Geometric Group Theory

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Ω1 Day 1 : Introductions

This is for the Directed Reading Program hosted by the Mathematics Graduate Student Association. This specific group is lead by Vivian He, and we will be reading about Geometric Group Theory. We've decided on the books

- Office Hours with a Geometric Group Theorist - M. Clay and D. Margalit
- Geometric Group Theory - C. Druţu and M. Kapovich

This Latex file will mostly keep track of the general content that we will be covering/already have covered, but I'll be adding some related examples and whatever that I learn about during this time, I'll try my best to keep them in the subsubsections. I'll also be trying to add my own solutions to the various exercises across both books. We'll be mostly following GGT but if it gets too intense we'll switch to Office hours. This weeks readings are Chapters 7-7.4 in GGT and 7.5 if you're up to it.

Ω1.1 GGT Chapter 7.1 - Finitely Generated Groups

Definition 1.1 (Finitely Generated)**.** We say a group, G, is *finitely generated* if it has a finite *generating set*,

Moreover we use $\operatorname{rank}(G)$ to denote the minimal number of generators for G.

Example 1.2

We have that $(\mathbb{Z}, +)$ is finitely generated by 1 and -1. Moreover we have that it can be generated by coprime integers p, q

Example 1.3

The Group $(\mathbb{Q}, +)$ is not finitely generated

Ω1.1.1 Wreath Product

Definition 1.4 (Group Complement)**.** Let K be a subgroup of G, then we say that Q is a the *complement* to K is $Q \cap K = \{1_G\}$ and $QK = G$

Definition 1.5 (Semi-Direct Product)**.** Define the *Semi-Direct Product* for two groups $(G, \cdot_G)$ and $(H, \cdot_H)$, and some homomorphism $\varphi : H \to G$ as $G \rtimes_\varphi H$. Which is the group who's underlying set is $G \times H = \{(g, h) | g \in G, h \in H\}$, and who's operation is defined to be

$$(g_1, h_1) \star (g_2, h_2) = (g_1(\varphi(h_1)g_2), h_1h_2)$$

Exercise 1.6. Prove that this is a group.

Definition 1.7 (Wreath Product)**.** YO THIS THING IS SO CONFUSING

Example 1.8 (Lamplighter Group)

We define the Lamp Lighter Group to be the *restricted* Wreath Product between $\mathbb{Z}^2 \wr \mathbb{Z}$

Proposition 1.9

Every Quotient, $\overline{G}$, of a finitely generated group, G, is finitely generated

We may take the generators of $\overline{G}$ to be the images of the generators of G

Proposition 1.10

If N is a normal subgroup of G, with N and $G \backslash N$ being finitely generated, then we have that G is also finitely generated

By assumption we have that $\{n_1, \ldots, n_k\}$ and $\{g_1N, \ldots, g_m\}$ generate N and $G \backslash N$ respectively. Then we have that

$$\{g_i, n_j | 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant k\}$$

Generates G

Ω1.1.2 Special Homomorphism

Definition 1.11 (Monomorphism)**.** A *monomorphism* is an injective homomorphism

Definition 1.12 (Epimorphism)**.** An *epimorphism* is a surjective homomorphism

Definition 1.13 (Automorphism)**.** An *automorphism* is a bijective homomorphism from a Group (more generally an algebraic structure) to itself

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Ω1.1.3 Exact Sequences

Definition 1.14 (Exact Sequences)**.** A sequence of groups and group homomorphisms is said to be *exact* if the the image of each homomorphism is equal to the kernel of the next

> **Remark 1.15.** There also exists Long Exact Sequences

We say an exact sequence is *short* if it is of form

$$1 \to A \xrightarrow{f} B \xrightarrow{g} C \to 1$$

Furthermore we have that f is a monomorohism and g is an epimorphism

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Lemma 1.16

If we have a short exact sequence of groups

$$1 \to A \xrightarrow{f} B \xrightarrow{g} C \to 1$$

such that A and C are finitely generated, then B is also finitely generated

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Ω1.1.4 Frattini Subgroups

Definition 1.17 (Non-Generator)**.** We say an element x is a *non-generator* of a group G, if for all generator sets S, the set $S\backslash\{x\}$ still generates G

Then the set of all non-generators forms a subgroup of G called the *Frattini Subgroup*

Definition 1.18 (Bounded Property)**.** A group G is said to have the *bounded property* (or to be *boundedly generated*) if there exists a finite subset $S = \{t_1, \dots, t_m\} \subset G$ such that for all elements $g \in G$, g can be written as

$$g = t_1^{k_1} \dots t_m^{k_m}$$

where $k_1, k_2, \dots, k_m$ are integers

We have that all finitely generated abelian groups have this property, we also have that all finite groups have this property too. But we have that non-abelian *free* groups do not have this property.

Ω1.2 GGT Chapter 7.2 - Free Groups

Let X be a set, then we call the elements in X *letters*, then defining the set

$$X^{-1} = \{a^{-1} | a \in X\}$$

to be the set of *inverse letters*. We can think of $X \cup X^{-1}$ as an *alphabet* then construct *words* from it

Definition 1.19 (Words)**.** A *word* in $X \cup X^{-1}$ is a finite string of letters from $X \cup X^{-1}$.

We notice that a word could be an empty string, in such a case we let 1 denote the empty word. Further more we define the *length* of a word to be the amount of letters in that word. Naturally, the empty word has length 0. Furthermore we use X^* to denote the set of words of the alphabet $X \cup X^{-1}$, where 1 is used to denote the empty word.

Definition 1.20 (Reduced)**.** We say a word in X^* is *reduced* if there are no pairs of consecutive letters of form $a^{-1}a$ or aa^{-1}. Then the *reduction* of a word is the deletion of all the consecutive letters of form $a^{-1}a$ or aa^{-1}.

Example 1.21

Given that a and b aren't inverses of each other. The word $abcc^{-1}ab$ is not reduced, and $abab$ is the reduction of the word

Example 1.22

The word abc is reduced

More generally we say that word is *cyclically reduced* if it is reduced and the first and last letters aren't inverses of each other.

Example 1.23

The word $abca^{-1}$ is reduced by not cyclically reduced

Definition 1.24 (Conjugation)**.** We may conjugate a word $w \in X^*$ by some letter $a \in X \cup X^{-1}$ by

$$w' = a^{-1}wa$$

Which results in a word w' with reduction of length greater than w

Furthermore we may define an equivalent relation on X^* such that we say two elements, $w \sim w'$ are equivalent if w can be obtained from w' by a finite sequence of reductions and their "inverse reduction"

Proposition 1.25

Every word $w \in X^*$ is equivalent to some *unique* reduced word in X^*

Moreover we could identify $X^*/\sim$ with $F(X)$

Definition 1.26 (Free Group)**.** The *free group* over X is the set $F(X)$ with operation $*$ defined by $w * w'$ is the unique reduced word equivalent to ww'

The cardinality of X is called the *rank* of $F(X)$. Although now we have two definitions for *rank*, the cardinality of a set, and the size of a minimal generator set for a finitely generated group. It actually turns out that both of these numbers are the same.

Exercise 1.27. A free group of rank at least 2 is not abelian. Thus, *free non-abelian* means 'free of rank at least 2.'

let G be a free group of rank at least 2, we notice that the operation on G is concatenation, so by letting $a, b \in G$ be arbitrary such that a and b aren't inverses of each other. Then it is so fucking obvious that ab isn't equal to ba. So G isn't abelian $\square$

Ω1.2.1 Semigroup

Definition 1.28 (Semigroup)**.** A set, S, with a binary operation $*$ that satisfies associativity

- $(x * y) * z = x * (y * z)$ for all $x, y, z \in S$

is called a *semigroup*

The *free semigroup*, $F^s(X)$, over X is defined synonymous to the definition of $F(X)$. But instead we take letters from only X, and not X^{-1}. With this construction we don't need to worry about reductions

Proposition 1.29

A map $\varphi : X \to G$ from a set X to a group G can be extended to a homomorphism $\Phi : F(X) \to G$, furthermore this extension is unique

We first extend φ to $X \cup X^{-1}$ by defining $\varphi(a^{-1}) = \varphi(a)^{-1}$. Then for every reduced word in $w \in F(x)$, define

$$\Phi(w) = \varphi(a_1)\dots\varphi(a_n)$$

Finally we define $\Phi(1_F) = 1_G$. Then we have our homomorphism. Then we prove Uniqueness. Let $\Psi(x) : F(X) \to G$ be a homomorphism such that $\Psi(x) = \varphi(x)$, then we have

$$\Psi(x) = \varphi(a_1)\dots\varphi(a_n) = \Phi(x)$$

So we have a homomorphism that is unique $\square$

Corollary 1.30

Every group is a *quotient group* of a free group

The idea is to apply the previous proposition but take X to be a generating set for G

Definition 1.31 (Split)**.** We say the short exact sequence

$$1 \to A \to B \to C \to 1$$

splits if $B = A \bigoplus C$

Lemma 1.32

Every short exact sequence *splits*

$$1 \to N \to G \xrightarrow{f} F(X) \to 1$$

Furthermore we have that there exists a subgroup of G ismoorphic to $F(X)$

Corollary 1.33

Every short exact sequence

$$1 \to G \to N \to \mathbb{Z} \to 1$$

splits

Ω1.3 GGT Chapter 7.3 - Presentation of Groups

Let G be a group with generating set S, then by Proposition 1.27 we may extend the inclusion map $i : S \to G$ to a unique epimorphism $\pi_S : F(S) \to G$, then we define the elements of $Ker(\pi_S)$ to *relators* (or *relations*) on the group G with generating set S. The idea behind this is that we basically define some word to be equal to the identity. Moreover if we define $R = \{r_i | i \in I\} \subset F(S)$ such that $Ker(\pi_s)$ is *normally generated* by R, then we have that the tuple (S, R), usually denoted $\langle S|R\rangle$, is a presentation of G. We further define the elements $r \in R$ to be *defining relators* of the presentation $\langle S|R\rangle$

Definition 1.34 (Finitely Presented)**.** if S, R are both finite, then we say that the pair (S, R) *finitely represent* G. Moreover, we say a group, G, is *finitely presented* if it's presentation is finite, which means that there are finite generators and finite relations

Definition 1.35 (Normally Generated)**.** A subset $X \subseteq G$ normally generated G if every $g \in G$ can be expressed as

$$g = a^{-1}xa \text{ for some } a \in G \text{ and } x \in X$$

Example 1.36

The cyclic group of order n can be *finitely* represented by

$$\langle a | a^n = 1\rangle$$

We have that our generating set is only one element, a, and our relation set is also one element a^n. So this group is *finitely represented.*

Definition 1.37 (Commutator)**.** For two elements g, h in a group G, we define their *commutator* by

$$[g, h] = g^{-1}h^{-1}gh$$

The significance of this is that in G, we have that g and h commute

You might've also noticed that representation's of groups aren't always unique, taking $n = 6$ in the above example yields C_6. But the following example is also C_6

Example 1.38

The Cyclic group of order 6 can be represented as

$$\langle x, y | x^2, y^3, [x, y]\rangle$$

Where $[x, y]$ is used to denote the commutator

Although this example seems super abstract and out of the blue, there is some logic to it. We first notice that since we have $[x, y]$ in our relation set, we have that x, y commute within our group. Then we may rewrite any element in G to

$$x^n y^m$$

where $n \in \{0, 1\}$ and $m \in \{0, 1, 2\}$. Then you may interpret x as a generator of C_2 and y as a generator of C_3. Then after taking their direct product it is trivial to see that it is isomorphic to C_6. Not the most rigorous explanation but this is how I understood it.

Remark 1.39. It is often convention to drop the " = 1" from the relation, so the above example could be also written as

$$\langle a | a^n \rangle$$

Exercise 1.40. Let p be a prime number, let $\langle S, R \rangle$ be a finite presentation for C_p. Prove or Disprove[1] : There exists proper subgroups $G < C_p$ and $H < C_p$ such that $GH = C_p$

Example 1.41

Let $A = \{a_1, a_2, \ldots\}$ be a set, then the free group $F(A)$ which is subject to $a_i^2 = 1$ is an *non-finitely presented* group

This should be obvious as both our generating set and relations set have infinite elements.

Remark 1.42. Sometimes finding a *finite presentation* of a finitely presented group is difficult, and sometimes algorithmically impossible

Furthermore we can create a group with representation by quotienting the free group of an alphabet by a set of relations

$$G := F(S)/Ker(\pi_S)$$

Then we have that $\langle S, R \rangle$ is a presentation of G. As a slight abuse of notation we simply write

$$G = \langle S, R \rangle$$

Remark 1.43. The generating set of groups is synonymous to a generating set for Vector Spaces

Definition 1.44 (Projection)**.** Let $G = \langle S, R \rangle$, then a word $w \in S$. projected into G is denoted $[w]$. Moreover we may defined projection equality by

$$[w] = [v] \qquad\qquad w \equiv_G v$$

Although a presentation seems rather abstract, it does have significant value. Because if G has presentation,$\langle S, R \rangle$, then

- every word in G can be written as the finite product $x_1 \ldots x_n$ with

$$x_i \in S \cup S^{-1} = \{s^{\pm 1} | s \in S\}$$

[1] I have no idea if this is true or not, just a fun question I thought of

- a word w in the alphabet $S \cup S^{-1}$ is equal to the identity element in G, $w \equiv_G 1$ if and only if the word w in $F(S)$ is equal a finite product conjugates of the words $r_i \in R$.

$$w = \prod_{i=1}^{m} u_i^{-1} r_i u_i$$

for $m \in \mathbb{N}, u_i \in F(S), r_i \in R$. I'm like 99% sure the book has a typo in it.

Lemma 1.45

If we have G has presentation $\langle S, R \rangle$, and H being a group. We have that $\psi : X \to H$ such that $\psi(r) = 1$ for all $r \in R$, then we may extend ψ to a group homomorphism $\psi : G \to H$

This was in the book, I have no idea what X is, so I can't really write a proof

Exercise 1.46. Show that the group $\bigoplus_{x \in X} \mathbb{Z}_2$ has presentation

$$\langle x \in X | x^2, [x, y], \forall x, y \in X \rangle$$

Proposition 1.47

if G has a finite presentation, $\langle S, R \rangle$. If $\langle X, T \rangle$ is an arbitrary presentation of G with X being finite, then there exists a subset of T, $T_0 \subseteq T$, such that $\langle X, T_0 \rangle$ is a presentation of G

Proof by reference to textbook! But this proposition can be reformulated regarding short exact sequences. Letting G be finitely presented and X to be finite. It follows that N is normally generated by $n_1, \ldots, n_k$ in the following short exact sequence

$$1 \to N \to F(X) \to G \to 1$$

We may further generalize to arbitrary short exact sequence

Lemma 1.48

Consider the short exact sequence

$$1 \to N \to K \to G \to 1$$

with K being finitely generated. then N is normally generated by finitely elements $n_1, \ldots, n_k \in N$

Proof by reference to textbook!

A list of important groups with finite presentation

Example 1.49 (Fundamental Group)

The Fundamental Group is presented by $\langle a_1, b_1, \ldots, a_n, b_n | [a_1, b_1], \ldots, [a_n, b_n] \rangle$

This group is only *somewhat* useful in Algebraic Topology

Example 1.50 (Right Angled Artin Group)

We first let $G = (V, E)$ be a finite graph, then the Right-angled Artin Graph is presented by $\langle V | [x_i, x_j]$ whenever $[x_i, x_j] \in E \rangle$

Ω1.3.1 Coxeter Graph

Definition 1.51 (Coxeter Graph)**.** Let $G = (V, E)$ be a simple finite graph. then for each edge $e \in E$, we may label it $m(e) \in \mathbb{Z}^+ \backslash 1$. Then we may call the pair

$$\Gamma := (G, m : E \to \mathbb{Z}^+ \backslash 1)$$

A *Coxeter Graph*

Example 1.52 (Coxeter Group)

Let Γ be a Coxeter Graph, then the corresponding Coxeter Group, C_Γ, is presented by

$$\left\langle x_i \in V | x_i^2, (x_i x_j)^{m(e)}, \text{whenever there exists an edge } e = [x_i, x_j] \right\rangle$$

Example 1.53 (Artin Groups)

I will add all these later!

Example 1.54 (Shephard Group)

I will add all these later!

Example 1.55 (Generalized von Dyck group)

I will add all these later!

Example 1.56 (Integer Heisenberg group)

I will add all these later!

Example 1.57 (Baumslag–Solitar group)
I will add all these later!

Up until now a lot of these Groups were defined combinatorically, meaning in terms of their presentations. But below we have several important classes of finitely presented groups which are defined *geometrically*

- $CAT(-1)$ Groups
  Groups G which act geometrically on $CAT(-1)$ metric spaces.
- $CAT(0)$ Groups
  Groups G which act geometrically on $CAT(0)$ metric spaces.
- Automatic groups
- Hyperbolic and relatively hyperbolic groups
  This will be covered in Chapter 11 of GGT, maybe we'll get there?
- Semihyperbolic groups

Theorem 1.58
Every finitely generated group is the fundamental group of a smooth closed manifold of dimension 4.

This is actually a crazy theorem, like who even came up with this? Are they math god?

Definition 1.59 (Laws in Groups)**.** An identity or Law is a non-trivial reduced word

$$w = w(x_1 \dots x_n)$$

in the letters $x_1, \dots, x_n$ and their inverses

I dont understand what this part means, gotta ask vivian!

Ω1.4 GGT Chapter 7.4 - The Rank of a Free Group Determines The Group

Proposition 1.60
Two free groups $F(X)$ and $F(Y)$ are isomorphic if X and Y have the same cardinality

Lemma 1.61
The quotient $\bar{F} := F \backslash N$ is isomorphic to $A = \mathbb{Z}_2^{\bigoplus X}$

The proofs for these two are kind of disgusting. But Proposition Proposition 1.58 implies that for every $n \in \mathbb{N}$, up to isomorphism, there exists only one Free Group of rank n. We use F_n to denote such a group

Corollary 1.62

For every $n \in \mathbb{N}$, we have that n is the smallest size for a generating set of F_n. Or rather bluntly, rank(F_n)=n

Theorem 1.63 (Nielsen–Schreier)

Any subgroup of a free group is a free group.

Ω1.5 GGT Chapter 7.5 - Free Constructions, Amalgams of Groups

The motivation for this Chapter is that **Amalgams** (amalgamated free products and HNN extensions) allow for us to build more complex groups when we're either given 2 groups, or a group and a pair of isomorphic subgroups

Definition 1.64 (Free Product)**.** We define the *free product* on groups $G_1 = \langle X_1|R_1\rangle$ and $G_2 = \langle X_2|R_2\rangle$. by the representation

$$G_1 \star G_2 = \langle G_1, G_2| \quad \rangle$$

Which is short hand for

$$\langle X_1 \sqcup X_2 | R_1 \sqcup R_2\rangle$$

Although this definition is fine, it isn't very flexible as we're restrained to dealing with the entire group.

Definition 1.65 (Amalgamated Free Product)**.** So we have a more general definition which requires us to take subgroups $H_1 \leqslant G_1$ and $H_2 \leqslant G_2$, along with an isomorphism $\phi : H_1 \to H_2$. Then we may define the *amalgamated free product* as

$$G_{1 \star H_1 \cong H_2} G_2 = \langle G_1, G_2 | \phi(h)h^{-1}, h \in H_1\rangle$$

the amalgamated free product is very similar to the "regular" free product. But in addition to the relations defined on G_1 and G_2, we further define the relation $\phi(h)h^{-1}$ for $h \in H_1$.

Definition 1.66 (HHN Extensions)**.** This is just a variation of the amalgamated free product, but with the extra hypothesis that $G_1 = G_2$. So let G be a group, H be a subgroup, and $\phi : H \to G$ be a monomorphism, then the *HNN Extension* of G via ϕ is defined as

$$G_{\star H,\phi} = \langle G, t | tht^{-1} = \phi(h), \forall h \in H\rangle$$

Where t is just some new generator for the group.

Again we may further generalize this notion of extension by considering multiple subgroups and multiple isomorphic embeddings

Definition 1.67 (Simultaneous HNN Extension)**.** Let J be an indexing set, then suppose that we are given a collection of subgroups H_j of G and isomorphic embeddings $\phi_j : H_j \to G$. Then we define the *simultaneous HNN extension* of G to be the group

$$G\star_{\phi_j:H_j\to G, j\in J} = \langle G, t_j, j \in J | t_j h t_j^{-1} = \phi_j(h), \forall h \in H_j, j \in J\rangle$$ [2]

For this next part we shift our focus onto **Graphs of Groups**. We no longer assume for our graphs to be simple, but we still assume that they are connected.

Definition 1.68 (Graph of Groups)**.** Let Γ be Graph. assign to each vertex, v, in Γ a group G_v; similarly assign each edge, e, to a group G_e. Then we may *orient* each edge, e, so that it's head is e_+ and tail is e_- (Directing each edge). Furthermore suppose that for each edge, e, there exists monomorphisms[3]

$$\phi_{e_+} : G_e \to G_{e_+} \qquad\qquad \phi_{e_-} : G_e \to G_{e_-}$$

Then we say that the oriented graph Γ together with the collection of vertex and edge groups, along with the monomorphism $G_{e_\pm}$ is called the *graph of groups*, $\mathcal{G}$, based on Γ.

[2] This might be the most disgusting line of latex that I have ever written

[3] You can drop the injectivity requirement, but this won't be explored in this reading

Ω2 Day 2

We had our meeting today and Sophia presented. We spent a lot of time proving that finitely generated groups were countable. And that Frattini Groups were actually Groups. We didn't end up proving it in the end, it's actually hard. But this week's readings are all of Chapter 15 in Office Hours. I will be presenting!

Ω2.0.1 Finitely Generated Groups are Countable

I already proved this

Ω2.0.2 Frattini Group

Let $\mathbb{F}$ be the Frattini subgroup of a group G. It is immediate that the group identity exists in $\mathbb{F}$. Now let $x, y \in \mathbb{F}$ be arbitrary, but then to obtain a contradiction suppose that $xy \notin \mathbb{F}$. Then it follows that there exists a generating set S such that $S \backslash xy$ doesn't generate G. But then we may notice that $(S \backslash xy) \cup \{x, y\}$ necessarily generates G. Then since we have that x, y are non generators, we must also have that $[(S \backslash xy) \cup \{x, y\}] \backslash \{x, y\} = S \backslash xy$ Generates G. But this contradicts that it doesn't generate G, so it must follow that xy is also a non generator.[4]

Ω2.1 Preamble

The idea behind the Lamplighter Group is that you should imagine a a town which has a road with infinite unlit street lamps. Then there's a guy, the lamplighter, that goes around to a finite number of lamps then lights them up/puts them out. Then this Lamplighter ends his journey by leaning on some streetlamp. We actually have that this situation is a group, call it L_2.

To show this we're gonna need to make this notion a bit more mathematically precise, instead of an infinite chain of Lamps, we'll instead take a copy of the integer line, with a lamp placed at every integer. Then we may say that an element in L_2 consists of tuples. The first element in the tuple consists of a finite set of integers, these represent the lamps that are illuminated, then the second element will be the terminal position of the lamplighter.

Example 2.1

$(\{-3, -1, 2, 6, 14\}, 67)$ represents the situation where the lamps placed at $-3, -1, 2, 6, 14$ are illuminated, with the lamplighter being at the lamp at integer 67.

There is another interpretation as we since each lamp is either on or off, we may better represent it by using elements from $\mathbb{Z}/2\mathbb{Z}$. Such that 1 implies that a lamp is on and 0 if it is off. We can present this data as an element of the group $\bigoplus_{i=-\infty}^{\infty} (\mathbb{Z}/2\mathbb{Z})_i$. This

[4] I'm not sure if this proof is actually correct, but Vivian is ghosting me on whether it is or not :(

is understood to be the infinite direct sum of copies of $\mathbb{Z}/2\mathbb{Z}$, where each element in this group is an infinite tuple, which can be indexed by integers, with the further condition that a finite number of entries are 1. Then you may realize that the Lamplighter group can be constructed using $\mathbb{Z}$ and (the much larger group) $\bigoplus_{i=-\infty}^{\infty}(\mathbb{Z}/2\mathbb{Z})_i$. Where the integer represents the final position of the lamplighter, and the infinite tuple represents which lamps are illuminated. But enough with the hand waivy math stuff.

Ω2.2 Office Hours Chapter 15.1 - Generators and Relators

Definition 2.2. We may define the lamplighter group as

$$L_2 = \langle a, t | a^2 \rangle$$

The intuition behind this representation is that the letter t represents when the lamplighter moves the right by 1 unit, t^{-1} for moving left. And a is to toggle the lamp at his current position. Multiplying two elements g and h in L_2 is just concattenating the two together forming gh. Multiplying two elements together using this model is pretty easy, but multiplying them with the first model discussed is a bit more challenging.

Exercise 2.3. Describe how to multiply group elements when they are represented as a pair (k, x) where $k \in \mathbb{Z}$ and $x \in \bigoplus(\mathbb{Z}/2\mathbb{Z})_i$, that is, give a formula for the product $(k, x) \cdot (l, y)$.

Group Presentation

We defined the relation set to be $a^2 = 1$ in L_2, this is to be interpretted as the lamplighter toggling the lamp, then toggling it again, which obviously does nothing. But we may complete the presentation by adding on an infinite collections of relations. We first imagine that the lamplighter starts at the origin, $0 \in \mathbb{Z}$. Then they move j spaces, turns on that light, then returns to the origin, this action is $t^j at^{-j}$. Then the lamplighter moves $k \neq j$ spaces, turns on that light then returns to the origin, again this action is $t^k at^{-k}$. Now we have that the lamps at j, k are both on, and the lamplighter is at the origin. We notice that these two actions may commute, as it doesn't matter if the lamplighter turns on the j lamp or the k lamp first. So we necssarily have

$$[t^j at^{-j}, t^k at^{-k}] \text{ for all } j, k \in \mathbb{Z} \text{ in our relations set}$$

Note in this group we have that the identity element is the lamplighter at the origin with no lamps lit up[5]. Thus we arrive at the final presentation

$$L_2{}^{6} = \langle a, t | a^2, [t^j at^{-j}, t^k at^{-k}] \text{ for all } j, k, \in \mathbb{Z}\rangle^{7}$$

[5] I'll be referring to this element as the Identity State
[6] Notice the 2 is used to signify that a lamp has 2 states, either on or off,
[7] This is an example of a non-finitely presented group

L_2 as Matrices

The idea behind this is to first encode a state of lit lamps into a polynomial in t^i and t^{-i} with coefficients in $\mathbb{Z}/2\mathbb{Z}$, where the only non-zero coefficients correspond to the i^{th} lamp being lit.

Example 2.4

Consider

$$(\ldots, 0, 0, 0, 1, \underbrace{1}_{0}, 0, 1, 1, 1, 0, 0, \ldots)$$

Where the underbrace under the 1 is the 0^{th} index. The corresponding polynomial to this state is

$$t^{-1} + 1 + t^2 + t^3 + t^4$$

Now we may think of an element in L_2 as a tuple, (P, k), where P is some polynomial of t^i and t^{-i}, with coefficients in $\mathbb{Z}/2\mathbb{Z}$, such that finitely many of them are 1. and an integer k. k is used to denote the position of the lamplighter, and P is used to denote the state of the lamps. Then we may multiply two elements, (P_1, k_1) and (P_2, k_2) from L_2 by

$$\begin{pmatrix} t^{k_1} & P_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{k_2} & P_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t^{k_1+k_2} & P_2 t^{k_1} + P_1 \\ 0 & 1 \end{pmatrix}$$

A Family of Groups

We can generalize L_2 to more than 2 states for each lamp. So we can generalize an element of L_n to an element from $\bigoplus_{i=-\infty}^{\infty} (\mathbb{Z}/n\mathbb{Z})_i$ and an integer k to denote the final position of the lamplighter. Then we also have that the presentation of L_n is as follow

Definition 2.5. We define a more generalized version of the Lamplighter Group, where each lamp has n states instead of 2

$$L_n = \langle a, t | a^n, [t^j a t^{-j}, t^k a t^{-k}] \text{ for all } j, k, \in \mathbb{Z} \rangle$$

Ω2.3 Office Hours Chapter 15.2 - Computing Word Length

The goal for this next section will be to give an explicit formula for the word lengths of elements within L_2. But right now we have some ambiguity with regards to the words associated with an element of L_2.

Example 2.6

If we have the lamps at -1,1,2 turned on, then the following two words are suitable to reach this state, assuming the lamplighter starts at the origin

- $tat^{-2}at^3$
- $t^{-1}at^2ata$

Where the first word has length 8, and the latter has length 7

But an efficient path of the lamplighter can be constructed for any element of L_2 using the following algorithm

Definition 2.7 (Efficient Algorithm).

1. Move the lamplighter to the smallest non-negative integer lamps and turn it on.
2. Keep on moving to the right until all non-negative integer lamps have been turned on
3. When all non-negative integer lamps have been turned on, have the lamplighter move to the origin, 0
4. Now have the lamplighter move left, illuminating the desired lamps as he comes across them
5. When all the desired negative-integer lamps have been illuminated, have the lamplighter move to his final position

We use $\gamma : L_2 \to \mathbb{Z}$ to denote the length of word created by this procedure

We have an analagous procedure where we instead illuminate the negative indexed bulbs, then the positive one. We use $\gamma'(n)$ to denote the length of the word constructed this way.

Exercise 2.8. When are γ' and γ both geodesics?

When the final position is 0. First we let m be the index of the largest positive lamp, and n be the index of the smallest negative lamp. But the idea is basically that γ illuminates all the positive bulbs in m "moves", then returns to the origin with another m, then lights up the negative bulbs in n moves, then returning the origin in an additional n moves. So in total γ did $2n + 2m$ moves. Then synonymously for γ', it lights up the negative bulbs and returns to the origin in $2n$ moves. Then it further lights up the positive bulbs then returns to the origin in $2m$ moves.

Exercise 2.9. When is only γ a geodesic? When is only γ' a geodesic?

γ is a geodesic when the ending position is negative. Using the same variables as above, and letting $k \in \mathbb{Z}^+$ be the distance between the origin and the ending position. γ follows the following procedure

1. Start at the origin then move $2m$ spaces to illuminate the positive bulbs then returning to the origin
2. Move n spaces to illuminate
3. moving $n-k$ spaces to the final position

So we have that γ constructs a word of length $2m+n+n-k = 2m+2n-k$. Then we consider the length of the word by γ'.

1. Start at the origin then move $2n$ spaces to illuminate the negative bulbs then returning to the origin
2. then move $2m$ spaces to illuminate the positive bulbs then returning to the origin
3. then moving k moves to the ending position

So we have that γ' constructs a word of length $2m+2n+k$. Then we have that $2m+2n-k < 2m+2n+k$[8], so we have that γ is a geodesic when the ending position is negative. $\square$

Writing Down the Efficient Paths

We begin by letting $a_k = t^k a^{-k}$ denote the conjugate of a by t^k. We start by considering the identity state, then we notice that a_k is the action of moving k spaces, turning on the lamp, then returning to the origin. We further notice that

- all a_i commutes for all $i \in \mathbb{Z}$
- Occurences of a_k in the product
$$\prod_{i=0}^{m} a_{b_i}$$
cancel in pairs. This corresponds to moving k spaces, then toggling the lamp, then returning to the origin, then you do that again.

Then we may further notice that when we multiply these conjugates together, in increasing order, say a_2a_7, this ensures that the lamplighter moves to 2, lights it up, then moves to 7. This is evident from the following

$$a_2a_7 = t^2at^{-2}t^7at^{-7} = t^2at^5at^{-7}$$

[8]You might've noticed that I haven't been counting the number of illuminating actions, a. We may actually disregard these letters because since γ and γ' represent the same state, they'll have the same amount of lamps illuminated. Thus same amount of a

Thus if we have that an element $g \in L_2$, which illumated bulbs at $i_1 < i_2 < \ldots < i_n$ and $-j_1 > -j_2 > \ldots > -j_m$ with $j_1 > 0$[9] and the lamplighter ending at position k, we may write g as:

$$g = a_{i_1} a_{i_2} \ldots a_{i_n} a_{-j_1} a_{-j_2} \ldots a_{-j_m} t^k$$

Computing Word Length of Group Elements

Definition 2.10. Given $g \in L_2$, then write down $\gamma(g)$ using the a_k notation, further let $g = a_{i_1} a_{i_2} \ldots a_{i_n} a_{-j_1} a_{-j_2} \ldots a_{-j_m} t^k$. We call the length of the reduction of this word $D(g)$

Theorem 2.11 (Taback-Cleary)
For $g = a_{i_1} a_{i_2} \ldots a_{i_k} a_{-j_1} a_{-j_2} \ldots a_{-j_l} t^m \in L_2$

$$D(g) = k + l + min\{2j_l + i_k + |m - i_k|, 2i_k + j_l + |m + j_l|\}$$

The intuition behind this theorem took me a bit. Each term is important.

- $k + l$ is the total number of illuminated bulbs
- $2j_l + i_k + |m - i_k|$ is when the ending position m is positive
- $2i_k + j_l + |m + j_l|$ is when the ending position m is negative

Ω2.4 Office Hours Chapter 15.3 - Dead End Elements

We now consider some geometric properites regarding the Cayley Graph with respect to the generating set $\{a, t\}$.

Dead End Elements

Imagine you're going on walk away from your house. and then you reach a point where regardless of which path you take, you'll be closer to your house. Synonymously, imagine you're walking on the vertices of a Cayley Graph by traversing it's edges, away from the identity. Then you reach a vertex where all edges either take you closer to the identity, or maintain your current distance. A natural question is if this is even possible, which it is. And these vertices are called dead end elements. Another perspective is considering the geodesic path between this element and the identity, if this geodesic can be extended and remain a geodesic, then it is not a dead end element. If it cannot be extended while still preserving "geodesic-ness" then it is a dead end element.

[9] I think this is a typo in the book

Example 2.12

Consider the Cayley Graph of $\mathbb{Z}$ with respect to the generating set $\{2,3\}$. We use $d(a,b)$ as a distance function between integers a and b. We first notice

- $d(0,1) = 2$, Since $3 - 2 = 1$
- $d(0,2) = 1$
- $d(0,3) = 1$

Then we claim that 1 is a dead end element. So it suffices to check that the 4 vertices (-2,-1,3,4) connected to 1 have distance of 2 or less from 0.

- $d(0,-2) = 1$
- $d(0,-1) = 2$ since $2 - 3 = -1$
- $d(0,3) = 1$
- $d(0,4) = 2$ since $2 + 2 = 4$

Thus we have that 1 is a dead end element

Backtracking from Dead End Elements

An intuitive understanding for this part is to imagine yourself in a maze, more precisely that you're stuck in a dead end. In such a case you would have to backtrack your steps before your on a path that does lead you to the end.

More mathematically suppose that $g \in G$ is a dead end element, and its word length with respect to a generating set S is n, that is $|g|_S = n$

Definition 2.13 (Depth)**.** We say a word of length n, g, has *depth* k if the shortest path from to any group element of length $n+1$ is of length $k+1$.

Intuitively $k+1$ is the minimum number of edges you must traverse in the Cayley Graph in order to reach an element that is farther than g from the identity. Relating this back to the lamplighter group, we have the following theorem

Theorem 2.14

The lamplighter group L_2 contains dead end elements of arbitrary depth with respect to the generating set $\{t, a\}$.

We first find a dead-end element within L_2. We first consider

$$d_m = a_0 a_1 \dots a_m a_{-1} a_{-2} \dots a_{-m}$$

[10] Then we further notice that d_m has word length $m + m + 4m + 1 = 6m + 1$. Then we have that our efficient path to d_m also has this length, thus it must be a geodesic. So now it suffices to show that $d_m a, d_m t, d_m t^{-1}$ are all closer to the identity than the d_m.

- $d_m a$

  The difference between this and d_m is that the bulb at the origin isn't lit, thus we may effectively drop the a_0 term in the definition of d_m to reach $d_m a$. Thus we have a shorter word

- $d_m t$

  The difference is that the lamplighter finishes at index 1, instead of the origin. So then we may consider the following, since a_k commute

  $$d_{-m}t = a_0 a_{-1} \dots a_{-m} a_1 a_2 \dots a_m t = a_0 a_{-1} \dots a_{-m} a_1 a_2 \dots t^m a t^{-m+1}$$

  Which is lesser length than d_m

- $d_m t^{-1}$

  We directly compute

  $$d_m = a_0 a_1 \dots a_m a_{-1} a_{-2} \dots a_{-m} t^{-1} = a_0 a_1 \dots a_m a_{-1} a_{-2} \dots t^{-m} a t^{m-1}$$

  Which is lesser length than d_m

So in any case we have that d_m is a dead end element.

Ω2.5 Office Hours Chapter 15.4 - Geometry of the Cayley Graph

Now onto some more interesting stuff. You might've noticed that I've thrown the word "Cayley Graph" around[11], but we haven't actually done anything with the Cayley Graph of L_2 yet. But to make our Cayley Graph beautiful we will change our generating set a bit

Definition 2.15 (Electricity Free)**.** The generating set $\{a, at, a^2t, \dots, a^{n-1}\}$ is the *Electricity Free* generating set, specifically $\{a, at\}$ is the electricity free generating set for L_2.

We notice that $(at)^{-1} = t^{-1}a^{-1}$, as this element satisfies being the left/right inverse to at.

[10] All the lamps within an m radius are lit

[11] There isn't a definition for this, but I gave a hyperlink :)

Exercise 2.16. Describe an algorithm to construct efficient paths $\eta(g)$ and $\eta'(g)$ using the electricity-free generating set $\{t, at\}$.

The idea here is that you spam t until the lamplighter is on an index which should be on, then you do at. Repeat this until all the positive indices are illuminated. Then return to the origin, and repeat for the negative indices. Then go to the final position

Exercise 2.17. Can you use these new paths $\eta(g)$ and $\eta'(g)$ to compute the word length of $g \in L_2$ with respect to $\{t, at\}$?

If the smallest negative index is m and the largest positive index is n, and the lamplighter's final position is k. then the distance function is given by

$$D(g) = \min\{2m + n + |n + k|, 2n + m + |m - k|\}$$

Diestel–Leader graphs

We actually have that the graphs generated by $\{a, at\}$ are pretty special, special enough to get their own name, the *Diestel-Leader graphs.* These graphs were first introduced to the question

Is every "nice" infinite graph Quasi-Isometric to the Cayley Graph of some finitely generated group?

Although "nice" is rather ambiguous, the most notable qualities are that the graph is connected and every vertex has equal degree.

Definition 2.18 (Diestel-Leader Graphs)**.** First consider two infinite trees[12] with fixed valence[13] of $d + 1$, call them T_1 and T_2. Then we orient each tree such that each vertex has 1 incoming edge and d outgoing edges[14]. Now fix a height function $h_i : T_i \to \mathbb{Z}$ for $i = 1, 2$. This function fixes a level on each tree by first identifying a vertex as level 0. Then every edge directed away from a vertex of height d leads to a vertex of height $d + 1$. And the edge that ends in height d starts with a height of $d - 1$. Then we let $T_1 \times T_2$ be the product of trees T_1 and T_2. So the ordered pair $(t_1, t_2) \in T_1 \times T_2$, for $t_1 \in T_1$ and $t_2 \in T_2$. Then we have that the Diestel-Leader Graph is a special subset of this product of trees. Furthermore we use $DL_2(d)$ to denote this special subset.

[12] A graph with no cycles
[13] degree
[14] We don't have a root in this tree

Vertices in the Diestel-Leader graph $DL_2(d)$

We have that the Vertices $DL_2(d)$ are the elements in $T_1 \times T_2$ for which the sum of the heights of both elements is 0. More formally

$$\{(t_1, t_2) | t_1 \in T_2 \text{ and } h_1(t_1) + h_2(t_2) = 0\}$$

An intuitive visualization for this is to draw T_1 such that the height is increasing in one direction, then to draw T_2 such that the height is decreasing in that same direction, while making sure to identify height d in T_1 with heigh $-d$ in T_2. The book has a pretty nice visual for how to do this.

Edges in the Diestel–Leader graph $DL_2(d)$

Thus we say (t_1, s_1) is connected to (t_2, s_2) if t_1 is connected to t_2 in T_1 and s_1 is connected to s_2 in T_2. But then further notice that by the height-sum requirement on the vertices, we must have that one of two things happen between two connected vertices

- increasing height by 1 in T_1 while decreasing height by 1 in T_2
- decreasing height by 1 in T_1 while increasing height by 1 in T_2

We may then further notice that that for any given vertex in $d \in DL_2(d)$, we have that exactly $2d$ vertices are connected by 1 edge to d.

Vertices in $\mathbf{DL_2(2)}$ as elements of $\mathbf{L_2}$

We now show that the Cayley Graph $\Gamma(L_2, \{a, at\})$ is the Diestel-Leader Graph $DL_2(2)$. To do this we must show a way of associating a vertex in $DL_2(2)$ with an element of L_2 while still preserving the overall structure. We may do this as follows

1. Consider a single infinite tree of valence 3[15] with a fixed height function. then we have that each vertex in this tree has 1 incoming edge, and 2 outgoing edges. Label the left outgoing edge with 0, and the right outgoing edge with 1.
2. Then pick some vertex, v, in our tree such that there is a unique downwards path, with each "step" decreasing the height by 1. Then we may read the labels off of the edges we traverse and concatenate them into a string

   $$\mathcal{A}_1 = a_1 a_2 a_3 a_4 \ldots$$

   where each $a_i = 0, 1$ and $a_j = 0$ for all j greater than some constant

[15] For the CS people, an infinite complete binary tree

3. Conversely, given some integer k and a string $\mathcal{A}_2 = a_1a_2a_3a_4\dots$ where each $a_i = 0, 1$ and $a_j = 0$ for all j greater than some constant. We may find some vertex, w, at height k such that a path downwawrds from w is $\mathcal{A}$. Furthermore, we have that this vertex is unique

4. Then to describe any vertex of $DL_2(2)$, all we need is two strings $\mathcal{A}_1$ and $\mathcal{A}_2$, each of which is composed of elements only from $\{0, 1\}$, and such that the entries are 0 past a certain point. And an integer k. The first string, $\mathcal{A}_1$ is used to describe a vertex at height k in T_1 and $\mathcal{A}_2$ is used to describe a vertex at height $-k$ in T_2

DL₂(2) as a Cayley graph of L₂

Now we will exhibit a bijection between the vertices of $DL_2(2)$ and the elements of L_2. The idea is to consider the tuple $(\mathcal{A}_1, \mathcal{A}_2, k)$ where k is used to denote the position of the lamplighter. We then construct a polynomial of variables t and t^{-1} with coefficients in $\mathbb{Z}/2\mathbb{Z}$. Then we let the sequence $\{a_i\}$ be the coeffiecient to t^i. We then divide $\{a_i\}$ into two seperate sequences : $\mathcal{A}_1 = \{a_i\}_{i<k}$, and $\mathcal{A}_2 = \{a_i\}_{i\geq k}$. Where the former sequence must be read in reverse order. We also require that both sequences are eventually constantly 0.

Remark 2.19. The book gave the example of

- $\mathcal{A}_1 = (1, 0, 1, 0, 0, \dots)$
- $\mathcal{A}_2 = (1, 0, 0, 0, \dots)$
- $k = 2$

and had this element correspond to the matrix

$$\begin{pmatrix} t^2 & t^{-1} + t + t^2 \\ 0 & 1 \end{pmatrix}$$

But the idea is that starting from the $k'th$ position, start reading entries off of $\mathcal{A}_2$, such that the current position is the first entry, then continue reading them as you go right. And you read $\mathcal{A}_1$ in a similar fashion but travel left and start with the lamp directly left to the position.

You finish the proof that $DL_2(2)$ is the Cayley Graph of L_2 with an electric-free generating set is completed by showing that this identification extends to a graph isomorphism between the two graphs

Moving Around in This Cayley Graph

The goal is to move around this Cayley Graph as easily as we can move around $\Gamma(\mathbb{Z} \times \mathbb{Z}, \{(1,0),(0,1)\})$, hint : we can move around this integer lattice really easily. For $DL_2(2)$ we would like to develop a similar intuitive sense for movement. And how vertices who differ by a single edge differ in their $(\mathcal{A}_1, \mathcal{A}_2, k)$ coordinate. Specifically if we're at $(\mathcal{A}_1, \mathcal{A}_2, k)$ and wish to travel on the edge labelled with "t", what is the coordinate of the vertex we end up at? To properly describe this we may convert $(\mathcal{A}_1, \mathcal{A}_2, k)$ into matrix form, as well as the action of t.

$$\begin{pmatrix} t^k & P \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t^{k+1} & P \\ 0 & 1 \end{pmatrix}$$

Although it is immediate to see how this affects the matrix representation, what about the representation in $DL_2(2)$? It is immediate to see that the coordinate of the lamplighter moved from k to $k+1$. Then we obtain $(\mathcal{B}_1, \mathcal{B}_2, k+1)$ as our new coordinate, where

- $\mathcal{B}_1 = \{a_i\}_{i<k+1}$
- $\mathcal{B}_2 = \{a_i\}_{i \geq k+1}$

Now we have a precise difference between an element g and gt, and we are able to locate the latter given we know where the former is, and we can do it as follows :

- In T_1, we may proceed along the edge labeled a_k to a new vertex that is at height $k+1$
- In T_2, Simply remove the initial edge in the path, so that the new truncated path begins at a vertex of height $-(k+1)$

Thus we have that this new vertex corresponds to $(\mathcal{B}_1, \mathcal{B}_2, k+1)$ and is connected to $(\mathcal{A}_1, \mathcal{A}_2, k)$ by one edge. Moving along an electricity-free generated graph is synonymous, given some g in the electricity free graph, we may multiply it by at to get

$$g(at) = \begin{pmatrix} t^{k+1} & P + t^k \\ 0 & 1 \end{pmatrix}$$

After a similar argument as before, we have that the vertex gat has the same T_2 coordinate as gt, and it shares the same parent vertex in T_1. We also have that the lamp at t^k has been turned on, this only changes $\mathcal{B}_1$ by adding 1 onto it's first entry[16], and it's subsequent entries are exactly $\mathcal{A}_1$

[16]Mod 2 of course

Ω2.6 Office Hours Chapter 15.5 - Generalizations

Wreath Product

We actually have that the lamplighter group, L_2, is the wreath product between $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}$. and it is denoted as $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$. We actually have that the wreath product is a very general construction

Definition 2.20. Let F be a finite group and let G be a finitely generated group. Then we have that the elements in $F \wr G$ comprises of a finite collection of elements from F, analagous to the state of the lamps, and an element from G, analagous to the final placement of the lamplighter.

We may actually further extend this notion by replacing F with any finitely generated group

Lamplighters and Traveling Salesmen

Although it is super easy to find efficient paths in L_2, we may further consider infinite lamps on the place, specifically $\mathbb{Z}/2\mathbb{Z} \wr (\mathbb{Z} \times \mathbb{Z})$. So say we're given an element of this group, $g = ((x_1, y_1,), \ldots, (x_n, y_n))$ and a final position (x, y). To find a minimal lengthed word with respect to some generating set, we must find a minimal path that visit all of these $n+1$ vertices. This is actually a very hard problem

> **Remark 2.21** (Traveling Salesman Problem)**.** The *Traveling Salesman Problem* is as follows. Given a list of cities and a map (so that you know how far apart the cities are), determine a minimal path for a salesman who must visit each city exactly once. This question is very hard