac{\Psi \vdash \tau \quad \Psi \vdash [\tau/a]A \leq B}{\Psi \vdash \forall a. A \leq B} \leq_{\forall L} \quad \frac{\Psi, a \vdash A \leq B}{\Psi \vdash A \leq \forall a. B} \leq_{\forall R}
 \end{array}$$

Figure 3.2: Well-formedness of Declarative Types and Declarative Subtyping

and function type $A \rightarrow B$. We allow nested universal quantifiers to appear in types, but not in monotypes. Contexts Ψ collect a list of type variables.

In Figure 3.2, we give the well-formedness and subtyping relation for the declarative system, which is identical to the subtyping relation introduced in Subsection 2.3.1.

3.1.2 FINDING SOLUTIONS FOR VARIABLE INSTANTIATION

The declarative system specifies the behavior of subtyping relations, but is not directly implementable: the rule $\leq_{\forall L}$ requires guessing a monotype τ . The core problem that an algorithm for polymorphic subtyping needs to solve is to find an algorithmic way to compute the monotypes, instead of guessing them. An additional challenge is that the declarative rule \leq_{\rightarrow} splits one judgment into two, and the (partial) solutions found for existential variables when processing the first judgment should be transferred to the second judgement.

DUNFIELD AND KRISHNASWAMI'S APPROACH An elegant algorithmic solution to computing the monotypes is presented by Dunfield and Krishnaswami [2013]. Their algorithmic subtyping judgement has the form:

$$\Psi \vdash A \leq B \dashv \Phi$$

A notable difference to the declarative judgement is the presence of a so-called *output context* Φ , which refines the *input context* Ψ with solutions for existential variables found while processing the two types being compared for subtyping. Both Ψ and Φ are *ordered contexts* with the same structure. Ordered contexts are particularly useful to keep track of the correct scoping for variables, and are a notable different to older type-inference algorithms [Damas and Milner 1982] that use global unification variables or constraints collected in a set.

Output contexts are useful to transfer information across judgements in Dunfield and Krishnaswami's approach. For example, the algorithmic rule corresponding to $\leq \rightarrow$ in their approach is:

$$\frac{\Psi \vdash B_1 <: A_1 \dashv \Phi \quad \Phi \vdash [\Phi]A_2 <: [\Phi]B_2 \dashv \Phi'}{\Psi \vdash A_1 \rightarrow A_2 <: B_1 \rightarrow B_2 \dashv \Phi'} <:\rightarrow$$

The information gathered by the output context when comparing the input types of the functions for subtyping is transferred to the second judgement by becoming the new input context, while any solution derived from the first judgment is applied to the types of the second judgment.

EXAMPLE If we want to show that $\forall a. a \rightarrow a$ is a subtype of $1 \rightarrow 1$, the declarative system will guess the proper $\tau = 1$ for Rule $\leq \forall L$:

$$\frac{\cdot \vdash 1 \quad \cdot \vdash 1 \rightarrow 1 \leq 1 \rightarrow 1}{\cdot \vdash \forall a. a \rightarrow a \leq 1 \rightarrow 1} \leq \forall L$$

Dunfield and Krishnaswami introduce an *existential variable*—denoted with $\hat{\alpha}, \hat{\beta}$ —whenever a monotype τ needs to be guessed. Below is a sample derivation of their algorithm; we omit the full set of algorithmic rules due to lack of space:

$$\frac{\frac{\frac{}{\hat{\alpha} \vdash 1 \leq \hat{\alpha} \dashv \hat{\alpha} = 1} \text{InstRSolve} \quad \frac{}{\hat{\alpha} = 1 \vdash 1 \leq 1 \vdash \hat{\alpha} = 1} <:\text{Unit}}{\hat{\alpha} \vdash \hat{\alpha} \rightarrow \hat{\alpha} \leq 1 \rightarrow 1 \dashv \hat{\alpha} = 1} <:\rightarrow}{\cdot \vdash \forall a. a \rightarrow a \leq 1 \rightarrow 1 \dashv \cdot} <:\forall L$$

The first step applies Rule $<:\forall L$, which introduces a fresh existential variable, $\hat{\alpha}$, and opens the left-hand-side \forall -quantifier with it. Next, Rule $<:\rightarrow$ splits the judgement in two. For the first branch, Rule **InstRSolve** satisfies $1 \leq \hat{\alpha}$ by solving $\hat{\alpha}$ to 1, and stores the solution

in its output context. The output context of the first branch is used as the input context of the second branch, and the judgment is updated according to current solutions. Finally, the second branch becomes a base case, and Rule \leq_{Unit} finishes the derivation, makes no change to the input context and propagates the output context back.

Dunfield and Krishnaswami's algorithmic specification is elegant and contains several useful ideas for a mechanical formalization of polymorphic subtyping. For example *ordered contexts* and *existential variables* enable a purely inductive formulation of polymorphic subtyping. However the binding/scoping structure of their algorithmic judgement is still fairly complicated and poses challenges when porting their approach to a theorem prover.

3.1.3 THE WORKLIST APPROACH

We inherit Dunfield and Krishnaswami's ideas of ordered contexts, existential variables and the idea of solving those variables, but drop output contexts. Instead our algorithmic rule has the form:

$$\Gamma \vdash \Omega$$

where Ω is a list of judgments $A \leq B$ instead of a single one. This judgement form, which we call *worklist judgement*, simplifies two aspects of Dunfield and Krishnaswami's approach.

Firstly, as already stated, there are no output contexts. Secondly the form of the ordered contexts become simpler. The transfer of information across judgements is simplified because all judgements share the input context. Moreover the order of the judgements in the list allows information discovered when processing the earlier judgements to be easily transferred to the later judgements. In the worklist approach the rule for function types is:

$$\frac{\Gamma \vdash B_1 \leq A_1; A_2 \leq B_2; \Omega}{\Gamma \vdash A_1 \rightarrow A_2 \leq B_1 \rightarrow B_2; \Omega} \leq_{a \rightarrow}$$

The derivation of the previous example with the worklist approach is:

$$\begin{array}{c} \frac{}{\cdot \vdash \cdot} \text{a_nil} \\ \frac{}{\cdot \vdash 1 \leq 1; \cdot} \leq_{a \text{unit}} \\ \frac{}{\hat{a} \vdash 1 \leq \hat{a}; \hat{a} \leq 1; \cdot} \leq_{a \text{solve_ex}} \\ \frac{}{\hat{a} \vdash \hat{a} \rightarrow \hat{a} \leq 1 \rightarrow 1; \cdot} \leq_{a \rightarrow} \\ \frac{}{\cdot \vdash \forall a. a \rightarrow a \leq 1 \rightarrow 1; \cdot} \leq_{a \forall L} \end{array}$$

To derive $\cdot \vdash \forall a. a \rightarrow a \leq 1 \rightarrow 1$ with the worklist approach, we first introduce an existential variable and change the judgement to $\hat{\alpha} \vdash \hat{\alpha} \rightarrow \hat{\alpha} \leq 1 \rightarrow 1; \cdot$. Then, we split the judgment in two for the function types and the derivation comes to $\hat{\alpha} \vdash 1 \leq \hat{\alpha}; \hat{\alpha} \leq 1; \cdot$. When the first judgment is solved with $\hat{\alpha} = 1$, we immediately remove $\hat{\alpha}$ from the context, while propagating the solution as a substitution to the rest of the judgment list, resulting in $\cdot \vdash 1 \leq 1; \cdot$, which finishes the derivation in two trivial steps.

With this form of eager propagation, solutions no longer need to be recorded in contexts, simplifying the encoding and reasoning in a proof assistant.

KEY RESULTS Both the declarative and algorithmic systems are formalized in Abella. We have proven 3 important properties for this algorithm: *decidability*, ensuring that the algorithm always terminates; and *soundness* and *completeness*, showing the equivalence of the declarative and algorithmic systems.

3.2 A WORKLIST ALGORITHM FOR POLYMORPHIC SUBTYPING

This section presents our algorithm for polymorphic subtyping. A novel aspect of our algorithm is the use of worklist judgments: a form of judgement that facilitates the propagation of information.

3.2.1 SYNTAX AND WELL-FORMEDNESS OF THE ALGORITHMIC SYSTEM

Figure 4.4 shows the syntax and the well-formedness judgement.

EXISTENTIAL VARIABLES In order to solve the unknown types τ , the algorithmic system extends the declarative syntax of types with *existential variables* $\hat{\alpha}$. They behave like unification variables, but are not globally defined. Instead, the ordered *algorithmic context*, inspired by Dunfield and Krishnaswami [2013], defines their scope. Thus the type τ represented by the corresponding existential variable is always bound in the corresponding declarative context Ψ .

WORKLIST JUDGEMENTS The form of our algorithmic judgements is non-standard. Our algorithm keeps track of an explicit list of outstanding work: the list Ω of (reified) *algorithmic judgements* of the form $A \leq B$, to which a substitution can be applied once and for all to propagate the solution of an existential variable.

Type variables	a, b
Existential variables	$\hat{\alpha}, \hat{\beta}$
Algorithmic types	$A, B, C ::= 1 \mid a \mid \hat{\alpha} \mid \forall a. A \mid A \rightarrow B$
Algorithmic context	$\Gamma ::= \cdot \mid \Gamma, a \mid \Gamma, \hat{\alpha}$
Algorithmic judgments	$\Omega ::= \cdot \mid A \leq B; \Omega$
$\boxed{\Gamma \vdash A}$	
$\frac{}{\Gamma \vdash 1} \text{wf}_{\text{aunit}} \quad \frac{a \in \Gamma}{\Gamma \vdash a} \text{wf}_{\text{avar}} \quad \frac{\hat{\alpha} \in \Gamma}{\Gamma \vdash \hat{\alpha}} \text{wf}_{\text{aexvar}}$ $\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \rightarrow B} \text{wf}_{\text{a}\rightarrow} \quad \frac{\Gamma, a \vdash A}{\Gamma \vdash \forall a. A} \text{wf}_{\text{a}\forall}$	

Figure 3.3: Syntax and Well-Formedness Judgement for the Algorithmic System.

HOLE NOTATION To facilitate context manipulation, we use the syntax $\Gamma[\Gamma_M]$ to denote a context of the form $\Gamma_L, \Gamma_M, \Gamma_R$ where Γ is the context $\Gamma_L, \bullet, \Gamma_R$ with a hole (\bullet). Hole notations with the same name implicitly share the same Γ_L and Γ_R . A multi-hole notation like $\Gamma[\hat{\alpha}][\hat{\beta}]$ means $\Gamma_1, \hat{\alpha}, \Gamma_2, \hat{\beta}, \Gamma_3$.

3.2.2 ALGORITHMIC SUBTYPING

The algorithmic subtyping judgement, defined in Figure 4.5, has the form $\Gamma \vdash \Omega$, where Ω collects multiple subtyping judgments $A \leq B$. The algorithm treats Ω as a worklist. In every step it takes one task from the worklist for processing, possibly pushes some new tasks on the worklist, and repeats this process until the list is empty. This last and single base case is handled by Rule `a_nil`. The remaining rules all deal with the first task in the worklist. Logically we can discern 3 groups of rules.

Firstly, we have five rules that are similar to those in the declarative system, mostly just adapted to the worklist style. For instance, Rule $\leq_{\text{a}\rightarrow}$ consumes one judgment and pushes two to the worklist. A notable difference with the declarative Rule $\leq_{\forall\text{L}}$ is that Rule $\leq_{\text{a}\forall\text{L}}$ requires no guessing of a type τ to instantiate the polymorphic type $\forall a. A$, but instead introduces an existential variable $\hat{\alpha}$ to the context and to A . In accordance with the declarative system, where the monotype τ should be bound in the context Ψ , here $\hat{\alpha}$ should only be solved to a monotype bound in Γ . More generally, for any algorithmic context $\Gamma[\hat{\alpha}]$, the algorithmic variable $\hat{\alpha}$ can only be solved to a monotype that is well-formed with respect to Γ_L .

$$\boxed{\Gamma \vdash \Omega}$$

$$\frac{}{\Gamma \vdash \cdot} \leq_{\text{a nil}}$$

$$\frac{\Gamma \vdash \Omega}{\Gamma \vdash 1 \leq 1; \Omega} \leq_{\text{a unit}} \quad \frac{a \in \Gamma \quad \Gamma \vdash \Omega}{\Gamma \vdash a \leq a; \Omega} \leq_{\text{a var}} \quad \frac{\hat{\alpha} \in \Gamma \quad \Gamma \vdash \Omega}{\Gamma \vdash \hat{\alpha} \leq \hat{\alpha}; \Omega} \leq_{\text{a exvar}}$$

$$\frac{\Gamma \vdash B_1 \leq A_1; A_2 \leq B_2; \Omega}{\Gamma \vdash A_1 \rightarrow A_2 \leq B_1 \rightarrow B_2; \Omega} \leq_{\text{a } \rightarrow}$$

$$\frac{\hat{\alpha} \text{ fresh} \quad \Gamma, \hat{\alpha} \vdash [\hat{\alpha}/a]A \leq B; \Omega}{\Gamma \vdash \forall a. A \leq B; \Omega} \leq_{\text{a } \forall \text{L}} \quad \frac{b \text{ fresh} \quad \Gamma, b \vdash A \leq B; \Omega}{\Gamma \vdash A \leq \forall b. B; \Omega} \leq_{\text{a } \forall \text{R}}$$

$$\frac{\hat{\alpha} \notin FV(A) \cup FV(B) \quad \Gamma[\hat{\alpha}_1, \hat{\alpha}_2] \vdash \hat{\alpha}_1 \rightarrow \hat{\alpha}_2 \leq A \rightarrow B; [\hat{\alpha}_1 \rightarrow \hat{\alpha}_2/\hat{\alpha}]\Omega}{\Gamma[\hat{\alpha}] \vdash \hat{\alpha} \leq A \rightarrow B; \Omega} \leq_{\text{a instL}}$$

$$\frac{\hat{\alpha} \notin FV(A) \cup FV(B) \quad \Gamma[\hat{\alpha}_1, \hat{\alpha}_2] \vdash A \rightarrow B \leq \hat{\alpha}_1 \rightarrow \hat{\alpha}_2; [\hat{\alpha}_1 \rightarrow \hat{\alpha}_2/\hat{\alpha}]\Omega}{\Gamma[\hat{\alpha}] \vdash A \rightarrow B \leq \hat{\alpha}; \Omega} \leq_{\text{a instR}}$$

$$\frac{\Gamma[\hat{\alpha}][] \vdash [\hat{\alpha}/\hat{\beta}]\Omega}{\Gamma[\hat{\alpha}][\hat{\beta}] \vdash \hat{\alpha} \leq \hat{\beta}; \Omega} \leq_{\text{a solve_ex}} \quad \frac{\Gamma[\hat{\alpha}][] \vdash [\hat{\alpha}/\hat{\beta}]\Omega}{\Gamma[\hat{\alpha}][\hat{\beta}] \vdash \hat{\beta} \leq \hat{\alpha}; \Omega} \leq_{\text{a solve_ex}'}$$

$$\frac{\Gamma[a][] \vdash [a/\hat{\beta}]\Omega}{\Gamma[a][\hat{\beta}] \vdash a \leq \hat{\beta}; \Omega} \leq_{\text{a solve_var}} \quad \frac{\Gamma[a][] \vdash [a/\hat{\beta}]\Omega}{\Gamma[a][\hat{\beta}] \vdash \hat{\beta} \leq a; \Omega} \leq_{\text{a solve_var}'}$$

$$\frac{\Gamma[] \vdash [1/\hat{\alpha}]\Omega}{\Gamma[\hat{\alpha}] \vdash \hat{\alpha} \leq 1; \Omega} \leq_{\text{a solve_unit}} \quad \frac{\Gamma[] \vdash [1/\hat{\alpha}]\Omega}{\Gamma[\hat{\alpha}] \vdash 1 \leq \hat{\alpha}; \Omega} \leq_{\text{a solve_unit}'}$$

Figure 3.4: Algorithmic Subtyping

Secondly, Rules $\leq_{\text{a instL}}$ and $\leq_{\text{a instR}}$ partially instantiate existential types $\hat{\alpha}$, to function types. The domain and range of the new function type are undetermined: they are set to two fresh existential variables $\hat{\alpha}_1$ and $\hat{\alpha}_2$. To make sure that $\hat{\alpha}_1 \rightarrow \hat{\alpha}_2$ has the same scope as $\hat{\alpha}$, the new variables $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are inserted in the same position in the context where the old variable $\hat{\alpha}$ was. To propagate the instantiation to the remainder of the worklist, $\hat{\alpha}$ is substituted for $\hat{\alpha}_1 \rightarrow \hat{\alpha}_2$ in Ω . The *occurs-check* side-condition is necessary to prevent a diverging infinite instantiation. For example $1 \rightarrow \hat{\alpha} \leq \hat{\alpha}$ would diverge with no such check.

Thirdly, in the remaining six rules an existential variable can be immediately solved. Each of the six similar rules removes an existential variable from the context, performs a substitution on the remainder of the worklist and continues.

$$\begin{array}{c}
 \frac{}{\hat{\alpha}_1 \vdash \cdot \text{a_nil}} \\
 \frac{}{\hat{\alpha}_1 \vdash 1 \leq 1; \cdot} \leq_{\text{aunit}} \\
 \frac{}{\hat{\alpha}_1, \hat{\alpha}_2 \vdash \hat{\alpha}_1 \leq \hat{\alpha}_2; 1 \leq 1; \cdot} \leq_{\text{a solve_ex}} \\
 \frac{}{\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta} \vdash \hat{\alpha}_1 \leq \hat{\beta}; \hat{\beta} \leq \hat{\alpha}_2; 1 \leq 1; \cdot} \leq_{\text{a solve_ex}} \\
 \frac{}{\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta} \vdash \hat{\beta} \rightarrow \hat{\beta} \leq \hat{\alpha}_1 \rightarrow \hat{\alpha}_2; 1 \leq 1; \cdot} \leq_{\text{a} \rightarrow} \\
 \frac{}{\hat{\alpha}, \hat{\beta} \vdash \hat{\beta} \rightarrow \hat{\beta} \leq \hat{\alpha}; 1 \leq 1; \cdot} \leq_{\text{a instR}} \\
 \frac{}{\hat{\alpha} \vdash \forall a. a \rightarrow a \leq \hat{\alpha}; 1 \leq 1; \cdot} \leq_{\text{a} \forall \text{L}} \\
 \frac{}{\hat{\alpha} \vdash \hat{\alpha} \rightarrow 1 \leq (\forall a. a \rightarrow a) \rightarrow 1; \cdot} \leq_{\text{a} \rightarrow} \\
 \frac{}{\cdot \vdash \forall a. a \rightarrow 1 \leq (\forall a. a \rightarrow a) \rightarrow 1; \cdot} \leq_{\text{a} \forall \text{L}}
 \end{array}
 \quad
 \begin{array}{c}
 \frac{\text{stuck}}{\hat{\alpha}, b \vdash \hat{\alpha} \leq b; \cdot} ? \\
 \frac{}{\hat{\alpha} \vdash \hat{\alpha} \leq \forall b. b; \cdot} \leq_{\text{a} \forall \text{R}} \\
 \frac{}{\hat{\alpha} \vdash 1 \leq 1; \hat{\alpha} \leq \forall b. b; \cdot} \leq_{\text{aunit}} \\
 \frac{}{\hat{\alpha} \vdash 1 \rightarrow \hat{\alpha} \leq 1 \rightarrow \forall b. b; \cdot} \leq_{\text{a} \rightarrow} \\
 \frac{}{\cdot \vdash \forall a. 1 \rightarrow a \leq 1 \rightarrow \forall b. b; \cdot} \leq_{\text{a} \forall \text{L}}
 \end{array}$$

Figure 3.5: Successful and Failing Derivations for the Algorithmic Subtyping Relation

The algorithm on judgment list is designed to share the context across all judgments. However, the declarative system does not share a single context in its derivation. This gap is filled by strengthening and weakening lemmas of both systems, where most of them are straightforward to prove, except for the strengthening lemma of the declarative system, which is a little trickier.

EXAMPLE We illustrate the subtyping rules through a sample derivation in the left of Figure 3.5, which shows that that $\forall a. a \rightarrow 1 \leq (\forall a. a \rightarrow a) \rightarrow 1$. Thus the derivation starts with an empty context and a judgment list with only one element.

In step 1, we have only one judgment, and that one has a top-level \forall on the left hand side. So the only choice is rule $\leq_{\text{a} \forall \text{L}}$, which opens the universally quantified type with an unknown existential variable $\hat{\alpha}$. Variable $\hat{\alpha}$ will be solved later to some monotype that is well-formed within the context before $\hat{\alpha}$. That is, the empty context \cdot in this case. In step 2, rule $\leq_{\text{a} \rightarrow}$ is applied to the worklist, splitting the first judgment into two. Step 3 is similar to step 1, where the left-hand-side \forall of the first judgment is opened according to rule $\leq_{\text{a} \forall \text{L}}$ with a fresh existential variable. In step 4, the first judgment has an arrow on the left hand side, but the right-hand-side type is an existential variable. It is obvious that $\hat{\alpha}$ should be solved to a monotype of the form $\sigma \rightarrow \tau$. Rule instR implements this, but avoids guessing σ and τ by “splitting” $\hat{\alpha}$ into two existential variables, $\hat{\alpha}_1$ and $\hat{\alpha}_2$, which will be solved to some σ and τ later. Step 5 applies Rule $\leq_{\text{a} \rightarrow}$ again. Notice that after the split, $\hat{\beta}$ appears in two judgments. When the first $\hat{\beta}$ is solved during any step of derivation, the next $\hat{\beta}$ will be substituted by that solution. This propagation mechanism ensures the consistent solution of the variables, while keeping the context as simple as possible. Steps 6 and 7 solve existential variables. The existential variable that is right-most in the context is always solved in terms

of the other. Therefore in step 6, $\hat{\beta}$ is solved in terms of $\hat{\alpha}_1$, and in step 7, $\hat{\alpha}_2$ is solved in terms of $\hat{\alpha}_1$. Additionally, in step 6, when $\hat{\beta}$ is solved, the substitution $[\hat{\alpha}_1/\hat{\beta}]$ is propagated to the rest of the judgment list, and thus the second judgment becomes $\hat{\alpha}_1 \leq \hat{\alpha}_2$. Steps 8 and 9 trivially finish the derivation. Notice that $\hat{\alpha}_1$ is not instantiated at the end. This means that any well-scoped instantiation is fine.

A FAILING DERIVATION We illustrate the role of ordered contexts through another example: $\forall a. 1 \rightarrow a \leq 1 \rightarrow \forall b. b$. From the declarative perspective, a should be instantiated to some τ first, then b is introduced to the context, so that $b \notin FV(\tau)$. As a result, we cannot find τ such that $\tau \leq b$. The right of Figure 3.5 shows the algorithmic derivation, which also fails due to the scoping— $\hat{\alpha}$ is introduced earlier than b , thus it cannot be solved to b .

3.3 METATHEORY

This section presents the 3 main meta-theoretical results that we have proved in Abella. The first two are soundness and completeness of our algorithm with respect to Odersky and Läufer’s declarative subtyping. The third result is our algorithm’s decidability.

3.3.1 TRANSFER TO THE DECLARATIVE SYSTEM

To state the correctness of the algorithmic subtyping rules, Figure 3.6 introduces two *transfer* judgements to relate the declarative and the algorithmic system. The first judgement, transfer of contexts $\Gamma \rightarrow \Psi$, removes existential variables from the algorithmic context Γ to obtain a declarative context Ψ . The second judgement, transfer of the judgement list $\Gamma \mid \Omega \rightsquigarrow \Omega'$, replaces all occurrences of existential variables in Ω by well-scoped mono-types. Notice that this judgment is not decidable, i.e. a pair of Γ and Ω may be related with multiple Ω' . However, if there exists some substitution that transforms Ω to Ω' , and each subtyping judgment in Ω' holds, we know that Ω is potentially satisfiable.

The following two lemmas generalize Rule $\rightsquigarrow_{\text{exvar}}$ from substituting the first existential variable to substituting any existential variable.

Lemma 3.1 (Insert). *If $\Gamma \rightarrow \Psi$ and $\Psi \vdash \tau$ and $\Gamma, \Gamma_1 \mid [\tau/\hat{\alpha}]\Omega \rightsquigarrow \Omega'$, then $\Gamma, \hat{\alpha}, \Gamma_1 \mid \Omega \rightsquigarrow \Omega'$.*

Lemma 3.2 (Extract). *If $\Gamma, \hat{\alpha}, \Gamma_1 \mid \Omega \rightsquigarrow \Omega'$, then $\exists \tau$ s.t. $\Gamma \rightarrow \Psi, \Psi \vdash \tau$ and $\Gamma, \Gamma_1 \mid [\tau/\hat{\alpha}]\Omega \rightsquigarrow \Omega'$.*

In order to match the shape of algorithmic subtyping relation for the following proofs, we define a relation $\Psi \vdash \Omega$ for the declarative system, meaning that all the declarative judgments hold under context Ψ .

$$\begin{array}{c}
 \boxed{\Gamma \rightarrow \Psi} \\
 \frac{}{\cdot \rightarrow \cdot} \rightarrow \cdot \quad \frac{\Gamma \rightarrow \Psi}{\Gamma, a \rightarrow \Psi, a} \rightarrow \text{var} \quad \frac{\Gamma \rightarrow \Psi}{\Gamma, \hat{\alpha} \rightarrow \Psi} \rightarrow \text{exvar} \\
 \boxed{\Gamma \mid \Omega \rightsquigarrow \Omega'} \\
 \frac{}{\cdot \mid \Omega \rightsquigarrow \Omega} \rightsquigarrow \cdot \quad \frac{\Gamma \mid \Omega \rightsquigarrow \Omega'}{\Gamma, a \mid \Omega \rightsquigarrow \Omega'} \rightsquigarrow \text{var} \quad \frac{\Gamma \rightarrow \Psi \quad \Psi \vdash \tau \quad \Gamma \mid [\tau/\hat{\alpha}]\Omega \rightsquigarrow \Omega'}{\Gamma, \hat{\alpha} \mid \Omega \rightsquigarrow \Omega'} \rightsquigarrow \text{exvar}
 \end{array}$$

Figure 3.6: Transfer Rules

Definition 1 (Declarative Subtyping Worklist).

$$\Psi \vdash \Omega := \forall (A \leq B) \in \Omega, \Psi \vdash A \leq B$$

3.3.2 SOUNDNESS

Our algorithm is sound with respect to the declarative specification. For any derivation of a list of algorithmic judgments $\Gamma \vdash \Omega$, we can find a valid transfer $\Gamma \mid \Omega \rightsquigarrow \Omega'$ such that all judgments in Ω' hold in Ψ , with $\Gamma \rightarrow \Psi$.

Theorem 3.3 (Soundness). *If $\Gamma \vdash \Omega$ and $\Gamma \rightarrow \Psi$, then there exists Ω' , s.t. $\Gamma \mid \Omega \rightsquigarrow \Omega'$ and $\Psi \vdash \Omega'$.*

The proof proceeds by induction on the derivation of $\Gamma \vdash \Omega$, finished off by appropriate applications of the insertion and extraction lemmas.

3.3.3 COMPLETENESS

Completeness of the algorithm means that any declarative derivation has an algorithmic counterpart.

Theorem 3.4 (Completeness). *If $\Psi \vdash \Omega'$ and $\Gamma \rightarrow \Psi$ and $\Gamma \mid \Omega \rightsquigarrow \Omega'$, then $\Gamma \vdash \Omega$.*

The proof proceeds by induction on the derivation of $\Psi \vdash \Omega'$. As the declarative system does not involve information propagation across judgments, the induction can focus on the subtyping derivation of the first judgment without affecting other judgments. The difficult cases correspond to the \leq_{aInstL} and \leq_{aInstR} rules. When the proof by induction on $\Psi \vdash \Omega'$ reaches the $\leq \rightarrow$ case, the first declarative judgment has a shape like $A_1 \rightarrow A_2 \leq B_1 \rightarrow B_2$. One of the possible cases for the first corresponding algorithmic judgement is

$\hat{\alpha} \leq A \rightarrow B$. However, the case analysis does not indicate that $\hat{\alpha}$ is fresh in A and B . Thus we cannot apply Rule $\leq_{\text{a inst L}}$ and make use of the induction hypothesis. The following lemma helps us out in those cases: it rules out subtypings with infinite types as solutions (e.g. $\hat{\alpha} \leq 1 \rightarrow \hat{\alpha}$) and guarantees that $\hat{\alpha}$ is free in A and B .

Lemma 3.5 (Prune Transfer for Instantiation). *If $\Psi \vdash A_1 \rightarrow A_2 \leq B_1 \rightarrow B_2; \Omega'$ and $\Gamma \rightarrow \Psi$ and $\Gamma \mid (\hat{\alpha} \leq A \rightarrow B; \Omega) \rightsquigarrow (A_1 \rightarrow A_2 \leq B_1 \rightarrow B_2; \Omega')$, then $\hat{\alpha} \notin FV(A) \cup FV(B)$.*

A similar lemma holds for the symmetric case $(A \rightarrow B \leq \hat{\alpha}; \Omega)$.

3.3.4 DECIDABILITY

The third key result for our algorithm is decidability.

Theorem 3.6 (Decidability). *Given any well-formed judgment list Ω under Γ , it is decidable whether $\Gamma \vdash \Omega$ or not.*

We have proven this theorem by means of a lexicographic group of induction measurements $\langle |\Omega|_{\forall}, |\Gamma|_{\hat{\alpha}}, |\Omega|_{\rightarrow} \rangle$ on the worklist Ω and algorithmic context Γ . The worklist measures $|\cdot|_{\forall}$ and $|\cdot|_{\rightarrow}$ count the number of universal quantifiers and function types respectively.

Definition 2 (Worklist Measures).

$$\begin{aligned} |1|_{\forall} &= |a|_{\forall} = |\hat{\alpha}|_{\forall} = 0 & |1|_{\rightarrow} &= |a|_{\rightarrow} = |\hat{\alpha}|_{\rightarrow} = 0 \\ |A \rightarrow B|_{\forall} &= |A|_{\forall} + |B|_{\forall} & |A \rightarrow B|_{\rightarrow} &= |A|_{\rightarrow} + |B|_{\rightarrow} + 1 \\ |\forall x.A|_{\forall} &= |A|_{\forall} + 1 & |\forall x.A|_{\rightarrow} &= |A|_{\rightarrow} \\ |\Omega|_{\forall} &= \sum_{A \leq B \in \Omega} |A|_{\forall} + |B|_{\forall} & |\Omega|_{\rightarrow} &= \sum_{A \leq B \in \Omega} |A|_{\rightarrow} + |B|_{\rightarrow} \end{aligned}$$

The context measure $|\cdot|_{\hat{\alpha}}$ counts the number of unsolved existential variables.

Definition 3 (Context Measure).

$$|\cdot|_{\hat{\alpha}} = 0 \quad |\Gamma, a|_{\hat{\alpha}} = |\Gamma|_{\hat{\alpha}} \quad |\Gamma, \hat{\alpha}|_{\hat{\alpha}} = |\Gamma|_{\hat{\alpha}} + 1$$

It is not difficult to see that all but two of the algorithm's rules decrease one of the three measures. The two exceptions are the Rules $\leq_{\text{a inst L}}$ and $\leq_{\text{a inst R}}$; both increment the number of existential variables and the number of function types without affecting the number of universal quantifiers. To handle these rules, we handle a special class of judgements, which we call *instantiation judgements* Ω_i , separately. They take the form:

Definition 4 (Ω_i).

$$\Omega_i := \cdot \mid \hat{\alpha} \leq A; \Omega'_i \mid A \leq \hat{\alpha}; \Omega'_i \quad \text{where } \hat{\alpha} \notin FV(A) \cup FV(\Omega'_i)$$

These instantiation judgements are these ones consumed and produced by the Rules $\leq_{\text{a inst L}}$ and $\leq_{\text{a inst R}}$. The following lemma handles their decidability.

Lemma 3.7 (Instantiation Decidability). *For any context Γ and judgment list Ω_i, Ω , it is decidable whether $\Gamma \vdash \Omega_i, \Omega$ if both of the conditions hold*

- 1) $\forall \Gamma', \Omega' \text{ s.t. } |\Omega'|_{\forall} < |\Omega_i, \Omega|_{\forall}$, *it is decidable whether $\Gamma' \vdash \Omega'$.*
- 2) $\forall \Gamma', \Omega' \text{ s.t. } |\Omega'|_{\forall} = |\Omega_i, \Omega|_{\forall} \text{ and } |\Gamma'|_{\hat{\alpha}} = |\Gamma|_{\hat{\alpha}} - |\Omega_i|$, *it is decidable whether $\Gamma' \vdash \Omega'$.*

In other words, for any instantiation judgment prefix Ω_i , the algorithm either reduces the number of \forall 's or solves one existential variable per instantiation judgment. The proof of this lemma is by induction on the measure $2 * |\Omega_i|_{\rightarrow} + |\Omega_i|$ of the instantiation judgment list.

In summary, the decidability theorem can be shown through a lexicographic group of induction measurements $\langle |\Omega|_{\forall}, |\Omega|_{\hat{\alpha}}, |\Omega|_{\rightarrow} \rangle$. The critical case is that, whenever we encounter an instantiation judgment at the front of the worklist, we refer to Lemma 3.7, which reduces the number of unsolved variables by consuming that instantiation judgment, or reduces the number of \forall -quantifiers. Other cases are relatively straightforward.

3.4 THE CHOICE OF ABELLA

We have chosen the Abella (v2.0.5) proof assistant [Gacek 2008] to develop our formalization. Our development is only based on the reasoning logic of Abella, and does not make use of its specification logic. Abella is particularly helpful due to its built-in support for variable bindings, and its λ -tree syntax [Miller 2000] is a form of HOAS, which helps with the encoding and reasoning about substitutions. For instance, the type $\forall x. x \rightarrow a$ is encoded as `all (x \ arrow x a)`, where `x \ arrow x a` is a lambda abstraction in Abella. An opening $[b/x](x \rightarrow a)$ is encoded as an application `(x \ arrow x a) b`, which can be simplified(evaluated) to `arrow b a`. Name supply and freshness conditions are controlled by the ∇ -quantifier. The expression `new! x, F` means that `x` is a unique variable in `F`, i.e. it is different from any other names occurring elsewhere. Such variables are called nominal constants. They can be of any type, in other words, every type may contain unlimited number of such atomic nominal constants.

ENCODING OF THE DECLARATIVE SYSTEM As a concrete example, our declarative context and well-formedness rules are encoded as follows.

```
Kind ty      type.
Type i       ty.           % the unit type
```

```

Type all   (ty → ty) → ty. % forall-quantifier
Type arrow ty → ty → ty.  % function type
Type bound ty → o.         % variable collection in contexts

Define env : olist → prop by
  env nil;
  nabla x, env (bound x :: E) := env E.

Define wft : olist → ty → prop by
  wft E i;
  nabla x, wft (E x) x := nabla x, member (bound x) (E x);
  wft E (arrow A B) := wft E A ∧ wft E B;
  wft E (all A) := nabla x, wft (bound x :: E) (A x).
    
```

We use the type `olist` just as normal list of `o` with two constructors, namely `nil : olist` and `(::) : o → olist → olist`, where `o` purely means “the element type of `olist`”. The `member : o → olist → prop` relation is also pre-defined. The second case of the relation `wft` states rule `wfdvar`. The encoding `(E x)` basically means that the context *may* contain `x`. If we write `(E x)` as `E`, then the context should not contain `x`, and both `wft E x` and `member (bound x) E` make no sense. Instead, we treat `E : ty → olist` as an *abstract structure* of a context, such as `x \ bound x :: bound a :: nil`. For the fourth case of the relation `wft`, the type $\forall x. A$ in our target language is expressed as `(all A)`, and its opening `A`, `(A x)`.

ENCODING OF THE ALGORITHMIC SYSTEM In terms of the algorithmic system, notably, Abella handles the $\leq_a \text{instL}$ and $\leq_a \text{instR}$ rules in a nice way:

```

% sub_alg_list : enva → [subty_judgment] → prop
Define subal : olist → olist → prop by
  subal E nil;
  subal E (subt i i :: Exp) := subal E Exp;
  % some cases omitted ...
  % <: instL
  nabla x, subal (E x) (subt x (arrow A B) :: Exp x) :=
    exists E1 E2 F, nabla x y z, append E1 (exvar x :: E2) (E x) ∧
      append E1 (exvar y :: exvar z :: E2) (F y z) ∧
      subal (F y z) (subt (arrow y z) (arrow A B) :: Exp (arrow y z));
  % <: instR is symmetric to <: instL, omitted here
  % other cases omitted ...
    
```

Thanks to the way Abella deals with nominal constants, the pattern `subt x (arrow A B)` implicitly states that $x \notin FV(A) \wedge x \notin FV(B)$. If the condition were not required, we would have encoded the pattern as `subt x (arrow (A x) (B x))` instead.

File(s)	SLOC	# of Theorems	Description
olist.thm, nat.thm	303	55	Basic data structures
higher.thm, order.thm	164	15	Declarative system
higher_alg.thm	618	44	Algorithmic system
trans.thm	411	46	Transfer
sound.thm	166	2	Soundness theorem
depth.thm	143	12	Definition of depth
complete.thm	626	28	Lemmas and Completeness theorem
decidable.thm	1077	53	Lemmas and Decidability theorem
Total	3627	267	(33 definitions in total)

Figure 3.7: Statistics for the proof scripts

3.4.1 STATISTICS AND DISCUSSION

Some basic statistics on our proof script are shown in Figure 3.7. The proof consists of 3627 lines of code with a total of 33 definitions and 267 theorems. We have to mention that Abella provides few built-in tactics and does not support user-defined ones, and we would reduce significant lines of code if Abella provided more handy tactics. Moreover, the definition of natural numbers, the plus operation and less-than relation are defined within our proof due to Abella's lack of packages. However, the way Abella deals with name bindings is very helpful for type system formalizations and substitution-intensive formalizations, such as this one.

4 HIGHER-RANK POLYMORPHISM

WORKLIST ALGORITHM

This chapter presents the first fully mechanized formalization of the metatheory for higher-ranked polymorphic type inference. The system that we formalize is the bidirectional type system by Dunfield and Krishnaswami [2013]. We chose DK’s type system because it is quite elegant, well-documented and it comes with detailed manually written proofs. Furthermore the system is adopted in practice by a few real implementations of functional languages, including PureScript and Unison. The DK type system has two variants: a declarative and an algorithmic one. The two variants have been *manually* proved to be *sound*, *complete* and *decidable*. We present a mechanical formalization in the Abella theorem prover [Gacek 2008] for DK’s declarative type system using a different algorithm.

CHALLENGES While our initial goal was to formalize both DK’s declarative and algorithmic versions, we faced technical challenges with the latter, prompting us to find an alternative formulation.

The first challenge that we faced were missing details as well as a few incorrect proofs and lemmas in DK’s formalization. While DK’s original formalization comes with very well written manual proofs, there are still several details missing. These complicate the task of writing a mechanically verified proof. Moreover some proofs and lemmas are wrong and, in some cases, it is not clear to us how to fix them.

Despite the problems in DK’s manual formalization, we believe that these problems do not invalidate their work and that their results are still true. In fact we have nothing but praise for their detailed and clearly written metatheory and proofs, which provided invaluable help to our own work. We expect that for most non-trivial manual proofs similar problems exist, so this should not be understood as a sign of sloppiness on their part. Instead it should be an indicator that reinforces the arguments for mechanical formalizations: manual formalizations are error-prone due to the multiple tedious details involved in them. There are several other examples of manual formalizations that were found to have similar problems. For example, Klein et al. [2012] mechanized formalizations in Redex for nine ICFP 2009 papers and all were found to have mistakes.

Another challenge was variable binding. Type inference algorithms typically do not rely simply on local environments but instead propagate information across judgments. While local environments are well-studied in mechanical formalizations, there is little work on how to deal with the complex forms of binding employed by type inference algorithms in theorem provers. To keep track of variable scoping, DK’s algorithmic version employs input and output contexts to track information that is discovered through type inference. However modeling output contexts in a theorem prover is non-trivial.

Due to those two challenges, our work takes a different approach by refining and extending the idea of *worklist judgments* [Zhao et al. 2018], proposed recently to mechanically formalize an algorithm for *polymorphic subtyping* [Odersky and Läufer 1996]. A key innovation in our work is how to adapt the idea of worklist judgments to *inference judgments*, which are not needed for polymorphic subtyping, but are necessary for type-inference. The idea is to use a *continuation passing style* to enable the transfer of inferred information across judgments. A further refinement to the idea of worklist judgments is the *unification between ordered contexts* [Dunfield and Krishnaswami 2013; Gundry et al. 2010] and *worklists*. This enables precise scope tracking of free variables in judgments. Furthermore it avoids the duplication of context information across judgments in worklists that occurs in other techniques [Abel and Pientka 2011; Reed 2009]. Despite the use of a different algorithm we prove the same results as DK, although with significantly different proofs and proof techniques. The calculus and its metatheory have been fully formalized in the Abella theorem prover [Gacek 2008].

4.1 OVERVIEW

This section starts with a discussion on DK’s declarative type system. Then it introduces several techniques that have been used in algorithmic formulations, and which have influenced our own algorithmic design. Finally we introduce the novelties of our new algorithm. In particular the support for inference judgments in worklist judgments, and a new form of worklist judgment that unifies *ordered contexts* and the worklists themselves.

4.1.1 DK’S DECLARATIVE SYSTEM

Subsection 2.3.1 introduces DK’s declarative subtyping and typing systems. We also duplicate the rules here for the convenience of the reader.

OVERLAPPING RULES A problem we found in the declarative system is that some of the rules overlap with each other. Declarative subtyping rules $\leq\forall\text{L}$ and $\leq\forall\text{R}$ both match the conclusion $\Psi \vdash \forall a. A \leq \forall a. B$. In such case, choosing $\leq\forall\text{R}$ first is always better, since

Type variables	a, b		
Types	A, B, C	$::=$	$1 \mid a \mid \forall a. A \mid A \rightarrow B$
Monotypes	τ, σ	$::=$	$1 \mid a \mid \tau \rightarrow \sigma$
Expressions	e	$::=$	$x \mid () \mid \lambda x. e \mid e_1 e_2 \mid (e : A)$
Contexts	Ψ	$::=$	$\cdot \mid \Psi, a \mid \Psi, x : A$

Figure 4.1: Syntax of Declarative System

$\boxed{\Psi \vdash A}$ Well-formed declarative type	
$\frac{}{\Psi \vdash 1} \text{wf}_{\text{dunit}}$	$\frac{a \in \Psi}{\Psi \vdash a} \text{wf}_{\text{dvar}}$
$\frac{\Psi \vdash A \quad \Psi \vdash B}{\Psi \vdash A \rightarrow B} \text{wf}_{\text{d}\rightarrow}$	$\frac{\Psi, a \vdash A}{\Psi \vdash \forall a. A} \text{wf}_{\text{d}\forall}$
$\boxed{\Psi \vdash e}$ Well-formed declarative expression	
$\frac{x : A \in \Psi}{\Psi \vdash x} \text{wf}_{\text{d}\text{tmvar}}$	$\frac{}{\Psi \vdash ()} \text{wf}_{\text{d}\text{tmunit}}$
$\frac{\Psi, x : A \vdash e}{\Psi \vdash \lambda x. e} \text{wf}_{\text{d}\text{abs}}$	
$\frac{\Psi \vdash e_1 \quad \Psi \vdash e_2}{\Psi \vdash e_1 e_2} \text{wf}_{\text{d}\text{app}}$	$\frac{\Psi \vdash A \quad \Psi \vdash e}{\Psi \vdash (e : A)} \text{wf}_{\text{d}\text{anno}}$
$\boxed{\Psi \vdash A \leq B}$ Declarative subtyping	
$\frac{a \in \Psi}{\Psi \vdash a \leq a} \leq_{\text{Var}}$	$\frac{}{\Psi \vdash 1 \leq 1} \leq_{\text{Unit}}$
$\frac{\Psi \vdash B_1 \leq A_1 \quad \Psi \vdash A_2 \leq B_2}{\Psi \vdash A_1 \rightarrow A_2 \leq B_1 \rightarrow B_2} \leq_{\rightarrow}$	
$\frac{\Psi \vdash \tau \quad \Psi \vdash [\tau/a]A \leq B}{\Psi \vdash \forall a. A \leq B} \leq_{\forall\text{L}}$	$\frac{\Psi, b \vdash A \leq B}{\Psi \vdash A \leq \forall b. B} \leq_{\forall\text{R}}$

Figure 4.2: Declarative Well-formedness and Subtyping

we introduce the type variable a to the context earlier, which gives more flexibility on the choice of τ . The declarative typing rule `DeclSub` overlaps with both `Decl \forall I` and `Decl \rightarrow I`. However, we argue that more specific rules are always the best choices, i.e. `Decl \forall I` and `Decl \rightarrow I` should have higher priority than `DeclSub`.

For example, $\Psi \vdash \lambda x. x \Leftarrow \forall a. a \rightarrow a$ succeeds if derived from Rule `Decl \forall I`:

$$\frac{\Psi, a, x : a \vdash x \Leftarrow a}{\Psi, a \vdash \lambda x. x \Leftarrow a \rightarrow a} \text{Decl}_{\rightarrow\text{I}}, \quad \frac{\Psi, a \vdash \lambda x. x \Leftarrow a \rightarrow a}{\Psi \vdash \lambda x. x \Leftarrow \forall a. a \rightarrow a} \text{Decl}_{\forall\text{I}}$$

$\boxed{\Psi \vdash e \Leftarrow A}$	e checks against input type A .
$\boxed{\Psi \vdash e \Rightarrow A}$	e synthesizes output type A .
$\boxed{\Psi \vdash A \bullet e \Rightarrow C}$	Applying a function of type A to e synthesizes type C .

$\frac{(x : A) \in \Psi}{\Psi \vdash x \Rightarrow A} \text{DeclVar}$	$\frac{\Psi \vdash e \Rightarrow A \quad \Psi \vdash A \leq B}{\Psi \vdash e \Leftarrow B} \text{DeclSub}$
$\frac{\Psi \vdash A \quad \Psi \vdash e \Leftarrow A}{\Psi \vdash (e : A) \Rightarrow A} \text{DeclAnno}$	$\frac{}{\Psi \vdash () \Leftarrow 1} \text{Decl1I} \quad \frac{}{\Psi \vdash () \Rightarrow 1} \text{Decl1I} \Rightarrow$
$\frac{\Psi, a \vdash e \Leftarrow A}{\Psi \vdash e \Leftarrow \forall a. A} \text{Decl} \forall I$	$\frac{\Psi \vdash \tau \quad \Psi \vdash [\tau/a] A \bullet e \Rightarrow C}{\Psi \vdash \forall a. A \bullet e \Rightarrow C} \text{Decl} \forall \text{App}$
$\frac{\Psi, x : A \vdash e \Leftarrow B}{\Psi \vdash \lambda x. e \Leftarrow A \rightarrow B} \text{Decl} \rightarrow I$	$\frac{\Psi \vdash \sigma \rightarrow \tau \quad \Psi, x : \sigma \vdash e \Leftarrow \tau}{\Psi \vdash \lambda x. e \Rightarrow \sigma \rightarrow \tau} \text{Decl} \rightarrow I \Rightarrow$
$\frac{\Psi \vdash e_1 \Rightarrow A \quad \Psi \vdash A \bullet e_2 \Rightarrow C}{\Psi \vdash e_1 e_2 \Rightarrow C} \text{Decl} \rightarrow E$	$\frac{\Psi \vdash e \Leftarrow A}{\Psi \vdash A \rightarrow C \bullet e \Rightarrow C} \text{Decl} \rightarrow \text{App}$

Figure 4.3: Declarative Typing

but fails when applied to DeclSub:

$$\begin{array}{c}
 \text{Impossible!} \\
 \frac{a \notin \text{FV}(\sigma)}{\Psi, a \vdash a \leq \sigma \quad \Psi, a \vdash \tau \leq a} ? \\
 \frac{\Psi \vdash \sigma \rightarrow \tau \quad \Psi, x : \sigma \vdash e \Leftarrow \tau}{\Psi \vdash \lambda x. x \Rightarrow \sigma \rightarrow \tau} \text{Decl} \rightarrow I \Rightarrow \quad \frac{\Psi, a \vdash \sigma \rightarrow \tau \leq a \rightarrow a}{\Psi \vdash \sigma \rightarrow \tau \leq \forall a. a \rightarrow a} \leq \rightarrow \\
 \frac{\Psi \vdash \lambda x. x \Rightarrow \sigma \rightarrow \tau \quad \Psi \vdash \sigma \rightarrow \tau \leq \forall a. a \rightarrow a}{\Psi \vdash \lambda x. x \Leftarrow \forall a. a \rightarrow a} \leq \forall R \quad \text{DeclSub}
 \end{array}$$

Rule Decl \rightarrow I is also better at handling higher-order types. When the lambda-expression to be inferred has a polymorphic input type, such as $\forall a. a \rightarrow a$, DeclSub may not derive some judgments. For example, $\Psi, id : \forall a. a \rightarrow a \vdash \lambda f. f \, id \, (f \, ()) \Leftarrow (\forall a. a \rightarrow a) \rightarrow 1$ requires the argument of the lambda-expression to be a polymorphic type, otherwise it could not be applied to both id and $()$. If Rule DeclSub was chosen for derivation, the type of its argument is restricted by Rule Decl \rightarrow I \Rightarrow , which is not a polymorphic type. By contrast, Rule Decl \rightarrow I keeps the polymorphic argument type $\forall a. a \rightarrow a$, and will successfully derive the judgment.

We will come back to this topic in Section 4.3.2 and formally derive a system without overlapping rules.

4.1.2 DK'S ALGORITHM

DK's algorithm version revolves around their notion of *algorithmic context*.

Algorithmic Contexts $\Gamma, \Delta, \Theta ::= \cdot \mid \Gamma, a \mid \Gamma, x : A \mid \Gamma, \hat{\alpha} \mid \Gamma, \hat{\alpha} = \tau \mid \Gamma, \blacktriangleright_{\hat{\alpha}}$

In addition to the regular (universally quantified) type variables a , the algorithmic context also contains *existential* type variables $\hat{\alpha}$. These are placeholders for monotypes τ that are still to be determined by the inference algorithm. When the existential variable is “solved”, its entry in the context is replaced by the assignment $\hat{\alpha} = \tau$. A context application on a type, denoted by $[\Gamma]A$, substitutes all solved existential type variables in Γ with their solutions on type A .

All algorithmic judgments thread an algorithmic context. They have the form $\Gamma \vdash \dots \dashv \Delta$, where Γ is the input context and Δ is the output context: $\Gamma \vdash A \leq B \dashv \Delta$ for subtyping, $\Gamma \vdash e \Leftarrow A \dashv \Delta$ for type checking, and so on. The output context is a functional update of the input context that records newly introduced existentials and solutions.

Solutions are incrementally propagated by applying the algorithmic output context of a previous task as substitutions to the next task. This can be seen in the subsumption rule:

$$\frac{\Gamma \vdash e \Rightarrow A \dashv \Theta \quad \Theta \vdash [\Theta]A \leq [\Theta]B \dashv \Delta}{\Gamma \vdash e \Leftarrow B \dashv \Delta} \text{DK_Sub}$$

The inference task yields an output context Θ which is applied as a substitution to the types A and B before performing the subtyping check to propagate any solutions of existential variables that appear in A and B .

MARKERS FOR SCOPING. The sequential order of entries in the algorithmic context, in combination with the threading of contexts, does not perfectly capture the scoping of all existential variables. For this reason the DK algorithm uses scope markers $\blacktriangleright_{\hat{\alpha}}$ in a few places. An example is given in the following rule:

$$\frac{\Gamma, \blacktriangleright_{\hat{\alpha}}, \hat{\alpha} \vdash [\hat{\alpha}/a]A \leq B \dashv \Delta, \blacktriangleright_{\hat{\alpha}}, \Theta}{\Gamma \vdash \forall a. A \leq B \dashv \Delta} \text{DK_}\leq\forall\text{L}$$

To indicate that the scope of $\hat{\alpha}$ is local to the subtyping check $[\hat{\alpha}/a]A \leq B$, the marker is pushed onto its input stack and popped from the output stack together with the subsequent part Θ , which may refer to $\hat{\alpha}$. (Remember that later entries may refer to earlier ones, but not vice versa.) This way $\hat{\alpha}$ does not escape its scope.

At first sight, the DK algorithm would seem a good basis for mechanization. After all it comes with a careful description and extensive manual proofs. Unfortunately, we ran into several obstacles that have prompted us to formulate a different, more mechanization-friendly algorithm.

BROKEN METATHEORY While going through the manual proofs of DK’s algorithm, we found several problems. Indeed, two proofs of lemmas—Lemma 19 (Extension Equality Preservation) and Lemma 14 (Subsumption)—wrongly apply induction hypotheses in several cases. Fortunately, we have found simple workarounds that fix these proofs without affecting the appeals to these lemmas.

More seriously, we have also found a lemma that simply does not hold: Lemma 29 (Parallel Admissibility)¹. See Appendix for a detailed explanation and counterexample. This lemma is a cornerstone of the two metatheoretical results of the algorithm, soundness and completeness with respect to the declarative system. In particular, both instantiation soundness (i.e. a part of subtyping soundness) and typing completeness directly require the broken lemma. Moreover, Lemma 54 (Typing Extension) also requires the broken lemma and is itself used 13 times in the proof of typing soundness and completeness. Unfortunately, we have not yet found a way to fix this problem.

COMPLEX SCOPING AND PROPAGATION Besides the problematic lemmas in DK’s metatheory, there are other reasons to look for an alternative algorithmic formulation of the type system that is more amenable to mechanization. Indeed, two aspects that are particularly challenging to mechanize are the scoping of universal and existential type variables, and the propagation of the instantiation of existential type variables across judgments. DK is already quite disciplined on these accounts, in particular compared to traditional constraint-based type-inference algorithms like Algorithm \mathcal{W} [Milner 1978] which features an implicit global scope for all type variables. Indeed, DK uses its explicit and ordered context Γ that tracks the relative scope of universal and existential variables and it is careful to only instantiate existential variables in a well-scoped manner.

Moreover, DK’s algorithm carefully propagates instantiations by recording them into the context and threading this context through all judgments. While this works well on paper, this approach is still fairly involved and thus hard to reason about in a mechanized setting. Indeed, the instantiations have to be recorded in the context and are applied incrementally to each remaining judgment in turn, rather than immediately to all remaining judgments at

¹Ningning Xie found the issue with Lemma 29 in 2016 while collaborating with the second author on an earlier attempt to mechanically formalize DK’s algorithm. The authors acknowledged the problem after we contacted them through email. Although they briefly mentioned that it should be possible to use a weaker lemma instead they did not go into details.

once. Also, as we have mentioned above, the stack discipline of the ordered contexts does not mesh well with the use of output contexts; explicit marker entries are needed in two places to demarcate the end of an existential variable's scope. Actually we found a scoping issue related to the subsumption rule DK_Sub , which might cause existential variables to leak across judgments. In Section 4.4.1 we give a detailed discussion.

The complications of scoping and propagation are compelling reasons to look for another algorithm that is easier to reason about in a mechanized setting.

4.1.3 JUDGMENT LISTS

To avoid the problem of incrementally applying a substitution to remaining tasks, we can find inspiration in the formulation of constraint solving algorithms. For instance, the well-known unification algorithm by Martelli and Montanari [1982] decomposes the problem of unifying two terms $s \doteq t$ into a number of related unification problems between pairs of terms $s_i \doteq t_i$. These smaller problems are not tackled independently, but kept together in a set S . The algorithm itself is typically formulated as a small-step-style state transition system $S \mapsto S'$ that rewrites the set of unification problems until it is in solved form or until a contradiction has been found. For instance, the variable elimination rule is written as:

$$x \doteq t, S \mapsto x \doteq t, [t/x]S \quad \text{if } x \notin t \text{ and } x \in S$$

Because the whole set is explicitly available, the variable x can be simultaneously substituted.

In the above unification problem, all variables are implicitly bound in the same global scope. Some constraint solving algorithms for Hindley-Milner type inference use similar ideas, but are more careful tracking the scopes of variables [Pottier and Rémy 2005]. However they have separate phases for constraint generation and solving. Recent unification algorithms for dependently-typed languages are also more explicit about scopes. For instance, Reed [2009] represents a unification problem as $\Delta \vdash P$ where P is a set of equations to be solved and Δ is a (modal) context. Abel and Pientka [2011] even use multiple contexts within a unification problem. Such a problem is denoted $\Delta \Vdash \mathcal{K}$ where the meta-context Δ contains all the typings of meta-variables in the constraint set \mathcal{K} . The latter consists of constraints like $\Psi \vdash M = N : C$ that are equipped with their individual context Ψ . While accurately tracking the scoping of regular and meta-variables, this approach is not ideal because it repeatedly copies contexts when decomposing a unification problem, and later it has to substitute solutions into these copies.

4.1.4 SINGLE-CONTEXT WORKLIST ALGORITHM FOR SUBTYPING

In recent work, Zhao et al. [2018] have shown how to mechanize a variant of DK’s subtyping algorithm and shown it correct with respect to DK’s declarative subtyping judgment. This approach overcomes some problems in DK’s formulation by using a *worklist*-based judgment of the form

$$\Gamma \vdash \Omega$$

where Ω is a worklist (or sequence) of subtyping problems of the form $A \leq B$. The judgment is defined by case analysis on the first element of Ω and recursively processes the worklist until it is empty. Using both a single common ordered context Γ and a worklist Ω greatly simplifies propagating the instantiation of type variables in one subtyping problem to the remaining ones.

Unfortunately, this work does not solve all problems. In particular, it has two major limitations that prevent it from scaling to the whole DK system.

SCOPING GARBAGE Firstly, the single common ordered context Γ does not accurately reflect the type and unification variables currently in scope. Instead, it is an overapproximation that steadily accrues variables, and only drops those unification variables that are instantiated. In other words, Γ contains “garbage” that is no longer in scope. This complicates establishing the relation with the declarative system.

NO INFERENCE JUDGMENTS Secondly, and more importantly, the approach only deals with a judgment for *checking* whether one type is the subtype of another. While this may not be so difficult to adapt to the *checking* mode of term typing $\Gamma \vdash e \Leftarrow A$, it clearly lacks the functionality to support the *inference* mode of term typing $\Gamma \vdash e \Rightarrow A$. Indeed, the latter requires not only the communication of unification variable instantiations from one typing problem to another, but also an inferred type.

4.1.5 ALGORITHMIC TYPE INFERENCE FOR HIGHER-RANKED TYPES: KEY IDEAS

Our new algorithmic type system builds on the work above, but addresses the open problems.

SCOPE TRACKING We avoid scoping garbage by blending the ordered context and the worklist into a single syntactic sort Γ , our algorithmic worklist. This algorithmic worklist interleaves (type and term) variables with *work* like checking or inferring types of expressions. The interleaving keeps track of the variable scopes in the usual, natural way: each variable is in scope of anything in front of it in the worklist. If there is nothing in front, the variable is

no longer needed and can be popped from the worklist. This way, no garbage (i.e. variables out-of-scope) builds up.

$$\begin{array}{ll} \text{Algorithmic judgment chain} & \omega ::= A \leq B \mid e \Leftarrow A \mid e \Rightarrow_a \omega \mid A \bullet e \Rightarrow_a \omega \\ \text{Algorithmic worklist} & \Gamma ::= \cdot \mid \Gamma, a \mid \Gamma, \hat{\alpha} \mid \Gamma, x : A \mid \Gamma \Vdash \omega \end{array}$$

For example, suppose that the top judgment of the following worklist checks the identity function against $\forall a. a \rightarrow a$:

$$\Gamma \Vdash \lambda x. x \Leftarrow \forall a. a \rightarrow a$$

To proceed, two auxiliary variables a and x are introduced to help the type checking:

$$\Gamma, a, x : a \Vdash x \Leftarrow a$$

which will be satisfied after several steps, and the worklist becomes

$$\Gamma, a, x : a$$

Since the variable declarations $a, x : a$ are only used for a judgment already processed, they can be safely removed, leaving the remaining worklist Γ to be further reduced.

Our worklist can be seen as an all-in-one stack, containing variable declarations and subtyping/ typing judgments. The stack is an enriched form of ordered context, and it has a similar variable scoping scheme.

INFERENCE JUDGMENTS To express the DK's inference judgments, we use a novel form of work entries in the worklist: our algorithmic judgment chains ω . In its simplest form, such a judgment chain is just a check, like a subtyping check $A \leq B$ or a term typecheck $e \Leftarrow A$. However, the non-trivial forms of chains capture an inference task together with the work that depends on its outcome. For instance, a type inference task takes the form $e \Rightarrow_a \omega$. This form expresses that we need to infer the type, say A , for expression e and use it in the chain ω by substituting it for the placeholder type variable a . Notice that such a binds a fresh type variable for the inner chain ω , which behaves similarly to the variable declarations in the context.

Take the following worklist as an example

$$\hat{\alpha} \Vdash \underline{(\lambda x. x) () \Rightarrow_a a \leq \hat{\alpha}}, x : \hat{\alpha}, \hat{\beta} \Vdash \underline{\hat{\alpha} \leq \hat{\beta}} \Vdash \underline{\hat{\beta} \leq 1}$$

There are three (underlined) judgment chains in the worklist, where the first and second judgment chains (from the right) are two subtyping judgments, and the third judgment chain, $(\lambda x. x) () \Rightarrow_a a \leq \hat{\alpha}$, is a sequence of an inference judgment followed by a subtyping judgment.

The algorithm first analyses the two subtyping judgments and will find the best solutions $\hat{\alpha} = \hat{\beta} = 1$ (please refer to Figure 4.5 for detailed derivations). Then we substitute every instance of $\hat{\alpha}$ and $\hat{\beta}$ with 1, so the variable declarations can be safely removed from the worklist. Now we reduce the worklist to the following state

$$\cdot \Vdash \underline{(\lambda x. x) () \Rightarrow_a a \leq 1, x : 1}$$

which has a term variable declaration as the top element. After removing the garbage term variable declaration from the worklist, we process the last remaining inference judgment $(\lambda x. x) () \Rightarrow ?$, with the unit type 1 as its result. Finally the last judgment becomes $1 \leq 1$, a trivial base case.

4.2 ALGORITHMIC SYSTEM

This section introduces a novel algorithmic system that implements DK's declarative specification. The new algorithm extends the idea of worklists proposed by Zhao et al. [2018] in two ways. Firstly, unlike Zhao et al. [2018]'s worklists, the scope of variables is precisely tracked and variables do not escape their scope. This is achieved by unifying algorithmic contexts and the worklists themselves. Secondly, our algorithm also accounts for the type system (and not just subtyping). To deal with inference judgments that arise in the type system we employ a *continuation passing style* to enable the transfer of inferred information across judgments in a worklist.

4.2.1 SYNTAX AND WELL-FORMEDNESS

Figure 4.4 shows the syntax and well-formedness judgments used by the algorithm. Similarly to the declarative system the well-formedness rules are unsurprising and merely ensure well-scopedness.

EXISTENTIAL VARIABLES The algorithmic system inherits the syntax of terms and types from the declarative system. It only introduces one additional feature. In order to find unknown types τ in the declarative system, the algorithmic system extends the declarative types A with *existential variables* $\hat{\alpha}, \hat{\beta}$. They behave like unification variables, but their scope is restricted by their position in the algorithmic worklist rather than being global. Any existential

Existential variables	$\hat{\alpha}, \hat{\beta}$
Algorithmic types	$A, B, C ::= 1 \mid a \mid \forall a. A \mid A \rightarrow B \mid \hat{\alpha}$
Algorithmic judgment chain	$\omega ::= A \leq B \mid e \Leftarrow A \mid e \Rightarrow_a \omega \mid A \bullet e \Rightarrow_a \omega$
Algorithmic worklist	$\Gamma ::= \cdot \mid \Gamma, a \mid \Gamma, \hat{\alpha} \mid \Gamma, x : A \mid \Gamma \Vdash \omega$

$\boxed{\Gamma \vdash A}$ Well-formed algorithmic type

$$\begin{array}{c}
 \frac{}{\Gamma \vdash 1} \text{wf_unit} \quad \frac{a \in \Gamma}{\Gamma \vdash a} \text{wf_var} \quad \frac{\hat{\alpha} \in \Gamma}{\Gamma \vdash \hat{\alpha}} \text{wf_exvar} \quad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \rightarrow B} \text{wf_}\rightarrow \quad \frac{\Gamma, a \vdash A}{\Gamma \vdash \forall a. A} \text{wf_}\forall
 \end{array}$$

$\boxed{\Gamma \vdash e}$ Well-formed algorithmic expression

$$\begin{array}{c}
 \frac{x : A \in \Gamma}{\Gamma \vdash x} \text{wf_tmvar} \quad \frac{}{\Gamma \vdash ()} \text{wf_tmunit} \quad \frac{\Gamma, x : A \vdash e}{\Gamma \vdash \lambda x. e} \text{wf_abs} \\
 \frac{\Gamma \vdash e_1 \quad \Gamma \vdash e_2}{\Gamma \vdash e_1 e_2} \text{wf_app} \quad \frac{\Gamma \vdash A \quad \Gamma \vdash e}{\Gamma \vdash (e : A)} \text{wf_anno}
 \end{array}$$

$\boxed{\Gamma \vdash \omega}$ Well-formed algorithmic judgment

$$\begin{array}{c}
 \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \leq B} \text{wf_}\leq \quad \frac{\Gamma \vdash e \quad \Gamma \vdash A}{\Gamma \vdash e \Leftarrow A} \text{wf_}\Leftarrow \\
 \frac{\Gamma \vdash e \quad \Gamma, a \vdash \omega}{\Gamma \vdash e \Rightarrow_a \omega} \text{wf_}\Rightarrow \quad \frac{\Gamma \vdash A \quad \Gamma, a \vdash \omega \quad \Gamma \vdash e}{\Gamma \vdash A \bullet e \Rightarrow_a \omega} \text{wf_}\Rightarrow\Rightarrow
 \end{array}$$

$\boxed{\text{wf } \Gamma}$ Well-formed algorithmic worklist

$$\begin{array}{c}
 \frac{}{\text{wf } \cdot} \text{wf_}\cdot \quad \frac{\text{wf } \Gamma}{\text{wf } \Gamma, a} \text{wf_}\mathbf{a} \quad \frac{\text{wf } \Gamma}{\text{wf } \Gamma, \hat{\alpha}} \text{wf_}\hat{\alpha} \quad \frac{\text{wf } \Gamma \quad \Gamma \vdash A}{\text{wf } \Gamma, x : A} \text{wf_}\mathbf{of} \quad \frac{\text{wf } \Gamma \quad \Gamma \vdash \omega}{\text{wf } \Gamma \Vdash \omega} \text{wf_}\omega
 \end{array}$$

Figure 4.4: Extended Syntax and Well-Formedness for the Algorithmic System

variable $\hat{\alpha}$ should only be solved to a type that is well-formed with respect to the worklist to which $\hat{\alpha}$ has been added. The point is that the monotype τ , represented by the corresponding existential variable, is always well-formed under the corresponding declarative context. In other words, the position of $\hat{\alpha}$'s reflects the well-formedness restriction of τ 's.

JUDGMENT CHAINS Judgment chains ω , or judgments for short, are the core components of our algorithmic type-checking. There are four kinds of judgments in our system: subtyping ($A \leq B$), checking ($e \Leftarrow A$), inference ($e \Rightarrow_a \omega$) and application inference ($A \bullet e \Rightarrow_a \omega$). Subtyping and checking are relatively simple, since their result is only success or failure. However both inference and application inference return a type that is used in subsequent judgments. We use a continuation-passing-style encoding to accomplish this. For example, the judgment chain $e \Rightarrow_a (a \leq B)$ contains two judgments: first we want to infer the type of the expression e , and then check if that type is a subtype of B . The *unknown* type of e is represented by a type variable a , which is used as a placeholder in the second judgment to denote the type of e .

WORKLIST JUDGMENTS Our algorithm has a non-standard form. We combine traditional contexts and judgment(s) into a single sort, the *worklist* Γ . The worklist is an *ordered* collection of both variable bindings and judgments. The order captures the scope: only the objects that come after a variable's binding in the worklist can refer to it. For example, $[\cdot, a, x : a \vdash x \Leftarrow a]$ is a valid worklist, but $[\cdot \vdash \underline{x} \Leftarrow \underline{a}, x : \underline{a}, a]$ is not (the underlined symbols refer to out-of-scope variables).

HOLE NOTATION We use the syntax $\Gamma[\Gamma_M]$ to denote the worklist $\Gamma_L, \Gamma_M, \Gamma_R$, where Γ is the worklist $\Gamma_L, \bullet, \Gamma_R$ with a hole (\bullet). Hole notations with the same name implicitly share the same structure Γ_L and Γ_R . A multi-hole notation splits the worklist into more parts. For example, $\Gamma[\hat{\alpha}][\hat{\beta}]$ means $\Gamma_1, \hat{\alpha}, \Gamma_2, \hat{\beta}, \Gamma_3$.

4.2.2 ALGORITHMIC SYSTEM

The algorithmic typing reduction rules, defined in Figure 4.5, have the form $\Gamma \longrightarrow \Gamma'$. The reduction process treats the worklist as a stack. In every step it pops the first judgment from the worklist for processing and possibly pushes new judgments onto the worklist. The syntax $\Gamma \longrightarrow^* \Gamma'$ denotes multiple reduction steps.

$$\frac{}{\Gamma \longrightarrow^* \Gamma} \longrightarrow^* \text{id} \quad \frac{\Gamma \longrightarrow \Gamma_1 \quad \Gamma_1 \longrightarrow^* \Gamma'}{\Gamma \longrightarrow^* \Gamma'} \longrightarrow^* \text{step}$$

$$\begin{array}{c}
 \boxed{\Gamma \longrightarrow \Gamma'} \qquad \qquad \qquad \Gamma \text{ reduces to } \Gamma'. \\
 \\
 \Gamma, a \longrightarrow_1 \Gamma \quad \Gamma, \hat{\alpha} \longrightarrow_2 \Gamma \quad \Gamma, x : A \longrightarrow_3 \Gamma \\
 \\
 \Gamma \Vdash 1 \leq 1 \longrightarrow_4 \Gamma \\
 \Gamma \Vdash a \leq a \longrightarrow_5 \Gamma \\
 \Gamma \Vdash \hat{\alpha} \leq \hat{\alpha} \longrightarrow_6 \Gamma \\
 \Gamma \Vdash A_1 \rightarrow A_2 \leq B_1 \rightarrow B_2 \longrightarrow_7 \Gamma \Vdash A_2 \leq B_2 \Vdash B_1 \leq A_1 \\
 \Gamma \Vdash \forall a. A \leq B \longrightarrow_8 \Gamma, \hat{\alpha} \Vdash [\hat{\alpha}/a]A \leq B \quad \text{when } B \neq \forall a. B' \\
 \Gamma \Vdash A \leq \forall b. B \longrightarrow_9 \Gamma, b \Vdash A \leq B \\
 \\
 \Gamma[\hat{\alpha}] \Vdash \hat{\alpha} \leq A \rightarrow B \longrightarrow_{10} [\hat{\alpha}_1 \rightarrow \hat{\alpha}_2/\hat{\alpha}](\Gamma[\hat{\alpha}_1, \hat{\alpha}_2] \Vdash \hat{\alpha}_1 \rightarrow \hat{\alpha}_2 \leq A \rightarrow B) \\
 \qquad \qquad \qquad \text{when } \hat{\alpha} \notin FV(A) \cup FV(B) \\
 \Gamma[\hat{\alpha}] \Vdash A \rightarrow B \leq \hat{\alpha} \longrightarrow_{11} [\hat{\alpha}_1 \rightarrow \hat{\alpha}_2/\hat{\alpha}](\Gamma[\hat{\alpha}_1, \hat{\alpha}_2] \Vdash A \rightarrow B \leq \hat{\alpha}_1 \rightarrow \hat{\alpha}_2) \\
 \qquad \qquad \qquad \text{when } \hat{\alpha} \notin FV(A) \cup FV(B) \\
 \\
 \Gamma[\hat{\alpha}][\hat{\beta}] \Vdash \hat{\alpha} \leq \hat{\beta} \longrightarrow_{12} [\hat{\alpha}/\hat{\beta}](\Gamma[\hat{\alpha}][]) \\
 \Gamma[\hat{\alpha}][\hat{\beta}] \Vdash \hat{\beta} \leq \hat{\alpha} \longrightarrow_{13} [\hat{\alpha}/\hat{\beta}](\Gamma[\hat{\alpha}][]) \\
 \Gamma[a][\hat{\beta}] \Vdash a \leq \hat{\beta} \longrightarrow_{14} [a/\hat{\beta}](\Gamma[a][]) \\
 \Gamma[a][\hat{\beta}] \Vdash \hat{\beta} \leq a \longrightarrow_{15} [a/\hat{\beta}](\Gamma[a][]) \\
 \Gamma[\hat{\beta}] \Vdash 1 \leq \hat{\beta} \longrightarrow_{16} [1/\hat{\beta}](\Gamma[]) \\
 \Gamma[\hat{\beta}] \Vdash \hat{\beta} \leq 1 \longrightarrow_{17} [1/\hat{\beta}](\Gamma[]) \\
 \\
 \Gamma \Vdash e \Leftarrow B \longrightarrow_{18} \Gamma \Vdash e \Rightarrow_a a \leq B \quad \text{when } e \neq \lambda x. e' \text{ and } B \neq \forall a. B' \\
 \Gamma \Vdash e \Leftarrow \forall a. A \longrightarrow_{19} \Gamma, a \Vdash e \Leftarrow A \\
 \Gamma \Vdash \lambda x. e \Leftarrow A \rightarrow B \longrightarrow_{20} \Gamma, x : A \Vdash e \Leftarrow B \\
 \Gamma[\hat{\alpha}] \Vdash \lambda x. e \Leftarrow \hat{\alpha} \longrightarrow_{21} [\hat{\alpha}_1 \rightarrow \hat{\alpha}_2/\hat{\alpha}](\Gamma[\hat{\alpha}_1, \hat{\alpha}_2], x : \hat{\alpha}_1 \Vdash e \Leftarrow \hat{\alpha}_2) \\
 \\
 \Gamma \Vdash x \Rightarrow_a \omega \longrightarrow_{22} \Gamma \Vdash [A/a]\omega \quad \text{when } (x : A) \in \Gamma \\
 \Gamma \Vdash (e : A) \Rightarrow_a \omega \longrightarrow_{23} \Gamma \Vdash [A/a]\omega \Vdash e \Leftarrow A \\
 \Gamma \Vdash () \Rightarrow_a \omega \longrightarrow_{24} \Gamma \Vdash [1/a]\omega \\
 \Gamma \Vdash \lambda x. e \Rightarrow_a \omega \longrightarrow_{25} \Gamma, \hat{\alpha}, \hat{\beta} \Vdash [\hat{\alpha} \rightarrow \hat{\beta}/a]\omega, x : \hat{\alpha} \Vdash e \Leftarrow \hat{\beta} \\
 \Gamma \Vdash e_1 e_2 \Rightarrow_a \omega \longrightarrow_{26} \Gamma \Vdash e_1 \Rightarrow_b (b \bullet e_2 \Rightarrow_a \omega) \\
 \\
 \Gamma \Vdash \forall a. A \bullet e \Rightarrow_a \omega \longrightarrow_{27} \Gamma, \hat{\alpha} \Vdash [\hat{\alpha}/a]A \bullet e \Rightarrow_a \omega \\
 \Gamma \Vdash A \rightarrow C \bullet e \Rightarrow_a \omega \longrightarrow_{28} \Gamma \Vdash [C/a]\omega \Vdash e \Leftarrow A \\
 \Gamma[\hat{\alpha}] \Vdash \hat{\alpha} \bullet e \Rightarrow_a \omega \longrightarrow_{29} [\hat{\alpha}_1 \rightarrow \hat{\alpha}_2/\hat{\alpha}](\Gamma[\hat{\alpha}_1, \hat{\alpha}_2] \Vdash \hat{\alpha}_1 \rightarrow \hat{\alpha}_2 \bullet e \Rightarrow_a \omega)
 \end{array}$$

Figure 4.5: Algorithmic Typing

In the case that $\Gamma \longrightarrow^* \cdot$ this corresponds to successful type checking.

Please note that when a new variable is introduced in the right-hand side worklist Γ' , we implicitly pick a fresh one, since the right-hand side can be seen as the premise of the reduction.

Rules 1-3 pop variable declarations that are essentially garbage. That is variables that are out of scope for the remaining judgments in the worklist. All other rules concern a judgment at the front of the worklist. Logically we can discern 6 groups of rules.

1. Algorithmic subtyping We have six subtyping rules (Rules 4-9) that are similar to their declarative counterparts. For instance, Rule 7 consumes a subtyping judgment and pushes two back to the worklist. Rule 8 differs from declarative Rule $\leq \forall L$ by introducing an existential variable $\hat{\alpha}$ instead of guessing the monotype τ instantiation. Each existential variable is later solved to a monotype τ with the same context, so the final solution τ of $\hat{\alpha}$ should be well-formed under Γ .

WORKLIST VARIABLE SCOPING Rules 8 and 9 involve variable declarations and demonstrate how our worklist treats variable scopes. Rule 8 introduces an existential variable $\hat{\alpha}$ that is only visible to the judgment $[\hat{\alpha}/a]A \leq B$. Reduction continues until all the subtyping judgments in front of $\hat{\alpha}$ are satisfied. Finally we can safely remove $\hat{\alpha}$ since no occurrence of $\hat{\alpha}$ could have leaked into the left part of the worklist. Moreover, the algorithm garbage-collects the $\hat{\alpha}$ variable at the right time: it leaves the environment immediately after being unreferenced completely for sure.

EXAMPLE Consider the derivation of the subtyping judgment $(1 \rightarrow 1) \rightarrow 1 \leq (\forall a. 1 \rightarrow 1) \rightarrow 1$:

$$\begin{aligned}
 & \cdot \vdash (1 \rightarrow 1) \rightarrow 1 \leq (\forall a. 1 \rightarrow 1) \rightarrow 1 \\
 \longrightarrow_7 & \cdot \Vdash 1 \leq 1 \Vdash \forall a. 1 \rightarrow 1 \leq 1 \rightarrow 1 \\
 \longrightarrow_8 & \cdot \Vdash 1 \leq 1, \hat{\alpha} \Vdash 1 \rightarrow 1 \leq 1 \rightarrow 1 \\
 \longrightarrow_7 & \cdot \Vdash 1 \leq 1, \hat{\alpha} \Vdash 1 \leq 1 \Vdash 1 \leq 1 \\
 \longrightarrow_4 & \cdot \Vdash 1 \leq 1, \hat{\alpha} \Vdash 1 \leq 1 \\
 \longrightarrow_4 & \cdot \Vdash 1 \leq 1, \hat{\alpha} \\
 \longrightarrow_2 & \cdot \Vdash 1 \leq 1 \\
 \longrightarrow_4 & \cdot
 \end{aligned}$$

First, the subtyping of two function types is split into two judgments by Rule 7: a covariant subtyping on the return type and a contravariant subtyping on the argument type. Then we apply Rule 8 to reduce the \forall quantifier on the left side. The rule introduces an existential variable $\hat{\alpha}$ to the context (even though the type $\forall a. 1 \rightarrow 1$ does not actually refer to the quantified type variable a). In the following 3 steps we satisfy the judgment $1 \rightarrow 1 \leq 1 \rightarrow 1$ by Rules 7, 4 and 4.

Now the existential variable $\hat{\alpha}$, introduced before but still unsolved, is at the top of the worklist and Rule 2 garbage-collects it. The process is carefully designed within the algorithmic rules: when $\hat{\alpha}$ is introduced earlier by Rule 8, we foresee the recycling of $\hat{\alpha}$ after all the judgments (potentially) requiring $\hat{\alpha}$ have been processed. Finally Rule 4 reduces one of the base cases and finishes the subtyping derivation.

2. Existential decomposition. Rules 10 and 11 are algorithmic versions of Rule $\leq \rightarrow$; they both partially instantiate $\hat{\alpha}$ to function types. The domain $\hat{\alpha}_1$ and range $\hat{\alpha}_2$ of the new function type are not determined: they are fresh existential variables with the same scope as $\hat{\alpha}$. We replace $\hat{\alpha}$ in the worklist with $\hat{\alpha}_1, \hat{\alpha}_2$. To propagate the instantiation to the rest of the worklist and maintain well-formedness, every reference to $\hat{\alpha}$ is replaced by $\hat{\alpha}_1 \rightarrow \hat{\alpha}_2$. The *occurs-check* condition prevents divergence as usual. For example, without it $\hat{\alpha} \leq 1 \rightarrow \hat{\alpha}$ would diverge.

3. Solving existentials Rules 12-17 are base cases where an existential variable is solved. They all remove an existential variable and substitute the variable with its solution in the remaining worklist. Importantly the rules respect the scope of existential variables. For example, Rule 12 states that an existential variable $\hat{\alpha}$ can be solved with another existential variable $\hat{\beta}$ only if $\hat{\beta}$ occurs after $\hat{\alpha}$.

One may notice that the subtyping relation for simple types is just equivalence, which is true according to the declarative system. The DK's system works in a similar way.

4. Checking judgments. Rules 18-21 deal with checking judgments. Rule 18 is similar to `Dec1Sub`, but rewritten in the continuation-passing-style. The side conditions $e \neq \lambda x. e'$ and $B \neq \forall a. B'$ prevent overlap with Rules 19, 20 and 21; this is further discussed at the end of this section. Rules 19 and 20 adapt their declarative counterparts to the worklist style. Rule 21 is a special case of `Dec1 \rightarrow I`, dealing with the case when the input type is an existential variable, representing a monotype *function* as in the declarative system (it must be a function type, since the expression $\lambda x. e$ is a function). The same instantiation technique as in Rules 10 and 11 applies. The declarative checking rule `Dec11I` does not have a direct counterpart in the algorithm, because Rules 18 and 24 can be combined to give the same result.

RULE 21 DESIGN CHOICE The addition of Rule 21 is a design choice we have made to simplify the side condition of Rule 18 (which avoids overlap). It also streamlines the algorithm and the metatheory as we now treat all cases where we can see that an existential variable should be instantiated to a function type (i.e., Rules 10, 11, 21 and 29) uniformly.

The alternative would have been to omit Rule 21 and drop the condition on e in Rule 18. The modified Rule 18 would then handle $\Gamma \Vdash \lambda x. e \Leftarrow \hat{\alpha}$ and yield $\Gamma \Vdash \lambda x. e \Rightarrow_a a \leq \hat{\alpha}$, which would be further processed by Rule 25 to $\Gamma, \hat{\beta}_1, \hat{\beta}_2 \Vdash \hat{\beta}_1 \rightarrow \hat{\beta}_2 \leq \hat{\alpha}, x : \hat{\beta}_1 \Vdash e \Leftarrow \hat{\beta}_2$. As a subtyping constraint between monotypes is simply equality, $\hat{\beta}_1 \rightarrow \hat{\beta}_2 \leq \hat{\alpha}$ must end up equating $\hat{\beta}_1 \rightarrow \hat{\beta}_2$ with $\hat{\alpha}$ and thus have the same effect as Rule 21, but in a more roundabout fashion.

In comparison, DK's algorithmic subsumption rule has no restriction on the expression e , and they do not have a rule that explicitly handles the case $\lambda x. e \Leftarrow \hat{\alpha}$. Therefore the only way to check a lambda function against an existential variable is by applying the subsumption rule, which further breaks into type inference of a lambda function and a subtyping judgment.

5. Inference judgments. Inference judgments behave differently compared with subtyping and checking judgments: they *return* a type instead of only accepting or rejecting. For the algorithmic system, where guesses are involved, it may happen that the output type of an inference judgment refers to new existential variables, such as Rule 25. In comparison to Rule 8 and 9, where new variables are only referred by the sub-derivation, Rule 25 introduces variables $\hat{\alpha}, \hat{\beta}$ that affect the remaining judgment chain. This rule is carefully designed so that the output variables are bound by earlier declarations, thus the well-formedness of the worklist is preserved, and the garbage will be collected at the correct time. By making use of the continuation-passing-style judgment chain, inner judgments always share the context with their parent judgment.

Rules 22-26 deal with type inference judgments, written in continuation-passing-style. When an inference judgment succeeds with type A , the algorithm continues to work on the inner-chain ω by assigning A to its placeholder variable a . Rule 23 infers an annotated expression by changing into checking mode, therefore another judgment chain is created. Rule 24 is a base case, where the unit type 1 is inferred and thus passed to its child judgment chain. Rule 26 infers the type of an application by firstly inferring the type of the function e_1 , and then leaving the rest work to an application inference judgment, which passes a , representing the return type of the application, to the remainder of the judgment chain ω .

Rule 25 infers the type of a lambda expression by introducing $\hat{\alpha}, \hat{\beta}$ as the input and output types of the function, respectively. After checking the body e under the assumption $x : \hat{\alpha}$, the return type might reflect more information than simply $\hat{\alpha} \rightarrow \hat{\beta}$ through propagation when

existential variables are solved or partially solved. The variable scopes are maintained during the process: the assumption of argument type $(x : \hat{\alpha})$ is recycled after checking against the function body; the existential variables used by the function type $(\hat{\alpha}, \hat{\beta})$ are only visible in the remaining chain $[\hat{\alpha} \rightarrow \hat{\beta}/a]\omega$. The recycling process of Rule 25 differs from DK's corresponding rule significantly, and we further discuss the differences in Section 4.4.1.

6. Application inference judgments Finally, Rules 27-29 deal with application inference judgments. Rules 27 and 28 behaves similarly to declarative rules $\text{Decl}\forall\text{App}$ and $\text{Decl} \rightarrow \text{App}$. Rule 29 instantiates $\hat{\alpha}$ to the function type $\hat{\alpha}_1 \rightarrow \hat{\alpha}_2$, just like Rules 10, 11 and 21.

EXAMPLE Figure 4.6 shows a sample derivation of the algorithm. It checks the application $(\lambda x. x) ()$ against the unit type. According to the algorithm, we apply Rule 18 (subsumption), changing to inference mode. Type inference of the application breaks into two steps by Rule 26: first we infer the type of the function, and then the application inference judgment helps to determine the return type. In the following 5 steps the type of the identity function, $\lambda x. x$, is inferred to be $\hat{\alpha} \rightarrow \hat{\alpha}$: checking the body of the lambda function (Rule 25), switching from check mode to inference mode (Rule 18), inferring the type of a term variable (Rule 22), solving a subtyping between existential variables (Rule 12) and garbage collecting the term variable x (Rule 3).

After that, Rule 28 changes the application inference judgment to a check of the argument against the input type $\hat{\alpha}$ and returns the output type $\hat{\alpha}$. Checking $()$ against the existential variable $\hat{\alpha}$ solves $\hat{\alpha}$ to the unit type 1 through Rules 18, 24 and 16. Immediately after $\hat{\alpha}$ is solved, the algorithm replaces every occurrence of $\hat{\alpha}$ with 1. Therefore the worklist remains $1 \leq 1$, which is finished off by Rule 4. Finally, the empty worklist indicates the success of the whole derivation.

In summary, our type checking algorithm accepts $(\lambda x. x) () \Leftarrow 1$.

NON-OVERLAPPING AND DETERMINISTIC REDUCTION An important feature of our algorithmic rules is that they are directly implementable. Indeed, although written in the form of reduction rules, they do not overlap and are thus deterministic.

Consider in particular Rules 8 and 9, which correspond to the declarative rules $\leq\forall\text{L}$ and $\leq\forall\text{R}$. While those declarative rules both match the goal $\forall a. A \leq \forall b. B$, we have eliminated this overlap in the algorithm by restricting Rule 8 ($B \neq \forall a. B'$) and thus always applying Rule 9 to $\forall a. A \leq \forall b. B$.

Similarly, the declarative rule DeclSub overlaps highly with the other checking rules. Its algorithmic counterpart is Rule 18. Yet, we have avoided the overlap with other algorithmic checking rules by adding side-conditions to Rule 18, namely $e \neq \lambda x. e'$ and $B \neq \forall a. B'$.

$$\begin{aligned}
 & \cdot \Vdash (\lambda x. x) () \Leftarrow 1 \\
 \longrightarrow_{18} & \cdot \Vdash (\lambda x. x) () \Rightarrow_a a \leq 1 \\
 \longrightarrow_{26} & \cdot \Vdash (\lambda x. x) \Rightarrow_b (b \bullet ()) \Rightarrow_a a \leq 1 \\
 \longrightarrow_{25} & \cdot, \hat{\alpha}, \hat{\beta} \Vdash \hat{\alpha} \rightarrow \hat{\beta} \bullet () \Rightarrow_a a \leq 1, x : \hat{\alpha} \Vdash x \Leftarrow \hat{\beta} \\
 \longrightarrow_{18} & \cdot, \hat{\alpha}, \hat{\beta} \Vdash \hat{\alpha} \rightarrow \hat{\beta} \bullet () \Rightarrow_a a \leq 1, x : \hat{\alpha} \Vdash x \Rightarrow_b b \leq \hat{\beta} \\
 \longrightarrow_{22} & \cdot, \hat{\alpha}, \hat{\beta} \Vdash \hat{\alpha} \rightarrow \hat{\beta} \bullet () \Rightarrow_a a \leq 1, x : \hat{\alpha} \Vdash \hat{\alpha} \leq \hat{\beta} \\
 \longrightarrow_{12} & \cdot, \hat{\alpha} \Vdash \hat{\alpha} \rightarrow \hat{\alpha} \bullet () \Rightarrow_a a \leq 1, x : \hat{\alpha} \\
 \longrightarrow_3 & \cdot, \hat{\alpha} \Vdash \hat{\alpha} \rightarrow \hat{\alpha} \bullet () \Rightarrow_a a \leq 1 \\
 \longrightarrow_{28} & \cdot, \hat{\alpha} \Vdash \hat{\alpha} \leq 1 \Vdash () \Leftarrow \hat{\alpha} \\
 \longrightarrow_{18} & \cdot, \hat{\alpha} \Vdash \hat{\alpha} \leq 1 \Vdash () \Rightarrow_a a \leq \hat{\alpha} \\
 \longrightarrow_{24} & \cdot, \hat{\alpha} \Vdash \hat{\alpha} \leq 1 \Vdash 1 \leq \hat{\alpha} \\
 \longrightarrow_{16} & \cdot \Vdash 1 \leq 1 \\
 \longrightarrow_4 & \cdot
 \end{aligned}$$

Figure 4.6: A Sample Derivation for Algorithmic Typing

These restrictions have not been imposed arbitrarily: we formally prove that the restricted algorithm is still complete. In Section 4.3.2 we discuss the relevant metatheory, with the help of a non-overlapping version of the declarative system.

4.3 METATHEORY

This section presents the metatheory of the algorithmic system presented in the previous section. We show that three main results hold: *soundness*, *completeness* and *decidability*. These three results have been mechanically formalized and proved in the Abella theorem prover [Gacek 2008].

4.3.1 DECLARATIVE WORKLIST AND TRANSFER

To aid formalizing the correspondence between the declarative and algorithmic systems, we introduce the notion of a declarative worklist Ω , defined in Figure 4.7. A declarative worklist Ω has the same structure as an algorithmic worklist Γ , but does not contain any existential variables $\hat{\alpha}$.

WORKLIST INSTANTIATION. The relation $\Gamma \rightsquigarrow \Omega$ expresses that the algorithmic worklist Γ can be instantiated to the declarative worklist Ω , by appropriately instantiating all existential

Declarative worklist $\Omega ::= \cdot \mid \Omega, a \mid \Omega, x : A \mid \Omega \Vdash \omega$

$$\boxed{\Gamma \rightsquigarrow \Omega} \quad \Gamma \text{ instantiates to } \Omega.$$

$$\frac{}{\Omega \rightsquigarrow \Omega} \rightsquigarrow \Omega \quad \frac{\Omega \vdash \tau \quad \Omega, [\tau/\hat{\alpha}]\Gamma \rightsquigarrow \Omega}{\Omega, \hat{\alpha}, \Gamma \rightsquigarrow \Omega} \rightsquigarrow \hat{\alpha}$$

Figure 4.7: Declarative Worklists and Instantiation

variables $\hat{\alpha}$ in Γ with well-scoped monotypes τ . The rules of this instantiation relation are shown in Figure 4.7 too. Rule $\rightsquigarrow \hat{\alpha}$ replaces the first existential variable with a well-scoped monotype and repeats the process on the resulting worklist until no existential variable remains and thus the algorithmic worklist has become a declarative one. In order to maintain well-scopedness, the substitution is applied to all the judgments and term variable bindings in the scope of $\hat{\alpha}$.

Observe that the instantiation $\Gamma \rightsquigarrow \Omega$ is not deterministic. From left to right, there are infinitely many possibilities to instantiate an existential variable and thus infinitely many declarative worklists that one can get from an algorithmic one. In the other direction, any valid monotype in Ω can be abstracted to an existential variable in Γ . Thus different Γ 's can be instantiated to the same Ω .

Lemmas 4.1 and 4.2 generalize Rule $\rightsquigarrow \hat{\alpha}$ from substituting the first existential variable to substituting any existential variable.

Lemma 4.1 (Insert). *If $\Gamma_L, [\tau/\hat{\alpha}]\Gamma_R \rightsquigarrow \Omega$ and $\Gamma_L \vdash \tau$, then $\Gamma_L, \hat{\alpha}, \Gamma_R \rightsquigarrow \Omega$.*

Lemma 4.2 (Extract). *If $\Gamma_L, \hat{\alpha}, \Gamma_R \rightsquigarrow \Omega$, then there exists τ s.t. $\Gamma_L \vdash \tau$ and $\Gamma_L, [\tau/\hat{\alpha}]\Gamma_R \rightsquigarrow \Omega$.*

DECLARATIVE TRANSFER. Figure 4.8 defines a relation $\Omega \longrightarrow \Omega'$, which transfers all judgments in the declarative worklists to the declarative type system. This relation checks that every judgment entry in the worklist holds using a corresponding conventional declarative judgment. The typing contexts of declarative judgments are recovered using an auxiliary erasure function $\|\Omega\|$ ². The erasure function simply drops all judgment entries from the worklist, keeping only variable and type variable declarations.

²In the proof script we do not use the erasure function, for the declarative system and well-formedness judgments automatically fit the non-erased declarative worklist just as declarative contexts.

$\boxed{\ \Omega\ }$	Judgment erasure.
$\begin{aligned} \ \cdot\ &= \cdot \\ \ \Omega, a\ &= \ \Omega\ , a \\ \ \Omega, x : A\ &= \ \Omega\ , x : A \\ \ \Omega \vdash \omega\ &= \ \Omega\ \end{aligned}$	
$\boxed{\Omega \longrightarrow \Omega'}$	Declarative transfer.
$\begin{aligned} \Omega, a &\longrightarrow \Omega \\ \Omega, x : A &\longrightarrow \Omega \\ \Omega \vdash A \leq B &\longrightarrow \Omega && \text{when } \ \Omega\ \vdash A \leq B \\ \Omega \vdash e \Leftarrow A &\longrightarrow \Omega && \text{when } \ \Omega\ \vdash e \Leftarrow A \\ \Omega \vdash e \Rightarrow_a \omega &\longrightarrow \Omega \vdash [A/a]\omega && \text{when } \ \Omega\ \vdash e \Rightarrow A \\ \Omega \vdash A \bullet e \Rightarrow_a \omega &\longrightarrow \Omega \vdash [C/a]\omega && \text{when } \ \Omega\ \vdash A \bullet e \Rightarrow C \end{aligned}$	

Figure 4.8: Declarative Transfer

4.3.2 NON-OVERLAPPING DECLARATIVE SYSTEM

DK's declarative system, shown in Figures 2.8 and 2.9, has a few overlapping rules. In contrast, our algorithm has removed all overlap; at most one rule applies in any given situation. This discrepancy makes it more difficult to relate the two systems.

To simplify matters, we introduce an intermediate system that is still declarative in nature, but has no overlap. This intermediate system differs only in a few rules from DK's declarative system:

1. Restrict the shape of B in the rule $\forall L$ subtyping rule:

$$\frac{B \neq \forall b. B' \quad \Psi \vdash \tau \quad \Psi \vdash [\tau/a]A \leq B}{\Psi \vdash \forall a. A \leq B} \forall L'$$

2. Drop the redundant rule Decl1I , which can be easily derived by a combination of DeclSub , $\text{Decl1I} \Rightarrow$ and $\leq \text{Unit}$:

$$\frac{\frac{}{\Psi \vdash () \Rightarrow 1} \text{Decl1I} \Rightarrow \quad \frac{}{\Psi \vdash 1 \leq 1} \leq \text{Unit}}{\Psi \vdash () \Leftarrow 1} \text{DeclSub}$$

3. Restrict the shapes of e and A in the subsumption rule Dec1Sub:

$$\frac{e \neq \lambda x. e' \quad A \neq \forall a. A' \quad \Psi \vdash e \Rightarrow A \quad \Psi \vdash A \leq B}{\Psi \vdash e \Leftarrow B} \text{Dec1Sub}'$$

The resulting declarative system has no overlapping rules and more closely resembles the algorithmic system, which contains constraints of the same shape.

We have proven soundness and completeness of the non-overlapping declarative system with respect to the overlapping one to establish their equivalence. Thus the restrictions do not change the expressive power of the system. Modification (2) is relatively easy to justify, with the derivation given above: the rule is redundant and can be replaced by a combination of three other rules. Modifications (1) and (3) require inversion lemmas for the rules that overlap. Firstly, Rule $\forall L$ overlaps with Rule $\forall R$ for the judgment $\Psi \vdash \forall a. A \leq \forall b. B$. The following inversion lemma for Rule $\forall R$ resolves the overlap:

Lemma 4.3 (Invertibility of $\forall R$). *If $\Psi \vdash A \leq \forall a. B$ then $\Psi, a \vdash A \leq B$.*

The lemma implies that preferring Rule $\forall R$ does not affect the derivability of the judgment. Therefore the restriction $B \neq \forall b. B'$ in $\forall L'$ is valid.

Secondly, Rule Dec1Sub overlaps with both Dec1 $\forall I$ and Dec1 $\rightarrow I$. We have proven two inversion lemmas for these overlaps:

Lemma 4.4 (Invertibility of Dec1 $\forall I$). *If $\Psi \vdash e \Leftarrow \forall a. A$ then $\Psi, a \vdash e \Leftarrow A$.*

Lemma 4.5 (Invertibility of Dec1 $\rightarrow I$). *If $\Psi \vdash \lambda x. e \Leftarrow A \rightarrow B$ then $\Psi, x : A \vdash e \Leftarrow B$.*

These lemmas express that applying the more specific rules, rather than the more general rule Dec1Sub, does not reduce the expressive power.

The proofs of the above two lemmas rely on an important property of the declarative system, the *subsumption lemma*. To be able to formulate this lemma, Figure 4.9 introduces the *context subtyping relation* $\Psi \leq \Psi'$. Context Ψ subsumes context Ψ' if they bind the same variables in the same order, but the types A of the term variables x in the former are subtypes of types A' assigned to those term variables in the latter. Now we can state the subsumption lemma:

Lemma 4.6 (Subsumption). *Given $\Psi' \leq \Psi$:*

1. *If $\Psi \vdash e \Leftarrow A$ and $\Psi \vdash A \leq A'$ then $\Psi' \vdash e \Leftarrow A'$;*
2. *If $\Psi \vdash e \Rightarrow B$ then there exists B' s.t. $\Psi \vdash B' \leq B$ and $\Psi' \vdash e \Rightarrow B'$;*

$$\boxed{\Psi' \leq \Psi}$$

$$\frac{}{\cdot \leq \cdot} \text{CtxSubEmpty} \quad \frac{\Psi' \leq \Psi}{\Psi', a \leq \Psi, a} \text{CtxSubTyVar} \quad \frac{\Psi' \leq \Psi \quad \Psi \vdash A' \leq A}{\Psi', x : A' \leq \Psi, x : A} \text{CtxSubTmVar}$$

Figure 4.9: Context Subtyping

3. If $\Psi \vdash A \bullet e \Rightarrow C$ and $\Psi \vdash A' \leq A$, then there exists C' s.t. $\Psi \vdash C' \leq C$ and $\Psi' \vdash A' \bullet e \Rightarrow C'$.

This lemma expresses that any derivation in a context Ψ has a corresponding derivation in any context Ψ' that it subsumes.

We have tried to follow DK's manual proof of this lemma, but we discovered several problems in their reasoning that we have been unable to address. Fortunately we have found a different way to prove the lemma. The details of this issue can be found in Chapter 4.5.2.

THREE-WAY SOUNDNESS AND COMPLETENESS THEOREMS We now have three systems that can be related: DK's overlapping declarative system, our non-overlapping declarative system, and our algorithmic system. We have already established the first relation, that the two declarative systems are equivalent. In what follows, we will establish the soundness of our algorithm directly against the original overlapping declarative system. However, we have found that showing completeness of the algorithm is easier against the non-overlapping declarative system. Of course, as a corollary, it follows that our algorithm is also complete with respect to DK's declarative system.

4.3.3 SOUNDNESS

Our algorithm is sound with respect to DK's declarative system. For any worklist Γ that reduces successfully, there is a valid instantiation Ω that transfers all judgments to the declarative system.

Theorem 4.7 (Soundness). *If $\text{wf} \Gamma$ and $\Gamma \longrightarrow^* \cdot$, then there exists Ω s.t. $\Gamma \rightsquigarrow \Omega$ and $\Omega \longrightarrow^* \cdot$.*

The proof proceeds by induction on the derivation of $\Gamma \longrightarrow^* \cdot$. Interesting cases are those involving existential variable instantiations, including Rules 10, 11, 21 and 29. Applications of Lemmas 4.1 and 4.2 help analyse the full instantiation of those existential variables. For example, when $\hat{\alpha}$ is solved to $\hat{\alpha}_1 \rightarrow \hat{\alpha}_2$ in the algorithm, applying the Extract lemma gives two instantiations $\hat{\alpha}_1 = \sigma$ and $\hat{\alpha}_2 = \tau$. It follows that $\hat{\alpha} = \sigma \rightarrow \tau$, which enables the induction hypothesis and finishes the corresponding case. Some immediate corollaries which show the soundness for specific judgment forms are:

Corollary 4.8 (Soundness, single judgment form). *Given wf Γ :*

1. *If $\Gamma \Vdash A \leq B \longrightarrow^* \cdot$.
then there exist A', B', Ω s.t. $\Gamma \Vdash A \leq B \rightsquigarrow \Omega \Vdash A' \leq B'$ and $\|\Omega\| \vdash A' \leq B'$;*
2. *If $\Gamma \Vdash e \Leftarrow A \longrightarrow^* \cdot$.
then there exist A', Ω s.t. $\Gamma \Vdash e \Leftarrow A \rightsquigarrow \Omega \Vdash e \Leftarrow A'$ and $\|\Omega\| \vdash e \Leftarrow A'$;*
3. *If $\Gamma \Vdash e \Rightarrow_a \omega \longrightarrow^* \cdot$ for any ω
then there exists Ω, ω', A s.t. $\Gamma \rightsquigarrow \Omega$ and $\|\Omega\| \vdash e \Rightarrow A$;*
4. *If $\Gamma \Vdash A \bullet e \Rightarrow_a \omega \longrightarrow^* \cdot$ for any ω
then there exists Ω, ω', A', C s.t. $\Gamma \Vdash A \bullet e \Rightarrow_a \omega \rightsquigarrow \Omega \Vdash A' \bullet e \Rightarrow_a \omega'$ and
 $\|\Omega\| \vdash A' \bullet e \Rightarrow C$.*

4.3.4 COMPLETENESS

The completeness of our algorithm means that any derivation in the declarative system has an algorithmic counterpart. We explicitly relate between an algorithmic context Γ and a declarative context Ω to avoid potential confusion.

Theorem 4.9 (Completeness). *If wf Γ and $\Gamma \rightsquigarrow \Omega$ and $\Omega \longrightarrow^* \cdot$, then $\Gamma \longrightarrow^* \cdot$.*

We prove completeness by induction on the derivation of $\Omega \longrightarrow^* \cdot$ and use the non-overlapping declarative system. Since the declarative worklist is reduced judgment by judgment (shown in Figure 4.8), the induction always analyses the first judgment by a small step. As the algorithmic system introduces existential variables, a declarative rule may correspond to multiple algorithmic rules, and thus we analyse each of the possible cases.

Most cases are relatively easy to prove. The Insert and Extract lemmas are applied when the algorithm uses existential variables, but transferred to a monotype for the declarative system, such as Rules 6, 8, 10, 11, 12-17, 21, 25, 27 and 29.

Algorithmic Rules 10 and 11 require special treatment. When the induction reaches the $\leq \rightarrow$ case, the first judgment is of shape $A_1 \rightarrow A_2 \leq B_1 \rightarrow B_2$. One of the corresponding algorithmic judgments is $\hat{\alpha} \leq A \rightarrow B$. However, the case analysis does not imply that $\hat{\alpha}$ is fresh in A and B , therefore Rule 10 cannot be applied and the proof gets stuck. The following lemma helps us out in those cases: the success in declarative subtyping indicates the freshness of $\hat{\alpha}$ in A and B in its corresponding algorithmic judgment. In other words, the declarative system does not accept infinite types. A symmetric lemma holds for $A \rightarrow B \leq \hat{\alpha}$.

Lemma 4.10 (Prune Transfer for Instantiation). *If $(\Gamma \Vdash \hat{\alpha} \leq A \rightarrow B) \rightsquigarrow (\Omega \Vdash C \leq A_1 \rightarrow B_1)$ and $\|\Omega\| \vdash C \leq A_1 \rightarrow B_1$, then $\hat{\alpha} \notin FV(A) \cup FV(B)$.*

The following corollary is derived immediately from Theorem 4.9.

Corollary 4.11 (Completeness, single judgment form). *Given wf Γ containing no judgments:*

1. *If $\Omega \vdash A' \leq B'$ and $\Gamma \Vdash A \leq B \rightsquigarrow \Omega \Vdash A' \leq B'$
then $\Gamma \Vdash A \leq B \longrightarrow^*$.;*
2. *If $\Omega \vdash e \Leftarrow A'$ and $\Gamma \Vdash e \Leftarrow A \rightsquigarrow \Omega \Vdash e \Leftarrow A'$
then $\Gamma \Vdash e \Leftarrow A \longrightarrow^*$.;*
3. *If $\Omega \vdash e \Rightarrow A$ and $\Gamma \Vdash e \Rightarrow_a 1 \leq 1 \rightsquigarrow \Omega \Vdash e \Rightarrow_a 1 \leq 1$
then $\Gamma \Vdash e \Rightarrow_a 1 \leq 1 \longrightarrow^*$.;*
4. *If $\Omega \vdash A' \bullet e \Rightarrow C$ and $\Gamma \Vdash A \bullet e \Rightarrow_a 1 \leq 1 \rightsquigarrow \Omega \Vdash A' \bullet e \Rightarrow_a 1 \leq 1$
then $\Gamma \Vdash A \bullet e \Rightarrow_a 1 \leq 1 \longrightarrow^*$.*

4.3.5 DECIDABILITY

Finally, we show that our algorithm is decidable:

Theorem 4.12 (Decidability). *Given wf Γ , it is decidable whether $\Gamma \longrightarrow^* \cdot$ or not.*

Our decidability proof is based on a lexicographic group of induction measures $\langle |\Gamma|_e, |\Gamma|_{\Leftarrow}, |\Gamma|_{\forall}, |\Gamma|_{\hat{\alpha}}, |\Gamma|_{\rightarrow} + |\Gamma| \rangle$ on the worklist Γ . Formal definitions of these measures can be found in Figure 4.10. The first two, $|\cdot|_e$ and $|\cdot|_{\Leftarrow}$, measure the total size of terms and the total difficulty of judgments, respectively. In the latter, check judgments count for 2, inference judgments for 1 and function inference judgments for 3. Another two measures, $|\cdot|_{\forall}$ and $|\cdot|_{\rightarrow}$, count the total number of universal quantifiers and function types, respectively. Finally, $|\cdot|_{\hat{\alpha}}$ counts the number of existential variables in the worklist, and $|\cdot|$ is simply the length of the worklist.

It is not difficult to see that all but two algorithmic reduction rules decrease the group of measures. (The result of Rule 29 could be directly reduced by Rule 28, which decreases the measures.) The two exceptions are Rules 10 and 11. Both rules increase the number of existential variables without decreasing the number of universal quantifiers. However, they are both immediately followed by Rule 7, which breaks the subtyping problem into two smaller problems of the form $\hat{\alpha} \leq A$ and $A \leq \hat{\alpha}$ which we call *instantiation judgments*.

We now show that entirely reducing these smaller problems leaves the worklist in a state with an overall smaller measure. Our starting point is a worklist Γ, Γ_i where Γ_i are instantiation judgments.

$$\Gamma_i := \cdot \mid \Gamma_i, \hat{\alpha} \leq A \mid \Gamma_i, A \leq \hat{\alpha} \quad \text{where } \hat{\alpha} \notin FV(A) \cup FV(\Gamma_i)$$

4 Higher-Rank Polymorphism Worklist Algorithm

$\boxed{ \Gamma _e, \omega _e, e _e}$ Size measure.	$\boxed{ \Gamma _{\forall}, \omega _{\forall}, A _{\forall}}$ Polymorphism measure.
$ \Gamma _e = \sum_{\omega \in \Gamma} \omega _e$	$ \Gamma _{\Leftrightarrow} = \sum_{\omega \in \Gamma} \omega _{\forall}$
$ A \leq B _e = 0$	$ A \leq B _{\forall} = A _{\forall} + B _{\forall}$
$ e \Leftarrow A _e = e _e$	$ e \Leftarrow A _{\forall} = A _{\forall}$
$ e \Rightarrow \omega _e = e _e + \omega _e$	$ e \Rightarrow \omega _{\forall} = \omega _{\forall}$
$ A \bullet e \Rightarrow_a \omega _e = e _e + \omega _e$	$ A \bullet e \Rightarrow_a \omega _{\forall} = A _{\forall} + \omega _{\forall}$
$ x _e = () _e = 1$	$ 1 _{\forall} = a _{\forall} = \hat{\alpha} _{\forall} = 1$
$ \lambda x. e _e = e _e + 1$	$ A \rightarrow B _{\forall} = A _{\forall} + B _{\forall}$
$ e_1 e_2 _e = e_1 _e + e_2 _e + 1$	$ \forall a. A _{\forall} = A _{\forall} + 1$
$ e : A _e = e _e + 1$	
$\boxed{ \Gamma _{\Leftrightarrow}, \omega _{\Leftrightarrow}}$ Judgment measure.	$\boxed{ \Gamma _{\rightarrow}, \omega _{\rightarrow}, A _{\rightarrow}}$ Function measure.
$ \Gamma _{\Leftrightarrow} = \sum_{\omega \in \Gamma} \omega _e$	$ \Gamma _{\rightarrow} = \sum_{\omega \in \Gamma} \omega _{\forall}$
$ A \leq B _{\Leftrightarrow} = 0$	$ A \leq B _{\forall} = A _{\forall} + B _{\forall}$
$ e \Leftarrow A _{\Leftrightarrow} = 2$	$ e \Leftarrow A _{\forall} = A _{\forall}$
$ e \Rightarrow \omega _{\Leftrightarrow} = \omega _{\Leftrightarrow} + 1$	$ e \Rightarrow \omega _{\forall} = \omega _{\forall}$
$ A \bullet e \Rightarrow_a \omega _{\Leftrightarrow} = \omega _{\Leftrightarrow} + 3$	$ A \bullet e \Rightarrow_a \omega _{\forall} = A _{\forall} + \omega _{\forall}$
$\boxed{ \Gamma _{\hat{\alpha}}}$ Existential measure.	$ 1 _{\rightarrow} = a _{\rightarrow} = \hat{\alpha} _{\rightarrow} = 1$
$ \Gamma _{\hat{\alpha}} = \#\hat{\alpha} \in \Gamma$	$ A \rightarrow B _{\rightarrow} = A _{\rightarrow} + B _{\rightarrow} + 1$
	$ \forall a. A _{\rightarrow} = A _{\rightarrow}$

Figure 4.10: Worklist measures

Fully reducing these instantiation judgments at the top of the worklist has a twofold impact. Firstly, new entries may be pushed onto the worklist which are not instantiation judgments. This only happens when Γ_i contains a universal quantifier that is reduced by Rule 8 or 9. The new entries then are of the form Γ_{\leq} :

$$\Gamma_{\leq} := \cdot \mid \Gamma_{\leq}, a \mid \Gamma_{\leq}, \hat{\alpha} \mid \Gamma_{\leq}, A \leq B$$

Secondly, reducing the instantiation judgments may also affect the remainder of the worklist Γ , by solving existing existentials and introducing new ones. This worklist update is captured in the update judgment $\Gamma \rightarrow \Gamma'$ defined in Figure 4.11. For instance, an existential variable

$$\begin{array}{c}
 \boxed{\Gamma \rightarrow \Gamma'} \quad \Gamma \text{ updates to } \Gamma'. \\
 \frac{}{\Gamma \rightarrow \Gamma} \rightarrow \text{id} \quad \frac{|A|_{\forall} = 0 \quad \Gamma_L, [A/\hat{\alpha}] \Gamma_R \rightarrow \Gamma'}{\Gamma_L, \hat{\alpha}, \Gamma_R \rightarrow \Gamma'} \rightarrow \text{solve} \quad \frac{\Gamma_L, \hat{\alpha}, \Gamma_R \rightarrow \Gamma'}{\Gamma_L, \Gamma_R \rightarrow \Gamma'} \rightarrow \hat{\alpha}
 \end{array}$$

Figure 4.11: Worklist Update

instantiation, $\Gamma_L, \hat{\alpha}, \Gamma_R \rightarrow \Gamma_L, \hat{\alpha}_1, \hat{\alpha}_2, [\hat{\alpha}_1 \rightarrow \hat{\alpha}_2/\hat{\alpha}] \Gamma_R$, can be derived as a combination of the three rules that define the update relation.

The good news is that worklist updates do not affect the three main worklist measures:

Lemma 4.13 (Measure Invariants of Worklist Extension). *If $\Gamma \rightarrow \Gamma'$ then $|\Gamma|_e = |\Gamma'|_e$ and $|\Gamma|_{\Leftrightarrow} = |\Gamma'|_{\Leftrightarrow}$ and $|\Gamma|_{\forall} = |\Gamma'|_{\forall}$.*

Moreover, we can characterize the reduction of the instantiation judgments as follows.

Lemma 4.14 (Instantiation Decidability). *For any well-formed algorithmic worklist (Γ, Γ_i) :*

- 1) *If $|\Gamma_i|_{\forall} = 0$, then there exists Γ'
 s.t. $(\Gamma, \Gamma_i) \rightarrow^* \Gamma'$ and $|\Gamma'|_{\hat{\alpha}} = |\Gamma|_{\hat{\alpha}} - |\Gamma_i|$ and $\Gamma \rightarrow \Gamma'$.*
- 2) *If $|\Gamma_i|_{\forall} > 0$, then there exist Γ', Γ_{\leq}
 s.t. $(\Gamma, \Gamma_i) \rightarrow^* (\Gamma', \Gamma_{\leq})$ and $|\Gamma_{\leq}|_{\forall} = |\Gamma_i|_{\forall} - 1$ and $\Gamma \rightarrow \Gamma'$.*

Hence, reducing the instantiation judgment prefix Γ_i either decreases the number of universal quantifiers or eliminates one existential variable per instantiation judgment. The proof of this lemma proceeds by induction on the measure $2 * |\Gamma_i|_{\rightarrow} + |\Gamma_i|$ of the instantiation judgment list Γ_i .

Let us go back to the whole algorithm and summarize our findings. The decidability theorem is shown through a lexicographic group of induction measures $\langle |\Gamma|_e, |\Gamma|_{\Leftrightarrow}, |\Gamma|_{\forall}, |\Gamma|_{\hat{\alpha}}, |\Gamma|_{\rightarrow} + |\Gamma| \rangle$ which is decreased by nearly all rules. In the exceptional case that the measure does not decrease immediately, we encounter an instantiation judgment at the top of the worklist. We can then make use of Lemma 4.14 to show that $|\Gamma|_{\hat{\alpha}}$ or $|\Gamma|_{\forall}$ decreases when that instantiation judgment is consumed or partially reduced. Moreover, Lemma 4.13 establishes that no higher-priority measure component increases. Hence, in the exceptional case we have an overall measure decrease too.

Combining all three main results (soundness, completeness and decidability), we conclude that the declarative system is decidable by means of our algorithm.

Corollary 4.15 (Decidability of Declarative Typing). *Given wf Ω , it is decidable whether $\Omega \rightarrow^* \cdot$ or not.*

4.3.6 ABELLA AND PROOF STATISTICS

We have chosen the Abella (v2.0.7-dev³) proof assistant [Gacek 2008] to develop our formalization. Abella is designed to help with formalizations of programming languages, due to its built-in support for variable binding and the λ -tree syntax [Miller 2000], which is a form of HOAS. Nominal variables, or ∇ -quantified variables, are used as an unlimited name supply, which supports explicit freshness control and substitutions. Although Abella lacks packages, tactics and support for modules, its higher-order unification and the ease of formalizing substitution-intensive relations are very helpful.

While the algorithmic rules are in a small-step style, the proof script rewrites them into a big-step style for easier inductions. In addition, we do prove the equivalence of the two representations.

STATISTICS OF THE PROOF The proof script consists of 7,977 lines of Abella code with a total of 60 definitions and 596 theorems. Figure 4.1 briefly summarizes the contents of each file. The files are linearly dependent due to limitations of Abella.

TRANSLATION TABLE FOR THE PROOF SCRIPTS In the proof scripts, we use textual relational definitions rather than the symbolic ones used in the paper. The mapping, shown in Table 4.2, should be helpful for anyone who reads the script.

4.4 DISCUSSION

This section discusses some insights that we gained from our work and contrasts the scoping mechanisms we have employed with those in DK’s algorithm. We also discuss a way to improve the precision of their scope tracking. Furthermore we discuss and sketch an extension of our algorithm with an elaboration to a target calculus, and discuss an extension of our algorithm with scoped type variables [Peyton Jones and Shields 2004].

4.4.1 CONTRASTING OUR SCOPING MECHANISMS WITH DK’S

A nice feature of our worklists is that, simply by interleaving variable declarations and judgment chains, they make the scope of variables precise. DK’s algorithm deals with garbage collecting variables in a different way: through type variable or existential variable markers (as discussed in Section 4.1.2). Despite the sophistication employed in DK’s algorithm to

³We use a development version because the developers just fixed a serious bug that accepts a simple proof of false, which also affects our proof. Specifically, our scripts compile against commit 92829a of Abella’s GitHub repository.

Table 4.1: Statistics for the proof scripts

File(s)	LOC	#Thm	Description
olist.thm, nat.thm	311	57	Basic data structures
typing.thm	245	7	Declarative & algorithmic system, debug examples
decl.thm	226	33	Basic declarative properties
order.thm	235	27	The $ \cdot _{\forall}$ measure; decl. subtyping strengthening
alg.thm	679	80	Basic algorithmic properties
trans.thm	616	53	Worklist instantiation and declarative transfer; Lemmas 4.1 , 4.2
declTyping.thm	909	70	Non-overlapping declarative system; Lemmas 4.3 , 4.4 , 4.5 , 4.6
soundness.thm	1,107	78	Soundness theorem; aux. lemmas on transfer
depth.thm	206	14	The $ \cdot _{\rightarrow}$ measure; Lemma 4.10
dcl.thm	380	12	Non-overlapping declarative worklist
completeness.thm	1,124	61	Completeness theorem; aux. lemmas and relations
inst_decidable.thm	837	45	Other worklist measures; Lemma 4.14
decidability.thm	983	57	Decidability theorem and corollary
smallStep.thm	119	2	The equivalence between big-step and small-step
<i>Total</i>	7,977	596	(60 definitions in total)

Table 4.2: Translation Table for the Proof Scripts

Symbol	Abella textual relation	File (.thm)	Description
A	<code>ty, wft, wfta</code>	<code>typing</code>	The "type" type, <code>decl./alg.</code> well-formedness
e	<code>tm, wftm</code>	<code>typing</code>	The "term" type, (alg.) well-formedness
Ψ, Γ	<code>olist, wfj</code>	<code>typing</code>	olist is Abella's built-in list type, i.e. [o]
ω	<code>judgment, wfjg</code>	<code>typing</code>	Judgment chain and its well-formedness
$\Gamma \rightarrow \Gamma'$	<code>reduction</code>	<code>smallStep</code>	Algorithmic reduction: paper version
$\Gamma \rightarrow^* \cdot$	<code>judge</code>	<code>typing</code>	(judge Γ): a success reduction on Γ
$\Gamma \rightsquigarrow \Omega$	<code>tex</code>	<code>trans</code>	"transfer existential variables"
$\Omega \rightarrow^* \cdot$	<code>dc, dcl</code>	<code>trans, dcl</code>	Declarative chain representation
$\Psi' \leq \Psi$	<code>esub</code>	<code>declTyping</code>	Declarative context subtyping
$\Gamma \rightarrow \Gamma'$	<code>jExt</code>	<code>inst_decidable</code>	"Judgment extension"
Γ_i	<code>iexp</code>	<code>inst_decidable</code>	instantiation judgments ($\Gamma \vdash_{wf} \Gamma_i$)
$\Gamma \leq$	<code>subExp</code>	<code>inst_decidable</code>	subtyping judgments
$ \cdot _e$	<code>tmSize, tmSizeJ, tmSize1</code>	<code>declTyping, decidability</code>	Size measure for term, judgment chain or context
$ \cdot _{\Leftrightarrow}$	<code>m_judgeJ, m_judge</code>	<code>decidability</code>	Judgment measure for judgment chain or context
$ \cdot _{\hat{\alpha}}$	<code>nVar</code>	<code>inst_decidable</code>	Existential measure for context
$ \cdot _{\forall}$	<code>order, orderJ, order1</code>	<code>order, inst_decidable</code>	Polymorphism measure for type, judgment chain or context
$ \cdot _{\rightarrow}$	<code>depth, depthJ, depth1</code>	<code>order, decidability</code>	Function measure for type, judgment chain or context

keep scoping precise, there is still a chance that unused existential variables leak their scope to an output context and accumulate indefinitely. For example, the derivation of the judgment $(\lambda x. x) () \Leftarrow 1$ is as follows

$$\begin{array}{c}
 \frac{\dots x \Rightarrow \hat{\alpha} \dots \quad \dots \hat{\alpha} \leq \hat{\beta} \dots}{\Gamma, \hat{\alpha}, \hat{\beta}, x : \hat{\alpha} \vdash x \Leftarrow \hat{\beta} \dashv \Gamma_1, x : \hat{\alpha}} \quad \frac{\dots () \Leftarrow \hat{\alpha} \dots}{\Gamma_1 \vdash \hat{\alpha} \rightarrow \hat{\alpha} \bullet () \Rightarrow \hat{\alpha} \dashv \Gamma_2} \\
 \frac{\Gamma \vdash \lambda x. x \Rightarrow \hat{\alpha} \rightarrow \hat{\beta} \dashv \Gamma_1 \quad \Gamma_1 \vdash \hat{\alpha} \rightarrow \hat{\alpha} \bullet () \Rightarrow \hat{\alpha} \dashv \Gamma_2}{\Gamma \vdash (\lambda x. x) () \Rightarrow \hat{\alpha} \dashv \Gamma_2} \\
 \frac{\Gamma_2 \vdash 1 \leq 1 \dashv \Gamma_2}{\Gamma \vdash (\lambda x. x) () \Leftarrow 1 \dashv \Gamma, \hat{\alpha} = 1, \hat{\beta} = \hat{\alpha}}
 \end{array}$$

where $\Gamma_1 := (\Gamma, \hat{\alpha}, \hat{\beta} = \hat{\alpha})$ solves $\hat{\beta}$, and $\Gamma_2 := (\Gamma, \hat{\alpha} = 1, \hat{\beta} = \hat{\alpha})$ solves both $\hat{\alpha}$ and $\hat{\beta}$.

If the reader is not familiar with DK's algorithm, he/she might be confused about the inconsistent types across judgment. As an example, $(\lambda x. x) ()$ synthesizes $\hat{\alpha}$, but the second premise of the subsumption rule uses 1 for the result. This is because a context application $[\Gamma, \hat{\alpha} = 1, \hat{\beta} = \hat{\alpha}] \hat{\alpha} = 1$ happens between the premises.

The existential variables $\hat{\alpha}$ and $\hat{\beta}$ are clearly not used after the subsumption rule, but according to the algorithm, they are kept in the context until some parent judgment recycles a block of variables, or to the very end of a type inference task. In that sense, DK's algorithm does not control the scoping of variables precisely.

Two rules we may blame for not garbage collecting correctly are the inference rule for lambda functions and an application inference rule:

$$\frac{\Gamma, \hat{\alpha}, \hat{\beta}, x : \hat{\alpha} \vdash e \Leftarrow \hat{\beta} \dashv \Delta, x : \hat{\alpha}, \Theta}{\Gamma \vdash \lambda x. e \Rightarrow \hat{\alpha} \rightarrow \hat{\beta} \dashv \Delta} \text{DK}_{\rightarrow \text{I} \Rightarrow} \quad \frac{\Gamma, \hat{\alpha} \vdash [\hat{\alpha}/a] A \bullet e \Rightarrow C \dashv \Delta}{\Gamma \vdash \forall a. A \bullet e \Rightarrow C \dashv \Delta} \text{DK}_{\forall \text{App}}$$

In contrast, Rule 25 of our algorithm collects the existential variables right after the second judgment chain, and Rule 27 collects one existential variable similarly:

$$\begin{array}{c}
 \Gamma \Vdash \lambda x. e \Rightarrow_a \omega \longrightarrow_{25} \Gamma, \hat{\alpha}, \hat{\beta} \Vdash [\hat{\alpha} \rightarrow \hat{\beta}/a] \omega, x : \hat{\alpha} \Vdash e \Leftarrow \hat{\beta} \\
 \Gamma \Vdash \forall a. A \bullet e \Rightarrow_a \omega \longrightarrow_{27} \Gamma, \hat{\alpha} \Vdash [\hat{\alpha}/a] A \bullet e \Rightarrow_a \omega
 \end{array}$$

It seems impossible to achieve a similar outcome in DK's system by only modifying these two rules. Taking $\text{DK}_{\rightarrow \text{I} \Rightarrow}$ as an example, the declaration or solution for $\hat{\alpha}$ and $\hat{\beta}$ may be

referred to by subsequent judgments. Therefore leaving $\hat{\alpha}$ and $\hat{\beta}$ in the output context is the only choice, when the subsequent judgments cannot be consulted.

To fix the problem, one possible modification is on the algorithmic subsumption rule, as garbage collection for a checking judgment is always safe:

$$\frac{\Gamma, \blacktriangleright_{\hat{\alpha}} \vdash e \Rightarrow A \dashv \Theta \quad \Theta \vdash [\Theta]A \leq [\Theta]B \dashv \Delta, \blacktriangleright_{\hat{\alpha}}, \Delta'}{\Gamma \vdash e \Leftarrow B \dashv \Delta} \text{DK_Sub}$$

Here we employ the markers in a way they were originally not intended for. We create a dummy fresh existential $\hat{\alpha}$ and add a marker to the input context of the inference judgment. After the subtyping judgment is processed we look for the marker and drop everything afterwards. We pick this rule because it is the only one where a checking judgment calls an inference judgment. That is the point where our algorithm recycles variables—right after a judgment chain is satisfied. After applying this patch, to the best of our knowledge, DK's algorithm behaves equivalently to our algorithm in terms of variable scoping. However, this exploits markers in a way they were not intended to be used and seems ad-hoc.

4.4.2 ELABORATION

Type-inference algorithms are often extended with an associated elaboration. For example, for languages with implicit polymorphism, it is common to have an elaboration to a variant of System F [Reynolds 1983], which recovers type information and explicit type applications. Therefore a natural question is whether our algorithm can also accommodate such elaboration. While our algorithmic reduction does not elaborate to System F, we believe that it is not difficult to extend the algorithm with a (type-directed) elaboration. We explain the rough idea as follows.

Since the judgment form of our algorithmic worklist contains a collection of judgments, elaboration expressions are also generated as a list of equal length to the number of judgments (*not judgment chains*) in the worklist. As usual, subtyping judgments translate to coercions (denoted by f and represented by System F functions), all three other types of judgments translate to terms in System F (denoted by t).

Let Φ be the elaboration list, containing translated type coercions and terms:

$$\Phi ::= \cdot \mid \Phi, f \mid \Phi, t$$

Then the form of our algorithmic judgment becomes:

$$\Gamma \hookrightarrow \Phi$$

We take Rule 18 as an example, rewriting small-step reduction in a relational style,

$$\frac{\Gamma \Vdash e \Rightarrow_a a \leq B \hookrightarrow \Phi, f, t}{\Gamma \Vdash e \Leftarrow B \hookrightarrow \Phi, ft} \text{ Translation_18}$$

As is shown in the conclusion of the rule, a checking judgment at the top of the worklist corresponds to a top element for elaboration. The premise shows that one judgment chain may relate to more than one elaboration elements, and that the outer judgment, being processed before inner ones, elaborates to the top element in the elaboration list.

Moreover, existential variables need special treatment, since they may be solved at any point, or be recycled if not solved within their scopes. If an existential variable is solved, we not only propagate the solution to the other judgments, but also replace occurrences in the elaboration list. If an existential variable is recycled, meaning that it is not constrained, we can pick any well-formed monotype as its solution. The unit type 1, as the simplest type in the system, is a good choice.

4.4.3 LEXICALLY-SCOPED TYPE VARIABLES

We have further extended the type system with support for lexically-scoped type variables [Peyton Jones and Shields 2004]. Our Abella formalization for this extension proves all the metatheory we discuss in Section 4.3.

From a practical point of view, this extension allows the implementation of a function to refer to type variables from its type signature. For example,

$$(\lambda x. \lambda y. (x : a)) : \forall a b. a \rightarrow b \rightarrow a$$

has an annotation $(x : a)$ that refers to the type variable a in the type signature. This is not a surprising feature, since the declarative system already accepts similar programs

$$\frac{\Psi, a \vdash e \Leftarrow A}{\Psi \vdash \forall a. A \quad \Psi \vdash e \Leftarrow \forall a. A} \text{ Decl}\forall\text{I} \quad \frac{}{\Psi \vdash (e : \forall a. A) \Rightarrow \forall a. A} \text{ DeclAnno}$$

The main issue is the well-formedness condition. Normally $\Psi \vdash (e : A)$ follows from $\Psi \vdash e$ and $\Psi \vdash A$. However, when $A = \forall a. A'$, the type variable a is not in scope at e ,

therefore $\Psi \vdash e$ is not derivable. To address the problem, we add a new syntactic form that explicitly binds a type variable simulatenously in a function and its annotation.

$$\text{Expressions} \quad e ::= \dots \mid \Lambda a. e : A$$

This new type-lambda syntax $\Lambda a. e : A$ actually annotates its body e with $\forall a. A$, while making a visible inside the body of the function. The well-formedness judgments are extended accordingly:

$$\frac{\Psi, a \vdash e \quad \Psi, a \vdash A}{\Psi \vdash \Lambda a. e : A} \text{wf}_d\Lambda \quad \frac{\Gamma, a \vdash e \quad \Gamma, a \vdash A}{\Gamma \vdash \Lambda a. e : A} \text{wf}_\Lambda$$

Corresponding rules are introduced for both the declarative system and the algorithmic system:

$$\frac{\Psi, a \vdash A \quad \Psi, a \vdash e \Leftarrow A}{\Psi \vdash \Lambda a. e : A \Rightarrow \forall a. A} \text{Decl}\Lambda$$

$$\Gamma \Vdash \Lambda a. e : A \Rightarrow_b \omega \longrightarrow_{30} \Gamma \Vdash [(\forall a. A)/b]\omega, a \Vdash e \Leftarrow A$$

In practice, programmers would not write the syntax $\Lambda a. e : A$ directly. The Scoped-TypeVariables extension of Haskell is effective only when the type signature is explicitly universally quantified (which the compiler translates into an expression similar to $\Lambda a. e : A$); otherwise the program means the normal syntax $e : \forall a. A$ and may not later refer to the type variable a .

We have proven all three desired properties for the extended system, namely soundness, completeness and decidability.

4.5 TECHNICAL DISCUSSIONS ON DK'S MANUAL PROOFS

4.5.1 BROKEN LEMMAS IN DK'S PAPERS

We found false lemmas in the manual proofs of DK' two papers [Dunfield and Krishnaswami 2013] and [Dunfield and Krishnaswami 2019].

- In the first paper, Lemma 29 on page 9 of its appendix says:

Lemma 4.16 (Parallel Admissibility of [Dunfield and Krishnaswami 2013]).

If $\Gamma_L \longrightarrow \Delta_L$ and $\Gamma_R \longrightarrow \Delta_L, \Delta_R$ then:

1. $\Gamma_L, \hat{\alpha}, \Gamma_R \longrightarrow \Delta_L, \hat{\alpha}, \Delta_R$

2. If $\Delta_L \vdash \tau'$ then $\Gamma_L, \hat{\alpha}, \Gamma_R \longrightarrow \Delta_L, \hat{\alpha} = \tau', \Delta_R$.
3. If $\Gamma_L \vdash \tau$ and $\Delta_L \vdash \tau'$ and $[\Delta_L]\tau = [\Delta_L]\tau'$, then $\Gamma_L, \hat{\alpha} = \tau, \Gamma_R \longrightarrow \Delta_L, \hat{\alpha} = \tau', \Delta_R$.

We give a counter-example to this lemma:

Pick $\Gamma_L = \cdot, \Gamma_R = \hat{\beta}, \Delta_L = \hat{\beta}, \Delta_R = \cdot$, then both conditions $\cdot \longrightarrow \hat{\beta}$ and $\hat{\beta} \longrightarrow \hat{\beta}$ hold, but the first conclusion $\hat{\alpha}, \hat{\beta} \longrightarrow \hat{\beta}, \hat{\alpha}$ does not hold.

- In the second paper, as an extended work to the first paper, Lemma 26 on page 22 of its appendix says:

Lemma 4.17 (Parallel Admissibility of [Dunfield and Krishnaswami 2019]).

If $\Gamma_L \longrightarrow \Delta_L$ and $\Gamma_L, \Gamma_R \longrightarrow \Delta_L, \Delta_R$ then:

1. $\Gamma_L, \hat{\alpha} : \kappa, \Gamma_R \longrightarrow \Delta_L, \hat{\alpha} : \kappa, \Delta_R$
2. If $\Delta_L \vdash \tau' : \kappa$ then $\Gamma_L, \hat{\alpha} : \kappa, \Gamma_R \longrightarrow \Delta_L, \hat{\alpha} : \kappa = \tau', \Delta_R$.
3. If $\Gamma_L \vdash \tau : \kappa$ and $\Delta_L \vdash \tau'$ types and $[\Delta_L]\tau = [\Delta_L]\tau'$, then $\Gamma_L, \hat{\alpha} : \kappa = \tau, \Gamma_R \longrightarrow \Delta_L, \hat{\alpha} : \kappa = \tau', \Delta_R$.

A similar counter-example is given:

Pick $\Gamma_L = \cdot, \Gamma_R = \hat{\beta} : \star, \Delta_L = \hat{\beta} : \star, \Delta_R = \cdot$, then both conditions $\cdot \longrightarrow \hat{\beta} : \star$ and $\hat{\beta} : \star \longrightarrow \hat{\beta} : \star$ hold, but the first conclusion $\hat{\alpha} : \kappa, \hat{\beta} : \star \longrightarrow \hat{\beta} : \star, \hat{\alpha} : \kappa$ does not hold.

4.5.2 A FIX TO DK'S SUBSUMPTION LEMMA

Let's first recall the lemma:

Lemma 4.18 (Subsumption). Given $\Psi' \leq \Psi$:

1. If $\Psi \vdash e \Leftarrow A$ and $\Psi \vdash A \leq A'$ then $\Psi' \vdash e \Leftarrow A'$;
2. If $\Psi \vdash e \Rightarrow B$ then there exists B' s.t. $\Psi \vdash B' \leq B$ and $\Psi' \vdash e \Rightarrow B'$;
3. If $\Psi \vdash A \bullet e \Rightarrow C$ and $\Psi \vdash A' \leq A$, then there exists C' s.t. $\Psi \vdash C' \leq C$ and $\Psi' \vdash A' \bullet e \Rightarrow C'$.

PROBLEMS IN DK'S MANUAL PROOF We tried to follow DK's manual proof for this lemma. While being written in a clear format and providing enough details, the proof is not fully accepted by the proof assistant. Specifically, the first two applications of induction hypotheses on page 22 are not accepted. Either of them seems to use a slightly different “induction hypothesis” than the true one.

A FIX TO THE PROOF We make use of our worklist measures for the induction. Recall that $|e|_e$ measures term size; and the judgment measure counts checking as 2, inference as 1 and application inference as 3; and $|A|_\forall$ counts the number of \forall 's in a type.

The proof is by a nested mutual induction on the lexicographical order of the measures

$$\langle |e|_e, |\cdot|_{\Leftrightarrow}, |A|_\forall + |A'|_\forall \rangle,$$

where the second measure is 2 for checking (1), 1 for inference (2) and 3 for application inference (3); and the third measure does not apply to case (2) since no A is used.

All but two cases can be finished easily following the declarative typing derivation, and the proof shares a similar structure to DK's. One tricky case related to Rule $\text{Dec1}\forall\text{I}$ indicates that A has the shape $\forall a. A_0$, thus the subtyping relation derives from either $\leq\forall\text{L}$ or $\leq\forall\text{R}$. For each of the case, the third measure $|A|_\forall + |A'|_\forall$ decreases (the $\leq\forall\text{L}$ case requires a type substitution lemma obtaining $\Psi \vdash e \Leftarrow [\tau/a]A_0$ from the typing derivation).

Another tricky case is $\text{Dec1}\rightarrow\text{I}$. When the subtyping relation is derived from $\leq\rightarrow$, a simple application of induction hypothesis finishes the proof. When the subtyping relation is derived from $\leq\forall\text{R}$, $|A'|_\forall$ decreases, and thus the induction hypothesis finishes this case.

5 HIGHER-RANK POLYMORPHISM WITH OBJECT-ORIENTED SUBTYPING

5.1 OVERVIEW

In this chapter, we present a worklist algorithm which further supports subtyping, by introducing the top and bottom types. Such type inference algorithms are known to be hard to design, due to the complication of unifying variables under subtyping inequality rather than equality. Therefore, algorithmic existential variables that appear in subtyping judgments might have more solutions than before, and the eager substitution rules miss some possibilities. Our new *backtracking* algorithm extends the existing one by some overlapping rules that non-deterministically try different sorts of solutions instead of a single sort, improving the rate of success guessing.

Formalization of the algorithm in the Abella theorem prover shows soundness of the algorithm with respect to our declarative specification. Although the backtracking algorithm is still incomplete in some corner cases, the proof script indicates that the only source of incompleteness comes from higher-ranked subtyping relations. Following that discovery, we formally proved a partial completeness of our subtyping algorithm under the rank-1 restriction, and is at least comparable to local type inference algorithms.

5.1.1 TYPE INFERENCE IN PRESENCE OF SUBTYPING

In all previous chapters, “subtyping” refers to a relation that compares the degree of polymorphism between two types. In addition to that, we also include the top and bottom types. Both of the types have practical uses, especially in object-oriented programming languages. The top type, \top , is the super type of any type, i.e. any type is more general than \top and thus can be considered as an instance of type \top . In typical object-oriented programming languages, the `Object` class, as the base class of any class, is the \top type. In contrast, the bottom type, \perp , is dual to \top . An instance of \perp can be casted to a value of any type, which is usually impossible, except when that is a `null` pointer value (`(void *)` in C++, for example). Another practical use for bottom types is for exceptions. The type `Exception` $\rightarrow \perp$ given to the `raise` function may pass type checkers naturally. The type `Exception` $\rightarrow \forall a. a$ is also a

reasonable choice, which in fact reveals that \perp behaves almost identically as $\forall a. a$. From the theoretical point of view, both of them represent “falsity”.

A number of systems and algorithms are proposed for type inference with subtyping. Since subtyping constraints cannot be easily reduced in presence of subtyping, it is natural to extend the syntax of types so that unsolved constraints are carried with them, and the idea is studied by previous work [Eifrig et al. 1995; Trifonov and Smith 1996]. For example, the following function

```
select p v d = if (p v) then v else d
```

is inferred to have type

$$(\alpha \rightarrow \text{bool}) \rightarrow \alpha \rightarrow \beta \rightarrow \gamma \mid \alpha \leq \gamma, \beta \leq \gamma$$

However, this representation usually contains a large set of constraints, which might confuse programmers. Besides, subtyping comparison between constraint types is also not easy.

MLsub MLsub [Dolan and Mycroft 2017], as introduced in Section 2.4, is based on a type system with lattices and polar types. Inspired by [Pottier 1998], they separated input and output types into polar types, which greatly simplifies the subtyping judgment. With further help of their biunification algorithm, constraints are no longer carried with types, instead they are expressed directly on types. For example, the `select` function defined above has type

$$(\alpha \rightarrow \text{bool}) \rightarrow \alpha \rightarrow \beta \rightarrow \alpha \sqcup \beta$$

in MLsub. The \sqcup symbol in the output type indicate that any of the types might be returned when executed. On the other hand, \sqcap symbol might occur on an input type, when that argument is used as different types, meaning that the type must be able to convert to all these types.

Nevertheless, there are still drawbacks in practice. Firstly, the type inference algorithm produces types that have comparable size to previous systems with constraint types. Although a simplification algorithm is provided, it is not guaranteed to produce the most simple form.

Secondly, MLsub always infers an expression and returns its principle type, but some of the types are not easy to understand. For example, the `twice` function defined as `twice f x = f (f x)` has type $(\alpha \sqcup \beta \rightarrow \alpha) \rightarrow \beta \rightarrow \alpha$, where normally people might expect $(\alpha \rightarrow \alpha) \rightarrow \alpha \rightarrow \alpha$.

Thirdly, MLsub tries to infer on almost every possible expression, even those that might just be a mistake made by the programmer. For instance, the function

```
positive f x = if x > 0 then f x else f 0
```

always checks the positivity of x before applying to f . However, a slight mistake is made in the following definition

```
positive' f x = if x > 0 then f x else f
```

then the algorithm might still accept the program by giving the type:

$$\alpha \sqcap (\beta \rightarrow \alpha) \rightarrow \beta \sqcap \text{int} \rightarrow \alpha$$

Type inference algorithms, as logical tools that help people program, should accept good programs and reject bad ones. Typically, such function is considered to be badly-written, but unfortunately, MLsub accepts it without giving any warnings.

5.1.2 JUDGMENT LIST AND EAGER SUBSTITUTION

The judgment list algorithm we discussed in the last chapter treat subtyping judgments mainly as equality unification. However, the algorithm may lose some solutions when top and bottom types are introduced. For example, the judgment $\hat{\alpha} \leq 1$ has a best solution $\hat{\alpha} := 1$ in the previous system, but now $\hat{\alpha} := \perp$ should also be allowed. As a result, several subtyping judgments cannot be reduced easily with a single eager substitution.

Similar incompleteness also affect the instantiation and existential variable solving rules

$$\begin{aligned} \Gamma[\hat{\alpha}] \Vdash \hat{\alpha} \leq A \rightarrow B &\longrightarrow_{10} [\hat{\alpha}_1 \rightarrow \hat{\alpha}_2 / \hat{\alpha}](\Gamma[\hat{\alpha}_1, \hat{\alpha}_2] \Vdash \hat{\alpha}_1 \rightarrow \hat{\alpha}_2 \leq A \rightarrow B) \\ \Gamma[\hat{\alpha}][\hat{\beta}] \Vdash \hat{\alpha} \leq \hat{\beta} &\longrightarrow_{12} [\hat{\alpha} / \hat{\beta}](\Gamma[\hat{\alpha}][\hat{\beta}]) \end{aligned}$$

The instantiation judgment $\hat{\alpha} \leq A \rightarrow B$ immediately split $\hat{\alpha}$ into $\hat{\alpha}_1 \rightarrow \hat{\alpha}_2$, which is no longer the case — $\hat{\alpha}$ can also be \perp . The existential variable solving rule $\hat{\alpha} \leq \hat{\beta}$ is even more problematic, because the shapes of the instantiations of both existential variables are unknown, there are infinitely many possibilities if no additional constraints present.

5.1.3 OUR SOLUTION: BACKTRACKING ALGORITHM

After examine the examples where the worklist algorithm behave incompletely, we propose some improvements to the worklist algorithm to accept some simple cases. We have observed that subtyping judgments like $\hat{\alpha} \leq A$ and $A \leq \hat{\alpha}$ have trivial solutions $\hat{\alpha} := \perp$ and $\hat{\alpha} := \top$ respectively, thus we can try these simple solutions before the more complicated analysis.

For example, for the judgment list $\Gamma \Vdash \hat{\alpha} \leq A \rightarrow B$, we first assign $\hat{\alpha} := \perp$ and continue to check other judgments. If the judgment reduction succeeds, then we know that $\hat{\alpha} := \perp$ is a possible solution. If the reduction failed, we continue to try the other possibility according

Type variables	a, b	
Types	A, B, C	$::= 1 \mid \top \mid \perp \mid a \mid \forall a. A \mid A \rightarrow B$
Monotypes	τ	$::= 1 \mid \top \mid \perp \mid a \mid \tau_1 \rightarrow \tau_2$
Expressions	e	$::= x \mid () \mid \lambda x. e \mid e_1 e_2 \mid (e : A)$
Context	Ψ	$::= \cdot \mid \Psi, a \mid \Psi, x : A$

Figure 5.1: Declarative Syntax

to the previous instantiation rule by splitting $\hat{\alpha}$ into two existential variables $\hat{\alpha}_1$ and $\hat{\alpha}_2$. Note that for the reduced judgment $\hat{\alpha}_1 \rightarrow \hat{\alpha}_2 \leq A \rightarrow B$, $\hat{\alpha}_1$ and $\hat{\alpha}_2$ might be solve to \top or \perp .

For the type system, there are also interesting changes to both the declarative system and algorithmic system. Two algorithmic typing rules also adopt the backtracking approach:

- Checking judgment $e \Leftarrow \hat{\alpha}$ now has a trivial solution $\hat{\alpha} := \top$, because $e \Leftarrow \top$ should be accepted for any (well-formed) expression.
- The introduction of top and bottom types come with another new declarative rule $\perp \bullet e \Rightarrow \perp$. Therefore the application inference judgment $\hat{\alpha} \bullet e \Rightarrow_a \omega$ can be satisfied by $\hat{\alpha} := \perp$, with output \perp applied to the rest of the judgment chain ω .

By applying the above modifications, the algorithm remains sound w.r.t our new declarative system, and at the same time, we address most issues regarding the top and bottom types, obtaining a more complete algorithm. Furthermore, we formally verified that the subtyping algorithm is complete under the rank-1 restriction. However, for rank-2 judgments like $\hat{\alpha} \leq \hat{\beta}$, completeness is impossible for any algorithm that reduces eagerly.

Following the formal statements, we conclude that the backtracking algorithm is complete on most cases other than those where higher-ranked subtyping judgments are involved. In those cases, we only consider the equality case and ignore any other possibilities, and we believe that this is a practical tradeoff. In another perspective, our algorithm do not try to solve completely when parametric polymorphism and subtyping polymorphism occur at the same time.

5.2 DECLARATIVE SYSTEM

SYNTAX The syntax of the declarative system, shown in Figure 5.1, is similar to the previous systems by having a primitive type 1, type variables a , polymorphic types $\forall a. A$ and function types $A \rightarrow B$. Additionally, top and bottom types are introduced to the type system.

$$\boxed{\Psi \vdash A \leq B}$$

$$\begin{array}{c}
 \frac{a \in \Psi}{\Psi \vdash a \leq a} \leq \text{Var} \quad \frac{}{\Psi \vdash 1 \leq 1} \leq \text{Unit} \quad \frac{\Psi \vdash B_1 \leq A_1 \quad \Psi \vdash A_2 \leq B_2}{\Psi \vdash A_1 \rightarrow A_2 \leq B_1 \rightarrow B_2} \leq \rightarrow \\
 \frac{\Psi \vdash \tau \quad \Psi \vdash [\tau/a]A \leq B}{\Psi \vdash \forall a. A \leq B} \leq \forall L \quad \frac{\Psi, b \vdash A \leq B}{\Psi \vdash A \leq \forall b. B} \leq \forall R \\
 \frac{}{A \leq \top} \leq \text{Top} \quad \frac{}{\perp \leq A} \leq \text{Bot}
 \end{array}$$

Figure 5.2: Declarative Subtyping

The well-formedness for the system is standard and almost identical to the previous systems, therefore we omit the formal definitions.

DECLARATIVE SUBTYPING Shown in Figure 5.2, the declarative subtyping extends the polymorphic subtyping relation originally proposed by Odersky and Läufer [1996] by adding rules $\leq \text{Top}$ and $\leq \text{Bot}$, defining the properties of the \top and \perp types, respectively. Although the new rules seem quite simple, they may increase the uncertainty on polymorphic instantiations. For example, the subtyping judgement

$$\forall a. a \rightarrow a \leq \perp \rightarrow \top$$

accepts any well-formed instantiation on the polymorphic type $\forall a. a \rightarrow a$.

DECLARATIVE TYPING The declarative typing rules, shown in Figure 5.3, extends DK's higher-rank type system in order to support the top and bottom types. Rule $\text{Dec1}\top$ allows any well-formed expression to check against \top . Rule $\text{Dec1}\perp\text{App}$ returns the \perp type when a function of \perp type is applied to any argument. All other rules remain exactly the same as our previous work.

It's worth mentioning that the design of the two new rules are driven by the subsumption property described in Section 5.4.1. They maintain the property in presence of a more powerful declarative subtyping, and we will discuss further later in that part.

$\Psi \vdash e \Leftarrow A$	e checks against input type A .
$\Psi \vdash e \Rightarrow A$	e synthesizes output type A .
$\Psi \vdash A \bullet e \Rightarrow C$	Applying a function of type A to e synthesizes type C .

$$\begin{array}{c}
 \frac{(x : A) \in \Psi}{\Psi \vdash x \Rightarrow A} \text{DeclVar} \qquad \frac{\Psi \vdash e \Rightarrow A \quad \Psi \vdash A \leq B}{\Psi \vdash e \Leftarrow B} \text{DeclSub} \\
 \\
 \frac{\Psi \vdash A \quad \Psi \vdash e \Leftarrow A}{\Psi \vdash (e : A) \Rightarrow A} \text{DeclAnno} \qquad \frac{}{\Psi \vdash () \Rightarrow 1} \text{Decl1I} \Rightarrow \\
 \\
 \frac{}{\Psi \vdash () \Leftarrow 1} \text{Decl1I} \qquad \frac{\Psi \vdash e}{\Psi \vdash e \Leftarrow \top} \text{Decl}\top \qquad \frac{\Psi \vdash e}{\Psi \vdash \perp \bullet e \Rightarrow \perp} \text{Decl}\perp\text{App} \\
 \\
 \frac{\Psi, a \vdash e \Leftarrow A}{\Psi \vdash e \Leftarrow \forall a. A} \text{Decl}\forall\text{I} \qquad \frac{\Psi \vdash \tau \quad \Psi \vdash [\tau/a]A \bullet e \Rightarrow C}{\Psi \vdash \forall a. A \bullet e \Rightarrow C} \text{Decl}\forall\text{App} \\
 \\
 \frac{\Psi, x : A \vdash e \Leftarrow B}{\Psi \vdash \lambda x. e \Leftarrow A \rightarrow B} \text{Decl}\rightarrow\text{I} \qquad \frac{\Psi \vdash \sigma \rightarrow \tau \quad \Psi, x : \sigma \vdash e \Leftarrow \tau}{\Psi \vdash \lambda x. e \Rightarrow \sigma \rightarrow \tau} \text{Decl}\rightarrow\text{I} \Rightarrow \\
 \\
 \frac{\Psi \vdash e_1 \Rightarrow A \quad \Psi \vdash A \bullet e_2 \Rightarrow C}{\Psi \vdash e_1 e_2 \Rightarrow C} \text{Decl}\rightarrow\text{E} \qquad \frac{\Psi \vdash e \Leftarrow A}{\Psi \vdash A \rightarrow C \bullet e \Rightarrow C} \text{Decl}\rightarrow\text{App}
 \end{array}$$

Figure 5.3: Declarative Typing

5.3 BACKTRACKING ALGORITHM

5.3.1 SYNTAX

The algorithmic syntax is shown in Figure 5.4. Compared with the declarative system, existential variables $\hat{\alpha}, \hat{\beta}$ are used as placeholders for unsolved mono-types. The judgment chain ω and worklist context Γ are defined in the same way as the ICFP work.

The well-formedness relation is almost the same as that of the ICFP work. The hole notation is also inherited.

THE SOLVE NOTATION We define a set of auxiliary substitution functions in the form $\{\hat{\alpha} := \tau_A\}$ to improve readability of our algorithmic definitions. Basically it refers to a global substitution when solving an existential variable.

A detailed definition of the notation is shown as follows:

$$\begin{array}{ll}
 \{\hat{\alpha} := \tau\} (\Gamma_L, \hat{\alpha}, \Gamma_R) = \Gamma_L, [\tau/\hat{\alpha}] \Gamma_R & \tau = 1, \top, \perp \text{ or } a \\
 \{\hat{\alpha} := \hat{\beta}\} (\Gamma_L, \hat{\alpha}, \Gamma_R) = \Gamma_L, [\hat{\beta}/\hat{\alpha}] \Gamma_R & \hat{\beta} \in \Gamma_L \\
 \{\hat{\alpha} := \hat{\alpha}_1 \rightarrow \hat{\alpha}_2\} (\Gamma_L, \hat{\alpha}, \Gamma_R) = \Gamma_L, \hat{\alpha}_1, \hat{\alpha}_2, [\hat{\alpha}_1 \rightarrow \hat{\alpha}_2/\hat{\alpha}] \Gamma_R
 \end{array}$$

Existential variables	$\hat{\alpha}, \hat{\beta}$
Types	$A, B, C ::= 1 \mid \top \mid \perp \mid a \mid \forall a. A \mid A \rightarrow B \mid \hat{\alpha}$
Algorithmic judgment chain	$\omega ::= A \leq B \mid e \Leftarrow A \mid e \Rightarrow_a \omega \mid A \bullet e \Rightarrow_a \omega$
Algorithmic worklist	$\Gamma ::= \cdot \mid \Gamma, a \mid \Gamma, \hat{\alpha} \mid \Gamma \Vdash \omega$

Figure 5.4: Algorithmic Syntax

5.3.2 ALGORITHMIC SUBTYPING

Figure 5.5 describes the algorithmic rules for subtyping. The relation is stated in a small-step “reduction” form, i.e. in each step, the worklist is analysed from the right-hand-side and reduced according to the top judgment. The overall procedure succeeds iff the worklist eventually reduces to \cdot (the empty worklist).

We categorize them into 7 groups according to their behavior:

1. Rules 1-3 are basic garbage collection rules. Given that the worklist Γ is well-formed, no reference of a variable should occur before its declaration. Therefore removing the declaration in the top position does not break well-formedness.

An existential variable that is unsolved in the top position indicates that it is not constrained, thus picking any well-formed mono-type as its solution is acceptable. In our algorithmic formalization, we simply drop the existential variable.

2. Rules 4-10 directly correspond to the declarative subtyping rules. With no top-level existential variables, there are nothing to guess immediately, and thus the algorithm behaves just like the declarative system.
3. Rule 11 is a base case in the algorithmic system. The declarative reflexivity property suggests that any solution is acceptable, thus the judgment holds without any constraint.
4. Rules 12-13 are important rules that requires backtracking techniques for implementation. These rules *overlaps* with all the remaining rules when solving an existential variable. In other words, they simply try if \top or \perp satisfies the constraints in parallel with other possibilities.

$\boxed{\Gamma \longrightarrow \Gamma'}$ Γ reduces to Γ' .

$$\begin{aligned}
 & \Gamma, a \longrightarrow_1 \Gamma \\
 & \Gamma, \hat{\alpha} \longrightarrow_2 \Gamma \\
 & \Gamma, x : A \longrightarrow_3 \Gamma \\
 & \Gamma \Vdash 1 \leq 1 \longrightarrow_4 \Gamma \\
 & \Gamma \Vdash a \leq a \longrightarrow_5 \Gamma \\
 & \Gamma \Vdash A_1 \rightarrow A_2 \leq B_1 \rightarrow B_2 \longrightarrow_6 \Gamma \Vdash A_2 \leq B_2 \Vdash B_1 \leq A_1 \\
 & \Gamma \Vdash \forall a. A \leq B \longrightarrow_7 \Gamma, \hat{\alpha} \Vdash [\hat{\alpha}/a]A \leq B \\
 & \Gamma \Vdash A \leq \forall b. B \longrightarrow_8 \Gamma, b \Vdash A \leq B \\
 & \Gamma \Vdash A \leq \top \longrightarrow_9 \Gamma \\
 & \Gamma \Vdash \perp \leq B \longrightarrow_{10} \Gamma \\
 & \Gamma[\hat{\alpha}] \Vdash \hat{\alpha} \leq \hat{\alpha} \longrightarrow_{11} \Gamma \\
 & \Gamma[\hat{\alpha}] \Vdash A \leq \hat{\alpha} \longrightarrow_{12} \{\hat{\alpha} := \top\} \Gamma[\hat{\alpha}] \\
 & \Gamma[\hat{\alpha}] \Vdash \hat{\alpha} \leq A \longrightarrow_{13} \{\hat{\alpha} := \perp\} \Gamma[\hat{\alpha}] \\
 & \Gamma[\hat{\alpha}] \Vdash \hat{\alpha} \leq A \rightarrow B \longrightarrow_{14} \{\hat{\alpha} := \hat{\alpha}_1 \rightarrow \hat{\alpha}_2\} (\Gamma[\hat{\alpha}] \Vdash \hat{\alpha} \leq A \rightarrow B) \\
 & \quad \text{when } \hat{\alpha} \notin FV(A \rightarrow B) \\
 & \Gamma[\hat{\alpha}] \Vdash A \rightarrow B \leq \hat{\alpha} \longrightarrow_{15} \{\hat{\alpha} := \hat{\alpha}_1 \rightarrow \hat{\alpha}_2\} (\Gamma[\hat{\alpha}] \Vdash A \rightarrow B \leq \hat{\alpha}) \\
 & \quad \text{when } \hat{\alpha} \notin FV(A \rightarrow B) \\
 & \Gamma[a][\hat{\beta}] \Vdash a \leq \hat{\beta} \longrightarrow_{16} \{\hat{\beta} := a\} \Gamma[a][\hat{\beta}] \\
 & \Gamma[a][\hat{\beta}] \Vdash \hat{\beta} \leq a \longrightarrow_{17} \{\hat{\beta} := a\} \Gamma[a][\hat{\beta}] \\
 & \Gamma[\hat{\beta}] \Vdash 1 \leq \hat{\beta} \longrightarrow_{18} \{\hat{\beta} := 1\} \Gamma[\hat{\beta}] \\
 & \Gamma[\hat{\beta}] \Vdash \hat{\beta} \leq 1 \longrightarrow_{19} \{\hat{\beta} := 1\} \Gamma[\hat{\beta}] \\
 & \Gamma[\hat{\alpha}][\hat{\beta}] \Vdash \hat{\alpha} \leq \hat{\beta} \longrightarrow_{20} \{\hat{\beta} := \hat{\alpha}\} \Gamma[\hat{\alpha}][\hat{\beta}] \\
 & \Gamma[\hat{\alpha}][\hat{\beta}] \Vdash \hat{\beta} \leq \hat{\alpha} \longrightarrow_{21} \{\hat{\beta} := \hat{\alpha}\} \Gamma[\hat{\alpha}][\hat{\beta}]
 \end{aligned}$$

Figure 5.5: Algorithmic Garbage Collection and Subtyping

5. Rules 14-15 compares an existential variable $\hat{\alpha}$ with a function type, resulting in solving the $\hat{\alpha}$ by $\hat{\alpha}_1 \rightarrow \hat{\alpha}_2$. The freshness condition rules out the possibility when there is a cyclic dependency. For example, the judgment

$$\hat{\alpha} \leq 1 \rightarrow \hat{\alpha}$$

is satisfied with either of these solutions to $\hat{\alpha}$:

$$\perp, 1 \rightarrow \perp, \top \rightarrow \perp, 1 \rightarrow 1 \rightarrow \perp, \dots$$

However, we argue that comparing $\hat{\alpha}$ with a function type that contains $\hat{\alpha}$ itself is hardly useful in practice, and most of the solutions are meaningless. Therefore, in our algorithm, only the \perp solution is considered with rule 13. The condition of rule 14 rejects the judgement for further analysis and thus does not produce more solutions. This is one source of incompleteness of our algorithm with respect to the declarative specification.

6. Rules 16-19 solve existential variables against a type variable or the unit type. For instance, the judgment

$$\hat{\alpha} \leq 1$$

only has two solutions: $\hat{\alpha} = 1$ or $\hat{\alpha} = \perp$. In similar cases, one of the solutions is produced by rule 12 or 13, and the other one is given by one of the rules in this group. Additional well-formedness checks are performed when type variables are encountered; a solution to an existential variable must be well-formed in the context before the existential variable is defined.

7. Rules 20-21 deals with subtyping judgments that compares two different existential variables. The only difference between them is the variable order. Similar to the type variable case in the previous group, existential variables must solve to another one defined earlier. With rules 12, 13, 20 and 21, a judgment like

$$\hat{\alpha} \leq \hat{\beta}$$

could possibly give any of the following solutions:

$$\hat{\alpha} = \hat{\beta} \text{ (or } \hat{\beta} = \hat{\alpha}) \text{ or } \hat{\alpha} = \perp \text{ or } \hat{\beta} = \top$$

Those are good attempts, but unfortunately, they do not cover the complete set of possibilities. The following example worklist

$$\widehat{\alpha}, \widehat{\beta} \Vdash \widehat{\beta} \leq 1 \rightarrow 1 \Vdash \widehat{\alpha} \leq \widehat{\beta}$$

has a solution $\widehat{\alpha} = 1 \rightarrow \top, \widehat{\beta} = 1 \rightarrow 1$ missed by our algorithm. Similar situations happen when the judgments are specifically ordered; if $\widehat{\beta} \leq 1 \rightarrow 1$ is the top-most judgment, the algorithm will not miss this solution.

Although such treatment for existential variable solving is incomplete in theory, a large number of practical unification is simply equality, and the algorithm completely handles these programs. For other programs that exploit complex guessing involving subtyping, the programmer may put type annotations when the type inference algorithm does not find the optimal solution.

5.3.3 ALGORITHMIC TYPING

$\boxed{\Gamma \longrightarrow \Gamma'}$ Γ reduces to Γ' (continued).

$$\begin{aligned}
 & \Gamma \Vdash e \Leftarrow B \longrightarrow_{22} \Gamma \Vdash e \Rightarrow_a a \leq B \quad \text{when } e \neq \lambda x. e' \text{ and } B \neq \forall a. B' \\
 & \Gamma \Vdash e \Leftarrow \forall a. A \longrightarrow_{23} \Gamma, a \Vdash e \Leftarrow A \\
 & \Gamma \Vdash \lambda x. e \Leftarrow A \rightarrow B \longrightarrow_{24} \Gamma, x : A \Vdash e \Leftarrow B \\
 & \Gamma[\widehat{\alpha}] \Vdash \lambda x. e \Leftarrow \widehat{\alpha} \longrightarrow_{25} \{\widehat{\alpha} := \widehat{\alpha}_1 \rightarrow \widehat{\alpha}_2\} \Gamma, x : \widehat{\alpha}_1 \Vdash e \Leftarrow \widehat{\alpha}_2 \\
 & \Gamma \Vdash e \Leftarrow \top \longrightarrow_{26} \Gamma \\
 & \Gamma[\widehat{\alpha}] \Vdash e \Leftarrow \widehat{\alpha} \longrightarrow_{27} \{\widehat{\alpha} := \top\} \Gamma[\widehat{\alpha}] \\
 & \Gamma \Vdash x \Rightarrow_a \omega \longrightarrow_{28} \Gamma \Vdash [A/a]\omega \quad \text{when } (x : A) \in \Gamma \\
 & \Gamma \Vdash (e : A) \Rightarrow_a \omega \longrightarrow_{29} \Gamma \Vdash [A/a]\omega \Vdash e \Leftarrow A \\
 & \Gamma \Vdash () \Rightarrow_a \omega \longrightarrow_{30} \Gamma \Vdash [1/a]\omega \\
 & \Gamma \Vdash \lambda x. e \Rightarrow_a \omega \longrightarrow_{31} \Gamma, \widehat{\alpha}, \widehat{\beta} \Vdash [\widehat{\alpha} \rightarrow \widehat{\beta}/a]\omega, x : \widehat{\alpha} \Vdash e \Leftarrow \widehat{\beta} \\
 & \Gamma \Vdash e_1 e_2 \Rightarrow_a \omega \longrightarrow_{32} \Gamma \Vdash e_1 \Rightarrow_b (b \bullet e_2 \Rightarrow_a \omega) \\
 & \Gamma \Vdash \forall a. A \bullet e \Rightarrow_a \omega \longrightarrow_{33} \Gamma, \widehat{\alpha} \Vdash [\widehat{\alpha}/a]A \bullet e \Rightarrow_a \omega \\
 & \Gamma \Vdash A \rightarrow C \bullet e \Rightarrow_a \omega \longrightarrow_{34} \Gamma \Vdash [C/a]\omega \Vdash e \Leftarrow A \\
 & \Gamma \Vdash \perp \bullet e \Rightarrow_a \omega \longrightarrow_{35} \Gamma \Vdash [\perp/a]\omega \\
 & \Gamma[\widehat{\alpha}] \Vdash \widehat{\alpha} \bullet e \Rightarrow_a \omega \longrightarrow_{36} \{\widehat{\alpha} := \widehat{\alpha}_1 \rightarrow \widehat{\alpha}_2\} \Gamma \Vdash \widehat{\alpha}_1 \rightarrow \widehat{\alpha}_2 \bullet e \Rightarrow_a \omega \\
 & \Gamma[\widehat{\alpha}] \Vdash \widehat{\alpha} \bullet e \Rightarrow_a \omega \longrightarrow_{37} \{\widehat{\alpha} := \perp\} \Gamma[\widehat{\alpha}] \Vdash [\perp/a]\omega
 \end{aligned}$$

Figure 5.6: Algorithmic Typing

The algorithmic typing rules split into three groups, according to the category of the top-most judgment.

1. Checking mode. Rules 22-27 reduce top-level checking judgments. Rules 22, 23, 24 and 26 directly reflect how the declarative system behaves. Rule 25 splits $\hat{\alpha}$ into $\hat{\alpha}_1 \rightarrow \hat{\alpha}_2$ to mimic the same logic as rule 24, since a lambda expression must be of an function type. Rule 27 is another backtracking rule, which overlaps with all other checking rules. Solving $\hat{\alpha}$ to \top in such case reflects the declarative rule $\text{Dec1}\top$ when the unknown type is \top . Furthermore, we eliminate the algorithmic counterpart of declarative rule Dec11I , because a combination of Rules 22 and 30 already accepts the judgment $() \Leftarrow 1$.
2. Inference mode. Rules 28-32 reduce inference judgments. Like the work presented in Chapter 4, encoding of the return type is by an explicit substitution on the binder of a judgment chain. Rules 28, 29, 30 and 32 corresponds to the declarative system straightforwardly. Rule 31 guesses an unannotated lambda with a mono function type $\hat{\alpha} \rightarrow \hat{\beta}$. Writing in a slightly different way may help improve readability:

$$\Gamma \Vdash \lambda x. e \Rightarrow_a \omega \longrightarrow_{31} \Gamma, \hat{\alpha}, \hat{\beta} \Vdash [\hat{\alpha} \rightarrow \hat{\beta}/a] \omega \Vdash \lambda x. e \Leftarrow \hat{\alpha} \rightarrow \hat{\beta}$$

Rule 32 illustrates how the judgment chain works, with a continuation-passing-style encoding of the type inference task.

3. Application inference mode. Rules 33-37 reduce application inference judgments. Each of these judgments accepts an input function type and an argument expression and produces the expected return type. Rules 33, 34 and 35 are direct translation from the declarative rules. Rule 33, specifically, enables implicit parametric polymorphism via existential variable solving. Rules 36 and 37 deal with cases where a single existential variable $\hat{\alpha}$ as a function type. Rule 36 splits $\hat{\alpha}$ into a function unknown type $\hat{\alpha}_1 \rightarrow \hat{\alpha}_2$. Rule 37 tries the solution $\hat{\alpha} = \perp$ and returns the \perp type.

5.4 METATHEORY

In this section we present several properties formally verified. For the declarative system, the typing subsumption and subtyping transitivity lemmas are discussed in detail. The algorithmic system is proven to be sound with respect to the declarative system via a transfer relation. A partial completeness theorem is shown under the rank-1 restriction. We then briefly describe the challenges we face when proving termination. Lastly, proof statistics of Abella are discussed.

5.4.1 DECLARATIVE PROPERTIES

An important desired property for a type system is *checking subsumption*, which basically says that any expression can check against any super type of its actual type. Since our bidirectional type system defines the checking mode, inference mode and application inference mode mutually, we formalize the generalized *typing subsumption*.

Definition 5 (Worklist Subtyping). Worklist subtyping compares the type of variables bound in the worklist. $\Gamma <: \Gamma'$ iff each binding in Γ is converted to one with a super type.

$$\begin{array}{c} \frac{}{\cdot <: \cdot} <: \text{nil} \quad \frac{\Gamma <: \Gamma'}{\Gamma, a <: \Gamma', a} <: \text{ty} \\[10pt] \frac{\Gamma' \vdash A \leq B \quad \Gamma <: \Gamma'}{\Gamma, x : A <: \Gamma', x : B} <: \text{of} \quad \frac{\Gamma <: \Gamma'}{\Gamma \Vdash \omega <: \Gamma' \Vdash \omega} <: \omega \end{array}$$

A basic property of worklist subtyping is that they acts similarly when viewed as contexts.

Lemma 5.1 (Worklist Subtyping Equivalence). *Given $\Gamma <: \Gamma'$, $\Gamma \vdash A \leq B \iff \Gamma' \vdash A \leq B$.*

Lemma 5.2 (Typing Subsumption). *Given $\Gamma <: \Gamma'$,*

- 1) *If $\Gamma' \vdash e \Leftarrow A$ and $\Gamma' \vdash A \leq B$, then $\Gamma \vdash e \Leftarrow B$;*
- 2) *If $\Gamma' \vdash e \Rightarrow A$, then $\exists B$ s.t. $\Gamma' \vdash B \leq A$ and $\Gamma \vdash e \Rightarrow B$.*
- 3) *If $\Gamma' \vdash C \bullet e \Rightarrow A$ and $\Gamma' \vdash D \leq C$, then $\exists B$ s.t. $\Gamma' \vdash B \leq A$ and $\Gamma \vdash D \bullet e \Rightarrow B$.*

Proof. By induction on the following size measure (lexicographical order on a 3-tuple):

- Checking ($e \Leftarrow A$): $\langle |e|, 1, |A|_{\forall} + |B|_{\forall} \rangle$
- Inference ($e \Rightarrow A$): $\langle |e|, 0, 0 \rangle$
- Application inference ($A \bullet e \Rightarrow C$): $\langle |e|, 2, |C|_{\forall} + |D|_{\forall} \rangle$

Most of the cases are straightforward. When rule $\leq \forall L$ is applied for the subtyping predicate like $\Gamma' \vdash A \leq B$, a mono-type substitution is performed on $\forall a. A$, resulting in $[\tau/a]A$. Since τ is a mono-type, the result type reduces the number of \forall 's, and thus reduces the size measure. \square

The two new declarative rules $\text{Decl}\top$ and $\text{Decl}\perp\text{App}$ are discovered through it and verified against it. Without these rules, for any checking relation $\Psi \vdash e \Leftarrow A$, one definitely expects that e may be checked by any super type of A , therefore $\Psi \vdash e \Leftarrow \top$ should hold. Similarly, a more general type \perp should behave as if it was any function types, or simply the most general form $\top \rightarrow \perp$, which accepts any input and returns the \perp type, resulting in the derivation $\Psi \vdash \perp \bullet e \Rightarrow \perp$.

To the best of the authors' knowledge, they are the minimal set of rules that make the lemma hold.

SUBTYPING TRANSITIVITY The transitivity lemma for declarative subtyping is a commonly expected property. The proof depends on the following subtyping derivation size relation and an auxiliary lemma.

Definition 6 (Subtyping Derivation Size).

$$\begin{aligned} |1 \leq 1| &= 0 \\ |a \leq a| &= 0 \\ |A \leq \top| &= 0 \\ |\perp \leq B| &= 0 \\ |A_1 \rightarrow A_2 \leq B_1 \rightarrow B_2| &= |B_1 \leq A_1| + |A_2 \leq B_2| + 1 \\ |\forall a. A \leq B| &= |[\tau/a]A \leq B| + 1 \\ |A \leq \forall a. B| &= |A \leq B| + 1 \end{aligned}$$

Lemma 5.3 (Monotype Subtyping Substitution). *If $\Psi \vdash \tau$ and $\Psi, a, \Psi_R \vdash A \leq B$, then $\Psi, [\tau/a]\Psi_R \vdash [\tau/a]A \leq [\tau/a]B$.*

Proof. A routine induction on the subtyping relation $\Psi, a, \Psi_R \vdash A \leq B$ finishes the proof. \square

Corollary 5.4 (Monotype Subtyping Substitution for Type Variables). *If $\Psi \vdash \tau$ and $\Psi, a \vdash A \leq B$, then $\Psi \vdash [\tau/a]A \leq [\tau/a]B$.*

Lemma 5.5 (Subtyping Transitivity). *If $\Psi \vdash A \leq B$ and $\Psi \vdash B \leq C$ then $\Psi \vdash A \leq C$.*

Proof. Induction on $\langle |B|_{\forall}, |A \leq B| + |B \leq C| \rangle$. Most cases preserve the first element of the size measures $|B|_{\forall}$, and are relatively easy to prove. The difficult case is when B is a polymorphic type, when the conditions are $\Psi \vdash A \leq \forall a. B$ and $\Psi \vdash \forall a. B \leq C$. They are derived through rules $\leq\forall\text{L}$ and $\leq\forall\text{R}$, respectively. Therefore, we have $\Psi, a \vdash A \leq B$ and $\Psi \vdash [\tau/a]B \leq C$. To exploit the induction hypothesis, the contexts should be unified. By

$$\begin{array}{c}
 \text{Declarative worklist} \quad \Omega ::= \cdot \mid \Omega, a \mid \Omega, x : A \mid \Omega \Vdash \omega \\
 \\
 \boxed{\Gamma \rightsquigarrow \Omega} \qquad \qquad \qquad \Gamma \text{ instantiates to } \Omega. \\
 \\
 \frac{}{\Omega \rightsquigarrow \Omega} \rightsquigarrow \Omega \quad \frac{\Omega \vdash \tau \quad \Omega, [\tau/\hat{\alpha}]\Gamma \rightsquigarrow \Omega}{\Omega, \hat{\alpha}, \Gamma \rightsquigarrow \Omega} \rightsquigarrow \hat{\alpha}
 \end{array}$$

Figure 5.7: Declarative Worklists and Instantiation

Corollary 5.4, $\Psi \vdash A \leq [\tau/a]B$. Notice that the freshness condition is implicit for rule $\leq \forall L$. Clearly, $||[\tau/a]B|_{\forall} < |\forall a. B|_{\forall}$, i.e. the first size measure decreases. By induction hypothesis we get $\Psi \vdash A \leq C$ and finishes this case. \square

5.4.2 TRANSFER

Follow the approach of Section 4.3, the transfer relation and the declarative instantiation relation are defined in Figure 5.7.

Similarly, Lemmas 5.6 and 5.7 generalizing Rule $\rightsquigarrow \hat{\alpha}$ hold as well.

Lemma 5.6 (Insert). *If $\Gamma_L, [\tau/\hat{\alpha}]\Gamma_R \rightsquigarrow \Omega$ and $\Gamma_L \vdash \tau$, then $\Gamma_L, \hat{\alpha}, \Gamma_R \rightsquigarrow \Omega$.*

Lemma 5.7 (Extract). *If $\Gamma_L, \hat{\alpha}, \Gamma_R \rightsquigarrow \Omega$, then there exists τ s.t. $\Gamma_L \vdash \tau$ and $\Gamma_L, [\tau/\hat{\alpha}]\Gamma_R \rightsquigarrow \Omega$.*

Figure 5.8 defines a relation $\Omega \longrightarrow \Omega'$, checking that every judgment entry in the worklist holds using a corresponding declarative judgment.

5.4.3 SOUNDNESS

Our algorithm is sound with respect to the declarative system. For any worklist Γ that reduces successfully, there is a valid instantiation Ω that transfers all judgments to the declarative system.

Theorem 5.8 (Soundness). *If $\text{wf } \Gamma$ and $\Gamma \longrightarrow^*$, then there exists Ω s.t. $\Gamma \rightsquigarrow \Omega$ and $\Omega \longrightarrow^*$.*

5.4.4 PARTIAL COMPLETENESS OF SUBTYPING: RANK-1 RESTRICTION

The algorithm is incomplete due to the subtyping rules 14, 15, 20 and 21. However, subtyping is complete with respect to the declarative system in a rank-1 setting.

$\ \Omega\ $	Judgment erasure.
$\ \cdot\ = \cdot$	
$\ \Omega, a\ = \ \Omega\ , a$	
$\ \Omega, x : A\ = \ \Omega\ , x : A$	
$\ \Omega \vdash \omega\ = \ \Omega\ $	
$\Omega \longrightarrow \Omega'$	Declarative transfer.
$\Omega, a \longrightarrow \Omega$	
$\Omega, x : A \longrightarrow \Omega$	
$\Omega \vdash A \leq B \longrightarrow \Omega$	when $\ \Omega\ \vdash A \leq B$
$\Omega \vdash e \Leftarrow A \longrightarrow \Omega$	when $\ \Omega\ \vdash e \Leftarrow A$
$\Omega \vdash e \Rightarrow_a \omega \longrightarrow \Omega \vdash [A/a]\omega$	when $\ \Omega\ \vdash e \Rightarrow A$
$\Omega \vdash A \bullet e \Rightarrow_a \omega \longrightarrow \Omega \vdash [C/a]\omega$	when $\ \Omega\ \vdash A \bullet e \Rightarrow C$

Figure 5.8: Declarative Transfer

DECLARATIVE RANK-1 RESTRICTION Rank-1 types are also named type schemes in Hindley-Milner type system.

Declarative Type Schemes $\sigma ::= \forall a. \sigma \mid \tau$

In other words, the universal quantifiers only appear in the top level of all polymorphic types.

For declarative subtyping, a judgment must be of form $\sigma_1 \leq \sigma_2$.

5.4.5 ALGORITHMIC RANK-1 RESTRICTION (PARTIAL COMPLETENESS)

The algorithmic mono-types and type schemes are defined as following:

Algorithmic Mono-types $\tau_A ::= 1 \mid \top \mid \perp \mid a \mid A \rightarrow B \mid \hat{\alpha}$
 Algorithmic Type Schemes $\sigma_A ::= \forall a. \sigma_A \mid \tau_A$

Starting from the declarative judgment $\sigma_1 \leq \sigma_2$, the algorithmic derivation might involve different other kinds of judgments. The following derivation, as an example, shows how a rank-1 judgment derives.

$$\begin{aligned}
 & \cdot \Vdash \forall a. a \rightarrow a \leq \forall b. (b \rightarrow b) \rightarrow (b \rightarrow b) \\
 \longrightarrow_8 & b \Vdash \forall a. a \rightarrow a \leq (b \rightarrow b) \rightarrow (b \rightarrow b) \\
 \longrightarrow_7 & b, \hat{\alpha} \Vdash \hat{\alpha} \rightarrow \hat{\alpha} \leq (b \rightarrow b) \rightarrow (b \rightarrow b) \\
 \longrightarrow_6 & b, \hat{\alpha} \Vdash \hat{\alpha} \leq b \rightarrow b \Vdash b \rightarrow b \leq \hat{\alpha} \\
 \longrightarrow & \dots
 \end{aligned}$$

In this derivation, we begin from a judgment of the form $\sigma \leq \sigma$. After rule 8 is applied, the judgment becomes $\sigma \leq \tau$, since the right-hand-side polymorphic type is reduced to a declarative mono-type. Then, rule 7 introduces existential variables to the left-hand-side, resulting in a judgment like $\tau_A \leq \tau$, or $\sigma_A \leq \tau$ in a more general case. Finally, rule 6 breaks a judgment between functions into two sub-judgments, which swaps the positions of the argument types and creates a judgment like $\tau \leq \tau_A$. Notice that σ_A is not possible to occur to the right because the function type may not contain any polymorphic types as its argument type.

After a detailed analysis on the judgments derivations, we found that the only possible judgments that a rank-1 declarative subtyping judgment might step to belong to the following two categories:

$$\sigma_A \leq \sigma \quad \text{or} \quad \tau \leq \sigma_A$$

All the possible judgment types shown above fall into these categories. For example, $\tau_A \leq \tau$ is a special form of $\sigma_A \leq \sigma$, and $\tau \leq \tau_A$ belongs to $\tau \leq \sigma_A$.

An interesting observation is that $\hat{\alpha} \leq \hat{\beta}$ does not belong to either category, neither does $\hat{\alpha} \leq A \rightarrow B$ when $\hat{\alpha} \in \text{FV}(A \rightarrow B)$. Therefore, in the rank-1 setting, both cases of incompleteness never occur, and our algorithm is complete.

Theorem 5.9 (Completeness of Rank-1 Subtyping). *Given $\Psi \vdash \sigma_1 \leq \sigma_2$,*

- *If $\Gamma \Vdash \sigma_A \leq \sigma \rightsquigarrow \Psi \Vdash \sigma_1 \leq \sigma_2$
then $\Gamma \Vdash \sigma_A \leq \sigma \longrightarrow^* \cdot$;*
- *If $\Gamma \Vdash \tau \leq \sigma_A \rightsquigarrow \Psi \Vdash \sigma_1 \leq \sigma_2$
then $\Gamma \Vdash \tau \leq \sigma_A \longrightarrow^* \cdot$.*

5.4.6 TERMINATION

The measure used in Chapter 4 no longer works because subtyping judgments like

$$\hat{\alpha} \leq \perp \rightarrow \top$$

cause $\hat{\alpha}$ to split into $\hat{\alpha}_1 \rightarrow \hat{\alpha}_2$, without solving any part of it, resulting in an increased number of existential variables and possibly increased complexity of the worklist through the size-increasing substitution $\{\hat{\alpha} := \hat{\alpha}_1 \rightarrow \hat{\alpha}_2\}$.

We have performed a large set of tests on generated subtyping judgments that consist of algorithmic mono types, and all judgements terminated within a reasonable number of derivation depth. Unfortunately, we have not yet found any formal proof for the termination statement.

5.4.7 FORMALIZATION IN THE ABELLA PROOF ASSISTANT

We have chosen the Abella (v2.0.7-dev¹) proof assistant [Gacek 2008] to develop our formalization. Equipped with HOAS, Abella eases the formalization and proof tasks a lot compared with various libraries in Coq. Additionally, our algorithm heavily uses eager substitutions, and Abella greatly simplifies relevant proofs thanks to its built-in substitution representation and higher-order unification algorithms.

STATISTICS OF THE PROOF The proof script consists of 7,301 lines of Abella code with a total of 48 definitions and 592 theorems. Figure 5.1 briefly summarizes the contents of each file. The files are linearly dependent due to limitations of Abella.

5.5 DISCUSSION

5.5.1 A COMPLETE ALGORITHM UNDER MONOTYPE GUESSING RESTRICTIONS

The incompleteness of the algorithm is mainly due to incomplete guesses for existential variable instantiations when dealing with subtyping. If the declarative system did not treat the bottom and top types as monotypes in the first place, the algorithm would not need to guess those types. Formally, if monotypes are defined as follows,

$$\text{Monotypes} \quad \tau' ::= 1 \mid a \mid \tau_1 \rightarrow \tau_2$$

then unification based on equality regains completeness between monotypes. Incomplete examples mentioned before like $\hat{\alpha} \leq \hat{\beta}$ can be safely treated as $\hat{\alpha} = \hat{\beta}$, because of the lemma:

Lemma 5.10 (Subtyping between Monotypes is Equality).

If $\Psi \vdash \tau'_1 \leq \tau'_2$, then $\tau'_1 = \tau'_2$.

¹We use a forked version <https://github.com/JimmyZJX/abella> by only enhancing the Abella prover with a handy “applies” tactic.

Table 5.1: Statistics for the proof scripts

File(s)	LOC	#Thm	Description
olist.thm, nat.thm	311	57	Basic data structures
typing.thm	273	7	Declarative & algorithmic system, debug examples
decl.thm	241	33	Basic declarative properties
order.thm	274	27	The $ \cdot _{\forall}$ measure; decl. subtyping strengthening
alg.thm	699	82	Basic algorithmic properties
trans.thm	635	53	Worklist instantiation and declarative transfer; Lemmas 4.1, 4.2
declTyping.thm	1,087	76	Non-overlapping declarative system; Lemmas 4.3, 4.4, 4.5, 4.6
soundness.thm	1,206	81	Soundness theorem; aux. lemmas on transfer
dcl.thm	417	12	Non-overlapping declarative worklist
scheme.thm	1,113	98	Type scheme (rank-1 restriction)
completeness.thm	1,045	63	Completeness theorem; aux. lemmas and relations
<i>Total</i>	7,301	592	(48 definitions in total)

On the meantime, algorithmic subtyping rules 12 and 13, and typing rules 27 and 37 are removed from the algorithm under the monotype restriction, since these rules solves existential variables to \top or \perp . As a result, both soundness and completeness are successfully mechanized. It is not a surprising finding, given that the algorithm reduce judgments in a similar way compared with the one in the previous chapter. However, it still reveals the fact that if we do not try to guess any subtyping relation, the algorithm should remain simple and complete.

5.5.2 LAZY SUBSTITUTION AND NON-TERMINATING LOOPS

One major source of incompleteness comes from rule 20 and 21, where two different existential variables are compared. A natural idea is to delay the solving procedure until all the corresponding constraints are collected. Constraint collection can be implemented by the following new worklist definition, where the existential variables also come with a set of upper bounds and a set of lower bounds. Both of the bounds consist of algorithmic monotypes.

$$\text{Algorithmic worklist} \quad \Gamma ::= \cdot \mid \Gamma, a \mid \Gamma, \overline{\tau_A} \leq \hat{\alpha} \leq \overline{\tau_A} \mid \Gamma \Vdash \omega$$

In presence of the bound collections, the algorithm behaves slightly differently on variable solving rules. For example, rules 20 and 21 add the corresponding existential variable as part of a bound rather than performing a global substitution.

$$\begin{aligned}\Gamma[\hat{\alpha}][L \leq \hat{\beta} \leq U] \Vdash \hat{\alpha} \leq \hat{\beta} &\longrightarrow_{20'} \Gamma[\hat{\alpha}][L \cup \{\hat{\alpha}\} \leq \hat{\beta} \leq U] \\ \Gamma[\hat{\alpha}][L \leq \hat{\beta} \leq U] \Vdash \hat{\beta} \leq \hat{\alpha} &\longrightarrow_{21'} \Gamma[\hat{\alpha}][L \leq \hat{\beta} \leq U \cup \{\hat{\alpha}\}]\end{aligned}$$

The change of existential variable declaration also result in the change of the algorithmic rule on existential variable garbage collection. When $\hat{\alpha}$ is to be recycled, we need to propagate the subtyping check to the bounds: any type that is a subtype of $\hat{\alpha}$ should be subtype of any super type of $\hat{\alpha}$, according to the transitivity of subtyping.

$$\Gamma, \{l_i\}^n \leq \hat{\alpha} \leq \{u_j\}^m \longrightarrow_{2'} \Gamma \Vdash_{1 \leq i \leq n, 1 \leq j \leq m} l_i \leq u_j$$

Luckily, such check is sufficient. In other words, we can always find a proper solution for $\hat{\alpha}$ if any lower bound is subtype of any upper bound.

This new approach fixes the incompleteness caused by rules 20 and 21, however, the algorithm might loop forever due to the (new version of) instantiation rules 14 and 15. For example, the following derivation results in infinite instantiations:

$$\begin{aligned}&\hat{\alpha}, \hat{\beta} \Vdash \hat{\alpha} \leq 1 \rightarrow \hat{\beta} \Vdash \hat{\beta} \leq 1 \rightarrow \hat{\alpha} \\ \longrightarrow &\hat{\alpha}, \hat{\beta}_1, \hat{\beta}_2, \{\} \leq \hat{\beta} \leq \{\hat{\beta}_1 \rightarrow \hat{\beta}_2\} \Vdash \hat{\alpha} \leq 1 \rightarrow \hat{\beta} \Vdash \hat{\beta}_1 \rightarrow \hat{\beta}_2 \leq 1 \rightarrow \hat{\alpha} \\ \longrightarrow &\dots \\ \longrightarrow &\hat{\alpha}, \{1\} \leq \hat{\beta}_1 \leq \{\}, \{\} \leq \hat{\beta}_2 \leq \{\hat{\alpha}\}, \{\} \leq \hat{\beta} \leq \{\hat{\beta}_1 \rightarrow \hat{\beta}_2\} \Vdash \hat{\alpha} \leq 1 \rightarrow \hat{\beta} \\ \longrightarrow &\dots\end{aligned}$$

PART II

RELATED AND FUTURE WORK

6 RELATED WORK

Throughout the thesis we have already discussed much of the closest related work. In this section we summarize the key differences and novelties, and discuss some other related work.

6.1 HIGHER-RANKED POLYMORPHIC TYPE INFERENCE ALGORITHMS

6.1.1 PREDICATIVE ALGORITHMS

Higher-ranked polymorphism is a convenient and practical feature of programming languages. Since full type-inference for System F is undecidable [Wells 1999], various decidable partial type-inference algorithms were developed. The declarative system of this paper, proposed by Dunfield and Krishnaswami [2013], is *predicative*: \forall 's only instantiate to monotypes. The monotype restriction on instantiation is considered reasonable and practical for most programs, except for those that require sophisticated forms of higher-order polymorphism. In those cases, the bidirectional system accepts guidance through type annotations, which allow polymorphic types. Such annotations also improve readability of the program, and are not much of a burden in practice.

DK's algorithm is shown to be sound, complete and decidable in 70 pages of manual proofs. Though carefully written, some of the proofs are incorrect (see discussion in Section 4.5.1 and 4.5.2), which creates difficulties when formalizing them in a proof assistant. In their follow-up work [Dunfield and Krishnaswami 2019] enrich the bidirectional higher-rank system with existentials and indexed types. With a more complex declarative system, they developed a proof of over 150 pages. It is even more difficult to argue its correctness for every single detail within such a big development. Unfortunately, we find that their Lemma 26 (Parallel Admissibility) appears to have the same issue as lemma 29 in [Dunfield and Krishnaswami 2013]: the conclusion is false. We also discuss the issue in more detail in Section 4.5.1.

Peyton Jones et al. [2007] developed another higher-rank predicative bidirectional type system. Their subtyping relation is enriched with *deep skolemisation*, which is more general than ours and allows more valid relations. In comparison to DK's system, they do not use the application inference judgment, resulting in a complicated mechanism for implicit in-

stantiation taken care by the unification process for the algorithm. A manual proof is given, showing that the algorithm is sound and complete with respect to their declarative specification.

In a more recent work, Xie and Oliveira [2018] proposed a variant of a bidirectional type inference system for a predicative system with higher-ranked types. Type information flows from arguments to functions with an additional *application* mode. This variant allows more higher-order typed programs to be inferred without additional annotations. Following the new mode, the let-generalization of the Hindley-Milner system is well supported as a syntactic sugar. The formalization includes some mechanized proofs for the declarative type system, but all proofs regarding the algorithmic type system are manual.

6.1.2 IMPREDICATIVE ALGORITHMS

Impredicative System F allows instantiation with polymorphic types, but unfortunately its subtyping system is already undecidable [Tiuryn and Urzyczyn 1996]. Works on partial impredicative type-inference algorithms [Le Botlan and Rémy 2003; Leijen 2008; Vytiniotis et al. 2008] navigate a variety of design tradeoffs for a decidable algorithm. As a result, such algorithms tend to be more complicated, and thus less adopted in practice.

A work proposed *Guarded Impredicative Polymorphism* [Serrano et al. 2018], as an improvement on GHC’s type inference algorithm with impredicative instantiation. They make use of local information in n -ary applications to infer polymorphic instantiations with a relatively simple specification and unification algorithm. Although not all impredicative instantiations can be handled well, their algorithm is already quite useful in practice.

A recent follow-up work, the Quick Look [Serrano et al. 2020], takes a more conservative approach. The algorithm will try its best to infer impredicative instantiations, and only use the instantiation when it is the best one. In order to do so, the algorithm also need to analyse all the argument types during a function call, and make use of the *guardedness* property to decide the principality of the inferred instantiation. When Quick Look cannot give the best instantiation, it will instead try predicative inference as HM. This conservative approach makes sure that the algorithm does not infer bad types and lead the subsequent type checking in wrong directions.

FreezeML [Emrich et al. 2020] is another recent work that extends the ML type system with impredicative instantiations. A special syntax of expression, the explicit freezing $\lceil x \rceil$, is added to guide the type system. Frozen variables are prevented to be instantiated by the type system, therefore variables of polymorphic types may force the type inference algorithm to instantiate impredicatively. In combination with the let-generalization rule, programmers may also encode explicit generalization and explicit instantiation. This work provides a nice

means for programmers to control how type inference algorithm deals with impredicative instantiations.

6.2 TYPE INFERENCE ALGORITHMS WITH SUBTYPING

6.3 TECHNIQUES USED IN TYPE INFERENCE ALGORITHMS

6.3.1 ORDERED CONTEXTS IN TYPE INFERENCE

Gundry et al. [2010] revisit algorithm \mathcal{W} and propose a new unification algorithm with the help of ordered contexts. Similar to DK’s algorithm, information of meta variables flow from input contexts to output contexts. Not surprisingly, its information increase relation has a similar role to DK’s context extension. Our algorithm, in contrast, eliminates output contexts and solution records ($\hat{\alpha} = \tau$), simplifying the information propagation process through immediate substitution by collecting all the judgments in a single worklist.

6.3.2 THE ESSENCE OF ML TYPE INFERENCE

Constraint-based type inference is adopted by Pottier and Rémy [2005] for ML type systems, which do not employ higher-ranked polymorphism. An interesting feature of their algorithm is that it keeps precise scoping of variables, similarly to our approach. Their algorithm is divided into constraint generation and solving phases (which are typical of constraint-based algorithms). Furthermore an intermediate language is used to describe constraints and their constraint solver utilizes a stack to track the state of the solving process. In contrast, our algorithm has a single phase, where the judgment chains themselves act as constraints, thus no separate constraint language is needed.

6.3.3 LISTS OF JUDGMENTS IN UNIFICATION

Some work [Abel and Pientka 2011; Reed 2009] adopts a similar idea to this paper in work on unification for dependently typed languages. Similarly to our work the algorithms need to be very careful about scoping, since the order of variable declarations is fundamental in a dependently typed setting. Their algorithms simplify a collection of unification constraints progressively in a single-step style. In comparison, our algorithm mixes variable declarations with judgments, resulting in a simpler judgment form, while processing them in a similar way. One important difference is that contexts are duplicated in their unification judgments, which complicates the unification process, since the information of each local context needs to be synchronized. Instead we make use of the nature of ordered context to

control the scopes of unification variables. While their algorithms focus only on unification, our algorithm also deals with other types of judgments like synthesis. A detailed discussion is in Section 4.1.3.

6.4 MECHANICAL FORMALIZATION OF POLYMORPHIC TYPE SYSTEMS

Since the publication of the POPLMARK challenge [Aydemir et al. 2005], many theorem provers and packages provide new methods for dealing with variable binding [Aydemir et al. 2008; Chlipala 2008; Urban 2008]. More and more type systems are formalized with these tools. However, mechanizing certain algorithmic aspects, like unification and constraint solving, has received very little attention and is still challenging. Moreover, while most tools support local (input) contexts in a neat way, many practical type-inference algorithms require more complex binding structures with output contexts or various forms of constraint solving procedures.

Algorithm \mathcal{W} , as one of the classic type inference algorithms for polymorphic type systems, has been manually proven to be sound and complete with respect to the Hindley-Milner type system [Damas and Milner 1982; Hindley 1969; Milner 1978]. After around 15 years, the algorithm was formally verified by Naraschewski and Nipkow [1999] in Isabelle/HOL [Nipkow et al. 2002]. The treatment of new variables was tricky at that time, while the overall structure follows the structure of Damas’s manual proof closely. Later on, other researchers [Dubois 2000; Dubois and Menissier-Morain 1999] formalized algorithm \mathcal{W} in Coq [The Coq development team 2017]. Nominal techniques [Urban 2008] in Isabelle/HOL have been developed to help programming language formalizations, and are used for a similar verification [Urban and Nipkow 2008]. Moreover, Garrigue [Garrigue 2015] mechanized a type inference algorithm, with the help of locally nameless [Charguéraud 2012], for Core ML extended with structural polymorphism and recursion.

PART III

EPILOGUE

7 CONCLUSION AND FUTURE WORK

7.1 CONCLUSION

In this paper we have shown how to mechanise an algorithmic subtyping relation for higher-order polymorphism, together with its proofs of soundness, completeness and decidability, in the Abella proof assistant. In ongoing work we are extending our mechanisation with a bidirectional type inference algorithm. The main difficulty there is communicating the instantiations of existential variables from the subtyping algorithm to the type inference. To make this possible we are exploring a continuation passing style formulation, which generalises the worklist approach. Another possible extension is to have the algorithm return an explicit witness for the subtyping as part of type-directed elaboration into System F.

In this paper we have shown how to mechanise an algorithmic subtyping relation for higher-order polymorphism, together with its proofs of soundness, completeness and decidability, in the Abella proof assistant. In ongoing work we are extending our mechanisation with a bidirectional type inference algorithm. The main difficulty there is communicating the instantiations of existential variables from the subtyping algorithm to the type inference. To make this possible we are exploring a continuation passing style formulation, which generalises the worklist approach. Another possible extension is to have the algorithm return an explicit witness for the subtyping as part of type-directed elaboration into System F.

7.2 FUTURE WORK

ELABORATION

OPTIMIZATION

OO CLASS HIERARCHY

BOUNDED QUANTIFICATION / F-BOUNDED QUANTIFICATION

BIBLIOGRAPHY

[Citing pages are listed after each reference.]

Andreas Abel and Brigitte Pientka. 2011. Higher-order dynamic pattern unification for dependent types and records. In *International Conference on Typed Lambda Calculi and Applications*. Springer, 10–26. [cited on pages 38, 43, and 94]

Brian Aydemir, Arthur Charguéraud, Benjamin C. Pierce, Randy Pollack, and Stephanie Weirich. 2008. Engineering Formal Metatheory. In *Proceedings of the 35th Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL '08)*. [cited on page 95]

Brian E Aydemir, Aaron Bohannon, Matthew Fairbairn, J Nathan Foster, Benjamin C Pierce, Peter Sewell, Dimitrios Vytiniotis, Geoffrey Washburn, Stephanie Weirich, and Steve Zdancewic. 2005. Mechanized metatheory for the masses: The POPLmark challenge. In *The 18th International Conference on Theorem Proving in Higher Order Logics*. [cited on pages 7 and 95]

Arthur Charguéraud. 2012. The Locally Nameless Representation. *Journal of Automated Reasoning* 49, 3 (01 Oct 2012), 363–408. [cited on page 95]

Paul Chiusano and Runar Bjarnason. 2015. Unison. <http://unisonweb.org> [cited on page 6]

Adam Chlipala. 2008. Parametric Higher-order Abstract Syntax for Mechanized Semantics. In *Proceedings of the 13th ACM SIGPLAN International Conference on Functional Programming (ICFP '08)*. [cited on page 95]

Luis Damas and Robin Milner. 1982. Principal Type-schemes for Functional Programs. In *Proceedings of the 9th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL '82)*. [cited on pages 4, 25, and 95]

- N.G de Bruijn. 1972. Lambda calculus notation with nameless dummies, a tool for automatic formula manipulation, with application to the Church-Rosser theorem. *Indagationes Mathematicae (Proceedings)* 75, 5 (1972), 381–392. [https://doi.org/10.1016/1385-7258\(72\)90034-0](https://doi.org/10.1016/1385-7258(72)90034-0) [cited on page 8]
- Disciple Development Team. 2017. The Disciplined Disciple Compiler. <http://disciple.ouroborus.net/> [cited on page 7]
- Stephen Dolan and Alan Mycroft. 2017. Polymorphism, Subtyping, and Type Inference in MLsub. In *Proceedings of the 44th ACM SIGPLAN Symposium on Principles of Programming Languages (Paris, France) (POPL 2017)*. Association for Computing Machinery, New York, NY, USA, 60–72. <https://doi.org/10.1145/3009837.3009882> [cited on pages 20 and 73]
- Catherine Dubois. 2000. Proving ML type soundness within Coq. *Theorem Proving in Higher Order Logics* (2000), 126–144. [cited on pages 7 and 95]
- Catherine Dubois and Valerie Menissier-Morain. 1999. Certification of a type inference tool for ML: Damas–Milner within Coq. *Journal of Automated Reasoning* 23, 3 (1999), 319–346. [cited on pages 7 and 95]
- Joshua Dunfield and Neelakantan R. Krishnaswami. 2013. Complete and Easy Bidirectional Typechecking for Higher-rank Polymorphism. In *Proceedings of the 18th ACM SIGPLAN International Conference on Functional Programming (ICFP ’13)*. [cited on pages 5, 6, 17, 23, 24, 27, 37, 38, 69, and 92]
- Joshua Dunfield and Neelakantan R. Krishnaswami. 2019. Sound and Complete Bidirectional Typechecking for Higher-rank Polymorphism with Existentials and Indexed Types. *Proc. ACM Program. Lang.* 3, POPL, Article 9 (Jan. 2019), 28 pages. [cited on pages 69, 70, and 92]
- Jonathan Eifrig, Scott Smith, and Valery Trifonov. 1995. Type Inference for Recursively Constrained Types and its Application to OOP. *Electronic Notes in Theoretical Computer Science* 1 (1995), 132 – 153. [https://doi.org/10.1016/S1571-0661\(04\)80008-2](https://doi.org/10.1016/S1571-0661(04)80008-2) MFPS XI, Mathematical Foundations of Programming Semantics, Eleventh Annual Conference. [cited on page 73]
- Frank Emrich, Sam Lindley, Jan Stolarek, James Cheney, and Jonathan Coates. 2020. FreezeML: Complete and Easy Type Inference for First-Class Polymorphism. In *Proceedings of the 41st ACM SIGPLAN Conference on Programming Language Design and Implementation (London, UK) (PLDI 2020)*. Association for Computing Machinery, New

- York, NY, USA, 423–437. <https://doi.org/10.1145/3385412.3386003> [cited on page 93]
- Phil Freeman. 2017. PureScript. <http://www.purescript.org/> [cited on pages 6 and 7]
- Andrew Gacek. 2008. The Abella Interactive Theorem Prover (System Description). In *Proceedings of IJCAR 2008 (Lecture Notes in Artificial Intelligence)*. [cited on pages 8, 34, 37, 38, 54, 63, and 88]
- Jacques Garrigue. 2015. A certified implementation of ML with structural polymorphism and recursive types. *Mathematical Structures in Computer Science* 25, 4 (2015), 867–891. [cited on pages 7 and 95]
- Andrew Gill, John Launchbury, and Simon L. Peyton Jones. 1993. A Short Cut to Deforestation. In *Proceedings of the Conference on Functional Programming Languages and Computer Architecture (FPCA '93)*. [cited on page 5]
- Adam Gundry, Conor McBride, and James McKinna. 2010. Type Inference in Context. In *Proceedings of the Third ACM SIGPLAN Workshop on Mathematically Structured Functional Programming (MSFP '10)*. [cited on pages 38 and 94]
- Roger Hindley. 1969. The principal type-scheme of an object in combinatory logic. *Transactions of the american mathematical society* 146 (1969), 29–60. [cited on pages 4 and 95]
- Mark P. Jones. 1995. Functional Programming with Overloading and Higher-Order Polymorphism. In *Advanced Functional Programming (Lecture Notes in Computer Science 925)*. [cited on page 5]
- A. J. Kfoury and J. Tiuryn. 1992. Type Reconstruction in Finite Rank Fragments of the Second-Order λ -Calculus. *Inf. Comput.* 98, 2 (June 1992), 228–257. [https://doi.org/10.1016/0890-5401\(92\)90020-G](https://doi.org/10.1016/0890-5401(92)90020-G) [cited on page 14]
- Casey Klein, John Clements, Christos Dimoulas, Carl Eastlund, Matthias Felleisen, Matthew Flatt, Jay A. McCarthy, Jon Raskind, Sam Tobin-Hochstadt, and Robert Bruce Findler. 2012. Run Your Research: On the Effectiveness of Lightweight Mechanization. In *Proceedings of the 39th Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (Philadelphia, PA, USA) (POPL '12)*. 285–296. [cited on page 37]
- Ralf Lämmel and Simon Peyton Jones. 2003. Scrap Your Boilerplate: A Practical Design Pattern for Generic Programming. In *Proceedings of the 2003 ACM SIGPLAN International Workshop on Types in Languages Design and Implementation (TLDI '03)*. [cited on page 5]

- John Launchbury and Simon L. Peyton Jones. 1995. State in Haskell. *LISP and Symbolic Computation* 8, 4 (1995), 293–341. [cited on page 5]
- Didier Le Botlan and Didier Rémy. 2003. MLF: Raising ML to the Power of System F. In *Proceedings of the Eighth ACM SIGPLAN International Conference on Functional Programming (ICFP '03)*. [cited on pages 5 and 93]
- Daan Leijen. 2008. HMF: Simple Type Inference for First-class Polymorphism. In *Proceedings of the 13th ACM SIGPLAN International Conference on Functional Programming (ICFP '08)*. [cited on pages 5 and 93]
- Xavier Leroy et al. 2012. The CompCert verified compiler. *Documentation and user's manual*. INRIA Paris-Rocquencourt (2012). [cited on page 7]
- Alberto Martelli and Ugo Montanari. 1982. An Efficient Unification Algorithm. *ACM Trans. Program. Lang. Syst.* 4, 2 (April 1982), 258–282. [cited on page 43]
- Dale Miller. 2000. Abstract Syntax for Variable Binders: An Overview. In *CL 2000: Computational Logic (Lecture Notes in Artificial Intelligence)*. [cited on pages 34 and 63]
- Robin Milner. 1978. A theory of type polymorphism in programming. *Journal of computer and system sciences* 17, 3 (1978), 348–375. [cited on pages 4, 13, 42, and 95]
- Wolfgang Naraschewski and Tobias Nipkow. 1999. Type inference verified: Algorithm W in Isabelle/HOL. *Journal of Automated Reasoning* 23, 3 (1999), 299–318. [cited on pages 7 and 95]
- Tobias Nipkow, Lawrence C Paulson, and Markus Wenzel. 2002. *Isabelle/HOL: a proof assistant for higher-order logic*. Vol. 2283. Springer Science & Business Media. [cited on page 95]
- Martin Odersky and Konstantin Läufer. 1996. Putting Type Annotations to Work. In *Proceedings of the 23rd ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL '96)*. [cited on pages 5, 6, 15, 17, 23, 38, and 76]
- Simon Peyton Jones and Mark Shields. 2004. Lexically-scoped type variables. (2004). <http://research.microsoft.com/en-us/um/people/simonpj/papers/scoped-tyvars/> Draft. [cited on pages 63 and 68]
- Simon Peyton Jones, Dimitrios Vytiniotis, Stephanie Weirich, and Mark Shields. 2007. Practical type inference for arbitrary-rank types. *Journal of functional programming* 17, 1 (2007), 1–82. [cited on pages 5, 6, 18, and 92]

- Benjamin C. Pierce and David N. Turner. 2000. Local Type Inference. *ACM Trans. Program. Lang. Syst.* 22, 1 (Jan. 2000), 1–44. <https://doi.org/10.1145/345099.345100> [cited on page 6]
- Andrew M. Pitts. 2003. Nominal logic, a first order theory of names and binding. *Information and Computation* 186, 2 (2003), 165–193. [https://doi.org/10.1016/S0890-5401\(03\)00138-X](https://doi.org/10.1016/S0890-5401(03)00138-X) Theoretical Aspects of Computer Software (TACS 2001). [cited on page 8]
- François Pottier. 1998. *Type inference in the presence of subtyping: from theory to practice*. Ph.D. Dissertation. INRIA. [cited on page 73]
- François Pottier and Didier Rémy. 2005. *Advanced Topics in Types and Programming Languages*. The MIT Press, Chapter The Essence of ML Type Inference, 387–489. [cited on pages 43 and 94]
- Jason Reed. 2009. Higher-order Constraint Simplification in Dependent Type Theory. In *Proceedings of the Fourth International Workshop on Logical Frameworks and Meta-Languages: Theory and Practice (LFMTP '09)*. [cited on pages 38, 43, and 94]
- John C. Reynolds. 1983. Types, Abstraction and Parametric Polymorphism. In *Proceedings of the IFIP 9th World Computer Congress*. [cited on pages 3 and 67]
- Alejandro Serrano, Jurriaan Hage, Simon Peyton Jones, and Dimitrios Vytiniotis. 2020. A Quick Look at Impredicativity. *Proc. ACM Program. Lang.* 4, ICFP, Article 89 (Aug. 2020), 29 pages. <https://doi.org/10.1145/3408971> [cited on page 93]
- Alejandro Serrano, Jurriaan Hage, Dimitrios Vytiniotis, and Simon Peyton Jones. 2018. Guarded Impredicative Polymorphism. In *Proceedings of the 39th ACM SIGPLAN Conference on Programming Language Design and Implementation (PLDI 2018)*. [cited on pages 5 and 93]
- The Coq development team. 2017. The Coq proof assistant. <https://coq.inria.fr/> [cited on page 95]
- Jerzy Tiuryn and Pawel Urzyczyn. 1996. The subtyping problem for second-order types is undecidable. In *Proceedings 11th Annual IEEE Symposium on Logic in Computer Science*. [cited on pages 18 and 93]
- Valery Trifonov and Scott Smith. 1996. Subtyping constrained types. In *Static Analysis*, Radhia Cousot and David A. Schmidt (Eds.). Springer Berlin Heidelberg, Berlin, Heidelberg, 349–365. [cited on page 73]

- Christian Urban. 2008. Nominal techniques in Isabelle/HOL. *Journal of Automated Reasoning* 40, 4 (2008), 327–356. [cited on page 95]
- Christian Urban and Tobias Nipkow. 2008. Nominal verification of algorithm W. *From Semantics to Computer Science. Essays in Honour of Gilles Kahn* (2008), 363–382. [cited on pages 7 and 95]
- Dimitrios Vytiniotis, Stephanie Weirich, and Simon Peyton Jones. 2008. FPH: First-class Polymorphism for Haskell. In *Proceedings of the 13th ACM SIGPLAN International Conference on Functional Programming (ICFP '08)*. [cited on pages 5 and 93]
- Joe B Wells. 1999. Typability and type checking in System F are equivalent and undecidable. *Annals of Pure and Applied Logic* 98, 1-3 (1999), 111–156. [cited on pages 5, 13, and 92]
- Ningning Xie and Bruno C. d. S. Oliveira. 2018. Let Arguments Go First. In *Programming Languages and Systems*, Amal Ahmed (Ed.). Springer International Publishing, Cham, 272–299. [cited on page 93]
- Jinxu Zhao, Bruno C. d. S. Oliveira, and Tom Schrijvers. 2018. Formalization of a Polymorphic Subtyping Algorithm. In *ITP (Lecture Notes in Computer Science, Vol. 10895)*. Springer, 604–622. [cited on pages 10, 38, 44, and 46]
- Jinxu Zhao, Bruno C. d. S. Oliveira, and Tom Schrijvers. 2019. A Mechanical Formalization of Higher-Ranked Polymorphic Type Inference. *Proc. ACM Program. Lang.* 3, ICFP, Article 112 (July 2019), 29 pages. <https://doi.org/10.1145/3341716> [cited on page 10]